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PERRON–FROBENIUS THEOREM FOR NONNEGATIVE TENSORS*

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Abstract. We generalize the Perron–Frobenius Theorem for nonnegative matrices to the class of nonnegative tensors.

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1. Introduction

The Perron–Frobenius Theorem is a fundamental result for nonnegative matrices. It has numerous applications, not only in many branches of mathematics, such as Markov chains, graph theory, game theory, and numerical analysis, but in various fields of science and technology, e.g., economics, operational research, and recently, page rank in the internet, as well. Its infinite dimensional extension is known as the Krein Rutman Theorem for positive linear compact operators, which has also been widely applied to Partial Differential Equations, Fixed Point Theory, and Functional Analysis.

In late studies of numerical multilinear algebra [7, 4, 1], eigenvalue problems for tensors have been brought to special attention. In particular, the Perron–Frobenius Theorem for nonnegative tensors is related to measuring higher order connectivity in linked objects [5] and hypergraphs [6]. The purpose of this paper is to extend Perron–Frobenius Theorem to nonnegative tensors. It is well known that Perron–Frobenius Theorem has the following two forms:

THEOREM 1.1. *(Weak Form) If A is a nonnegative square matrix, then*

1. *$r(A)$, the spectral radius of A, is an eigenvalue.*
2. *There exists a nonnegative vector $x_0 \neq 0$ such that*

$$Ax_0 = r(A)x_0. \tag{1.1.1}$$

We recall the following definition of irreducibility of A: a square matrix A is said to be reducible if it can be placed into block upper-triangular form by simultaneous row/column permutations. A square matrix that is not reducible is said to be irreducible.

THEOREM 1.2. *(Strong Form) If A is an irreducible nonnegative square matrix, then*

1. *$r(A) > 0$ is an eigenvalue,*
2. *there exists a nonnegative vector $x_0 > 0$, i.e., where all components of x_0 are positive, such that $Ax_0 = r(A)x_0$,*

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3. *(uniqueness) if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = r(A)$,*
4. *$r(A)$ is a simple eigenvalue of A, and*
5. *if λ is an eigenvalue of A, then $|\lambda| \le r(A)$.*

We shall extend these results to nonnegative tensors. But first, let us recall some definitions on tensors. An order-m n-dimensional tensor $\mathcal{C}$ is a set of n^m real entries

$$\mathcal{C} = (c_{i_1 \cdots i_m}), \quad c_{i_1 \cdots i_m} \in \mathbb{R}, \quad 1 \le i_1, \ldots, i_m \le n. \tag{1.2.1}$$

$\mathcal{C}$ is called nonnegative (or, respectively, positive) if $c_{i_1 \cdots i_m} \ge 0$ (or, respectively, $c_{i_1 \cdots i_m} > 0$). To an n-vector $x = (x_1, \ldots, x_n)$, real or complex, we define an n-vector:

$$\mathcal{C}x^{m-1} := \left(\sum_{i_2, \ldots, i_m = 1}^{n} c_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \le i \le n}. \tag{1.2.2}$$

Suppose $\mathcal{C}x^{m-1} \neq 0$; a pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue and an eigenvector if they satisfy

$$\mathcal{C}x^{m-1} = \lambda x^{[m-1]}, \tag{1.2.3}$$

where $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})$. When m is even and $\mathcal{C}$ is symmetric, this was introduced by Qi [7]; when m is odd, Lim [4] used $(x_1^{m-1} \operatorname{sgn} x_1, \ldots, x_n^{m-1} \operatorname{sgn} x_n)$ on the right-hand side instead, and the notion has been generalized by Chang, Pearson, and Zhang [1].

Unlike matrices, the eigenvalue problem for tensors is nonlinear, involving finding nontrivial solutions of polynomial systems in several variables. This feature requires us to employ different methods in generalizations.

The main results of this paper are stated as follows:

THEOREM 1.3. *If $\mathcal{A}$ is a nonnegative tensor of order m and dimension n, then there exist $\lambda_0 \ge 0$ and a nonnegative vector $x_0 \neq 0$ such that*

$$\mathcal{A}x^{m-1} = \lambda_0 x_0^{[m-1]}. \tag{1.3.1}$$

THEOREM 1.4. *If $\mathcal{A}$ is an irreducible nonnegative tensor of order m and dimension n, then the pair (λ_0, x_0) in Equation 1.3.1 satisfies*

1. *$\lambda_0 > 0$ is an eigenvalue.*
2. *$x_0 > 0$, i.e., all components of x_0 are positive.*
3. *If λ is an eigenvalue with nonnegative eigenvector, then $\lambda = \lambda_0$. Moreover, the nonnegative eigenvector is unique up to a multiplicative constant.*
4. *If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \le \lambda_0$.*

However, unlike matrices, such λ_0 is not necessarily a simple eigenvalue for tensors in general. We shall present an example to demostrate this distinction. Furthermore, some additional conditions will be imposed to ensure the simplicity of the eigenvalue λ_0.

In the paper of Lim [4], some of the above conclusions in Theorem 1.4 were obtained. Here, however, we shall study this problem more systematically in a more self-contained manner via a different approach.

We organize our paper as follows: Section 2 is devoted to proving the main theorems except (4) of Theorem 1.4. In Section 3, we discuss the simplicity of λ_0. In Section 4, we study an extended version of Collatz's minimax theorem, from which assertion (4) of Theorem 1.4 will follow as a direct consequence. In the last section, Section 5, various extensions of the main results will be given.

2. Proofs of the main theorems

Let $X=\mathbb{R}^n$. Then X has a positive cone $P=\{(x_1,\dots,x_n)\in X \mid x_i\geq 0, 1\leq i\leq n\}$. The interior of P is denoted $\operatorname{int}P=\{(x_1,\dots,x_n)\in P \mid x_i>0, 1\leq i\leq n\}$. An order is induced by P: $\forall x,y\in X$, we define $x\leq y$ if $y-x\in P$, and $x<y$ if $x\leq y$ and $x\neq y$.

An order-m tensor $\mathcal{C}$ is hence associated with a nonlinear $(m-1)$ homogeneous operator $C:X\to X$ by $Cx=\mathcal{C}x^{m-1}$, $\forall x\in X$, i.e.,

$$C(tx)=t^{m-1}Cx, \quad \forall x\in X, \quad \forall t\in\mathbb{R}^1. \tag{2.1.1}$$

It is obvious that if $\mathcal{C}$ is nonnegative (or, respectively, positive), i.e, all entries are nonnegative (or, respectively, positive), then the associated nonlinear operator is $C: P\to P$ (or $C:P\setminus\{0\}\to\operatorname{int}P$). Moreover, if $\mathcal{C}$ is nonnegative, then

$$Cx\leq Cy, \quad \forall x\leq y, \quad \forall x,y\in P. \tag{2.1.2}$$

We are now ready for the proof of Theorem 1.3.

Proof. We reduce the problem to a fixed-point problem as follows. Let $D=\{(x_1,\dots,x_n)\in X \mid x_i\geq 0, 1\leq i\leq n, \sum_{i=1}^n x_i=1\}$ be a closed convex set. One may assume $\mathcal{A}x^{m-1}\neq 0$ $\forall x\in D$. For otherwise, there exists at least one $x_0\in D$ so that $\mathcal{A}x_0^{m-1}=0$. Let $\lambda_0=0$; then (λ_0,x_0) is a solution to 1.2.3, and we are done. Then the following map $F:D\to D$ is well defined:

$$F(x)_i=\frac{(\mathcal{A}x^{m-1})_i^{\frac{1}{m-1}}}{\sum_{j=1}^n(\mathcal{A}x^{m-1})_j^{\frac{1}{m-1}}}, \qquad 1\leq i\leq n, \tag{2.1.3}$$

where $(\mathcal{A}x^{m-1})_i$ is the i-th component of $\mathcal{A}x^{m-1}$. $F:D\to D$ is clearly continuous. According to the Brouwer's Fixed Point Theorem, $\exists\, x_0\in D$ such that $F(x_0)=x_0$, i.e.,

$$\mathcal{A}x_0^{m-1}=\lambda_0 x_0^{[m-1]},$$

where

$$\lambda_0=\left(\sum_{j=1}^n(\mathcal{A}x_0^{m-1})_j^{\frac{1}{m-1}}\right)^{m-1}. \tag{2.1.4}$$

□

We now turn to Theorem 1.4. If $\mathcal{A}$ is positive then we can use similar arguments used in positive matrices to establish conclusions (1) – (3) in Theorem 1.4 based on Theorem 1.3.

Our purpose in the remaining of this section is to introduce a condition on tensors which lies in between positivity and nonnegativity to ensure that similar results hold as in the Perron–Frobenius Theorem for matrices.

DEFINITION 2.1. *(Reducibility) A tensor $\mathcal{C}=(c_{i_1\cdots i_m})$ of order m and dimension n is called reducible if there exists a nonempty proper index subset $I\subset\{1,\dots,n\}$ such that*

$$c_{i_1\cdots i_m}=0, \quad \forall i_1\in I, \quad \forall i_2,\dots,i_m\notin I.$$

If $\mathcal{C}$ is not reducible, then we call $\mathcal{C}$ irreducible.

LEMMA 2.2. *If a nonnegative tensor $\mathcal{C}$ of order m and dimension n is irreducible, then*

$$\sum_{i_2,\ldots,i_m=1}^{n} c_{ii_2\cdots i_m} > 0, \quad \forall 1 \le i \le n.$$

Proof. Suppose not, then there exists i_0 so that $\sum_{i_2,\ldots,i_m=1}^{n} c_{i_0 i_2\cdots i_m} = 0$. Since $\mathcal{C}$ is nonnegative, $c_{i_0 i_2\cdots i_m} = 0\ \forall i_2,\ldots,i_m$. In particular, if we let $I = \{i_0\}$, then $c_{i_1 i_2\cdots i_m} = 0$, $\forall i_1 \in I$ and $\forall i_2,\ldots,i_m \notin I$; this contradicts irreducibility. ∎

We are now ready for the proof of Theorem 1.4.

Proof. 1° First, we prove $x_0 \in \operatorname{int} P$. Note $P \setminus \operatorname{int} P = \partial P = \cup_{I\in\Lambda} F_I$, where Λ is the set of all index subsets I of $\{1,\ldots,n\}$ and

$$F_I = \{(x_1,\ldots,x_n) \in P \mid x_i = 0\ \forall i \in I, \text{ and } x_j \neq 0\ \forall j \notin I\}.$$

Suppose $x_0 \notin \operatorname{int} P$; since $x_0 \neq 0$, there must be a maximal proper index subset $I \in \Lambda$ such that $x_0 \in F_I$, i.e., such that $(x_0)_i = 0\ \forall i \in I$ and $(x_0)_j > 0\ \forall j \notin I$. Let $\delta = \operatorname{Min}\{(x_0)_j \mid j \notin I\}$; we then have that $\delta > 0$. Since x_0 is an eigenvector, $Ax_0 \in F_I$, i.e.,

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} (x_0)_{i_2} \cdots (x_0)_{i_m} = 0, \quad \forall i \in I.$$

It follows that

$$\delta^{m-1} \sum_{i_2,\ldots,i_m \notin I} a_{ii_2\cdots i_m} \le \sum_{i_2,\ldots,i_m \notin I} a_{ii_2\cdots i_m} (x_0)_{i_2} \cdots (x_0)_{i_m} = 0, \quad \forall i \in I,$$

hence we have $a_{ii_2\cdots i_m} = 0 \quad \forall i \in I, \quad \forall i_2,\ldots,i_m \notin I$, i.e., $\mathcal{A}$ is reducible, a contradiction.

2° Combining 1° and Lemma 2.2, we have $\lambda_0 > 0$.

3° We now prove that the eigenvalue corresponding to the positive eigenvector is unique, i.e., if (λ,x) and $(\mu,y) \in \mathbb{R}\times P$ are solutions of 1.3.1, then $\lambda = \mu$. According to 1° and 2°, for such solutions $x,y \in \operatorname{int} P$ and $\lambda,\mu > 0$. $\forall z \in \operatorname{int} P$ and $\forall w \notin P$, we define $\delta_z(w) = \operatorname{Max}\{s \in \mathbb{R}_+ \mid z + sw \in P\}$; then $\delta_z(w) > 0$, $z + tw \in P$ for $0 \le t \le \delta_z(w)$, and $z + tw \notin P$ for $t > \delta_z(w)$. Applying these to $(z,w) = (x,-y)$, we have $x - ty \in P$ for $0 \le t \le \delta_z(-y)$. By definition and by 2.1.1 and 2.1.2,

$$\lambda x^{[m-1]} = \mathcal{A}x^{m-1} \ge \delta_x(-y)^{m-1} \mathcal{A}y^{m-1} = \mu \delta_x(-y)^{m-1} y^{[m-1]}; \tag{2.2.1}$$

it follows that $x \ge (\frac{\mu}{\lambda})^{\frac{1}{m-1}} \delta_x(-y) y$, so $\mu \le \lambda$.

Likewise, if we interchange x and y, it follows that $y \ge (\frac{\lambda}{\mu})^{\frac{1}{m-1}} \delta_y(-x) x$, and thus $\lambda \le \mu$. We have hence proved $\lambda = \mu$. Therefore, the only eigenvalue corresponding to the positive eigenvector is λ_0.

4° We prove that the positive eigenvector is unique up to a multiplicative constant, i.e., if $x_0, x \in P \setminus \{0\}$ satisfies $\mathcal{A}x_0^{m-1} = \lambda_0 x_0^{[m-1]}$ and $\mathcal{A}x^{m-1} = \lambda_0 x^{[m-1]}$, then $x = kx_0$ for some constant k. It is known that $x_0 \in \operatorname{int} P$; by the definition of $\delta_{x_0}(-x)$, we have $x_0 - tx \in P$ for $0 \le t \le \delta_{x_0}(-x)$ and $x_0 - tx \notin P$ for $t > \delta_{x_0}(-x)$. This implies $x_0 -$

$t_0x \in \partial P$, where $t_0 = \delta_{x_0}(-x)$. So there exists a nonempty maximal index subset $I \subset \{1,\dots,n\}$ such that $x_0 - t_0x \in F_I$. If $I = \{1,\dots,n\}$, then $x_0 = t_0x$, and we are done. Otherwise, I is a nonempty proper subset. There exist $\epsilon > 0$ and $\delta > 0$ such that

$$\begin{aligned} &(x_0)_i \geq \delta \quad \forall i \in \{1,2,....,n\}, \\ &0 < t_0x_i = (x_0)_i \qquad \forall i \in I, \\ &0 < \frac{t_0x_i}{(x_0)_i} < 1-\epsilon \quad \forall i \notin I, \end{aligned}$$

and then $\forall i \in I$

$$\sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\cdots i_m}[(x_0)_{i_2}\cdots(x_0)_{i_m} - t_0^{m-1}x_{i_2}\cdots x_{i_m}] = \lambda_0[(x_0)_i^{m-1} - (t_0x_i)^{m-1}] = 0.$$

We have

$$t_0^{m-1}x_{i_2}\cdots x_{i_m} \leq (x_0)_{i_2}\cdots(x_0)_{i_m} \quad \forall i_2,\dots,i_m,$$

and

$$t_0^{m-1}x_{i_2}\cdots x_{i_m} \leq (1-\epsilon)^{m-1}(x_0)_{i_2}\cdots(x_0)_{i_m} \quad \forall i_2,\dots,i_m \notin I.$$

It follows that

$$\begin{aligned} &\delta^{m-1}(1-(1-\epsilon)^{m-1}) \sum_{i_2,\dots,i_m \notin I} a_{ii_2\cdots i_m} \\ &\leq \sum_{i_2,\dots,i_m \notin I} a_{ii_2\cdots i_m}[(x_0)_{i_2}\cdots(x_0)_{i_m} - t_0^{m-1}x_{i_2}\cdots x_{i_m}] \\ &\leq \sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\cdots i_m}[(x_0)_{i_2}\cdots(x_0)_{i_m} - t_0^{m-1}x_{i_2}\cdots x_{i_m}] = 0 \quad \forall i \in I, \end{aligned}$$

thus $a_{ii_2\cdots i_m} = 0 \quad \forall i \in I$, and $\forall i_2,\dots,i_m \notin I$, i.e., $\mathcal{A}$ is reducible, a contradiction. □

REMARK 2.3. By the same argument used in 1° of the proof of Theorem 1.4, the following improvement also holds: Assume $\mathcal{A}$ is an irreducible nonnegative tensor; if $x_0 \in P \setminus \{0\}$ is a solution of the inequality $\mathcal{A}x^{m-1} \leq \lambda x^{[m-1]}$, then $x_0 \in \operatorname{int} P$.

3. The simplicity of the eigenvalue λ_0

For a matrix (i.e., $m = 2$) A, an eigenvalue λ is called algebraically simple if λ is a simple root of the characteristic polynomial $\det(A - \lambda I)$, and is called geometrically simple if $\dim \operatorname{Ker}(A - \lambda I) = 1$. We will generalize these notions to the tensor setting. Since the operator A associated with a tensor $\mathcal{A}$ is nonlinear but homogeneous, we can define the geometric multiplicity of an eigenvalue of $\mathcal{A}$ as follows:

DEFINITION 3.1. *Let λ be an eigenvalue of*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{3.1.1}$$

We say λ has geometric multiplicity q if the maximum number of linearly independent eigenvectors corresponding to λ equals q. If $q = 1$, then λ is called geometrically simple.

It is worth noting the geometric multiplicity for a real eigenvalue λ of a real matrix A is independent of the base field being real or complex, i.e.,

$$\dim_{\mathbb{R}}\{x\in\mathbb{R}^n|(A-\lambda I)x=0\}=\dim_{\mathbb{C}}\{z\in\mathbb{C}^n|(A-\lambda I)z=0\}.$$

This is due to the fact that if $z=x+iy\in\mathbb{R}^n+i\mathbb{R}^n$ satisfies $(A-\lambda I)z=0$, then $x,y\in \mathrm{Ker}(A-\lambda I)\bigcap\mathbb{R}^n$.

As for higher order tensors, since $\mathcal{A}x^{m-1}$ is $m-1$ homogeneous, we still have real geometric multiplicity $\leq$ complex geometric multiplicity, but equality does not hold in general. This can be seen from the following example:

EXAMPLE 3.2. Let $m=3$ and $n=2$. Consider $\mathcal{A}=(a_{ijk})$ where $a_{111}=a_{222}=1$, $a_{122}=a_{211}=\epsilon$ for $0<\epsilon<1$, and $a_{ijk}=0$ for other (ijk). Then the eigenvalue problem becomes:

$$\begin{cases} x_1^2+\epsilon x_2^2=\lambda x_1^2,\\ \epsilon x_1^2+x_2^2=\lambda x_2^2. \end{cases} \tag{3.2.1}$$

We have $\lambda=1+\varepsilon$, with eigenvectors $u_1=(1,1)$ and $u_2=(1,-1)$, and $\lambda=1-\varepsilon$ with eigenvectors $u_3=(1,i)$, and $u_4=(1,-i)$. In this example we see that

- real geometric multiplicity of $\lambda=1+\varepsilon$ equals complex geometric multiplicity 2.
- real geometric multiplicity of $\lambda=1-\varepsilon$ is 0, and complex geometric multiplicity is 2.

The same example also shows the nonnegative irreducible tensor $\mathcal{A}$ has a positive eigenvalue $1+\varepsilon$ with unique positive eigenvector (up to a multiplicative constant), which is not geometrically simple, neither in $\mathbb{R}$ nor in $\mathbb{C}$.

EXAMPLE 3.3. Let $m=4,n=2$, and $\mathcal{A}=(a_{ijkl})$, with $a_{1222}=a_{2111}=1$ and $a_{ijkl}=0$ elsewhere. Then after computation, we see there are two eigenvalues: $\lambda=\pm1$, with eigenvectors $(x,\pm x),(x,\pm\exp\frac{2\pi i}{3}x),(x,\pm\exp\frac{4\pi i}{3}x)$. Therefore both $\lambda=\pm1$ are all real geometrically simple, but with complex geometrical multiplicity 3.

In the following, we shall seek a sufficient condition to ensure the real geometric simplicity of λ_0. If m is odd, there are two different types of eigenvalue problems, which impose the same constraints on P:

1. $\mathcal{A}x^{m-1}=\lambda(x_1^{m-1},\ldots,x_n^{m-1})$,
2. $\mathcal{A}x^{m-1}=\lambda(\mathrm{sgn}x_1x_1^{m-1},\ldots,\mathrm{sgn}x_nx_n^{m-1})$.

THEOREM 3.4. *Let m be odd, and let $\mathcal{A}$ be an irreducible nonnegative tensor of order m and dimension n. If $\mathcal{A}x^{m-1}$ is invariant under any one of the transformations $(x_1,...,x_n)\to(\pm x_1,...,\pm x_n)$ except the identity and its reflection, then λ_0 is not geometrically simple for problem 1. If all terms in $\mathcal{A}x^{m-1}$ are monomials of $x_1^2,\ldots,x_n^2$, i.e., $a_{i_1i_2\cdots i_m}\neq0$ only if the numbers of indices appearing in $\{i_2,\ldots,i_m\}$ are all even, $\forall i_1$, then λ_0 is real geometrically simple for problem 2.*

Proof. (1) Let T be the transformation under which $\mathcal{A}x^{m-1}$ is invariant. By assumption, if $x_0=(x_1^0,\ldots,x_n^0)\in\mathrm{int}P$ is a solution of (1), then Tx_0 is also a solution of (1) corresponding to the same eigenvalue λ_0, so λ_0 is not geometrically simple.

(2) By assumption, $\mathcal{A}x^{m-1}\geq0$, $\forall x\in\mathbb{R}^n$, which implies that all solutions of (2) must be in P. Using assertion (3) of Theorem 1.4, we see that $x=kx_0$, i.e., λ_0 is real geometrically simple. ∎

We next examine the case when m is even. We introduce a condition on $\mathcal{C}$ to ensure that the associated nonlinear operator C is increasing, i.e.,

$$x \le y \Rightarrow Cx \le Cy. \tag{3.4.1}$$

Comparing with 2.1.2, there is no restriction: $x, y \in P$ in 3.4.1.

DEFINITION 3.5. *(Condition (M)) A tensor $\mathcal{C} = c_{i_1 i_2 \cdots i_m}$ of order $m > 2$ dimension and n is said to satisfy Condition (M) if there exists a nonnegative matrix $D = (d_{ij})$ such that $c_{i_1 i_2 \cdots i_m} = d_{i_1 i_2} \delta_{i_2 \cdots i_m}$, where $\delta_{i_2 \cdots i_m}$ is the Kronecker delta.*

REMARK 3.6. For $m = 2$, Condition (M) is trivial, hence superfluous.

In fact, if m is even, Condition (M) on $\mathcal{C}$ implies that

$$\frac{\partial}{\partial x_j}(\mathcal{C}x^{m-1})_i = (m-1)\sum_{j=1}^{n} d_{ij} x_j^{m-2} \ge 0 \quad \forall i, j,$$

and then $Cx \le Cy$, $\forall x \le y$, $\forall x, y \in \mathbb{R}^n$. We now state and prove the following:

THEOREM 3.7. *Let m be even, and let $\mathcal{A}$ be an irreducible nonnegative tensor. If $\mathcal{A}$ satisfies Condition (M), then the eigenvalue λ_0 for the nonnegative eigenvector is real geometrically simple.*

REMARK 3.8. The special problem can, by setting $y = x^{[m-1]}$, be reduced to the problem for matrices, hence becomes a direct consequence of the Perron–Frobenius Theorem. However, we present the following proof since it will be useful for more general problems, see Section 5

Proof. We follow 4° in the proof of Theorem 1.4. We note that the only difference is that now, $x \in \mathbb{R}^n \setminus \{0\}$ but not $P \setminus \{0\}$. We still have that $t_0 = \delta_{x_0}(-x)$ such that $x_0 - tx \in P$ for $0 \le t \le t_0$ and $x_0 - tx \notin P$ for $t > t_0$. We want to show that $x_0 = t_0 x$. Suppose not; then $(x_0)_i \ge \delta > 0$, $\forall i$ and there exists a nonempty proper index subset I such that $t_0 x_i = (x_0)_i \quad \forall i \in I$ and such that $t_0 x_i < (1-\epsilon)(x_0)_i \quad \forall i \notin I$. It follows that $\forall i \in I$

$$\begin{aligned}
&\delta^{m-1}(1-(1-\epsilon)^{m-1}) \sum_{j \notin I} a_{ij\cdots j} \\
&\le \sum_{j \notin I} a_{ij\cdots j}[(x_0)_j^{m-1} - (t_0 x_i)^{m-1}] \\
&\le \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m}[(x_0)_{i_2} \cdots (x_0)_{i_m} - t_0^{m-1} x_{i_2} \cdots x_{i_m}] = 0,
\end{aligned}$$

so $a_{ij\cdots j} = 0 \quad \forall j \notin I$. Combining this with Condition (M), we obtain $a_{i_1 i_2 \cdots i_m} = 0 \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I$, which contradicts the irreducibility of $\mathcal{A}$. Therefore, $x_0 = t_0 x$, i.e., λ_0 is geometrically simple as desired. ∎

We next define the algebraic simplicity of the eigenvalue of 3.1.1. We follow the approach described in Cox et al. [2] (pp. 97–105) to define the characteristic polynomial $\psi_{\mathcal{A}}(\lambda)$ of $\mathcal{A}$ by

$$\psi_{\mathcal{A}}(\lambda) := \mathrm{Res}((\mathcal{A}x^{m-1})_1 - \lambda x_1^{m-1}, \ldots, (\mathcal{A}x^{m-1})_n - \lambda x_n^{m-1})),$$

where $\mathrm{Res}(P_1,\dots,P_n)$ is the resultant of n homogeneous polynomials $P_1,\dots,P_n$. For each $\mathcal{A}$, such $\psi_{\mathcal{A}}(\lambda)$ is unique up to a nonzero extraneous factor.

DEFINITION 3.9. *Let λ be an eigenvalue of 3.1.1. We say λ has algebraic multiplicity p if λ is a root of $\psi_{\mathcal{A}}(\lambda)$ of multiplicity p. And we call λ an algebraically simple eigenvalue if $p=1$.*

In Example 3.2, we know that $\lambda=1+\epsilon$ has geometric multiplicity 2 in both the real and complex fields. After computation we have

$$\begin{aligned}\psi_{\mathcal{A}}(\lambda)&=\det\begin{pmatrix}1-\lambda & 0 & \epsilon & 0\\ 0 & 1-\lambda & 0 & \epsilon\\ \epsilon & 0 & 1-\lambda & 0\\ 0 & \epsilon & 0 & 1-\lambda\end{pmatrix}\\ &=(\lambda-1+\epsilon)^2(\lambda-1-\epsilon)^2,\end{aligned}$$

which shows that the eigenvalue $\lambda_0=1+\epsilon$ also has algebraic multiplicity 2.

By definition, we see that complex geometric multiplicity $\leq$ algebraic multiplicity, and equality does not always hold; this can be seen in the next example.

EXAMPLE 3.10. Let $m=4$ and $n=2$. Consider $\mathcal{A}=(a_{ijkl})$ where $a_{1111}=a_{1112}=a_{2122}=a_{2222}=1$, and $a_{ijkl}=0$ for other $(ijkl)$. Then the eigenvalue problem becomes:

$$\begin{cases}x_1^3+x_1^2x_2=\lambda x_1^3,\\ x_1x_2^2+x_2^3=\lambda x_2^3.\end{cases}$$

We compute to see that

$$\begin{aligned}\psi_{\mathcal{A}}(\lambda)&=\det\begin{pmatrix}1-\lambda & 1 & 0 & 0 & 0 & 0\\ 0 & 1-\lambda & 1 & 0 & 0 & 0\\ 0 & 0 & 1-\lambda & 1 & 0 & 0\\ 0 & 0 & 1 & 1-\lambda & 0 & 0\\ 0 & 0 & 0 & 1 & 1-\lambda & 0\\ 0 & 0 & 0 & 0 & 1 & 1-\lambda\end{pmatrix}\\ &=\lambda(\lambda-2)(\lambda-1)^4,\end{aligned}$$

which shows the eigenvalues $\lambda=0,2$ are both algebraically and geometrically simple, with eigenvectors $u_1=(1,-1)$, and $u_2=(1,1)$, respectively, while $\lambda=1$ has algebraic multiplicity 4, but has only two linearly independent eigenvectors $u_3=(1,0)$ and $u_4=(0,1)$, so its geometric multiplicity is 2.

4. A minimax theorem

The following well-known [3] minimax theorem for irreducible nonnegative matrices will be extended to irreducible nonnegative tensors.

THEOREM 4.1. *(Collatz) Assume that A is an irreducible nonnegative $n\times n$ matrix; then*

$$\mathrm{Min}_{x\in\mathrm{int}\,P}\mathrm{Max}_{x_i>0}\frac{(Ax)_i}{x_i}=\lambda_0=\mathrm{Max}_{x\in\mathrm{int}\,P}\mathrm{Min}_{x_i>0}\frac{(Ax)_i}{x_i},\tag{4.1.1}$$

where λ_0 is the unique positive eigenvalue corresponding to the positive eigenvector.

In the remainder of this section, we will prove the following:

THEOREM 4.2. *Assume that $\mathcal{A}$ is an irreducible nonnegative tensor of order m and dimension n; then*

$$\mathrm{Min}_{x\in \mathrm{int}\, P}\, \mathrm{Max}_{x_i>0}\, \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} = \lambda_0 = \mathrm{Max}_{x\in \mathrm{int}\, P}\, \mathrm{Min}_{x_i>0}\, \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}, \tag{4.2.1}$$

where λ_0 is the unique positive eigenvalue corresponding to the positive eigenvector.

Before we proceed with the proof of Theorem 4.2, we first define the following two functions on $P\setminus\{0\}$:

$$\mu_*(x) = \mathrm{Min}_{x_i>0}\, \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} \quad \text{and} \quad \mu^*(x) = \mathrm{Max}_{x_i>0}\, \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}.$$

Clearly, $\mu_*(x)\le\mu^*(x)$. Note $\mu^*(x)$ may be $+\infty$ on the boundary $\partial P\setminus\{0\}$.

Since both $\mu_*(x)$ and $\mu^*(x)$ are positive 0-homogeneous functions, we can restrict them on the compact set

$$\Delta = \{x=(x_1,\dots,x_n)\in P \mid \sum_{i=1}^{n} x_i = 1\}.$$

Now, μ_* is continuous and bounded from above, and μ^* is continuous on $\Delta\bigcap \mathrm{int}\, P$ and is bounded from below, so there exist $x_*, x^*\in\Delta$ such that

$$r_* := \mu_*(x_*) = \mathrm{Max}_{x\in\Delta}\, \mu_*(x) = \mathrm{Max}_{x\in P\setminus\{0\}}\, \mu_*(x),$$
$$r^* := \mu^*(x^*) = \mathrm{Min}_{x\in\Delta}\, \mu^*(x) = \mathrm{Min}_{x\in P\setminus\{0\}}\, \mu^*(x).$$

Let $(\lambda_0, x_0)\in\mathbb{R}_+\times \mathrm{int}\, P$ be the positive eigenpair obtained in Theorem 1.4; we then have:

$$\mu^*(x^*)\le\mu^*(x_0) = \lambda_0 = \mu_*(x_0)\le\mu_*(x_*).$$

Therefore,

$$r^*\le\lambda_0\le r_*. \tag{4.2.3}$$

We shall prove they are indeed all equal. To do so, we modify 3° in the proof of Theorem 1.4 as follows:

LEMMA 4.3. *Let $\mathcal{A}$ be an irreducible nonnegative tensor of order m and dimension n. If (λ,x) and $(\mu,y)\in\mathbb{R}_+\times(P\setminus\{0\})$ satisfy $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$ and $\mathcal{A}y^{m-1}\ge\mu y^{[m-1]}$ (or, respectively, $\mathcal{A}y^{m-1}\le\mu y^{[m-1]}$), then $\mu\le\lambda$ (or, respectively, $\lambda\le\mu$).*

Proof. We first assume that $\mathcal{A}y^{m-1}\ge\mu y^{[m-1]}$. Since $x\in\mathrm{int}\, P$, we have $t_0 = \delta_x(-y)>0$ such that $x-ty\in P$ for $0\le t\le t_0$ and $x-ty\notin P$ for $t>t_0$. This implies that

$$\lambda x^{[m-1]} = \mathcal{A}x^{m-1}\ge t_0^{m-1}\mathcal{A}y^{m-1}\ge t_0^{m-1}\mu y^{[m-1]},$$

hence $x\ge(\frac{\mu}{\lambda})^{\frac{1}{m-1}} t_0 y$; consequently, $\mu\le\lambda$.

Next we assume that $\mathcal{A}y^{m-1}\le\mu y^{[m-1]}$. From the remark of Section 2, we have that $y\in\mathrm{int}\, P$, and if we interchange the roles of x and y in the previous paragraph, then we have that $\lambda\le\mu$. Our assertion now follows. ∎

We now return to the proof of Theorem 4.2:

Proof. After 4.2.3, it remains to show that $r_* \le \lambda_0 \le r^*$. By the definition of $\mu_*(x)$, we have

$$r_* = \mu_*(x_*) = \mathrm{Min}_{x_i>0} \frac{(\mathcal{A}x_*^{m-1})_i}{(x_*)_i^{m-1}}.$$

This means

$$\mathcal{A}x_*^{m-1} \ge r_* x_*^{[m-1]}.$$

Likewise,

$$\mathcal{A}x^{*m-1} \le r^* x^{*[m-1]}.$$

Our desired inequality follows from Lemma 4.3. ∎

Since μ_* is continuous on Δ and is 0-homogeneous, we have

COROLLARY 4.4.

$$\lambda_0 = \mathrm{Max}_{x\in P\setminus\{0\}} \mathrm{Min}_{x_i>0} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}.$$

We close this section by proving assertion (4) of Theorem 1.4:

Proof. Let $z \in \mathbb{C}^n \setminus \{0\}$ be a solution of $\mathcal{A}z^{m-1} = \lambda z^{[m-1]}$ for some $\lambda \in \mathbb{C}$. We wish to show that $|\lambda| \le \lambda_0$. Let $y_i = |z_i| \quad \forall i$ and set $y = (y_1, \ldots, y_n)$. Clearly, $y \in P \setminus \{0\}$. One has

$$|(\mathcal{A}z^{m-1})_i| = |\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} z_{i_2}\cdots z_{i_m}| \le \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} = (\mathcal{A}y^{m-1})_i.$$

This shows that

$$|\lambda| y_i^{m-1} = |\lambda||z_i|^{m-1} = |(\mathcal{A}z^{m-1})_i| \le (\mathcal{A}y^{m-1})_i \quad \forall i.$$

Applying Corollary 4.4, we have

$$|\lambda| \le \mathrm{Min}_{y_i>0} \frac{(\mathcal{A}y^{m-1})_i}{y_i^{m-1}} \le \mathrm{Max}_{x\in P\setminus\theta} \mathrm{Min}_{x_i>0} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} = \lambda_0.$$

∎

5. Some extensions

There are various ways of defining eigenvalues for tensors, e.g., there are H-eigenvalues, Z-eigenvalues, D-eigenvalues, etc. see [7, 8, 9, 4]. They are unified in [1]. In this section, we extend the above results to more general eigenvalue problems for tensors. Let $\mathcal{A}$ and $\mathcal{B}$ be two mth-order n-dimensional real tensors. Assume that both $\mathcal{A}x^{m-1}$ and $\mathcal{B}x^{m-1}$ are not identically zero. We say that $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an eigenpair (or eigenvalue and eigenvector) of $\mathcal{A}$ relative to $\mathcal{B}$, if the n-system of equations

$$(\mathcal{A} - \lambda\mathcal{B})x^{m-1} = 0 \tag{5.1.1}$$

possesses a solution.

The problem 1.3.1, called the H-eigenvalue problem, corresponds to the case where $\mathcal{B}=(\delta_{i_1 i_2 \ldots i_m})$ is the unit tensor. We next introduce a few more conditions on nonnegative tensors.

DEFINITION 5.1. *(Quasi-diagonal) A tensor $\mathcal{C}$ of order m and dimension n is said to be quasi-diagonal if for all nonempty proper index subsets $I \subset \{1,\ldots,n\}$, $c_{i_1,i_2,\ldots i_m}=0$ for $i_1 \notin I$ and $i_2,\ldots,i_m \in I$.*

EXAMPLE 5.2. For $m=2$, $\mathcal{C}$ is quasi-diagonal if and only if it is a diagonal matrix.

EXAMPLE 5.3. If $\mathcal{C}=(\delta_{i_1 \ldots i_m})$, where $\delta_{i_1 \cdots i_m}$ is the Kronecker delta, then $\mathcal{C}$ is quasi-diagonal.

LEMMA 5.4. *If a nonnegative tensor $\mathcal{C}$ of order m and dimension n is quasi-diagonal, then there exists $M>0$ such that for all nonempty proper index subsets $I \subset \{1,\ldots,n\}$, one has $\mathcal{C}e^I \le Me^I$, where $e^I=(e_1^I,\ldots,e_n^I)$ with*

$$e_i^I = \begin{cases} 1, & i \in I \\ 0, & i \notin I. \end{cases}$$

Proof. Let $M=\sum_{i_1,\ldots,i_m=1}^n c_{i_1 \cdots i_m}$. We verify that $(\mathcal{C}e^I)_i=0 \quad \forall i \notin I$ by computing

$$(\mathcal{C}(e^I)^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^n c_{i i_2 \cdots i_m} e_{i_2}^I \cdots e_{i_m}^I = \sum_{i_2,\ldots,i_m \in I} c_{i i_2 \cdots i_m} = 0, \quad \forall i \notin I,$$

provided by that $\mathcal{C}$ is quasi-diagonal. ∎

DEFINITION 5.5. *(Condition (E)) A nonnegative tensor $\mathcal{C}$ of order m and dimension n is said to satisfy Condition (E), if there exists a homeomorphism $\tilde{C}:\mathbb{R}^n \to \mathbb{R}^n$ such that (1) $\tilde{C}|_P = C|_P$, and (2) $\forall x,y \in P$, $x \le y$ implies $\tilde{C}^{-1}x \le \tilde{C}^{-1}y$.*

For $\mathcal{C}=(\delta_{i_1 \ldots i_m})$, $\tilde{C}$ is the identity operator, so Condition (E) is satisfied.

EXAMPLE 5.6. Let m be even, and let D be a positive definite matrix. If $\mathcal{C}$ is an mth-order n-dimensional tensor satisfying

$$\mathcal{C}x^{m-1} = Dx(Dx,x)^{\frac{m}{2}-1},$$

then $\mathcal{C}$ satisfies (1) in Condition (E). Indeed,

$$\tilde{C}^{-1}y = D^{-1}y(y,D^{-1}y)^{-\frac{m-2}{2(m-1)}}.$$

EXAMPLE 5.7. Let us consider the following example: let $C_k : P \to P$ be the nonlinear operator

$$C_k x = \begin{cases} x^{[2k-1]}|x|^{2(r-k)}, & m=2r, \\ x^{[2k]}|x|^{2(r-k)} & m=2r+1, \end{cases}$$

where $1 \le k \le r$. And let $\mathcal{C}_k=(c_{i_1,\ldots,i_m})$ be an mth-order n-dimensional nonnegative tensor corresponding to C_k, for example,

$$\sum c_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m} = \begin{cases} x_i^{2k-1}(x_1^2+\ldots.+x_n^2)^{r-k} & m=2r, \\ x_i^{2k}(x_1^2+\ldots.+x_n^2)^{r-k} & m=2r+1 \end{cases}$$

$$= \sum_{|\alpha|=r-k} \frac{(r-k)!}{\alpha_1!....\alpha_n!} x_1^{2\alpha_1}....x_n^{2\alpha_n} \begin{cases} x_i^{2k-1}, & m=2r, \\ x_i^{2k} & m=2r+1. \end{cases} \tag{5.7.1}$$

The left hand side equals

$$\sum_{|\beta|=m-1} \sum_{(i_2,...,i_m)\sim(1^{\beta_1},....,n^{\beta_n})} c_{ii_2....i_m} x_1^{\beta_1}....x_n^{\beta_n},$$

where $(1^{\beta_1},....,n^{\beta_n})$ means j is repeated for β_j times, $\forall 1 \le j \le n$, and $(i_2,....,i_m) \sim (i'_2,....,i'_m)$ means there exists a $\pi \in \mathfrak{S}_{m-1}$, the $m-1$ permutation group, such that $\pi(i_2,....,i_m) = (i'_2,....,i'_m)$. 5.7.1 implies that

$$\beta_j = 2\alpha_j + \delta_{ij} \begin{cases} 2k-1 & m=2r, \\ 2k & m=2r+1. \end{cases}$$

Therefore there exists a representation $\mathcal{C}_k$ of C_k such that $c_{i,i_2,....,i_m} \neq 0$ only if $\exists l \ge 2$ such that $i_l = i$. Consequently, $\mathcal{C}_k$ is quasi-diagonal.

Also, $\mathcal{C}_k$ satisfies Condition (E). In fact, define

$$\tilde{C}_k = \begin{cases} x^{[2k-1]}|x|^{2(r-k)} & m=2r, \\ x^{[2k]}|x|^{2(r-k)}\,\mathrm{Sgn}(x) & m=2r+1, \end{cases}$$

where we use the notation: $x^{[\alpha]}\mathrm{Sgn}(x) = (x_1^\alpha \mathrm{sgn}(x_1),....,x_n^\alpha \mathrm{sgn}(x_n))$. Then

$$\tilde{C}_k^{-1} = \begin{cases} y^{[\frac{1}{2k-1}]}(\Sigma_{i=1}^n |y_i|^{\frac{2}{2k-1}})^{-\frac{r-k}{m-1}} & m=2r \\ |y|^{[\frac{1}{2k}]}\mathrm{Sgn}(y)(\Sigma_{i=1}^n |y_i|^{\frac{1}{k}})^{-\frac{r-k}{m}} & m=2r+1. \end{cases} \tag{5.7.2}$$

Obviously, $\tilde{C}_k$ satisfies (1) and (2) in the definition of Condition (E).

REMARK 5.8. For $\mathcal{B} = \mathcal{C}_1$ the problem 5.1.1 corresponds to Z eigenvalue, and when m is even for $\mathcal{B} = C_{\frac{m}{2}}$ it corresponds to H eigenvalue.

We have the following general result:

THEOREM 5.9. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are nonnegative tensors, and that $\mathcal{B}$ satisfies (1) in Condition (E); then there exist $\lambda_0 \ge 0$ and a nonnegative vector $x_0 \neq 0$, such that*

$$\mathcal{A}x_0^{m-1} = \lambda_0 \mathcal{B}x_0^{m-1}. \tag{5.8.1}$$

If we further assume that $\mathcal{A}$ is irreducible and that $\mathcal{B}$ satisfies Condition (E) and is quasi-diagonal, then $x_0 \in \mathrm{int}\, P$, $\lambda_0 > 0$ and λ_0 is the unique eigenvalue with nonnegative eigenvectors. In particular, for $\mathcal{B} = \mathcal{C}_k$, the nonnegative eigenvector is unique up to a multiplicative constant.

Proof. We shall only sketch the proof, since it is parallel to the argument given in Section 2. Let A, B be the nonlinear operators corresponding to $\mathcal{A}, \mathcal{B}$, respectively. For the existence part, we define

$$F(x)_i = \frac{(\tilde{B}^{-1}Ax))_i}{\Sigma_{j=1}^n (\tilde{B}^{-1}Ax)_j}$$

and

$$\lambda_0 = (\Sigma_{j=1}^n (\tilde{B}^{-1}Ax)_j)^{m-1}$$

in place of 2.1.3 and 2.1.4. The subsequent argument is the same as for the counterpart in Section 2.

Next we follow step 1° in the proof of Theorem 1.4 to prove that $x_0 \in \operatorname{int} P$ by contradiction. Suppose not; then there exists a maximal proper index subset I such that $x_0 \in F_I$. From the equation

$$\mathcal{A}x_0^{m-1} = \lambda_0 \mathcal{B}x_0^{m-1}$$

and that $\mathcal{B}$ is quasi diagonal, it follows that $Bx_0 \in F_I$ and hence $Ax_0 \in F_I$. The following arguments are the same.

In step 3°, 2.2.1 is replaced by

$$\lambda Bx = Ax \geq \mu \delta_x (-y)^{m-1} By.$$

Since $\tilde{B}^{-1}$ is order-preserving in P and is positively $\frac{1}{m-1}$ homogeneous, we have

$$\lambda^{\frac{1}{m-1}} x \geq \mu^{\frac{1}{m-1}} \delta_x (-y) y.$$

Therefore $\mu \leq \lambda$. Again the rest is the same.

As to the uniqueness of the positive eigenvector (up to a multiplicative constant), we reduce the problem by changing variables. For $x \neq 0$, let

$$\xi = \begin{cases} x|x|^{-\frac{2(r-k)}{m-\alpha-1}} & m = 2r \\ x|x|^{-\frac{2(r-k)}{m-\alpha-2}}, & m = 2r+1, \end{cases}$$

where $\alpha = 2k$ when m is even, and $\alpha = 2k+1$ when m is odd. The problem is then reduced to:

$$A\xi = \lambda_0 \xi^{[\alpha]}.$$

We shall prove that the nonnegative eigenvector x_0 is unique by contradiction. Suppose not; there exist $x, y \in P \backslash \{0\}$ satisfying $Ax = \lambda_0 x$ and $Ay = \lambda_0 y$. Let ξ, η be the images of x, y under the above change of variables. Then by the argument in step 4° of the proof in Theorem 1.4, there exists $t > 0$ such that $\xi = t\eta$. This implies

$$x = t \left(\frac{|x|}{|y|} \right)^{\frac{2(r-k)}{m-\alpha-1}} y,$$

where $\alpha = 2k$ or $\alpha = 2k+1$ if $m = 2r$ or $m = 2r+1$, resp.

Lastly, the minimax theorem in Section 4 is also extended:

$$\operatorname{Min}_{x \in \operatorname{int} P} \operatorname{Max}_{x_i > 0} \frac{(\mathcal{A}x^{m-1})_i}{(\mathcal{C}_k x^{m-1})_i} = \lambda_0 = \operatorname{Max}_{x \in \operatorname{int} P} \operatorname{Min}_{x_i > 0} \frac{(\mathcal{A}x^{m-1})_i}{(\mathcal{C}_k x^{m-1})_i}.$$

The subsequent steps follow the last paragraph of Section 4. □

COROLLARY 5.10. *Theorem 1.3 holds for the D-eigenvalue problem. Theorem 1.4 holds for H-eigenvalue problem and the Z-eigenvalue problem.*

More generally, for $Bx = x^{[k]}\varphi(x)$, where $1 \leq k \leq [\frac{m}{2}]$ and $\varphi(x)$ is a positive $m-k-1$ homogeneous polynomial, the Perron–Frobenius theorem still holds.

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