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MULTICOMMODITY FLOWS ON ROAD NETWORKS*

M. HERTY†, C. KIRCHNER‡, S. MOUTARI§, AND M. RASCLE¶

Abstract. In this paper, we discuss the multicommodity flow for vehicular traffic on road networks. To model the traffic, we use the "Aw-Rascle" multiclass macroscopic model [3]. We describe a solution to the Riemann problem at junctions with a criterion of maximization of the total flux, taking into account the destination path of the vehicles. At such a junction, the *actual* distribution depends on the demands and the supplies on the incoming and outgoing roads, respectively. Furthermore, this new distribution scheme captures efficiently key merging characteristics of the traffic and in contrast to [M. Herty, S. Moutari and M. Rascle, *Networks and Heterogeneous Media*, 1, 275-294, 2006] leads to an easy computational model to solve approximately the homogenization problem described in [M. Herty, S. Moutari and M. Rascle, *Networks and Heterogeneous Media*, 1, 275-294, 2006], [M. Herty and M. Rascle, *SIAM J. Math. Anal.*, 38(2), 595-616, 2006]. Furthermore, we deduce the equivalent distribution scheme for the LWR multiclass model in [M. Garavello and B. Piccoli, *Commun. Math. Sci.*, 3, 261-283, 2005] and we compare the results with those obtained with the "Aw-Rascle" multiclass model for the same initial conditions.

Key words. Aw–Rascle model, multicommodity flow models, traffic networks

AMS subject classifications. 35LXX, 35L6

1. Introduction

A typical vehicular traffic system consists of the vehicle-driver pairs and the infrastructures, i.e., a collection of highways systems and all their operational elements. In a vehicular traffic system, a number of trips — defined by their destination path, the travel route, etc. — interact on the road network and generate various dynamics and phenomena. To study these traffic phenomena and the corresponding applications, many traffic models [8, 9, 14, 18, 20, 21, 24, 29] and simulation packages [19] have been suggested in the literature. However, most of the models of multicommodity traffic systems have been made under the framework of the LWR model [25, 9]. In this study, we are particularly interested in the impact of the destination path of the vehicles on traffic dynamics at road intersections when modelling with a class of "second order" models of traffic flow. In the present work, we consider a multiclass macroscopic model [3] derived from the "Aw-Rascle" (AR) second order model of traffic flow and we propose a definition of the solution to the associated Riemann problem at a junction. This solution is based on the maximization of the mass flux at the junction and the conservation of the pseudo-momentum (see below). The vehicles with the same destination are considered to belong to the same commodity. Each commodity is attached to each vehicle and is therefore a Lagrangian variable that we must keep track of in the whole flow and in particular through junctions (see also [9] for another definition of commodity). Given some initial boundary conditions, we make use of the supply and demand methods [6, 23] to compute the fluxes through a

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†Fachbereich Mathematik, TU Kaiserslautern, D-67653 Kaiserslautern, Germany (herty@rhrk.uni-kl.de).

‡Fachbereich Mathematik, TU Kaiserslautern, D-67653 Kaiserslautern, Germany (kirchner@mathematik.uni-kl.de).

§Laboratoire J. A. Dieudonné, Université de Nice-Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 2, France (salissou@math.unice.fr).

¶Laboratoire J. A. Dieudonné, Université de Nice-Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 2, France (rascle@math.unice.fr).

junction and then solve the associated Riemann problem. We then use homogenization techniques [16, 15] to obtain coupling conditions for the second moment of the Aw–Rascle equation. Other approaches conserving the second moment recently have been introduced in [12, 27].

2. Preliminaries

We model a road network as a finite, directed graph $G=(\mathcal{E},\mathcal{N})$, where we denote by $\mathcal{E}$ and $\mathcal{N}$ the set of arcs and vertices of the graph G, respectively. We label each arc in $\mathcal{E}$ by $e=1,\ldots,\mathbf{E}$ and assume that each arc corresponds to a road. Similarly, each vertex in $\mathcal{N}$ labeled $n=1,\ldots,\mathbf{N}$ corresponds to a junction. For an arbitrary given junction $n\in\mathcal{N}$ we denote by δ_n^- and δ_n^+ the set of all incoming and outgoing roads to n, respectively. In the sequel, we will use the indices k for the elements of δ_n^- and j for the elements of δ_n^+. Moreover, each road e is modelled by an interval $I_e:=[y_e^l,y_e^r]$.

2.1. The model. We consider a class of "second order" models for traffic flow on road networks given by the "Aw-Rascle" equations [3]. These consist of the conservation of mass:

$$\partial_t\rho_e+\partial_x(\rho_e v_e)=0, \tag{2.1}$$

where the function ρ_e describes the car density on road e and v_e the velocity. Furthermore, some additional quantities appear in the model. First, w_e is considered to be a car-specific property which influences the velocity and is assumed here to satisfy the relation

$$w_e=v_e+p_e(\rho_e) \tag{2.2}$$

for some known function $p_e(\rho)$ ("traffic pressure") with the properties

$$\text{for all } \rho\geq 0,\, \rho p_e''(\rho)+2p_e'(\rho)>0 \text{ and } p_e(\rho)\approx\rho^\gamma,\gamma>0,\rho\to 0. \tag{2.3}$$

In contrast with w_e, on each road e, we call a_e^i the local proportion of cars with the commodity i. Hence, the density of cars with commodity i is given by $\rho_e^i:=a_e^i\rho_e$. We assume that we have $i=1,\ldots,m$ commodities in the network.

Since we implicitly assume that each road is a single lane road, there is no passing between cars: for all i, a_e^i is thus a Lagrangian variable on each road e, but in principle has no influence on the velocity v_e. Let $\vec{a}_e:=(a_e^i)_{1\leq i\leq m}$, then we have

$$\partial_t w_e+v_e\partial_x w_e=0,\quad \partial_t\vec{a}_e+v_e\partial_x\vec{a}_e=\vec{0}. \tag{2.4}$$

Let us repeat that only w_e has an influence on v_e through (2.2), whereas $\vec{a}_e$ is passively advected on each road and of course modified at each junction. We have the following interpretation in mind: The value of the property w_e may distinguish cars and trucks having different acceleration and deceleration behavior. On the other hand, the function a_e^i describes the proportion of ρ_e at (x,t) with destination i. For the latter interpretation we add the constraint

$$\sum_i a_e^i=1 \quad \forall\ e\in\delta^\pm. \tag{2.5}$$

For a mathematical analysis we reformulate (2.1), (2.2), (2.4) and (2.5) in conservative form

$$\partial_t \rho_e + \partial_x(\rho_e v_e) = 0, \tag{2.6a}$$
$$\partial_t(\rho_e w_e) + \partial_x(\rho_e v_e w_e) = 0, \tag{2.6b}$$
$$\partial_t(\rho_e \vec{a}_e) + \partial_x(\rho_e v_e \vec{a}_e) = \vec{0}, \tag{2.6c}$$
$$w_e = v_e + p_e(\rho_e), \sum_i a_e^i = 1. \tag{2.6d}$$

We introduce the following notation: $U_e := \left(\rho_e, \rho_e w_e, \rho_e \vec{a}_e\right)^T$ and $F_e(U_e) = \left(\rho_e v_e, \rho_e v_e w_e, \rho_e v_e \vec{a}_e\right)^T$.

The multiclass version of the original "Aw-Rascle" model [2] was introduced in [3] for a single commodity, i.e., $m = 1$. We extended this to the multicommodity case by introducing the vector $\vec{a}_e$. However, the properties of the now $(m+2) \times (m+2)$ model are clearly similar to the properties of the commodity model introduced in [3]. In particular, for this model, the first characteristic field is genuinely nonlinear whereas the second and the m fields corresponding to a_e^i are linearly degenerate. The solution to the Riemann problem associated to the system (2.6) consists of two waves: rarefaction or shock wave associated with the first characteristic field, followed by a contact discontinuities associated with the other characteristic fields. Furthermore, shock curves and rarefaction curves coincide for (2.6). Therefore the system (2.6) belongs to the class of Temple systems [28] and its Riemann invariants are v_e, w_e and $\vec{a}_e$.

3. Solution to the Riemann problem

For later analysis we recall the notion of Riemann problems for (2.6): A Riemann problem of (2.6) is a Cauchy problem with piecewise constant initial data $U_e(x,0) = U_l$ for $x < 0$ and $U_e(x,0) = U_r$ for $x \geq 0$. We refer the reader to [3] for a general discussion of solutions to Riemann problems associated to (2.6). A derivation of the necessary conditions at the junction, i.e., the coupling conditions, can be found in [10, 16, 17]. We define weak solutions of the network problem in the following sense: A set of functions $\{U_e = (\rho_e, \rho_e v_e)\}_{e \in \mathcal{E}}$ is called a weak solution of (2.6) if and only if

$$\sum_{e=1}^{\mathbf{E}} \int_0^\infty \int_{y_e^l}^{y_e^r} \begin{pmatrix} \rho_e \\ \rho_e w_e \\ \rho_e \vec{a}_e \end{pmatrix} \cdot \partial_t \phi_e + \begin{pmatrix} \rho_e v_e \\ \rho_e v_e w_e \\ \rho_e v_e \vec{a}_e \end{pmatrix} \cdot \partial_x \phi_e dx dt$$
$$+ \int_{y_e^l}^{y_e^r} \begin{pmatrix} \rho_{e,0} \\ \rho_{e,0} w_{e,0} \\ \rho_{e,0} \vec{a}_{e,0} \end{pmatrix} \cdot \phi_e(x,0) dx = 0 \tag{3.1}$$

holds for any set of C^1- functions $\{\phi_e\}_{e \in \mathcal{E}} : I_e \times [0, +\infty) \to \mathbb{R}^3$ having compact support. Additionally, the family $\{\phi_e\}_{e \in \mathcal{E}}$ needs to be continuous across a junction n, i.e.,

$$\phi_k(y_k^r) = \phi_j(y_j^l), \ \forall k \in \delta_n^- \text{ and } \forall j \in \delta_n^+. \tag{3.2}$$

Here, $U_{e,0}(x) = \left(\rho_{e,0}(x), (\rho_{e,0} v_{e,0})(x), (\rho_{e,0} v_{e,0} \vec{a}_{e,0})(x)\right)$ is the initial data. Furthermore, the set of functions U_e satisfies for all e the relation

$$w_e(x,t) = v_e(x,t) + p_e^\dagger(\rho_e(x,t)), \tag{3.3}$$

where the function $p_e^\dagger(\cdot)$ is initially unknown.

On an outgoing road its explicit form depends on the mixture of the cars. On any incoming road $k \in \delta_n^-$ the relation $p_k^\dagger \equiv p_k$ is valid. The precise definition can be found e.g. in [16] and an explicit construction can be found in [15].

Next, we extend the previous discussion to the system, including property $\vec{a}_e$, $\forall\, e$. With the previous remarks in mind, we consider a single intersection n with incoming and outgoing arcs δ_n^- and δ_n^+, respectively. We assume the initial data $U_{e,0}$ for $e \in \delta_n^- \cup \delta_n^+$ to be constant. At the vertex n located at $x = x_0$ we consider the following (half-)Riemann problems

$$\partial_t U_e + \partial_x F_e(U_e) = 0, U_e(x,0) = \begin{pmatrix} U_e^- & x < x_0 \\ U_e^+ & x > x_0 \end{pmatrix}, \tag{3.4}$$

for each arc $e \in \delta_n^- \cup \delta_n^+$. Depending on the road, only one of the Riemann data in (3.4) is given, namely, if $e \in \delta_n^-$ then $U_e^- = U_{e,0}, x_0 = y_e^r$ and if $e \in \delta_n^+$ then $U_e^+ = U_{e,0}, x_0 = y_e^l$. We construct a weak solution to (2.6) such that the generated waves have non-positive speed (resp. non-negative speed) if $e \in \delta_n^-$ (resp. if $e \in \delta_n^+$). For each road the unknown state U_e^+ (resp. U_e^-) for $e \in \delta_n^-$ (resp. $e \in \delta_n^+$) will be determined in such a way that the coupling conditions

$$\sum_{e \in \delta_n^-} (\rho_e v_e)(y_e^r -, t) = \sum_{e \in \delta_n^+} (\rho_e v_e)(y_e^l +, t), \tag{3.5a}$$

$$\sum_{e \in \delta_n^-} (\rho_e v_e w_e)(y_e^r -, t) = \sum_{e \in \delta_n^+} (\rho_e v_e w_e)(y_e^l +, t), \tag{3.5b}$$

$$\sum_{e \in \delta_n^-} (\rho_e v_e \vec{a}_e)(y_e^r -, t) = \sum_{e \in \delta_n^+} (\rho_e v_e \vec{a}_e)(y_e^l +, t) \tag{3.5c}$$

are satisfied. Those coupling conditions are the conservation of the momentum of the conservation laws, i.e., conservation of mass and the "pseudo"-momentum $\rho v w$ as well as the property $\rho v \vec{a}$. Even in the case $\vec{a} \equiv 1$, $\forall\, e$, the conditions (3.5) are not sufficient to obtain a unique family of solutions $\{U_e\}_{e \in \mathcal{E}}$ at the intersection; see [17] for the discussion of the simpler case of a single commodity flow and the Lighthill-Whitham equation and [16] for single commodity Aw-Rascle flow. Therefore, we impose additional constraints which imply (3.5); see the definition of solutions at an intersection below. Of course, there is freedom in the modelling of additional conditions.

For notational convenience we denote by q_{jk} the initially unknown mass flux of cars going from road k to road j. We introduce the quantities $q_k := \sum_{j \in \delta_n^+} q_{jk}$ and $q_j := \sum_{k \in \delta_n^-} q_{jk}$ describing the total mass flux on incoming and outgoing roads k and j, respectively. Due to the construction of solutions to the (half-)Riemann problems we have $\rho_k v_k(y_k^r -, t) = q_k$ for $k \in \delta_n^-$ and $\rho_j v_j(y_j^l +, t) = q_j$ for $j \in \delta_n^+$ and $t > 0$.

We now present a straightforward approach to obtaining suitable coupling conditions, before turning to more sophisticated examples.

First, we denote by α_{jk} the percentage of cars on road k going to road j at an intersection n. The corresponding matrix $A := (\alpha_{jk})_{j \in \delta_n^+, k \in \delta_n^-, \forall n}$ is assumed to be *known*, see [4, 10, 16]. By definition we have

$$\sum_{j \in \delta_n^+} \alpha_{jk} = 1 \quad \forall k \in \delta_n^-. \tag{3.6}$$

Then, we obtain $\alpha_{jk} = q_{jk}/q_k$, and clearly, any solution $\{\rho_k v_k\}_{k\in\delta_n^\pm}$ which satisfies

$$q_j = \sum_{k\in\delta_n^-} \alpha_{jk} q_k \equiv \sum_{k\in\delta_n^-} q_{jk} \tag{3.7}$$

also complies with (3.5a).

Next, we introduce the quantity $\beta_{jk} := q_{jk}/q_j$. This quantity describes the mixture of cars on the outgoing road. Several approaches exist for determining β_{jk}, but in any case we have $\sum_{k\in\delta_n^-} \beta_{jk} = 1$. In [16] the proportions β_{jk} are assumed to be known and an existence result is valid for an arbitrary number of connected roads. In [15], however, the proportions β_{jk} are obtained from a maximization problem but the result is limited to vertices of degree three. To keep the discussion as simple as possible, we postulate the quantities β_{jk} to be known a priori. More precisely, as in [9], we assume that at any junction n, all the cars on the incoming roads with destination i will take the same path and therefore the same outgoing road.

We introduce a set of parameters $\gamma_j^i, i = 1,\ldots,m$, such that $\forall\ j \in \delta_n^+$ we have

$$\gamma_j^i = \begin{cases} 1 \ \text{ if } \ j \ \text{ belongs to a unique “shortest path” to reach the destination } \ i, \\ \\ 0 \ \text{ otherwise.} \end{cases} \tag{3.8}$$

The notion of “shortest path” depends on the context under consideration. The path can be the shortest according to the distance, the travel cost, the travel time, etc. The coefficients γ_j^i can be determined a priori according to the topology of the network, the prescribed behavior of the drivers, etc. In particular, we assume that they are *known* at any junction. A mathematical description is to say that j belongs to a path $\mathbb{P}$ of a predfined set of possible paths in the network.

A path P in the network G is a sequence of vertices $n_l \in \mathcal{N}$, $l = 1,\ldots,N-1$ such that n_l and n_{l+1} are connected by the directed arc $e_{l,l+1} \in \mathcal{E}$. The set of all paths in G is denoted by $\mathcal{P}$ and we note P^i the path that cars of commodity i should follow.

Remark 3.1. *We emphasize that the restriction to considering the shortest path as a condition for defining γ_j^i is for ease of presentation. Other choices are possible: for instance, the path with the actual shortest travel time to the destination i. Then the definition (3.8) becomes more complicated and can vary as time progresses.*

From a computational point of view defining appropriate values γ_j^i at the intersection for a given network is an NP-hard problem.

Next, we assume that the particular choice of γ_j^i and the proportions a_k^i define the coefficients α_{jk}, β_{jk} at an intersection and the proportions a_j^i on each outgoing road. We summarize all assumptions on the distribution at an intersection as follows:

A1 The behavioral rule (3.8) implies a distribution of the flux.

$$\alpha_{jk} = \frac{\sum\limits_{i=1}^{m} \gamma_j^i a_k^i (y_k^r-,t)}{\sum\limits_{i=1}^{m} a_k^i} = \sum_{i=1}^{m} \gamma_j^i a_k^i (y_k^r-,t), \quad \forall k \in \delta_n^-, \ \ \forall j \in \delta_n^+. \tag{3.9}$$

Hence, we require that

$$(\rho_j v_j)(y_j^l+,t) = \sum_{k\in\delta_n^-} \alpha_{jk}(\rho_k v_k)(y_k^r+,t)\,\forall j\in\delta_n^+.$$

A2 The behavioral rule (3.8) implies a mixture on the outgoing roads.

$$\beta_{jk} = \frac{\sum_{i=1}^m \gamma_j^i a_k^i(y_k^r-,t)}{\sum_{i=1}^m \sum_{e\in\delta_n^-} \gamma_j^i a_e^i(y_e^r-,t)}, \quad \forall k\in\delta_n^-, \quad \forall j\in\delta_n^+. \tag{3.10}$$

Then, we require that (3.13) holds:

$$(w_j)(y_j^l+,t) = \sum_{k\in\delta_v^-} \beta_{jk} w_k(y_k^r-,t).$$

A3 The behavioral rule (3.8) implies a distribution of the quantities a_j^i.

$$a_j^i(y_j^l+,t) = \frac{\sum_{k\in\delta_n^-} \gamma_j^i a_k^i(y_k^r-,t)}{\sum_{o=1}^m \sum_{k\in\delta_n^-} \gamma_j^o a_k^o(y_k^r-,t)} \quad \forall j\in\delta_n^+, \quad \forall i\in\{1,\dots,m\}. \tag{3.11}$$

The coupling conditions (3.9) and (3.11) are similar to the ones proposed in [9] where the dynamics on the network are governed by the LWR-equations. Moreover, it is easy to check that:

PROPOSITION 3.1. *Under the assumptions* **A1** – **A3** *the coupling conditions (3.5) are satisfied, since we have*

$$\sum_{j\in\delta_n^+} \alpha_{jk} = 1, \quad \sum_{k\in\delta_n^-} \beta_{jk} = 1, \quad \sum_{i=1}^m a_j^i = 1. \tag{3.12}$$

From the Lagrangian point of view it is intuitive to prescribe inflow conditions on the outgoing arcs. In fact, $w_k(y_k^r-,t) = v_{k,0} + p_k(\rho_{k,0}) =: w_k(U_{k,0})$ for all $t>0$ and $k\in\delta_n^-$. Assuming that (2.6) and (3.10) are satisfied, the quantity

$$w_j(y_j^l+,t) = \bar{w}_j = \sum_{k\in\delta_n^-} \beta_{jk} w_k(U_{k,0}) \tag{3.13}$$

is constant for all $t>0$. However, since the variable w is coupled by $p(\rho)$ to the dynamics of the system, one has to be careful of the precise relation of w_j and the function p_j for $j\in\delta_n^+$. The correct formulation includes the homogenization process. This can be understood as follows: Consider constant initial data $U_{j,0}$. Since w_l is a Lagrangian quantity and transported with the velocity v_l, at an intersection we therefore obtain that $w_k(t,0+) = w_{k,0}$ for $k\in\delta_v^-$. On the outgoing roads $j\in\delta_v^+$ we can have a mixture of quantities $w_{k,0}$ with cars from road k entering road j. This mixture is described by β_{jk}. This is described in condition (3.13). However,

w_j is not just a quantity which is transported but has itself an influence on the dynamics, since it has to satisfy $w = v + p(\rho)$. Hence, the mixture β_{jk} implies new flow conditions $p_j^\dagger$. We introduce $P_j(\xi) = p_j(1/\xi)$. Then the new flow conditions in Lagrangian coordinates are $1/\rho^*(X,t) = \sum_{k\in\delta_v^-} \beta_{jk} P_j^{-1}(w_{k,0} - v^*(X,t))$, which we choose to rewrite as $w = v + p_j^\dagger(1/\rho)$ for $w = \sum_{k\in\delta_v^-} \beta_{jk} w_{k,0} \equiv \bar{w}_j$. This reformulation holds true for $w \equiv \bar{w}_j$ only and defines the homogenized $p_j^\dagger$ for a given set of mixture rules and initial data $w_{k,0}$. The solution on the outgoing road $j \in \delta_v^+$ finally satisfies $w_j(t,0+) = v(t,0+) + p_j^\dagger(t,0+) \equiv \bar{w}_j$. We refer to Sec. 5 and Sec. 6 of [16] for a more precise mathematical derivation of the previous ideas.

We conclude this section with the definition of a solution for the multicommodity network model.

Definition 3.2. *Consider a junction n with an arbitrary number of incoming and outgoing roads and denote by $\{U_{e,0}\}_{e\in\delta_n^\pm}$ given constant initial data. The coefficients γ_j^i are given (for example determined according to the drivers traveling strategies).*

We call a family $\{U_e\}_{e\in\delta_n^\pm}$ an admissible solution at the junction if and only if it satisfies

(C1) $\forall e \in \delta^- \cup \delta^+$, $U_e(x,t)$ is a weak entropy solution in the sense of (3.1) of the network problem (3.3), where $p_e^\dagger \equiv p_e, \forall e \in \delta_n^-$.
On an outgoing road $j \in \delta^+$, the solution $U_j(x,t)$ is constructed as in page 12 of [15]: in the triangle $\{(x,t); y_j^l < x < y_j^l + t v_{j,0}\}$, U_j is the homogenized solution defined below with $p_j^\dagger \equiv p_j^*$, whereas for $x > y_j^l + t v_{j,0}$, $p_j^\dagger \equiv p_j$.

(C2) The flux distribution satisfies

$$(\rho_j v_j)(y_j^l+,t) = \sum_{k\in\delta_n^-} \alpha_{jk}(t)(\rho_k v_k)(y_k^r+,t) \quad \forall j \in \delta_n^+,$$

wherein

$$\alpha_{jk}(t) := \sum_{i=1}^m \gamma_j^i a_k^i(y_k^r-,t), \quad \forall k \in \delta_n^-, \quad \forall j \in \delta_n^+.$$

(C3) The car properties are mixed according to

$$(w_j)(y_j^l+,t) = \sum_{k\in\delta_v^-} \beta_{jk}(t) w_k(y_k^r-,t),$$

wherein

$$\beta_{jk}(t) = \frac{\sum_{i=1}^m \gamma_j^i a_k^i(y_k^r-,t)}{\sum_{i=1}^m \sum_{e\in\delta_n^-} \gamma_j^i a_e^i(y_e^r-,t)}, \quad \forall k \in \delta_n^-, \quad \forall j \in \delta_n^+.$$

(C4) The sum of the incoming fluxes $\sum_{k\in\delta^-} \rho_k v_k(y_k^r-,t) = \sum_{k\in\delta^-} q_k$ is maximal subject to (C2) and (C3).

Some remarks are in order. The conditions (C2) and (C3) inspired by the behavioral rule (3.8). In (C1) a weak entropy solution on each arc is a weak entropic solution in the sense of Lax. A family of functions $\{U_e\}$ satisfies (3.5) a.e. in t and furthermore $\sum_{i=1}^m a_e^i = 1$ for all roads e. Using Theorem 7.1 from [16], the existence of solutions to constant initial data $U_{e,0}$ is straightforward.

4. The coupling conditions: a general framework

In order to treat the case of multicommodity flows at a general junction, we present two distribution schemes of traffic flow based on the approximation of the homogenization coefficients β_{jk} on the outgoing roads (see Equ. (3.10) for instance). Furthermore, we compare the numerical results obtained with both schemes in Sec. 5.

To distribute the vehicle flux at the intersection we propose two approaches: The first one consists of first distributing the incoming flux on the outgoing roads and then merging or homogenizing the flux on the respective outgoing roads, whereas the second approach amounts to doing the converse; i.e., we first homogenize all the incoming fluxes and then we distribute the homogenized flux on the outgoing roads.

For both schemes we assume the following rule: at a given junction n the distribution of the flux on the outgoing roads is induced by the path with the actual shortest traveling time from n to the different destinations i. Before we describe the two distribution schemes, let us recall the notion of supply and demand functions.

The demand function $d_C(\rho)$ and the supply function $s_C(\rho)$ as introduced in [23] and [6] are the non-decreasing (see Fig. 4.1) and the non-increasing (Fig. 4.1) parts of the curve $w=$const, respectively. We remark that for any constant const≥ 0 there is a level curve $w=$cste and the demand $d=d_C$ and supply $s=s_C$ function depend on the chosen constant. However, the shape of the demand and supply function are similar to the ones shown in the figure below.

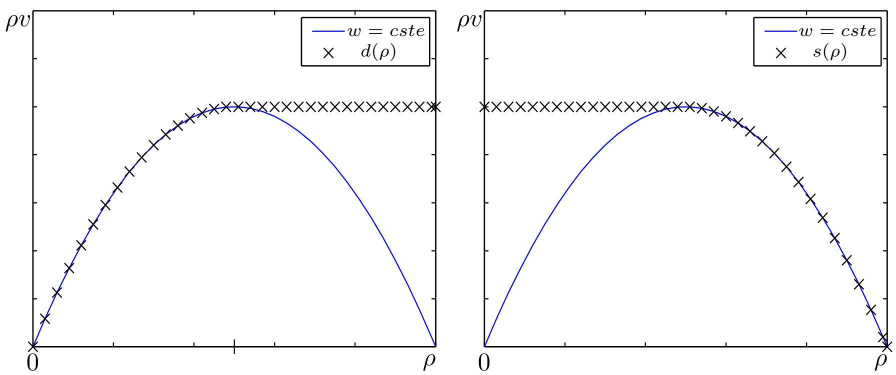

Fig. 4.1. *The demand (left) and supply (right) function for a given constant* cste

4.1. Scheme 1: Distribution before homogenization.

We first distribute the incoming flux on the outgoing roads and then we homogenize the total flux on each outgoing road. We assume that the coefficients γ_j^i are known.

Hence, on each outgoing road $j\in\delta_n^+$, near the junction, the property w_j belongs to the curve $w_j=$const where the constant w_j^* is given by

$$w_j^*(y_j^l+,t):=\sum_{k\in\delta_n^-}\beta_{jk}w_k(y_k^r-,t), \tag{4.1}$$

where β_{jk} are given by (3.10).

Then the quantities a_j^i on the outgoing roads are given by (3.11).

For the outflow and the inflow at a given junction, we have the following constraints in terms of the supply and the demand:

$$q_j = \sum_{i=1}^{m} q_j a_j^i \le s_j(y_j^l+,t) = s_j, \tag{4.2}$$

$$q_k = \sum_{i=1}^{m} q_k a_k^i \le d_k(y_k^r-,t) = d_k. \tag{4.3}$$

The supply on each outgoing road j is $s_j = s_j(y_j^l+,t) = \mathbf{s}(U_j^*)$, where $U_j^* = (\rho_j^*, v_j^*)$ is the state on the road j such that:

$$\rho_j^* = p_j^{-1}(w_j^* - v_j^*) \quad \text{and} \quad v_j^* = v_j^+. \tag{4.4}$$

The mentioned approximation consists precisely in replacing the exact equation $\rho_j^* = (p_j^\dagger)^{-1}(w_j^* - v_j^+)$, where $p_j^\dagger$ is the homogenized "pressure" see [16, 15], with (4.4).

In order to compute the optimal fluxes that satisfy the Rankine-Hugoniot jump conditions (3.5) on each road at a given junction, we solve the following problem:

$$\max \sum_{j\in\delta^-} q_j, \tag{4.5a}$$

subject to: (4.5b)

$$0 \le q_k a_k^i \le d_k a_k^i, \quad \forall k \in \delta^-, \forall i, \tag{4.5c}$$

$$0 \le q_j \le s_j \quad \forall j \in \delta^+, \tag{4.5d}$$

$$\sum_{j\in\delta^+} q_j a_j^i = \sum_{k\in\delta^-} q_k a_k^i, \quad \forall i. \tag{4.5e}$$

Note that here, the coefficient β_{jk} only plays a role through the supply s_j. The problem is a convex optimization problem on a bounded domain and has therefore at least one solution. Since the mixture rates β_{jk} are fixed by the choice of γ_{jk}^i, we conclude by applying Theorem 7.1 from [16] that there exists a unique solution. By construction of β_{jk} and α_{jk}, this solution is also a solution in the sense of Definition 3.2.

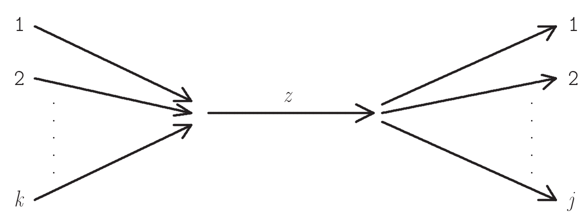

Fig. 4.2. *Decomposition of a junction*

4.2. Scheme 2: Homogenization before distribution. Contrary to what we did above, in this approach we first homogenize all the incoming fluxes and then distribute the homogenized flux on the outgoing roads. In other words, we decompose a junction into a juxtaposition of a "merging junction" and a "diverging junction". Let us label z the virtual intermediate road which is both the outgoing road for the "merging junction" and the incoming road for the "diverging junction," as depicted in Figure 4.2, see also [24].

Therefore, we have

$$a_z^i = \frac{\sum\limits_{k\in\delta_n^-} a_k^i}{\sum\limits_{o=1}^{m}\sum\limits_{k\in\delta_n^-} a_k^o}, \quad \forall i=1,\dots,m, \tag{4.6}$$

$$\beta_{zk} = \frac{\sum\limits_{i=1}^{m} a_k^i}{\sum\limits_{o=1}^{m}\sum\limits_{k\in\delta_n^-} a_k^o}, \quad \forall k\in\delta_n^-, \tag{4.7}$$

and the quantities a_j^i, $\forall i=1\dots m$, $\forall j\in\delta_n^+$ are determined by (3.11).

The traffic condition on the virtual intermediate road z is the same on all the outgoing roads connected to the junction and is given by the curve

$$\{w_j^*(y_j^l+,t) = w_z = \sum_{k\in\delta_n^-} \beta_{zk} w_k(y_k^r-,t)\}. \tag{4.8}$$

Therefore, on each outgoing road $j\in\delta^+$, the supply s_j connected to the junction is given by the supply $\mathbf{s}(U_j^*)$ at the intermediate state $U_j^*=(\rho_j^*,v_j^*)$ such that

$$\rho_j^* = p_j^{-1}(w_j^*-v_j^*) \;\; \text{and} \;\; v_j^* = v_j^+. \tag{4.9}$$

Instead of solving (4.5) we now consider the following maximization problem:

$$\max \sum_{j\in\delta_n^-} q_j, \tag{4.10a}$$

$$\text{subject to:} \tag{4.10b}$$

$$0 \le q_k a_k^i \le d_k a_k^i, \quad \forall k\in\delta^-, \forall i, \tag{4.10c}$$

$$0 \le q_j \le s_j \quad \forall j\in\delta^+, \tag{4.10d}$$

$$\sum_{j\in\delta_n^+} q_j a_j^i = \sum_{k\in\delta_n^-} q_k a_k^i, \quad \forall i. \tag{4.10e}$$

As in Subsec. 4.1 the solution to (4.10) enables us to define a solution at an intersection in the sense of Definition 3.2.

REMARK 4.1. *Note that although the optimization problems (4.5) and (4.10) formally look alike, they yield in general different optimal solutions. The supply s_j differs in the formulations due to the changes in the definitions of w_j^* in (4.1) and in (4.8). In (4.1) the quantities β_{jk}, cf. (3.10), are used, whereas in (4.8) we use β_{zk} as given in (4.7). This leads to different values ρ_j^* in (4.4) and (4.9), respectively.*

4.3. Coupling conditions for the LWR multicommodity model. In Sec. 5, we include a brief description of the coupling conditions we use for the LWR multiclass model [9] given by

$$\partial_t\rho_e + \partial_x v_e(\rho_e)\rho_e = 0, \tag{4.11a}$$

$$\partial\vec{a}_e + v_e\partial_x\vec{a}_e = 0, \tag{4.11b}$$

where the flux function $f(\rho) := v_e(\rho_e)\rho_e$ is a strictly concave function with a single maximum. The demand function is defined as the nondecreasing part of the flux–function f. The supply function is the nonincreasing part of the flux function f. We consider here the following setup of a diverging junction with two commodities. Then as in [4, 9, 17] we obtain a solution satisfying the coupling conditions (3.5a) and (3.5c) by considering the following optimization problem:

$$\begin{aligned}
&\max f(\rho_1^r),\\
&\text{subject to:}\\
&0 \le f(\rho_1^r) \le d_1,\\
&0 \le \alpha_{21} f(\rho_1^r) \le s_2,\\
&0 \le (1-\alpha_{21}) f(\rho_1^r) \le s_3.
\end{aligned}$$

Here ρ_1^r denotes the right state in the half-Riemann problem on the incoming road, $d_1 = d_1(\rho_l)$ denotes the demand on the incoming road 1 at the given state ρ_l, $s_2 = s_2(\rho_2^r)$ and $s_3 = s_3(\rho_3^r)$ denote the supplies for the given right states ρ_2^r and ρ_3^r on the outgoing roads 2 and 3 respectively. On the one hand, we can determine the value α_{21} using the coupling condition (3.9) as for the AR model, which enables us to compare the two different models.

Next, we slightly adjust our notation, considering now distribution rates γ_{jk}^i for distributing the ith commodity coming from road k and moving to road j. Clearly, this is just another convenient way to rewrite the path condition (3.8).

$$\alpha_{jk} = \frac{\sum_{i=1}^{m} \gamma_{jk}^i a_k^i(y_k^r-,t)}{\sum_{i=1}^{m} a_k^i} = \sum_{i=1}^{m} \gamma_{jk}^i a_k^i(y_k^r-,t).$$

For the computations in Sec. 5 we choose a value for γ_{21}^1 and a_1^1. Then γ_{31}^1 and a_1^2 are determined. It was shown in [4] that the optimization problem under consideration is well posed for all $\alpha_{21} \in [0,1]$ and constant left and right states $\rho_1^l, \rho_2^r, \rho_3^r$. The main difference between our approach and the one in [9] is that here the path from the source to the destination is not fixed in advance because of the variation of the control parameters γ_{jk}^i. Therefore drivers are allowed to change their itinerary according to the traffic conditions.

5. Numerical results

In this section we present numerical simulations for the different schemes constructed above for the AR model and the LWR model. First we solve the Riemann problem for the multicommody AR model with some given initial data using the distribution schemes proposed in subsections 4.1 and 4.2 respectively and then we compare the obtained results. In the second part of this section, we simulate the distribution scheme for the multicommodity LWR model presented in Subsec. 4.3. Moreover, we compare the results of the AR multicommodity model with those obtained by the multicommodity LWR model for the same intitial and distribution settings.

5.1. Investigation of the different distribution schemes. In this subsection we compare the results obtained by the distribution schemes presented in subsections 4.1 and 4.2 respectively. To this end we consider the junction depicted in Fig. 5.1. We assume only two commodities $i = 1$ and $i = 2$ on the incoming roads

1 and 2. To simplify the presentation, we assume that the outgoing road 3 belongs to the "shortest" path for commodity 1 and that drivers with commodity 2 can reach their destination faster via the outgoing road 4. Consequently, we require $a_3^1=1, a_3^2=0, a_4^1=0$, and $a_4^2=1$. In all our simulations we obtain this result for the values of quantities a_e^i independently of the distribution scheme used and the model used. The controls corresponding to the defined "shortest" paths read

$$\gamma_{13}^1=1, \quad \gamma_{13}^2=0, \quad \gamma_{23}^1=1, \quad \gamma_{23}^2=0.$$

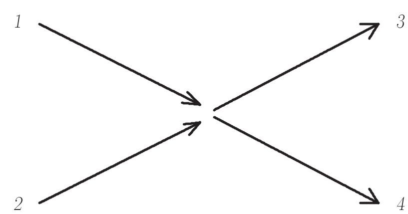

FIG. 5.1. *A junction with two incoming and two outgoing roads.*

For some given initial values of a_k^i, $k,i=1,2$, we define controls on the outgoing roads

$$\alpha_{3k} := \gamma_{k3}^1 a_k^1 + \gamma_{k3}^2 a_k^2.$$

The LWR coupling conditions are then

$$f_3(\rho_3^-) = \alpha_{31} f_1(\rho_1^+) + \alpha_{32} f_2(\rho_2^+),$$
$$f_4(\rho_4^-) = (1-\alpha_{31}) f_1(\rho_1^+) + (1-\alpha_{32}) f_2(\rho_2^+).$$

For the LWR model we solve for some given states $\rho_1^-, \rho_2^-, \rho_3^+$, and ρ_4^+ the following optimization problem:

$$\max q_1 + q_2, \tag{5.1a}$$
$$\text{subject to:}$$
$$\alpha_{31} q_1 + \alpha_{32} q_2 \le s_3(\rho_3^+), \tag{5.1b}$$
$$(1-\alpha_{31}) q_1 + (1-\alpha_{32}) q_2 \le s_4(\rho_4^+), \tag{5.1c}$$
$$q_1 \le d_1(\rho_1^-), \tag{5.1d}$$
$$q_2 \le d_2(\rho_2^-), \tag{5.1e}$$

where the s_3 and s_4 are the supplies on the outgoing roads and d_1 and d_2 are the demands on the incoming roads. In the following, we compare the results of solutions at a junction for the LWR optimization problem with those obtained by the distribution schemes for the AR model presented in subsections 4.1 and 4.2. We impose for the LWR model $v(\rho) = 2\cdot(1-\rho)$ and for the AR model we set $p_e(\rho) = \rho$, $\forall e$. The initial densities are the following

$$\rho_1^- = 0.6, \quad \rho_2^- = 0.7, \quad \rho_3^+ = 0.5, \quad \rho_4^+ = 0.4,$$

therefore

$$v_1^- = 0.8, \quad v_2^- = 0.6, \quad v_3^+ = 1, \quad v_4^+ = 1.2.$$

For the AR model, on the incoming roads 1 and 2 we have

$$w_1 = w_1^- = v_1^- + p_1(\rho_1^-), \quad w_2 = w_2^- = v_2^- + p_2(\rho_2^-).$$

In the LWR model the fundamental diagram is identical on each road and is given by the function

$$\rho \longmapsto \rho v(\rho) = 2(\rho - \rho^2). \tag{5.2}$$

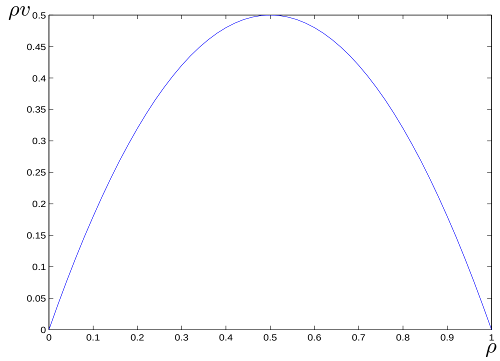

FIG. 5.2. *The fundamental diagram for the LWR model on each road anywhere.*

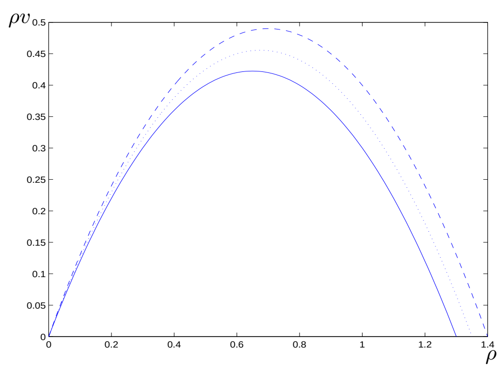

FIG. 5.3. *Fundamental diagrams for the AR model: on the incoming road 1, the dashed line; on the incoming road 2, the solid line; on an outgoing road connectedto the junction, the fundamental diagram follows from the homogenization of traffic on incoming roads and therefore lies between the fundamental diagrams on incoming roads, the dotted line.*

Whereas for the AR model it depends on the actual value of w on the considered road. Therefore, on an incoming road k, $k = 1,2$, we have

$$\rho \longmapsto \rho w_k - \rho p_k(\rho) = \rho w_k - \rho^2, \tag{5.3}$$

and on an outgoing road j, $j = 3,4$, *far away from the junction* we have

$$\rho \longmapsto \rho w_j - \rho p_j(\rho) = \rho w_j - \rho^2. \tag{5.4}$$

However, on an outgoing road j, $j=3,4$, *nearby the junction*, due to the traffic homogenization, the fundamental diagram is given by

$$\rho \longmapsto \rho w_j^* - \rho p_j(\rho) = \rho w_j^* - \rho^2, \tag{5.5}$$

where $w_j^* = \sum_{k=1}^{2} \beta_{jk} w_k$ with β_{jk} obtained by (3.10).

The values of the quantites a_k^i on the incoming roads 1 and 2 are for the first and second test, respectively,

$$\text{Test 1:} \quad a_1^1 = 0.6, \quad a_1^2 = 0.4, \quad a_2^1 = 0.3, \quad a_2^2 = 0.7, \tag{5.6}$$

$$\text{Test 2:} \quad a_1^1 = 0.2, \quad a_1^2 = 0.8, \quad a_2^1 = 0.9, \quad a_2^2 = 0.1. \tag{5.7}$$

We summarize the results for the LWR model and the AR model (the schemes *homogenize then distribute* (HD) and *distribute then homogenize* (DH)) in Table 5.1.

	LWR	HD	DH
q_1	0.5	0.36	0.36
q_2	0.42857	0.4225	0.4225
q_3	0.42857	0.34275	0.34275
q_4	0.5	0.43975	0.43975

	LWR	HD	DH
q_1	0.5	0.48	0.48
q_2	0.4	0.376	0.4
q_3	0.5	0.4344	0.455625
q_4	0.4	0.4216	0.424

TABLE 5.1. *Solution of the optimization problems for the different models and schemes. The left table corresponds to the initial data (5.6) and the right one to (5.7).*

We observe that in the three cases we have flow conservation through the junction: $q_1+q_2=q_3+q_4$. For the LWR model $q_1=f(\rho_1^+)$, and ρ_1^+ is obtained in a way such that the wave connecting ρ_1^- and ρ_1^+ has a nonpositive speed, cf. [4]. In Table 5.2 we report the density values corresponding to the fluxes of Table 5.1. The flux for the distribution schemes HD and DH are the same in both test cases, whereas the corresponding densities are not always the same.

	LWR	HD	DH
ρ_1^+	0.5	0.34	0.34
ρ_2^+	0.69	0.7	0.7
ρ_3^-	0.311	0.33	0.34
ρ_3^*	–	0.36	0.35
ρ_4^-	0.5	0.586	0.549
ρ_4^*	–	0.136	0.15

	LWR	HD	DH
ρ_1^+	0.5	0.6	0.6
ρ_2^+	0.6	0.8656	0.8
ρ_3^-	0.5	0.318	0.35
ρ_3^*	–	0.318	0.35
ρ_4^-	0.3	0.448	0.497
ρ_4^*	–	0.18	0.15

TABLE 5.2. *Complete solution to the Riemann problem at the junction depicted in Figure 5.1. The left table captures the results for the initial data (5.6), the right one for (5.7).*

In the first test setup the results from the LWR and the AR model are rather similar. We have the same traffic behavior. However, in the second example we find considerable disagreement. In the AR model we have a jam on road 2 indicated by a density $\rho_2^+ \geq 0.8 > 0.7 = \rho_2^-$, whereas in the LWR model the traffic still flows without backing up. That means the dynamics of multicommodity flows depends strongly on the used model. Additionally, in Table 5.2 we show that the mixing strategies at an intersection with more than three roads influences the solution to the Riemann problem.

5.2. Simulations on a test network. We now consider the network depicted in Figure 5.4. We simulate for this network the dynamics for a constant inflow using the LWR and the AR multicommodity, respectively. The two distribution schemes for the AR model do not produce different results, since the network under consideration has only nodes of at most degree three. For the LWR model we assume that $v(\rho) = 4\cdot(1-\rho)$.

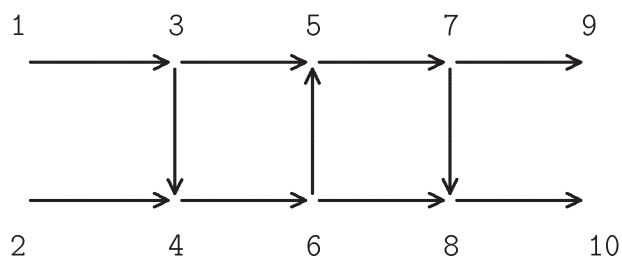

FIG. 5.4. *Example of road network with* 4 *commodities.*

We have only 2 commodities for the network considered in Figure 5.4, and they are summarized in Table 5.3. We also assume that the "shortest" paths are always the same and given by the node sequence stated in the third column of Table 5.3:

Commodities		"Shortest" Paths
Commodity 1:	Vehicles with destination 9	1-3-5-7-9, 2-4-6-5-7-9
Commodity 2:	Vehicles with destination 10	1-3-4-6-8-10, 2-4-6-8-10

TABLE 5.3. *Commodities for the network given in Figure 5.4.*

Due to the topology of the network of Figure 5.4 and the assumption in Table 5.3 among the control parameters γ, only four can be chosen freely; for instance, see Table 5.4 below.

$$\begin{array}{c} \gamma^1_{(1,3)(3,4)} \in [0,1] \\ \gamma^2_{(1,3)(3,4)} \in [0,1] \\ \gamma^1_{(4,6)(6,5)} \in [0,1] \\ \gamma^2_{(4,6)(6,5)} \in [0,1] \end{array}$$

TABLE 5.4. *Controls for the network in Figure 5.4.*

In particular, we can choose the value 0 or 1 for these controls. To model the shortest paths indicated in Table 5.3 we set $\gamma^1_{(1,3)(3,4)} = \gamma^2_{(4,6)(6,5)} = 0$ and $\gamma^2_{(1,3)(3,4)} = \gamma^1_{(4,6)(6,5)} = 1$.

For the simulation, we assume that the network is initially empty and the inflows on the incoming roads (1,3) and (2,4) are constant in time and given by $\rho_{(1,3)}(y^l_{(1,3)},t) = 0.2$ and $\rho_{(2,4)}(y^l_{(2,4)},t) = 0.25$, respectively. Furthermore, we assume that the supply is infinity on the outgoing roads (7,9) and (8,10), respectively. For the quantities a_e^i, we assume (initially and over time) the parameters given in Table 5.5.

Commodity 1	Commodity 2
$a^1_{(1,3)}=0.7$	$a^2_{(1,3)}=0.3$
$a^1_{(2,4)}=0.4$	$a^2_{(2,4)}=0.6$

Table 5.5. *Initial values of the quantities a_e^i for the network given in Figure 5.4.*

In Table 5.6 we compare the density and the quantities a_e^i in the steady state distributions in the steady state at $T=4$ for selected roads.

Arc (e)	ρ_e		a_e^1		a_e^2	
	LWR	AR	LWR	AR	LWR	AR
(1,3)	0.2	0.2	0.7	0.7	0.3	0.3
(2,4)	0.25	0.25	0.4	0.4	0.6	0.6
(4,6)	0.38	0.1198	0.32	0.2	0.68	0.8
(3,5)	0.1285	0.1373	1	1	0	0
(6,5)	0.08167	0.0233	1	1	0	0

Table 5.6. *Comparison at $T=4$ for the simulation with the LWR model and AR model for the network in Figure 5.4.*

As in the previous section, the models produce different results, in particular for roads (4,6), (3,5) and (6,5).

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