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STABILITY OF RECONSTRUCTION SCHEMES FOR SCALAR HYPERBOLIC CONSERVATION LAWS*

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Abstract. We study the numerical approximation of scalar conservation laws in dimension 1 via general reconstruction schemes within the finite volume framework. We exhibit a new stability condition, derived from an analysis of the spatial convolutions of entropy solutions with characteristic functions of intervals. We then propose a criterion that ensures the existence of some numerical entropy fluxes. The consequence is the convergence of the approximate solution to the unique entropy solution of the considered equation.

Key words. hyperbolic equations, numerical schemes, reconstruction schemes, entropy schemes

AMS subject classifications. 35L65, 65M12

1. Introduction

This paper deals with the scalar initial value problem of first order in dimension 1 in space

$$\partial_t u(t,x) + \partial_x f(u)(t,x) = 0, \qquad \text{for } t \in \mathbb{R}_+, x \in \mathbb{R}, \tag{1.1}$$

$$u(0,x) = u^0(x), \qquad \text{for } x \in \mathbb{R}, \tag{1.2}$$

where $f \in \mathcal{C}^1(\mathbb{R})$ and $u^0 \in L^\infty(\mathbb{R})$. Considering the weak form of (1.1, 1.2) allows multiple solutions. Therefore we restrict our study to the *entropy* solution of this problem, that is to say to a weak solution that satisfies the additional partial differential inequalities

$$\partial_t S(u)(t,x) + \partial_x G(u)(t,x) \leq 0 \qquad \text{for } t \in \mathbb{R}_+, x \in \mathbb{R}, \tag{1.3}$$

for every entropy-entropy flux pair (S,G), i.e. every pair of $C^1(\mathbb{R})$ functions (S,G) such that S is convex and $G' = S'f'$. It is known that there exists a unique entropy weak solution to (1.1, 1.2) (cf. [16, 8]). For example, the solution belongs to $L^\infty((0,T)\times\mathbb{R})$ for $T \in \mathbb{R}_+$ and furthermore is total variation decreasing. Thus $u(t,\cdot) \in BV(\mathbb{R})$ for all $t \in \mathbb{R}_+$ if $u^0 \in BV(\mathbb{R})$. Let us recall that u is the entropy solution to (1.1, 1.2) if and only if for every $k \in \mathbb{R}$

$$\partial_t S_k(u)(t,x) + \partial_x G_k(u)(t,x) \leq 0 \qquad \text{for } t \in \mathbb{R}_+, x \in \mathbb{R}, \tag{1.4}$$

with $S_k(u) = |u-k|$ and $G_k(u) = \text{sgn}(u-k)(f(u)-f(k))$.

We are here concerned with the numerical approximation of these entropy solutions in the standard framework of finite volume schemes.

Let $\Delta x \in \mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$ and $\Delta t \in \mathbb{R}_+^*$ be given positive real numbers, denoting respectively the space and time steps. We replace Equation (1.1) with the discrete in time and space equation

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(f_{j+1/2}^n - f_{j-1/2}^n), \quad n \in \mathbb{N}, j \in \mathbb{Z},$$

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with the numerical initial condition $(u_j^0)_{j\in\mathbb{Z}}$ given by

$$u_j^0 = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u^0(x)\,dx, \quad j\in\mathbb{Z}.$$

The terms $f_{j+1/2}^n$ $(j\in\mathbb{Z}, n\in\mathbb{N})$ are called the numerical fluxes and are to be computed in such a manner that the numerical approximation

$$\overline{u}_{\Delta x}^{\Delta t}(t,x) = \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{Z}} u_j^n \chi_{[n\Delta t,(n+1)\Delta t)}(t)\chi_{[(j-1/2)\Delta x,(j+1/2)\Delta x)}(x)$$

converges toward the entropy solution to (1.1, 1.2) as Δt and Δx tend to 0 (in a norm to be specified). In the following pages, we propose new conditions on the fluxes ensuring the convergence.

More precisely, this paper is an analysis of *reconstruction* schemes, whose spirit is to decompose the resolution, i.e. the computation of the fluxes $f_{j+1/2}^n$, into three steps:

- given the constant-in-cell solution at time step n, a reconstruction step consisting in reconstructing a new initial condition, and
- the exact computation of the solution with this reconstructed condition,
- a "projection" of this solution onto the mesh, that is, an L^2-projection on the space of constant-in-cell functions (to recover a constant-in-cell function).

The aim of the reconstruction step is to add some detail in the numerical solution in order to counter the loss of detail due to the "projection".

The paper is organized as follows:

First (Sec. 2), we state a theoretical result concerning entropy solutions: we show that the convolution of any entropy solution with the characteristic function of a bounded interval satisfies a local maximum principle. The result seems to be new.

This result is then used, in Sec. 3, to derive new conditions for a reconstruction scheme to be L^∞-decreasing and Total Variation Diminishing (TVD) *away from sonic points* (the technical assumption $f'\geq 0$ is necessary at this point of the paper). These conditions are sufficient for a scheme to converge to a weak solution to (1.1, 1.2). However, the limit solution is not necessarily the entropy solution.

Thus we focus in Sec. 3.3 on numerical entropy inequalities. We give a general condition on the reconstruction for the global scheme to be consistent with entropy inequalities. This implies convergence toward the entropy solution. This condition is a consequence of an inequality of Hardy.

There is a wealth of literature on reconstruction schemes and entropy conditions. Usually, the focus is on designing second order or high order schemes. This is not the aim here: the present paper is only devoted to convergence conditions. Among the wide amount of studies, the reader can refer to the classical references [27, 28], to [13] for a general study of discrete entropy conditions, to [9] for the geometric limiters theory (slope limiters), to [25] for the flux limiter theory, and to [5] for its extension to the Euler system [4, 20] for the study of MUSCL schemes and entropy. One can read [3] for a precise study of links between geometric reconstruction and decrease of numerical entropy, and [18] for high order approximation with entropy inequalities. We also also mention [22] for a general study of convergence and order.

2. Convolution of entropy solutions

We here state a preliminary result that will be used in Sec. 3 for controlling the stability of reconstruction schemes. This result is a stability result concerning the convolution in space of the entropy solution to Equations (1.1, 1.2) with the characteristic function of any bounded interval.

Let $u \in L^\infty((0,+\infty)\times\mathbb{R})$ be the unique entropy solution to the problem (1.1, 1.2). Let $\delta \in \mathbb{R}_+^*$. Let us denote by $[u]_\delta : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ the convolution of u with the characteristic function of $[-\delta/2, \delta/2]$ normalized by a factor $1/\delta$:

$$[u]_\delta(t,x) = \frac{1}{\delta}\left(u(t,\cdot) * \chi_{[-\delta/2,\delta/2]}(\cdot)\right)(x) = \frac{1}{\delta}\int_{x-\delta/2}^{x+\delta/2} u(t,y)\,dy.$$

Let us now define $c_1 = \inf_{x\in\mathbb{R}} f'(u^0(x))$, $c_2 = \sup_{x\in\mathbb{R}} f'(u^0(x))$. One has $-\infty < c_1 \le c_2 < +\infty$. We denote by $\mathscr{C}_u(t,x)$ the dependence interval of x at time t:

$$\mathscr{C}_u(t,x) = [x - c_2 t, x - c_1 t].$$

Then, the following theorem holds.

Theorem 2.1. *The convolved entropy solution $[u]_\delta$ satisfies*

$$\min_{y\in\mathscr{C}_u(t,x)} [u]_\delta(0,y) \le [u]_\delta(t,x) \le \max_{y\in\mathscr{C}_u(t,x)} [u]_\delta(0,y) \qquad \textit{for } (t,x)\in\mathbb{R}_+\times\mathbb{R}.$$

Corollary 2.2. *The convolved entropy solution $[u]_\delta$ satisfies: for every $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, there exists $y(t,x)\in\mathscr{C}_u(t,x)$ such that*

$$[u]_\delta(t,x) = [u]_\delta(0, y(t,x)).$$

This corollary is an immediate consequence of Thm. 2.1 and of the fact that $[u]_\delta(t,\cdot)$ is a Lipschitz-continuous function (with Lipschitz constant $2\|u^0\|_{L^\infty}/\delta$).

Proof. The present proof uses a parabolic regularization of (1.1) and consists of three main parts:

- we first show that the convolved solution to the problem with parabolic regularization satisfies a global maximum principle;
- we then deduce the same maximum principle for the convolution of the non-regularized entropy solution;
- we enforce the maximum principle by localizing it.

Regularized equation. Let $(\rho^\varepsilon)_{\varepsilon\in\mathbb{R}_+^*}$ be a $\mathcal{C}^\infty$-regularizing set. For $\varepsilon > 0$ (fixed) we define the regularized initial condition

$$u^{\varepsilon 0}(x) = \left(u^0 * \rho^\varepsilon\right)(x) = \int_{\mathbb{R}} u^0(y)\rho^\varepsilon(x-y)\,dy$$

and the regularized flux

$$f^\varepsilon(u) = (f * \rho^\varepsilon)(u) = \int_{\mathbb{R}} f(v)\rho^\varepsilon(u-v)\,dv.$$

We consider the solution u^ε of the following parabolic problem:

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \partial_x f^\varepsilon(u^\varepsilon)(t,x) = \varepsilon \partial^2_{x,x} u^\varepsilon(t,x) & \text{for } t\in\mathbb{R}_+, x\in\mathbb{R}, \\ u^\varepsilon(0,x) = u^{\varepsilon 0}(x) & \text{for } x\in\mathbb{R}. \end{cases} \tag{2.1}$$

It is well-known that this problem has a unique solution and that this solution belongs to $\mathcal{C}^\infty(\mathbb{R}_+\times\mathbb{R})$ (see [8], for example).

We now introduce the convolved (normalized) regularized solution (for $\delta>0$)

$$[u^\varepsilon]_\delta(t,x)=\frac{1}{\delta}\left(u^\varepsilon(t,\cdot)*\chi_{[-\delta/2,\delta/2]}(\cdot)\right)(x),$$

which of course belongs to $\mathcal{C}^\infty(\mathbb{R}_+\times\mathbb{R})$. Let us show that $[u^\varepsilon]_\delta$ is the solution to a partial differential equation. By performing the convolution of the terms of relation (2.1) with $\chi_{[-\delta/2,\delta/2]}$, we get

$$\frac{1}{\delta}\left[\chi_{[-\delta/2,\delta/2]}(\cdot)*(\partial_t u^\varepsilon(t,\cdot)+\partial_x f^\varepsilon(u^\varepsilon)(t,\cdot))\right](x)=\frac{\varepsilon}{\delta}\left[\chi_{[-\delta/2,\delta/2]}(\cdot)*\left(\partial^2_{x,x}u^\varepsilon(t,\cdot)\right)\right](x),$$

which equivalently reads

$$\partial_t[u^\varepsilon]_\delta(t,x)+\frac{f^\varepsilon(u^\varepsilon(t,x+\delta/2))-f^\varepsilon(u^\varepsilon(t,x-\delta/2))}{\delta}=\varepsilon\partial^2_{x,x}[u^\varepsilon]_\delta(t,x).$$

A key point is now to show that the above equation can be recast into an advection-diffusion problem. Let us first remark that $\delta\partial_x[u^\varepsilon]_\delta(t,x)=u^\varepsilon(t,x+\delta/2)-u^\varepsilon(t,x-\delta/2)$, so that whenever $u^\varepsilon(t,x+\delta/2)\neq u^\varepsilon(t,x-\delta/2)$, we have

$$\begin{aligned}&\frac{f^\varepsilon(u^\varepsilon(t,x+\delta/2))-f^\varepsilon(u^\varepsilon(t,x-\delta/2))}{\delta}\\&=\frac{f^\varepsilon(u^\varepsilon(t,x+\delta/2))-f^\varepsilon(u^\varepsilon(t,x-\delta/2))}{u^\varepsilon(t,x+\delta/2)-u^\varepsilon(t,x-\delta/2)}\partial_x[u^\varepsilon]_\delta(t,x).\end{aligned}$$

Furthermore, if $u^\varepsilon(t,x+\delta/2)=u^\varepsilon(t,x-\delta/2)$, $\partial_x[u^\varepsilon]_\delta(t,x)=0$. These considerations allow to define

$$v^\varepsilon_\delta(t,x)=\begin{cases}\dfrac{f^\varepsilon(u^\varepsilon(t,x+\delta/2))-f^\varepsilon(u^\varepsilon(t,x-\delta/2))}{u^\varepsilon(t,x+\delta/2)-u^\varepsilon(t,x-\delta/2)} & \text{if } u^\varepsilon(t,x+\delta/2)\neq u^\varepsilon(t,x-\delta/2),\\ (f^\varepsilon)'(u^\varepsilon(t,x+\delta/2)) & \text{if } u^\varepsilon(t,x+\delta/2)=u^\varepsilon(t,x-\delta/2),\end{cases}$$

which acts as a velocity for an advection-diffusion equation. Indeed,

$$\begin{cases}\partial_t[u^\varepsilon]_\delta(t,x)+v^\varepsilon_\delta(t,x)\partial_x[u^\varepsilon]_\delta(t,x)=\varepsilon\partial^2_{x,x}[u^\varepsilon]_\delta(t,x) \quad \text{for } (t,x)\in\mathbb{R}_+\times\mathbb{R}\\ [u^\varepsilon]_\delta(0,x)=\dfrac{1}{\delta}\left[u^{\varepsilon 0}*\chi_{[-\delta/2,\delta/2]}\right](x) \quad \text{for } x\in\mathbb{R}.\end{cases}$$

Let us repeat that $\partial_x[u^\varepsilon]_\delta(t,x)=0$ when $u^\varepsilon(t,x+\delta/2)=u^\varepsilon(t,x-\delta/2)$. It would therefore be possible to adopt other definitions for the velocity on such points. Indeed, the solution to the above advection-diffusion equation does not depend on $v^\varepsilon_\delta(t,x)$ at points (t,x) such that $u^\varepsilon(t,x+\delta/2)=u^\varepsilon(t,x-\delta/2)$. Nevertheless, the definition we use here plays a role in the regularity of the velocity, which we now study: we show that $v^\varepsilon_\delta\in\mathcal{C}^\infty(\mathbb{R}_+\times\mathbb{R})$. One can easily check that

$$v^\varepsilon_\delta(t,x)=\int_0^1 f^{\varepsilon\prime}(\theta u^\varepsilon(t,x+\delta/2)+(1-\theta)u^\varepsilon(t,x-\delta/2))\,d\theta,$$

which states the regularity of v^ε_δ and shows furthermore that v^ε_δ is bounded over $\mathbb{R}_+\times\mathbb{R}$. Let us now sum up what we know about $[u^\varepsilon]_\delta$:

- $[u^\varepsilon]_\delta\in\mathcal{C}^\infty(\mathbb{R}_+\times\mathbb{R})$, and

- $[u^\varepsilon]_\delta$ satisfies the uniformly parabolic partial differential equation

$$\partial_t [u^\varepsilon]_\delta(t,x) + v^\varepsilon_\delta(t,x)\partial_x [u^\varepsilon]_\delta(t,x) = \varepsilon \partial^2_{x,x}[u^\varepsilon]_\delta(t,x)$$

  with given velocity $v^\varepsilon_\delta(t,x) \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$.

We can conclude (see [23] or [14], for example) that $[u^\varepsilon]_\delta$ satisfies a *global* maximum principle, namely

$$\inf_{y\in\mathbb{R}} [u^\varepsilon]_\delta(0,y) \le [u^\varepsilon]_\delta(t,x) \le \sup_{y\in\mathbb{R}} [u^\varepsilon]_\delta(0,y) \quad \text{for } t\in\mathbb{R}_+. \tag{2.2}$$

Back to the non-regularized problem. First, $u^{\varepsilon 0}$ being a regularization of u^0, one has

$$\begin{aligned}\inf_{y\in\mathbb{R}} [u^\varepsilon]_\delta(0,y) &\ge \inf_{y\in\mathbb{R}} [u]_\delta(0,y),\\ \sup_{y\in\mathbb{R}} [u^\varepsilon]_\delta(0,y) &\le \sup_{y\in\mathbb{R}} [u]_\delta(0,y);\end{aligned}$$

this is a simple consequence of the fact that

$$[u^\varepsilon]_\delta(0,y) = \frac{1}{\delta}\left(u^0 * \rho^\varepsilon\right) * \chi_{[-\delta/2,\delta/2]}(y) = \frac{1}{\delta}\left(u^0 * \chi_{[-\delta/2,\delta/2]}\right) * \rho^\varepsilon(y) = [u]_\delta * \rho^\varepsilon(0,y).$$

Thus, combining this with (2.2), one obtains

$$\inf_{y\in\mathbb{R}} [u]_\delta(0,y) \le [u^\varepsilon]_\delta(t,x) \le \sup_{y\in\mathbb{R}} [u]_\delta(0,y) \quad \text{for } t\in\mathbb{R}_+.$$

Recall (see [8], for example) that the entropy solution u to (1.1, 1.2) is such that

$$\lim_{\varepsilon\to 0} u^\varepsilon = u \text{ in } \mathcal{C}^0\left([0,T], L^1_{\mathrm{loc}}(\mathbb{R})\right)$$

for $T\in\mathbb{R}_+$. Furthermore,

$$\left|[u^\varepsilon]_\delta(t,x) - [u]_\delta(t,x)\right| = \left|\int_{x-\delta/2}^{x+\delta/2} u^\varepsilon(t,y) - u(t,y)\,dy\right| \le \int_{x-\delta/2}^{x+\delta/2} |u^\varepsilon(t,y) - u(t,y)|\,dy,$$

so that we have, for $\delta > 0$:

$$\lim_{\varepsilon\to 0} [u^\varepsilon]_\delta(t,x) = [u]_\delta(t,x) \quad \text{for } (t,x)\in[0,\infty)\times\mathbb{R}.$$

We finally have the estimate

$$\inf_{y\in\mathbb{R}} [u]_\delta(0,y) \le [u]_\delta(t,x) \le \sup_{y\in\mathbb{R}} [u]_\delta(0,y) \quad \text{for } t \ge 0,$$

which is a global maximum principle.

Local maximum principle. One important property of entropy solutions of Equation (1.1) is the finite speed of propagation. Due to this property, the global maximum principle showed above is local. This ends the proof. ∎

3. Convergence of reconstruction schemes

As mentioned in the introduction, we consider finite volume approximations of (1.1, 1.2) of the form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(f_{j+1/2}^n - f_{j-1/2}^n), \quad n \in \mathbb{N}, j \in \mathbb{Z}, \tag{3.1}$$

where u_j^n is intended to represent the value of solution u in the space cell $C_j = [(j-1/2)\Delta x, (j+1/2)\Delta x)$ at time $n\Delta t$. The numerical initial condition u_j^0 is given by

$$u_j^0 = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u^0(x)\,dx, \quad j \in \mathbb{Z}. \tag{3.2}$$

We propose to compute the numerical fluxes $(f_{j+1/2}^n)_{n\in\mathbb{N}, j\in\mathbb{Z}}$ using a three-step procedure:

- given a constant-in-cell function, compute a reconstructed function that contains more details;
- compute the exact (entropy) solution at time Δt of (1.1) with the reconstructed function as initial condition;
- "project" this exact solution on the mesh in order to obtain a constant-in-cell function for the following time step.

Note that the last two steps are equivalent to computing the *fluxes* of the exact solution, which shows the finite volume form of the algorithm.

Each of these steps can be associated to an operator: we shall call $\mathcal{R}$, $\mathcal{E}$ and $\mathcal{P}$ respectively the reconstruction, the exact resolution and the projection operators. Let us provide a more precise definition of them.

Definition 3.1.

1 *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a constant-in-cell function.*
$\mathcal{R}u : \mathbb{R} \longrightarrow \mathbb{R}$ denotes the reconstruction of u (not a priori *constant-in-cell).*

2 *Let $t \in \mathbb{R}$ and $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a function in $L^\infty(\mathbb{R})$.*
$\mathcal{E}(t)u : \mathbb{R} \longrightarrow \mathbb{R}$ denotes the exact entropy solution at time t of Equation (1.1) with initial condition u.

3 *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a function in $L^\infty(\mathbb{R})$.*
$\mathcal{P}u : \mathbb{R} \longrightarrow \mathbb{R}$ denotes the projection of u on the mesh:

$$\mathcal{P}u(x) = \sum_{j\in\mathbb{Z}} u_j \chi_{C_j}(x)$$

where χ_{C_j} denotes the characteristic function of
$C_j = [(j-1/2)\Delta x, (j+1/2)\Delta x)$, and where $u_j = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u(x)\,dx$.

Let us now define the approximate solution $\overline{u}^n : \mathbb{R} \longrightarrow \mathbb{R}$ at the time step n by

$$\overline{u}^n(x) = \sum_{j\in\mathbb{Z}} u_j^n \chi_{C_j}(x).$$

The scheme is then defined by

$$\overline{u}^{n+1} = \mathcal{P}\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n. \tag{3.3}$$

For example, if $\mathcal{R}u = u$ for every constant-in-cell function u, the resulting scheme, which does not involve reconstruction, is the Godunov scheme. If $\mathcal{R}u$ is an affine-in-cell function for every constant-in-cell function u, the resulting scheme can be a MUSCL scheme as studied in [3]. Nevertheless, we do not use such characterizations of the reconstruction form in the following.

The last two steps (exact computation and projection) are solved in a single step by taking as numerical fluxes in Equation (3.1)

$$f_{j+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} f\left(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x)\right) ds, \quad n \in \mathbb{N}, j \in \mathbb{Z}. \tag{3.4}$$

This shows the finite volume form of the reconstruction scheme.

In this paper, we will not insist on the second step of the algorithm, i.e. the exact computation of the solution with the reconstructed initial condition. We assume that it is possible to compute it and focus only on the reconstruction step (the projection step being classical).

The equivalent formulations (3.3) and (3.1, 3.2, 3.4) will be alternatively used.

Before beginning the numerical analysis, let us introduce the following notation:

$$\begin{cases} m = \inf_{j\in\mathbb{Z}} u_j^0, \\ M = \sup_{j\in\mathbb{Z}} u_j^0. \end{cases} \tag{3.5}$$

3.1. Conservativity. We consider only conservative reconstructions such that

$$\frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \mathcal{R}u(x)\,dx = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u(x)\,dx = \mathcal{P}u(j\Delta x), \quad j \in \mathbb{Z} \tag{3.6}$$

for every constant-in-cell function u. The exact operator $\mathcal{E}(t)$ and the projection $\mathcal{P}$ being conservative, the whole scheme defined by (3.3) is consequently conservative.

3.2. L^∞-decrease and decrease of the total variation. We say that a numerical scheme of the form (3.1, 3.2) is L^∞-decreasing if and only if for every $u^0 \in L^\infty(\mathbb{R})$,

$$\sup_{j\in\mathbb{Z}} |u_j^{n+1}| \leq \sup_{j\in\mathbb{Z}} |u_j^n|, \quad n \in \mathbb{N}.$$

It is said that a numerical scheme of the form (3.1, 3.2) is Total Variation Diminishing (TVD) if and only if for every $u^0 \in BV(\mathbb{R})$,

$$\sum_{j\in\mathbb{Z}} |u_{j+1}^{n+1} - u_j^{n+1}| \leq \sum_{j\in\mathbb{Z}} |u_{j+1}^n - u_j^n|, \quad n \in \mathbb{N}.$$

In the sequel, the initial condition u^0 is supposed to belong to $BV(\mathbb{R})$.

Thm. 2.1 will help us in exhibiting a new condition for the finite volume reconstruction scheme to be L^∞-decreasing and TVD.

The same notation as in the previous section for the convolved solutions is used and Δx now shall play the role of δ:

$$[u]_{\Delta x} = \frac{1}{\Delta x}\chi_{[-\Delta x/2, \Delta x/2]} * u$$

for any $u \in L^\infty(\mathbb{R})$.

From now on, we assume that there is no sonic point in the computational domain, e.g.,

$$f'(u)>0 \quad \text{for } u\in[m,M]. \tag{3.7}$$

All the following results remain true for the case $f'(u)<0$, $u\in[m,M]$. The case of a change of sign of f' would require a (local) special treatment. One can think of a usual entropy flux, such as the entropy flux of Lax-Friedrichs or Engquist-Osher, locally where the sign of f' changes. This local modification is not the purpose here and it will not be developed.

PROPOSITION 3.2. *Assume that (3.6) (conservativity) and (3.7) (no sonic point in $\mathbb{R}$) hold. Assume that the CFL (Courant-Friedrichs-Lewy) condition $\max_{u\in[m,M]} f'(u)\Delta t\leq\Delta x$ is fulfilled. Assume that the reconstructed solution $\mathcal{R}\overline{u}^n$ satisfies, for $n\in\mathbb{N}$, $j\in\mathbb{Z}$,*

$$\min(u^n_{j-1},u^n_j)\leq[\mathcal{R}\overline{u}^n]_{\Delta x}((j-\theta)\Delta x)\leq\max(u^n_{j-1},u^n_j) \quad \textit{for all } \theta\in[0,1]. \tag{3.8}$$

Then, the scheme given by (3.1, 3.2, 3.4) or (3.3) is L^∞-decreasing and TVD.

This result is not obvious because the constraint (3.8) does not bound the total variation of the reconstructed solution. Let us insist on this fact: the result states that the important point is the boundedness of the total variation of the *mean value over one cell* of the reconstruction, and not of the reconstruction. The total variation of the unknown itself may dramatically increase during the reconstruction; according to the preceding result, it will then decrease enough after the exact step and the projection. Note that condition (3.8) does not bound the number of local extrema of the reconstructed solution, so that it covers schemes that are not *weakly non-oscillatory* in the sense of [15].

Proof. First note that the conservativity assumption implies that for $n\in\mathbb{N}$, $j\in\mathbb{Z}$, $[\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x)=u^n_j$. Thus, condition (3.8) is equivalent to

$$\begin{aligned}&\min([\mathcal{R}\overline{u}^n]_{\Delta x}((j-1)\Delta x),[\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x))\\ &\leq[\mathcal{R}\overline{u}^n]_{\Delta x}((j-\theta)\Delta x)\\ &\leq\max([\mathcal{R}\overline{u}^n]_{\Delta x}((j-1)\Delta x),[\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x)) \quad \text{for all } \theta\in[0,1].\end{aligned}$$

We here use the formulation (3.3) of the scheme. Under the CFL condition, the interval of dependence of $(\Delta t,j\Delta x)$ is included in $[(j-1)\Delta x,j\Delta x]$ (recall that $f'(u)>0$ for every $u\in[m,M]$). Note that $[(j-1)\Delta x,j\Delta x]$ is the set of convex combinations of $(j-1)\Delta x$ and $j\Delta x$: $[(j-1)\Delta x,j\Delta x]=\{x\in\mathbb{R}$ s.t. there exists $\theta\in[0,1]$ s.t. $x=\theta(j-1)\Delta x+(1-\theta)j\Delta x\}$. Thus, a direct consequence of Thm. 2.1 is that

$$\begin{aligned}&\min_{\theta\in[0,1]}[\mathcal{R}\overline{u}^n]_{\Delta x}(\theta(j-1)\Delta x+(1-\theta)j\Delta x)\\ &\leq[\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x)\\ &\leq\max_{\theta\in[0,1]}[\mathcal{R}\overline{u}^n]_{\Delta x}(\theta(j-1)\Delta x+(1-\theta)j\Delta x),\end{aligned}$$

and thus, by (3.8), we recover

$$\min(u^n_{j-1},u^n_j)\leq[\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x)\leq\max(u^n_{j-1},u^n_j)$$

arguing that $\theta(j-1)\Delta x+(1-\theta)j\Delta x=(j-\theta)\Delta x$. It remains to note that $[\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x)=\mathcal{P}\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n(j\Delta x)=u_j^{n+1}$, and we obtain

$$\min(u_{j-1}^n,u_j^n)\leq u_j^{n+1}\leq \max(u_{j-1}^n,u_j^n).$$

This is true for every $n\in\mathbb{N}$ and every $j\in\mathbb{Z}$, and, by a classical argument of incremental analysis of Le Roux and Harten (see [19] and [12]), the scheme is L^∞-decreasing and TVD. ∎

The conclusion of this section is the following convergence result.

Theorem 3.3. *Let us consider the scheme defined by (3.3) with constraints (3.6) and (3.8). Assume that (3.7) is satisified. Define the approximate solution as*

$$\overline{u}_{\Delta x}^{\Delta t}(t,x)=\sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{Z}}u_j^n\chi_{[n\Delta t,(n+1)\Delta t)}(t)\chi_{C_j}(x).$$

Then, for any sequences $(\Delta t_k)_{k\in\mathbb{N}}$, $(\Delta x_k)_{k\in\mathbb{N}}$ converging to 0 and that satisfy the CFL condition $\Delta t_k/\Delta x_k\leq\frac{1}{\max_{u\in[m,M]}f'(u)}$, there exists a sequence $\overline{u}_{\Delta x_{k_j}}^{\Delta t_{k_j}}(t,x)$ that converges in $L^\infty((0,T),L^1_{loc}(\mathbb{R}))$ for $T\in\mathbb{R}_+$ and whose limit is a weak solution to (1.1, 1.2).

This is a classical consequence of conservativity, L^∞-stability and of the decrease of the total variation: see [8] for example.

3.3. Numerical entropy inequalities. Thm. (3.3) does not imply entropy convergence, i.e. convergence toward the unique entropy solution to (1.1, 1.2). For this purpose, we need the scheme (that is to say: the reconstruction operator) to satisfy some entropy inequalities. This is the aim of this section. We first show a simple and constructive condition that ensures the existence of numerical entropy fluxes for *one strictly convex* entropy in Sec. 3.3.1. This is sufficient to ensure entropy convergence in the case of a strictly convex flux f, but not in the general case. A more formal condition is then given to obtain the entropy convergence for any $f\in\mathcal{C}^1(\mathbb{R})$ in Sec. 3.3.2.

3.3.1. Decrease of one entropy. Here is derived a sufficient condition on the reconstruction for the global scheme to be entropy consistent for one entropy. This is not sufficient for the scheme to be convergent toward the unique Krushkov entropy solution in the general case, but it is well-known that it is sufficient when f is strictly convex, provided that the chosen entropy is strictly convex too. Let S be the entropy and G the associated entropy flux.

A very useful technique to prove the entropy convergence of a numerical solution is to exhibit some *discrete entropy fluxes*.

Definition 3.4. *Let (S,G) be an entropy-entropy flux pair. It is said that scheme (3.1, 3.2, 3.4) has discrete entropy fluxes relatively to (S,G) if and only if for every $(u_j^n)_{j\in\mathbb{Z}}$ there exists $(G_{j+1/2}^n)_{j\in\mathbb{Z}}$ such that*

- *$G_{j+1/2}^n$ is consistent with G (in the classical sense of finite volume);*
- *One has:*

$$S_j^{n+1}\leq S_j^n-\frac{\Delta t}{\Delta x}\left(G_{j+1/2}^n-G_{j-/2}^n\right),\quad n\in\mathbb{N},\,j\in\mathbb{Z} \tag{3.9}$$

$$\textit{with} \qquad S_j^n = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{R}\overline{u}^n(x))\,dx. \tag{3.10}$$

Remark 3.5. *We here use the definition of [3], taking* $S_j^n = 1/\Delta x \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{R}\overline{u}^n(x))\,dx$ *instead of* $S_j^n = S(u_j^n)$. *From an algorithmic point of view, the schemes we will consider in the following shall make use of both* the unknown u_j^n *and* the entropy associated to it, S_j^n. *This is already the idea in [2].*

It seems reasonable, as the exact resolution is used, to take the exact flux (similarly to Equation (3.4)) as entropy flux $G_{j+1/2}^n$,

$$G_{j+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x))\,ds, \quad n\in\mathbb{N},\, j\in\mathbb{Z}. \tag{3.11}$$

Equation (3.9) acts like a constraint on the reconstruction procedure.

Remark 3.6. *In case of a stationary shock at $x_{j+1/2}$, the entropy flux $G(u)$ is discontinuous for all time at $x_{j+1/2}$, and formula (3.11) is ambiguous. This case is excluded thanks to the assumption that $f' \neq 0$.*

A stronger entropy inequality is also proposed in

Proposition 3.7. *Assume that the reconstructed solution $\mathcal{R}\overline{u}^n$ satisfies, for any $n\in\mathbb{N}$ and any $j\in\mathbb{Z}$,*

$$S_j^n \le \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^{n-1}(x))\,dx. \tag{3.12}$$

Then, the scheme given by (3.1, 3.2, 3.4), or (3.3) owns some discrete entropy fluxes relative to (S,G).

Proof. Because $\mathcal{E}(\Delta t)$ is the exact (entropy) operator, we get

$$\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n(x)) - S(\mathcal{R}\overline{u}^n(x))\,dx + \left(\int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x)) - G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j-1/2)\Delta x))\,ds\right) \le 0,$$

so that

$$\begin{aligned} S_j^{n+1} \le \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n(x))\,dx \le S_j^n \\ - \frac{1}{\Delta x}\left(\int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x)) - G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j-1/2)\Delta x))\,ds\right), \end{aligned}$$

and the discrete entropy fluxes are given by

$$G_{j+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x))\,ds, \quad n\in\mathbb{N},\, j\in\mathbb{Z}. \qquad \square$$

Remark 3.8. *The set of reconstructed solutions satisfying (3.12) is not empty and contains the non-reconstructed solution, thanks to Jensen's inequality:*

$$S(u_j^n) = S\left(\frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \mathcal{E}(\Delta t)\mathcal{R}\overline{u}^{n-1}(x)\,dx\right)$$
$$\leq \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^{n-1}(x))\,dx.$$

Thus, the set of reconstructions satisfying (3.9, 3.10, 3.11) contains the non-reconstructed solution.

The interest of the existence of numerical entropy fluxes stands in the following well-known theorem.

Theorem 3.9. *Let us consider the scheme defined by (3.3) with constraints (3.6), (3.8) and (3.9) (conservativity, stability, existence of entropy fluxes). Assume that (3.7) is fulfilled. Then, any weak solution to (1.1, 1.2) that is a limit point of approximate solutions as in Thm. 3.3 satisfies*

$$\partial_t S(u) + \partial_x G(u) \leq 0.$$

3.3.2. Decrease of any entropy. We here propose a way to ensure the decrease of any Krushkov entropy, that is, the decrease of any entropy of the form $S_k(u) = |u-k|$ with $k\in\mathbb{R}$, with entropy flux $G_k(u) = \mathrm{sgn}(u-k)(f(u)-f(k))$. This will be done with the help of the entropy inequality (3.12). By this we mean that we want to ensure that

$$\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} |\mathcal{R}\overline{u}^n(x)-k|\,dx$$
$$\leq \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \left|\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^{n-1}(x)-k\right|dx. \quad k\in\mathbb{R},\, j\in\mathbb{Z}, \tag{3.13}$$

This section is essentially based on a link between the Krushkov entropies and the theory of rearrangement. We make the use of the decreasing (resp. increasing) rearrangement of a function on a bounded interval. Following [21], let us recall:

Definition 3.10. *Let $f\in L^1((a,b),\mu)$ where μ is the Lebesgue measure on (a,b). The* decreasing *rearrangement $f_\downarrow$ and increasing rearrangement $f_\uparrow$ are is defined by*

$$f_\downarrow(x) = \sup\{y \text{ s.t. } \mu\{u \text{ s.t. } f(u)>y\} > x-a\} \quad x\in(a,b), \text{and}$$
$$f_\uparrow(x) = \inf\{y \text{ s.t. } \mu\{u \text{ s.t. } f(u)<y\} > x-a\}, \text{ where } x\in(a,b)$$

Remark 3.11. *It is said that two functions $f,g\in L^1((a,b),\mu)$ are* equimeasurable *if and only if $\mu\{u \text{ s.t. } f(u)>y\} = \mu\{u \text{ s.t. } g(u)>y\}$, $y\in\mathbb{R}$. The decreasing (resp. increasing) rearrangement $f_\downarrow$ (resp. $f_\uparrow$) is a decreasing (resp. increasing) function equimeasurable to f, i.e. $\mu\{u \text{ s.t. } f_\downarrow(u)>y\} = \mu\{u \text{ s.t. } f(u)>y\}$, $y\in\mathbb{R}$. Another straightforward property of these rearrangements is $\int_a^b f_\downarrow(x)\,dx = \int_a^b f_\uparrow(x)\,dx = \int_a^b f(x)\,dx$.*

In the following, we consider the decreasing or increasing rearrangement of a function with $a=(j-1/2)\Delta x$ and $b=(j+1/2)\Delta x$.

THEOREM 3.12. *The reconstruction operation satisfies (3.13) if and only if it is conservative (Equation (3.6)) and*

$$\int_{(j-1/2)\Delta x}^{y} (\mathcal{R}u)_{\downarrow}(x)\,dx \le \int_{(j-1/2)\Delta x}^{y} u_{\downarrow}(x)\,dx, \qquad y\in[(j-1/2)\Delta x,(j+1/2)\Delta x], \quad j\in\mathbb{Z}. \tag{3.14}$$

Symmetrically, the reconstruction operation satisfies (3.13) if and only if it is conservative and

$$\int_{(j-1/2)\Delta x}^{y} (\mathcal{R}u)_{\uparrow}(x)\,dx \ge \int_{(j-1/2)\Delta x}^{y} u_{\uparrow}(x)\,dx, \qquad y\in[(j-1/2)\Delta x,(j+1/2)\Delta x], \quad j\in\mathbb{Z}.$$

Proof. This is a direct consequence of a theorem by Hardy, Littlewood and Pòlya, found in [11], which says that if $f,g\in L^1((a,b),\mu)$, then

$$\int_a^y f_{\downarrow}(x)\,dx \le \int_a^y g_{\downarrow}(x)\,dx \quad \text{for } y\in(a,b)$$
$$\text{and } \int_a^b f_{\downarrow}(x)\,dx = \int_a^b g_{\downarrow}(x)\,dx$$

if and only if

$$\int_a^b |f(x)-k|\,dx \le \int_a^b |g(x)-k|\,dx \quad \text{for } k\in\mathbb{R}.$$

Let us also mention [10, 7, 1] for developments.

Thus the family of inequalities

$$\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} |\mathcal{R}u(x)-k|\,dx \le \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} |u(x)-k|\,dx \quad \text{for } k\in\mathbb{R}$$

is equivalent to

$$\int_{(j-1/2)\Delta x}^{y} (\mathcal{R}u)_{\downarrow}(x)\,dx \le \int_{(j-1/2)\Delta x}^{y} u_{\downarrow}(x)\,dx \quad \text{for } y\in[(j-1/2)\Delta x,(j+1/2)\Delta x]$$

$$\text{and } \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} (\mathcal{R}u)_{\downarrow}(x)\,dx = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_{\downarrow}(x)\,dx.$$

Now recall that

$$\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} (\mathcal{R}u)_{\downarrow}(x)\,dx = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \mathcal{R}u(x)\,dx$$

and

$$\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_{\downarrow}(x)\,dx = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u(x)\,dx.$$

Condition $\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} (\mathcal{R}u)_{\downarrow}(x)\,dx = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_{\downarrow}(x)\,dx$ thus may be rewritten as $\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \mathcal{R}u(x)\,dx = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u(x)\,dx$, which is exactly the conservativity Assumption (3.6).

The equivalence involving the increasing rearrangement can be shown in the same way or by remarking that $f_{\uparrow} = -(-f)_{\downarrow}$. ∎

Thm. 3.12 is of particular interest when the chosen reconstructed solution is either decreasing in $[(j-1/2)\Delta x, (j+1/2)\Delta x]$ (then, $(\mathcal{R}u)_{\downarrow} = \mathcal{R}u$) or increasing in $[(j-1/2)\Delta x, (j+1/2)\Delta x]$ (then, $(\mathcal{R}u)_{\uparrow} = \mathcal{R}u$).

Applying Thm. 3.9 with all Krushkov entropies leads to the convergence of approximate solutions toward the unique Krushkov solution to (1.1, 1.2).

As a conclusion of this paper on general reconstruction schemes, let us point out that

- condition (3.8) is weaker than the "no sawtooth" condition from [3] (indeed, saw-teeth are here allowed, the stability condition being required on the *convolution* of the saw-teeth), and
- condition (3.14) is necessary and sufficient for the reconstruction to verify conservativity and the decrease of every Krushkov entropy.

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