

FAST COMMUNICATION

MODERATE DISPERSION IN CONSERVATION LAWS WITH  
CONVEX FLUXES\*

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**Abstract.** We consider the weakly dissipative and weakly dispersive Burgers-Hopf-Korteweg-de-Vries equation with the diffusion coefficient  $\varepsilon$  and the dispersion rate  $\delta$  in the range  $\delta/\varepsilon \rightarrow 0$ . We study the traveling wave connecting  $u(-\infty) = 1$  to  $u(+\infty) = 0$  and show that it converges strongly to the entropic shock profile as  $\varepsilon, \delta \rightarrow 0$ .

**Key words.** traveling waves, moderate dispersion, Korteweg de Vries equation, entropy solutions, dissipative-dispersive scalar conservation laws

**AMS subject classifications.** 35B25, 35Q53, 35L67

1. Introduction

This paper is concerned with hyperbolic conservation laws approximated by a weakly dissipative and weakly dispersive equation; that is, we are interested in the limit as  $\varepsilon \rightarrow 0$ ,  $\delta \rightarrow 0$  of the solutions of the following problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(A(u)) &= \varepsilon \frac{\partial^2 u}{\partial x^2} - \delta \frac{\partial^3 u}{\partial x^3}, \\ u(t, 0) &= u^0(x). \end{aligned} \tag{1.1}$$

It is well known that when the parameter  $\varepsilon$  vanishes too fast compared to  $\delta$ , the dispersive effects dominate and produce oscillations. In that case the (weak) limit is not a weak solution to the conservation law with  $\varepsilon = \delta = 0$  ([9]). Therefore several authors have considered the *weak dispersion* case  $\delta = \alpha\varepsilon^2$ , with  $0 < \alpha < \infty$  fixed, showing that the solutions of (1.1) converge strongly ([7, 8, 11]) to weak solution to the equation with  $\varepsilon = \delta = 0$ . Such limits may lead to non-entropic solutions [1, 6, 7, 10, 11] for non-convex fluxes  $A(u)$ . However, for strictly convex fluxes and in this weak dispersion regime  $\delta = \alpha\varepsilon^2$  with  $\alpha > 0$  fixed, one expects that the (strong) limits always satisfy the family of Kruzkov entropies; this is proved in [10] for instance for traveling waves. For general solutions to the initial value problem in the weak dispersion regime, it is easy to prove that the square entropy satisfies the entropy inequality but there is no direct derivation of the full family of entropy inequalities. However, an indirect argument due to R. DiPerna indicates that for convex fluxes a single entropy inequality implies all the others; the original argument uses the *BV* regularity of solutions (such a bound is not available here) and this regularity assumption has been removed recently in [3, 4]. But we are not aware of any related result in the moderate dispersion regime.

The purpose of this paper is to further investigate the case of convex fluxes, on

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the simple example of the Burgers-Hopf equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = \varepsilon \frac{\partial^2 u}{\partial x^2} - \delta \frac{\partial^3 u}{\partial x^3}. \tag{1.2}$$

Then we sustain, by means of the study of traveling waves, the idea that the entropy inequalities are reached in the more general limit

$$\frac{\delta}{\varepsilon} \rightarrow 0 \quad (\text{moderate dispersion}), \tag{1.3}$$

and that the condition  $\delta = O(\varepsilon^2)$  is perhaps too stringent. More precisely we prove the

**THEOREM 1.1.** *There exists a (unique up to translation) traveling wave solution  $u(t, x) = S_{\varepsilon, \delta}(x - t/2)$  of (1.2) connecting the states  $u(-\infty) = 1$  to  $u(+\infty) = 0$  which converges strongly, as  $\varepsilon, \delta \rightarrow 0$ , together with (1.3), to the entropic shock profile  $\bar{u}(t, x) = \bar{S}(x - t/2)$  with*

$$\bar{S}(x) = \begin{cases} 1, & \text{for } x < 0, \\ 0, & \text{for } x > 0. \end{cases}$$

The proof of this theorem is given in the next section. We first recall several facts on traveling waves and rescale the problem to settle a clearer asymptotic problem. Then, we study the limiting case  $c = 0$  which serves as a basis for the expansions performed in the study of the general traveling wave. Our analysis indicates that when  $\delta = \alpha\varepsilon$  with  $\alpha > 0$  fixed then the traveling wave  $S_{\varepsilon, \delta}$  converges only weakly and convergence to the entropic shock breaks down at this level. In that sense, the result of Theorem 1.1 is sharp.

**2. Convergence of Burgers-KdV traveling wave solutions**

**2.1. Rescaled traveling waves.** We now look for traveling wave solutions of the Burgers-KdV equation

$$u_t + uu_x = \varepsilon u_{xx} - \delta u_{xxx} \tag{2.1}$$

that connect  $u = 1$  as  $x \rightarrow -\infty$  and  $u = 0$  as  $x \rightarrow +\infty$ , and move with the correct speed  $c_0 = 1/2$  (the fact that this is the only possible speed for a traveling wave connecting these two states follows immediately from the Rankine-Hugoniot condition), as is the case for an entropic shock of the Burgers equation. Such solutions have the form  $u(t, x) = S_{\varepsilon, \delta}(x - t/2)$ , where the function  $S_{\varepsilon, \delta}(x)$  solves

$$-\frac{1}{2}S'_{\varepsilon, \delta} + S_{\varepsilon, \delta}S'_{\varepsilon, \delta} = \varepsilon S''_{\varepsilon, \delta} - \delta S'''_{\varepsilon, \delta}, \quad S_{\varepsilon, \delta}(-\infty) = 1, \quad S_{\varepsilon, \delta}(+\infty) = 0. \tag{2.2}$$

Integrating between  $-\infty$  and  $x$  we obtain

$$-\frac{1}{2}S_{\varepsilon, \delta} + \frac{S_{\varepsilon, \delta}^2}{2} = \varepsilon S'_{\varepsilon, \delta} - \delta S''_{\varepsilon, \delta}, \quad S_{\varepsilon, \delta}(-\infty) = 1, \quad S_{\varepsilon, \delta}(+\infty) = 0. \tag{2.3}$$

We also rescale and reverse the direction:  $t = -x/\sqrt{\delta}$  and arrive at

$$S_{\varepsilon, \delta}(x) = S_c(-x/\sqrt{\delta}) \quad \text{with } c = \varepsilon/\sqrt{\delta}, \tag{2.4}$$

and

$$-\frac{1}{2}S_c + \frac{S_c^2}{2} = -cS'_c - S''_c, \quad S_c(-\infty) = 0, \quad S_c(+\infty) = 1. \tag{2.5}$$

Note that the function  $\phi(t) = 1 - S_c(t)$  satisfies the familiar Fisher-KPP equation

$$-c\phi' = \phi'' + \frac{1}{2}\phi(1 - \phi), \quad \phi(-\infty) = 1, \quad \phi(+\infty) = 0. \tag{2.6}$$

It follows that there are *three regimes* for the traveling front solutions of (1.2).

- The first one arises when  $c \geq c_* = \sqrt{2}$ , the minimal KPP speed. Then equation (2.6) admits monotonic traveling wave solutions. In terms of (2.1) this means that such solutions exist for  $\varepsilon^2/\delta \geq 2$ . After rescaling as in (2.4) and normalizing  $S_c(0) = 1/2$  we observe that  $S_{\varepsilon,\delta}(x)$  converges pointwise to the entropic shock profile

$$\bar{S}(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x > 0. \end{cases} \tag{2.7}$$

This regime has been widely studied, see [5] and the references therein.

- The second regime, when  $0 < c < c_*$ , is different — a traveling wave still exists for all  $\varepsilon > 0, \delta > 0$ , that is, for all  $c > 0$  but it is no longer monotonic in  $x$ . Nevertheless, as long as we keep  $\varepsilon/\sqrt{\delta} \geq c_o$  for any fixed  $c_o > 0$  and let  $\varepsilon, \delta \rightarrow 0$ , the wave  $S_{\varepsilon,\delta}$  converges to the shock (2.7), and here again an additional normalization is needed to fix the location of the traveling wave:

$$S'_c(x) > 0 \text{ for } x < 0 \quad \text{and} \quad S'_c(0) = 0, \tag{2.8}$$

a normalization we will use throughout the paper. This has been shown in [2], see also [7, 6, 10] for the same conclusions on the initial value problem. In terms of (2.6) this means that the speed  $c$  is bounded away from zero:  $c \geq c_o > 0$ .

- Here we are interested in what happens in the third regime, when  $c$  becomes small but so that  $\delta/\varepsilon$  vanishes. We would like to show that for the wave  $S_{\varepsilon,\delta}(x)$  the picture is as follows (arguing backward from  $+\infty$  to  $-\infty$ : for  $x > 0$  the solution decays to zero on the length scale  $\lambda = \sqrt{\delta}/c$ ; this region is followed by an interval of a comparable length where the solution oscillates between  $u = 0$  and  $u = 3/2$ , which is finally followed by a region where the solution oscillates on the length scale  $\lambda_1 = \sqrt{\delta}$  and approaches the value  $u = 1$  at the exponential rate  $k = 1/\lambda$ . This means that the traveling wave converges pointwise to a shock profile as  $\varepsilon, \delta \rightarrow 0$ , as long as  $\lambda \rightarrow 0$ . In terms of the parameters  $\varepsilon$  and  $\delta$  this translates into  $\delta/\varepsilon \rightarrow 0$  as opposed to the case  $\delta \leq c_o^2 \varepsilon^2$  studied previously in [2]. The main difference compared to the latter regime is that there is a region inside the “viscous shock” where oscillations are strong.

In terms of the wave  $S_c(t)$  we have to show the following: (i) in the first zone (the monotonicity region),  $t < 0$ , we have  $S'_c(t) > 0$  and there exists a constant  $B > 0$ , independent of  $c > 0$  so that  $0 < S_c(t) \leq B e^{t/B}$ ; (ii) in the second zone (the transient oscillations region), between  $x = 0$  and  $x = O(1/c)$ , the wave  $S_c(t)$  oscillates between the values 0 and 3/2 with the period  $O(c^{-1/4})$ ; (iii) in the third zone (the exponential damping region),  $x > B/c$ , we have  $|1 - S_c(x)| \leq B e^{-c(x-B/c)}$  but  $S_c$  is still an oscillatory function with the period equal to  $O(1)$  — and these bounds should hold with a constant  $B$  independent of  $c > 0$ . A profile of  $S_c(x)$  with  $c = .05$  is depicted in Figure 2.2 with translated abscissae.

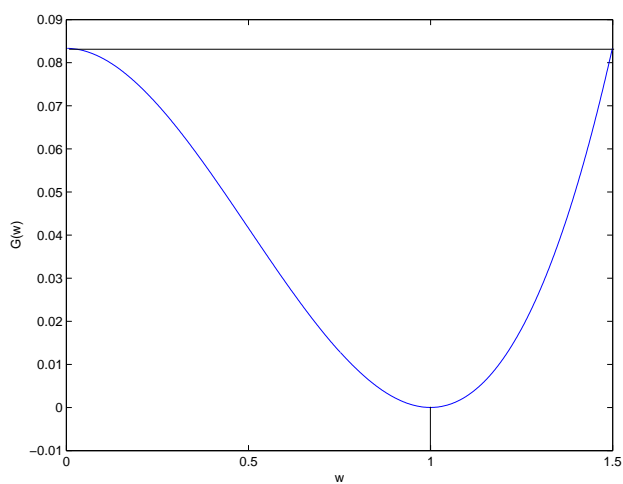


FIG. 2.1. The potential energy, the function  $G(w)$ .

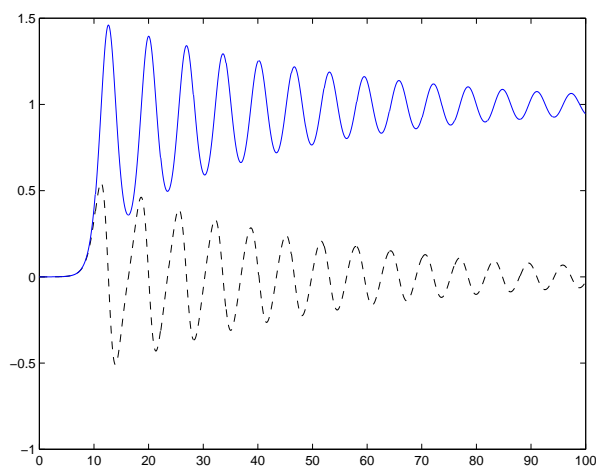


FIG. 2.2. Solid line: a profile of  $S_c(x)$  with  $c = .05$ ,  $w(0) = 5 \cdot 10^{-5}$ . Dashed line: the derivative  $S'_c(x)$ .

For this purpose, it is convenient to introduce the phase space variables  $w = S_c$  and  $v = -S'_c$  to rewrite (2.5)–(2.8) as

$$\begin{cases} w' = -v, \\ v' = -cv - \frac{1}{2}w(1-w), \\ w(-\infty) = v(-\infty) = 0, \quad v(0) = 0, \quad w(+\infty) = 1, \quad v(+\infty) = 0. \end{cases} \quad (2.9)$$

This system has an energy

$$H(v, w) = \frac{v^2}{2} + G(w), \quad \frac{d}{dt}H(v(t), w(t)) = -cv(t)^2, \quad (2.10)$$

with  $G'(w) = -w(1-w)/2$ . We choose the normalization so that  $\min G = G(1) = 0$ , which means that

$$G(w) = \frac{w^3}{6} - \frac{w^2}{4} + \frac{1}{12} = \frac{(1-w)^2(1+2w)}{12}.$$

This function is plotted in Figure 2.1.

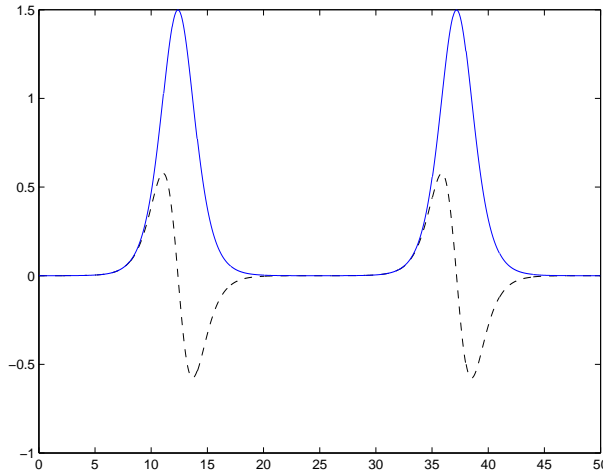


FIG. 2.3. Large energy periodic solution. Solid line: a profile of  $S_c(x)$  with  $c=0$ ,  $w(0) = 5 \cdot 10^{-5}$ . Dashed line: the derivative  $S'_c(x)$ .

**2.2. The periodic solutions for  $c=0$ .** In order to study the problem with  $c > 0$  we first recall the basic properties of the periodic solutions that exist when  $c=0$ . Then, the system (2.9) becomes

$$\begin{cases} w' = -v, \\ v' = -\frac{1}{2}w(1-w), \end{cases} \quad (2.11)$$

and the energy is conserved. The energy level,  $0 \leq H = H(0, w(0)) = G(w(0)) < 1/12 = G(0)$ , characterizes the solution.

**The period.** There exists a global solution (corresponding to  $T = \infty$ ) with the maximal energy  $H = 1/12$  and  $w > 0$ , which corresponds to a homoclinic orbit of (2.11) that connects  $(0, 0)$  to itself. Other solutions with energies  $0 \leq H < 1/12$  are periodic, with the period  $T$  given in terms of the energy  $H$  by

$$T = 2 \int_{w_1(H)}^{w_2(H)} \frac{dw}{\sqrt{2(H - G(w))}}. \quad (2.12)$$

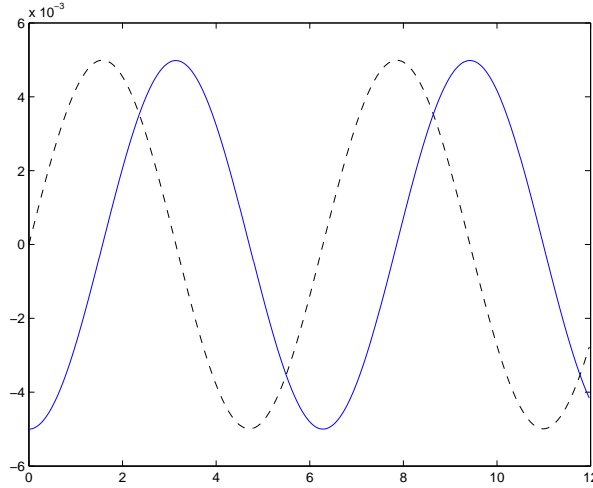


FIG. 2.4. *Small energy periodic solution. Solid line: a profile of  $S_c(x) - 1$  with  $c=0$ ,  $w(0) = 1 - 0.005$ . Dashed line: the derivative  $S'_c(x)$ .*

Here  $w_1(H) < w_2(H)$  are the two solutions of  $G(w_{1,2}(H)) = H$ , that is, the minimum and maximum values of the function  $w(t)$  on the trajectory. We have two limiting cases: (maximal energy)  $w_1(1/12) = 0$  and  $w_2(1/12) = 3/2$ , and (minimal energy)  $w_1(0) = w_2(0) = 1$ .

**Large energies,  $H \lesssim 1/12$ .** Such a periodic solution, with  $w(0) = 5 \cdot 10^{-5}$ , is depicted in Figure 2.3. The periodic solutions with  $H$  close to  $H = 1/12$  spend most of the time close to the minimal value  $w_1(H) \approx 0$  as can be seen from Figure 2.3 – the reason is that they “follow” the bound state for a long time. However, the time they spend between  $w = 1/2$  and  $w = w_2(H) \approx 3/2$  is uniformly (in  $H$ ) bounded from above. Indeed, assume that  $H > H(1/2, 0) = 1/24$ ,  $w(t_1) = 1/2$  with  $v(t_1) > 0$  and  $t_2$  is the first time larger than  $t_1$  such that  $w(t_2) = w_2(H)$ . Then we have

$$t_2 - t_1 = 2 \int_{1/2}^{w_2(H)} \frac{dw}{\sqrt{2(H - G(w))}} \leq C,$$

since  $G'(w_2(H))$  is bounded away from zero for  $H \geq 1/24$ . On the other hand, as  $G'(0) = 0$ , the period  $T(H)$  for  $H \geq 1/24$  is bounded from below:

$$T(H) = 2 \int_{w_1(H)}^{w_2(H)} \frac{dw}{\sqrt{2(H - G(w))}} \geq \frac{1}{\sqrt{2}} \int_{w_1(H)}^{1/2} \frac{dw}{\sqrt{2(H - G(w))}} \geq T_0 > 0.$$

Moreover, as  $H \rightarrow 1/12$  we have

$$T(H) = 2 \int_{w_1(H)}^{w_2(H)} \frac{dw}{\sqrt{2(H - G(w))}} \rightarrow +\infty.$$

For  $H \lesssim 1/12$ , one can also prove that

$$\begin{aligned} w_1(H) &= 2\sqrt{\frac{1}{12} - H} + \frac{4}{3} \left( \frac{1}{12} - H \right) + O\left(\left(\frac{1}{12} - H\right)^{3/2}\right), \\ w_2(H) &= \frac{8}{3} \left( \frac{1}{12} - H \right) + O\left(\left(\frac{1}{12} - H\right)^2\right), \\ T(H) &= -2\sqrt{2}\ln(w_1(H)) + O(1). \end{aligned} \tag{2.13}$$

Indeed, we have, by a simple Taylor expansion,  $w_1(H) = O(\sqrt{\frac{1}{12} - H})$ . Then we obtain from the expression (2.12) for the period that

$$\begin{aligned} T(H) &= 2 \int_{w_1}^{\cdot} \frac{dw}{\sqrt{2(H - \frac{1}{12} + \frac{w^2}{4} - \frac{w^3}{6})}} = 2 \int_0^{\cdot} \frac{ds}{\sqrt{s(w_1 - w_1^2) + s^2(\frac{1}{2} - w_1) - \frac{s^3}{3}}} \\ &= 2 \int_0^{\cdot/w_1} \frac{du}{\sqrt{u(1 - w_1) + u^2(\frac{1}{2} - w_1) - \frac{u^3}{3}w_1}} \\ &\approx 2 \int_0^{\cdot/w_1} \frac{du}{\sqrt{u + \frac{u^2}{2}}} \approx -2\sqrt{2}\ln w_1. \end{aligned}$$

**Small energies,  $H \gtrsim 0$ .** The situation is different for small energies: the period is bounded both from below and from above. Indeed, note that  $G(w)$  is convex for  $w \geq 1/2$  so that

$$H - G(w) \leq -G'(w_1(H))(w - w_1) \quad \text{for } w_1(H) \leq w \leq 1,$$

$$H - G(w) \leq G'(w_2(H))(w_2 - w) \quad \text{for } 1 \leq w \leq w_2(H).$$

Using these bounds we obtain by an elementary calculation

$$T = 2 \int_{w_1(H)}^{w_2(H)} \frac{dw}{\sqrt{2(H - G(w))}} \geq C \left( \frac{1}{\sqrt{w_1(H)}} + \frac{1}{\sqrt{w_2(H)}} \right) \geq T_1 > 0.$$

On the other hand, we have an upper bound for  $G(w)$ :

$$G(w) \leq H - \frac{H}{1 - w_1(H)}(w - w_1(H)) \quad \text{for } w_1(H) \leq w \leq 1,$$

and

$$G(w) \leq H + \frac{H}{w_2(H) - 1}(w - w_2(H)) \quad \text{for } 1 \leq w \leq w_2(H).$$

It follows with the help of another elementary computation that

$$T = 2 \int_{w_1(H)}^{w_2(H)} \frac{dw}{\sqrt{2(H - G(w))}} \leq C \left( \frac{1 - w_1}{\sqrt{H}} + \frac{w_2 - 1}{\sqrt{H}} \right) \leq T_2 < +\infty.$$

In fact, the asymptotic behavior for small energy is clear. The solution approaches the harmonic oscillator  $1 - w(x) \approx (1 - w(0)) \cos(2\pi t)$ .

**2.3. Traveling waves for a small  $c > 0$ .** As we have mentioned, existence of traveling wave solutions for all  $0 < c < c_* = \sqrt{2}$  has been established in [2]. It has been shown that they look as follows. In the first zone  $S_c(t)$  is increasing for  $t < 0$  and has its first maximum at  $x_0 = 0$ . Then,  $S_c(t)$  oscillates with a decreasing amplitude in the following sense: there exists infinite sequences of maxima  $0 = x_0 < x_1 < \dots < x_n < \dots$  and minima  $0 < y_1 < y_2 < \dots < y_n < \dots$  with  $x_{n-1} < y_n < x_n$  so that  $1 < S_c(x_{n+1}) < S_c(x_n) < 3/2$  and  $0 < S_c(y_n) < S_c(y_{n+1}) < 1$ ; see Figure 2.2 for a typical profile (with translated abscissae). Our task is to estimate the differences of  $S_c$  at successive maxima and minima, the distances  $L_n = x_{n+1} - x_n$  and  $l_n = y_{n+1} - y_n$ , and the energy drop  $H_c(x_{n+1}) - H_c(x_n)$ , where  $H_c(t)$  denotes the energy of  $S_c$  as introduced in (2.10). We do that in distinguishing two additional zones (transitory oscillations and exponential damping).

**Transitory oscillations.** Here, we wish to prove

LEMMA 2.1. For  $c > 0$  small enough, and as long as  $\frac{1}{24} \leq H(y_n) < \frac{1}{12}$ , for some constant  $K$ , we have

$$y_{n+1} - y_n \leq a_n := \frac{K}{(cn)^{1/4}}, \quad H(y_{n+1}) - H(y_n) \leq -Kc.$$

*Proof.* We denote by  $K$  a universal constant independent of  $c$  throughout the proof. We need a preliminary and straightforward estimate on solutions  $(w_c(t), v_c(t))$  with  $c > 0$  by those  $(w_0(t_0; t), v_0(t_0; t))$  obtained with  $c = 0$  and the same initial data at  $t = t_0$ ,  $(w_c(t_0), v_c(t_0)) = (w_0(t_0; t_0), v_0(t_0; t_0))$ . This estimate is

$$|v_c(t) - v_0(t_0; t)| + |w_c(t) - w_0(t_0; t)| \leq cC_T \sup_{|s-t_0| \leq T} |v_c(s)|, \text{ for all } |t - t_0| \leq T. \quad (2.14)$$

This is a consequence of the Gronwall lemma applied to the system (2.9), where the term  $-cv$  is considered as a source.

Next, we can consider the energy drop between  $-\infty$  and  $x_0 = 0$ . It is estimated as

$$\frac{1}{12} - H_c(0) = c \int_{-\infty}^0 v_c^2(t) dt.$$

We claim that  $H_c(0) \leq 1/12 - cM$  with a constant  $M$  independent of  $0 < c < c_0$ , for some  $c_0$  small enough. Indeed, we have for  $-1 \leq t \leq 0$ :

$$|v_c(t) - v_0(0; t)| + |w_c(t) - w_0(0; t)| \leq Kc \sup_{-1 \leq s \leq 0} |v_c(s)| \leq K'c, \text{ for all } -1 \leq t \leq 0.$$

This means that

$$\int_{-\infty}^0 v_c^2(t) dt \geq \int_{-1}^0 v_c^2(t) dt \geq \int_{-1}^0 v_0^2(0; t) dt - Kc \geq K,$$

and thus

$$H_c(0) \leq \frac{1}{12} - Kc. \quad (2.15)$$

We also have the following fact: if  $H_c(x_n) \geq 1/24$  then  $x_n - x_{n-1} \geq T_0$  with  $T_0$  independent of  $n$ . On the other hand, we have

$$H_c(x_{n-1}) - H_c(x_n) = c \int_{x_{n-1}}^{x_n} v_c^2(t) dt. \quad (2.16)$$



By the same argument as above we deduce that

$$H_c(x_n) \leq H_c(x_{n-1}) - Kc \quad \text{if } H_c(x_n) \geq 1/24.$$

This means that

$$H_c(x_n) \leq \frac{1}{12} - Kcn \quad \text{if } H_c(x_n) \geq 1/24.$$

As a consequence, we have  $w_c \geq K\sqrt{cn}$  for  $x_{n-1} \leq t \leq x_n$  and, in particular,  $w_c(y_n) \geq K\sqrt{cn}$ .

Let  $z_n \in (y_n, x_n)$  be the first point to the right of  $y_n$  where  $w_c = 3/4$ . In order to estimate  $|x_n - y_n|$  we will now first estimate the distance between  $y_n$  and  $z_n$ . On the interval between  $y_n$  and  $z_n$  the function  $w_c$  satisfies

$$-w_c'' = cw_c' - \frac{1}{2}w_c(1 - w_c),$$

and  $w_c \geq K\sqrt{cn}$ . Therefore, on this interval  $(y_n, z_n)$  we have

$$w_c'' + cw_c' \geq K\sqrt{cn}, \quad w_c(y_n) \geq K\sqrt{nc}, \quad w_c'(y_n) = 0, \quad w_c' \geq 0.$$

It follows that

$$w_c'(t) \geq \frac{K\sqrt{n}}{\sqrt{c}} \left(1 - e^{-c(t-y_n)}\right) \quad \text{for } y_n \leq t \leq z_n. \tag{2.17}$$

Therefore, as long as (2.17) holds and  $t \leq y_n + \frac{1}{100c}$  we get

$$\begin{aligned} w_c &\geq K\sqrt{cn} + \frac{K\sqrt{n}}{\sqrt{c}}(t - y_n) - \frac{K\sqrt{n}}{c\sqrt{c}} \left(1 - e^{-c(t-y_n)}\right) \\ &\geq K\sqrt{cn} + \frac{K\sqrt{n}}{\sqrt{c}}(t - y_n) - \frac{K\sqrt{n}}{c\sqrt{c}} \left(1 - 1 + c(t - y_n) - \frac{c^2}{4}(t - y_n)^2\right) \\ &= K\sqrt{cn}(1 + (t - y_n)^2). \end{aligned}$$

It follows that

$$z_n - y_n \leq \frac{K}{(cn)^{1/4}}.$$

On the other hand, it is straightforward to compute that  $|z_n - x_{n+1}| \leq T_0$  with  $T_0$  uniform in the energies. We see that

$$x_n - y_{n-1} \leq \frac{K}{(cn)^{1/4}}.$$

A similar computation shows that

$$y_n - x_n \leq \frac{K}{(cn)^{1/4}},$$

so that

$$y_n - y_{n-1} \leq a_n = \frac{K}{(cn)^{1/4}},$$

provided that  $H(y_n) \geq 1/24$ . □

We can now conclude the analysis of the transitory zone. Because we have seen that energy drops by  $Kc$  between  $y_n$  and  $y_{n+1}$ , we conclude that the number  $N$  of oscillations before the energy  $H = 1/24$  is reached is bounded by  $N_c = K/c$ . Therefore, the total time it takes to reach this energy level is bounded by

$$L = \sum_{n=1}^{N=K/c} a_n = \sum_{n=1}^{N=K/c} \frac{K}{(cn)^{1/4}} \leq \frac{K}{c^{1/4}} \left(\frac{K}{c}\right)^{3/4} \leq \frac{K}{c}.$$

**Exponential damping.** The third zone is when the energy  $H_c(t)$  is smaller than  $1/24$ . There, using the inequality

$$H'_c(t) = -cv^2 \geq -cH(t),$$

we deduce the lower bound

$$H(t) \geq H(t_0)e^{-c(t-t_0)}.$$

Our purpose is to prove the reverse inequality.

We begin by arguing as before for (2.14)

$$|v_c(t) - v_0(t_0; t)| + |w_c(t) - w_0(t_0; t)| \leq c \sqrt{H_c(t_0)} e^{M(t-t_0)}. \tag{2.18}$$

This proves, again by continuity as  $c \rightarrow 0$  on the rescaled quantities  $v_c(t)/\sqrt{H_c(x_n)}$ ,  $w_c(t)/\sqrt{H_c(x_n)}$ , that

$$x_{n+1} - x_n = O(1), \quad \text{and} \quad y_{n+1} - y_n = O(1). \tag{2.19}$$

Finally, with the similar argument and using again (2.16), we have a bound for energy drop over each oscillation:

$$H_c(x_n) \leq (1 - Kc)H_c(x_{n-1}).$$

This means that, in this third zone, energy decays exponentially at the rate  $cK$ :

$$H_c(x_n) \leq Ke^{-Kc(x_n - K/c)},$$

in other words,  $w_c(t) \rightarrow 1$  exponentially as we have claimed. This finishes the proof of Theorem 1.1.

**2.4. The general nonlinearities.** Our results may be generalized to any strictly convex flux  $f(u)$  and boundary conditions  $u(-\infty) = u_l$  and  $u(+\infty) = u_r$ :

$$u_t + (f(u))_x = \varepsilon u_{xx} - \delta u_{xxx}, \quad u(-\infty) = u_l, \quad u(+\infty) = u_r. \tag{2.20}$$

We look for a traveling wave that moves with the speed  $s = (f(u_r) - f(u_l))/(u_r - u_l)$ :

$$-sS'_{\varepsilon, \delta} + [f(S_{\varepsilon, \delta})]' = \varepsilon S''_{\varepsilon, \delta} - \delta S'''_{\varepsilon, \delta}.$$

Integrating between  $-\infty$  and  $x$  we get

$$-sS_{\varepsilon, \delta} + su_l + f(S_{\varepsilon, \delta}) - f(u_l) = \varepsilon S'_{\varepsilon, \delta} - \delta S''_{\varepsilon, \delta}.$$

Again, after rescaling  $x$  by  $\sqrt{\delta}$  and setting  $c = \varepsilon/\sqrt{\delta}$  we arrive at

$$cS'_c - S''_c = F(S_c), \quad S_c(-\infty) = u_l, \quad S_c(+\infty) = u_r.$$

The nonlinearity  $F$  has the form

$$F(\phi) = -s\phi + su_l + f(\phi) - f(u_l). \tag{2.21}$$

Note that  $F(u_l) = F(u_r) = 0$  — this follows from the Rankine-Hugoniot condition on the speed  $s$ . Moreover, for  $\phi$  between  $u_l$  and  $u_r$  we have

$$F(\phi) = f(\phi) - f(u_l) - \frac{f(u_r) - f(u_l)}{u_r - u_l}(\phi - u_l) \leq 0,$$

since  $f$  is convex. Thus, the situation is again reduced to the KPP since the nonlinearity  $F(\phi)$  is convex in  $\phi$ .

The situation is, of course, completely different in the non-convex case as traveling waves may not exist even if  $\delta = 0$  — that is, in the absence of dispersion. Indeed, if we look for a traveling wave solution of (2.20) with  $\delta = 0$  we arrive simply at

$$cS'_c = F(S_c), \quad S_c(-\infty) = u_l, \quad S_c(+\infty) = u_r,$$

with  $F$  given by (2.21). This equation may not have a solution if there exists a point  $\phi$  between  $u_l$  and  $u_r$  such that  $F(\phi) = 0$ , that is, if the line joining  $(u_l, f(u_l))$  and  $(u_r, f(u_r))$  intersects the graph of  $f(u)$  between  $u_l$  and  $u_r$ . For that to happen the function  $f$  has to be non-convex.

### 3. Conclusion

We have studied the oscillatory traveling waves for the dissipative-dispersive Burgers-Hopf equation and estimated precisely the periods and damping rate. From this study, we deduce that when  $\delta/\varepsilon$  vanishes, the traveling wave converges to an entropic shock. We conjecture that the entropic solutions should be reached in this regime even for the initial value problem.

It is therefore natural to ask the question of the regime  $\delta = \varepsilon$  (or maybe larger). It is easy to be convinced that the corresponding traveling wave converges only weakly and thus does not reach the entropic shock in the limit. Indeed, in the third region of our analysis, the oscillation length of  $S_c$  is of length  $O(1)$  and the damping rate toward 1 is  $e^{-ct}$ . Once rescaled to the actual traveling wave  $S_{\varepsilon, \delta}$  oscillates with the period of the order  $\sqrt{\delta}$  and the damping rate toward 1 is now  $e^{-x}$ . In other words in the third region,  $S_{\varepsilon, \delta}$  converges weakly to 1. It is plausible that the second region remains of size  $O(1)$  and a smooth transition is generated in the weak limit.

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