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A LOGARITHMIC FOURTH-ORDER PARABOLIC EQUATION AND RELATED LOGARITHMIC SOBOLEV INEQUALITIES*

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Abstract. A logarithmic fourth-order parabolic equation in one space dimension with periodic boundary conditions is studied. This equation arises in the context of fluctuations of a stationary nonequilibrium interface and in the modeling of quantum semiconductor devices. The existence of global-in-time non-negative weak solutions and some regularity results are shown. Furthermore, we prove that the solution converges exponentially fast to its mean value in the "entropy norm" and in the Fisher information, using a new optimal logarithmic Sobolev inequality for higher derivatives. In particular, the rate is independent of the solution and the constant depends only on the initial value of the entropy.

Key words. Cauchy problem, higher-order parabolic equations, existence of global-in-time solutions, long-time behavior, Fisher information, entropy–entropy production method, logarithmic Sobolev inequality, Poincaré inequality.

AMS subject classifications. 35K35, 35K55, 35B40

1. Introduction

This paper is concerned with the study of some properties of weak solutions to a nonlinear fourth-order equation with periodic boundary conditions and related logarithmic Sobolev inequalities. More precisely, we consider the problem

$$u_t+(u(\log u)_{xx})_{xx}=0, \quad u(\cdot,0)=u_0\geq 0 \quad \text{in } S^1, \tag{1.1}$$

where S^1 is the one-dimensional torus parametrized by a variable $x\in[0,L]$ for some fixed $L>0$.

Recently equation (1.1) has attracted the interest of many mathematicians since it possesses some remarkable properties. For instance, it is a one-homogeneous equation which is a simple example of a generalization of the heat equation to higher-order operators. The solutions are non-negative and there are several Lyapunov functionals. A formal calculation shows that the *entropy* is non-increasing:

$$\frac{d}{dt}\int_{S^1} u(\log u-1)dx+\int_{S^1} u\left|(\log u)_{xx}\right|^2 dx=0. \tag{1.2}$$

Another example of a Lyapunov functional is $\int_{S^1}(u-\log u)dx$ which formally yields

$$\frac{d}{dt}\int_{S^1}(u-\log u)dx+\int_{S^1}\left|(\log u)_{xx}\right|^2 dx=0. \tag{1.3}$$

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The last property is used to prove that solutions of (1.1) are non-negative. Indeed, a Poincaré inequality shows that $\log u$ is bounded in $H^2(S^1)$ and hence in $L^\infty(S^1)$, which implies that $u \geq 0$ in $S^1 \times (0,\infty)$. We prove this result rigorously in section 2. Notice that the equation is of higher order and no maximum principle argument can be employed. More Lyapunov functionals of (1.1) have been found in [4, 5]; for a systematic study we refer to [12].

Equation (1.1) has been first derived in the context of fluctuations of a stationary non-equilibrium interface [8]. It also appears as a zero-temperature zero-field approximation of the quantum drift-diffusion model for semiconductors [1] which can be derived by a quantum moment method from a Wigner-BGK equation [7]. The first analytical result has been presented in [4]; there the existence of local-in-time classical solutions with periodic boundary conditions has been proved. A global-in-time existence result with homogeneous Dirichlet-Neumann boundary conditions has been obtained in [13]. However, up to now, no global-in-time existence result is available for the problem (1.1). Our proof is an adaption of the method of [13]; we present the complete proof since we need the approximation scheme for the subsequent sections.

The long-time behavior of solutions has been studied in [5] using periodic boundary conditions under restrictive regularity conditions on the initial data, in [15] with homogeneous Dirichlet-Neumann boundary conditions and finally, in [11] employing non-homogeneous Dirichlet-Neumann boundary conditions. In particular, it has been shown that the solutions converge exponentially fast to their steady state in various norms and, see [13], in terms of the entropy. The decay rate has been numerically computed in [6]. We also mention the work [14] in which a positivity-preserving numerical scheme for the quantum drift-diffusion model has been proposed.

Concerning the multi-dimensional problem, there exists only the work [10] in which the existence of global-in-time weak solutions has been proved.

In the last years the question of non-negative or positive solutions of fourth-order parabolic equations has also been investigated in the context of lubrication-type equations, like the thin film equation

$$u_t + (f(u)u_{xxx})_x = 0$$

(see, e.g., [2, 3]), where typically, $f(u) = u^\alpha$ for some $\alpha > 0$. This equation is of degenerate type which makes the analysis easier than for (1.1), at least concerning the positivity property. Notice that (1.1) is *not* of degenerate type.

In this paper we show the following results. First, the existence of global-in-time weak solutions is shown under a rather weak condition on the initial datum u_0. We only assume that $u_0 \geq 0$ is measurable and such that $\int_{S^1}(u_0 - \log u_0)dx < \infty$. Compared to [4], we do not impose any smallness condition on u_0. We are able to prove that the solution is non-negative. The existence proof is based on a semi-discrete formulation of (1.1). The semi-discrete problem has a strictly positive and smooth solution. This property enables us to prove the long-time behavior of solutions rigorously.

Our second result is concerned with regularity issues. We prove that, if $\sqrt{u_0} \in H^1(S^1)$,

$$\sqrt{u} \in L^\infty(0,T;H^1(S^1)) \cap L^2(0,T;H^3(S^1)) \quad \text{for all } T > 0.$$

Although one might obtain more regularity results from this (see Remark 3.4), we are interested in applying this property in order to show an exponential decay rate of the Fisher information $\int_{S^1} |(\sqrt{u})_x(\cdot,t)|^2 dx$.

The third and main result of this paper is the exponential time decay of the solutions, i.e., we show that the solution constructed in Theorem 2.1 converges exponentially fast to its mean value $\bar{u} = \int u(x,t)dx/L$: for all $t > 0$,

$$\int_{S^1} u(x,t) \log\left(\frac{u(x,t)}{\bar{u}}\right) dx \le e^{-M_1 t} \int_{S^1} u_0 \log\left(\frac{u_0}{\bar{u}}\right) dx, \tag{1.4}$$

$$\int_{S^1} |(\sqrt{u})_x(\cdot,t)|^2 dx \le e^{-M_2 t} \int_{S^1} |(\sqrt{u_0})_x|^2 dx, \tag{1.5}$$

where $M_1 = 32\pi^4/L^4$, $M_2 = 16\mu\pi^4/L^4$, and $\mu = 1.646169\ldots$ The constant M_1 is easily obtained by linearization in the asymptotic regime. It also shows up in [5] (as the decay rate in a different norm) but only in the more restrictive H^1 setting. In [5], an exponential convergence rate in the L^p norm has been given. Then, in principle, the decay rate in the "entropy norm" (1.4) could be derived by letting $p \to 1$. However, the decay rate of [5] contains the factor $p-1$ which vanishes in the limit such that no decay rate in the "entropy norm" can be deduced. In [15], the exponential decay of the relative entropy is established, but with a rate which depends on the initial data. Here, M_1 and M_2 are independent of the solution and the constant on the right-hand side of (1.4) is optimal; it is simply the initial value of the relative entropy. Thus, both decay results (1.4) and (1.5) are new.

Our proof is based on the entropy–entropy production method. For the proof of (1.4), we show that the entropy production term $\int u|(\log u)_{xx}|^2 dx$ in (1.2) can be bounded from below by the entropy itself yielding

$$\frac{d}{dt}\int_{S^1} u \log\left(\frac{u}{\bar{u}}\right) dx + M_1 \int_{S^1} u \log\left(\frac{u}{\bar{u}}\right) dx \le 0.$$

Then Gronwall's inequality gives (1.4). For the proof of (1.5) we first need to show that the following entropy–entropy production inequality holds for some $\mu > 0$,

$$\frac{d}{dt}\int_{S^1} |(\sqrt{u})_x(\cdot,t)|^2 dx + \mu \int_{S^1} |(\sqrt{u})_{xxx}|^2 dx \le 0,$$

and then we apply the Poincaré inequality and Gronwall's lemma. The proof of the above inequality is based on the algorithmic entropy construction method recently developed in [12].

The lower bound for the entropy production in (1.2) is obtained through a logarithmic Sobolev inequality in S^1. We show (see Theorem 4.1) that any function $u \in H^n(S^1)$ ($n \in \mathbb{N}$) satisfies

$$\int_{S^1} u^2 \log\left(\frac{u^2}{\|u\|^2_{L^2(S^1)}}\right) dx \le 2\left(\frac{L}{2\pi}\right)^{2n} \int_{S^1} \left|u^{(n)}\right|^2 dx, \tag{1.6}$$

where $\|u\|^2_{L^2(S^1)} = \int u^2 dx/L$, and the constant is *optimal*. As already mentioned in the case $n=2$, the proof of this result uses the entropy–entropy production method.

Entropy estimates are interesting for the following reason. The L^1 norm of a solution u to (1.1) is preserved by the evolution. It is therefore natural to look for a convergence of u to its average $\bar{u}$ measured in L^1 rather than in L^p, $p > 1$. As noted in many papers, the limit of such L^p estimates as $p \to 1$, $p > 1$, is the entropy rather than the L^1 norm itself. The convergence of u to $\bar{u}$ is then a consequence of the standard Csiszár-Kullback inequality. Exactly as for the heat equation, $p=1$ looks as a threshold from the point of view of the existence theory and for the optimality of the estimates on the asymptotic behaviour. This work is a step towards a deeper understanding of both entropy methods and higher-order equations.

The paper is organized as follows. In section 2 the existence of weak solutions is proved. Section 3 is concerned with the regularity result. Then, section 4 is devoted to the proof of the optimal logarithmic Sobolev inequality (1.6). Finally, in section 5, the exponential time decay (1.4) and (1.5) are shown.

2. Existence of solutions

THEOREM 2.1. *Let $u_0: S^1 \to \mathbb{R}$ be a nonnegative measurable function such that $\int_{S^1}(u_0 - \log u_0)dx < \infty$ and let $T>0$. Then there exists a global weak solution u of (1.1) satisfying*

$$u \in L^{5/2}(0,T;W^{1,1}(S^1)) \cap W^{1,10/9}_{\mathrm{loc}}(0,T;H^{-2}(S^1)),$$
$$u \geq 0 \quad in\ S^1 \times (0,\infty), \quad \log u \in L^2(0,T;H^2(S^1)),$$

and for all $T>0$ and all smooth test functions ϕ,

$$\int_0^T \langle u_t, \phi\rangle_{H^{-2},H^2} dt + \int_0^T \int_{S^1} u(\log u)_{xx}\phi_{xx} dx dt = 0.$$

The initial datum is satisfied in the sense of $H^{-2}(S^1) := (H^2(S^1))^*$.

In order to prove this theorem, we first transform (1.1) by introducing the new variable $u = e^y$ as in [13]. Then (1.1) becomes

$$(e^y)_t + (e^y y_{xx})_{xx} = 0, \quad y(\cdot,0) = y_0 \quad \text{in } S^1, \tag{2.1}$$

where $y_0 = \log u_0$. In order to prove the existence of solutions to this equation, we semi-discretize (2.1) in time. For this, let $T>0$, and let $0 = t_0 < t_1 < \cdots < t_N = T$ with $t_k = k\tau$ be a partition of $[0,T]$ with $\tau = T/N$. Furthermore, let $y_{k-1} \in H^2(S^1)$ with $\int \exp(y_{k-1})dx = \int u_0 dx$ and $\int(\exp(y_{k-1}) - y_{k-1})dx \leq \int (u_0 - \log u_0)dx$ given. Then we solve recursively the elliptic equations

$$\frac{1}{\tau}(e^{y_k} - e^{y_{k-1}}) + (e^{y_k}(y_k)_{xx})_{xx} = 0 \quad \text{in } S^1. \tag{2.2}$$

LEMMA 2.2. *There exists a solution $y_k \in C^\infty(S^1)$ of (2.2).*

Proof. Set $z = y_{k-1}$. We consider first for given $\varepsilon > 0$ the equation

$$(e^y y_{xx})_{xx} - \varepsilon y_{xx} + \varepsilon y = \frac{1}{\tau}(e^z - e^y) \quad \text{in } S^1. \tag{2.3}$$

In order to prove the existence of a solution to this approximate problem we employ the Leray-Schauder theorem. For this, let $w \in H^1(S^1)$ and $\sigma \in [0,1]$ be given, and consider

$$a(y,\phi) = F(\phi) \quad \text{for all } \phi \in H^2(S^1), \tag{2.4}$$

where for all $y, \phi \in H^2(S^1)$,

$$a(y,\phi) = \int_{S^1}(e^w y_{xx}\phi_{xx} + \varepsilon y_x \phi_x + \varepsilon y \phi)dx, \quad F(\phi) = \frac{\sigma}{\tau}\int_{S^1}(e^z - e^w)\phi dx.$$

Clearly, $a(\cdot,\cdot)$ is bilinear, continuous and coercive on $H^2(S^1)$ and F is linear and continuous on $H^2(S^1)$. (Here we need the additional ε-terms.) Therefore, the Lax-Milgram lemma provides the existence of a solution $y \in H^2(S^1)$ of (2.4). This defines a fixed-point operator $S : H^1(S^1) \times [0,1] \to H^1(S^1)$, $(w,\sigma) \mapsto y$. It holds $S(w,0) = 0$ for all $w \in H^1(S^1)$. Moreover, the functional S is continuous and compact (since the embedding $H^2(S^1) \subset H^1(S^1)$ is compact). We need to prove a uniform bound for all fixed points of $S(\cdot,\sigma)$.

Let y be a fixed point of $S(\cdot,\sigma)$, i.e., $y \in H^2(S^1)$ solves for all $\phi \in H^2(S^1)$

$$\int_{S^1} (e^y y_{xx}\phi_{xx} + \varepsilon y_x \phi_x + \varepsilon y \phi)dx = \frac{\sigma}{\tau}\int_{S^1}(e^z - e^y)\phi dx. \tag{2.5}$$

Using the test function $\phi = 1 - e^{-y}$ yields

$$\begin{aligned}&\int_{S^1} y_{xx}^2 dx - \int_{S^1} y_{xx} y_x^2 dx + \varepsilon \int_{S^1} e^{-y} y_x^2 dx + \varepsilon \int_{S^1} y(1-e^{-y})dx\\ &= \frac{\sigma}{\tau}\int_{S^1}(e^z - e^y)(1-e^{-y})dx.\end{aligned}$$

The second term on the left-hand side vanishes since $y_{xx}y_x^2 = (y_x^3)_x/3$. The third and fourth term on the left-hand side are non-negative. Furthermore, with the inequality $e^x \geq 1 + x$ for all $x \in \mathbb{R}$,

$$(e^z - e^y)(1 - e^{-y}) \leq (e^z - z) - (e^y - y).$$

We obtain

$$\frac{\sigma}{\tau}\int_{S^1}(e^y - y)dx + \int_{S^1} y_{xx}^2 dx \leq \frac{\sigma}{\tau}\int_{S^1}(e^z - z)dx.$$

As z is given, this provides a uniform bound for y_{xx} in $L^2(S^1)$. Moreover, the inequality $e^x - x \geq |x|$ for all $x \in \mathbb{R}$ implies a (uniform) bound for y in $L^1(S^1)$ and for $\int y dx$. Now we use the Poincaré inequality

$$\left\| u - \int_{S^1} u \frac{dx}{L}\right\|_{L^2(S^1)} \leq \frac{L}{2\pi}\|u_x\|_{L^2(S^1)} \leq \left(\frac{L}{2\pi}\right)^2 \|u_{xx}\|_{L^2(S^1)} \quad \text{for all } u \in H^2(S^1).$$

Recall that $\|u\|_{L^2(S^1)}^2 = \int_{S^1} u^2 dx/L$. Then the above estimates provide a (uniform in ε) bound for y and y_x in $L^2(S^1)$ and thus for y in $H^2(S^1)$. This shows that all fixed points of the operator $S(\cdot,\sigma)$ are uniformly bounded in $H^1(S^1)$. We notice that we even obtain a uniform bound for y in $H^2(S^1)$ which is independent of ε. The Leray-Schauder fixed-point theorem finally ensures the existence of a fixed point of $S(\cdot,1)$, i.e., of a solution $y \in H^2(S^1)$ to (2.3).

Next we show that the limit $\varepsilon \to 0$ can be performed in (2.3) and that the limit function satisfies (2.2). Let y_ε be a solution of (2.3). The above estimate shows that y_ε is bounded in $H^2(S^1)$ uniformly in ε. Thus there exists a subsequence (not relabeled) such that, as $\varepsilon \to 0$,

$$\begin{aligned}&y_\varepsilon \rightharpoonup y && \text{weakly in } H^2(S^1),\\ &y_\varepsilon \to y && \text{strongly in } H^1(S^1) \text{ and in } L^\infty(S^1).\end{aligned}$$

We conclude that $e^{y_\varepsilon} \to e^y$ in $L^2(S^1)$ as $\varepsilon \to 0$. In particular, $e^{y_\varepsilon}(y_\varepsilon)_{xx} \rightharpoonup e^y y_{xx}$ weakly in $L^1(S^1)$. The limit $\varepsilon \to 0$ in (2.5) can be performed proving that y solves (2.2).

Moreover, using the test function $\phi\equiv 1$ in the weak formulation of (2.2) shows that $\int \exp(y_k)dx = \int \exp(y_{k-1})dx = \int u_0 dx$.

It remains to prove that the solution y_k of (2.2) lies in $C^\infty(S^1)$. Writing $u=e^{y_k}$, we can reformulate (2.2) as

$$u_{xxxx} = \left(\frac{2u_x^2 u_{xx}}{u} - \frac{u_x^3}{u^2}\right)_x - \frac{1}{\tau}(u - e^{y_{k-1}}). \tag{2.6}$$

Since $y_k \in H^2(S^1) \hookrightarrow W^{1,\infty}(S^1)$, also $u \in H^2(S^1) \hookrightarrow W^{1,\infty}(S^1)$ and u is strictly positive in S^1. In particular, $1/u \in L^\infty(S^1)$. Thus $u_x^2 u_{xx}/u \in L^2(S^1)$ and $u_x^3/u^2 \in L^\infty(S^1)$. By (2.6) this implies that $u_{xxxx} \in H^{-1}(S^1)$ and therefore $u \in H^3(S^1)$. Again by (2.6) this yields $u_{xxxx} \in L^2(S^1)$ and $u \in H^4(S^1)$. Continuing this procedure leads to $u \in H^n(S^1)$ for all $n \in \mathbb{N}$ and hence $u \in C^\infty(S^1)$. Finally, since u is strictly positive, this shows that $y_k = \log u \in C^\infty(S^1)$. ∎

For the proof of Theorem 2.1 we need further uniform estimates for the finite sequence $(y^{(N)})$. For this, let $y^{(N)}$ be defined by $y^{(N)}(x,t) = y_k(x)$ for $x \in S^1$, $t \in (t_{k-1}, t_k]$, $1 \le k \le N$. Then we have shown in the proof of Lemma 2.2 that there exists a constant $c>0$ depending neither on τ nor on N such that

$$\|y^{(N)}\|_{L^2(0,T;H^2(S^1))} + \|y^{(N)}\|_{L^\infty(0,T;L^1(S^1))} + \|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(S^1))} \le c. \tag{2.7}$$

To pass to the limit in the approximating equation, we need further compactness estimates on $e^{y^{(N)}}$. Here we proceed similarly as in [11].

Lemma 2.3. *The following estimates hold:*

$$\|y^{(N)}\|_{L^{5/2}(0,T;W^{1,\infty}(S^1))} + \|e^{y^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(S^1))} \le c, \tag{2.8}$$

where $c>0$ does not depend on τ and N.

Proof. We obtain from the Gagliardo-Nirenberg inequality and (2.7):

$$\|y^{(N)}\|_{L^{5/2}(0,T;L^\infty(S^1))} \le c\|y^{(N)}\|^{3/5}_{L^\infty(0,T;L^1(S^1))}\|y^{(N)}\|^{2/5}_{L^1(0,T;H^2(S^1))} \le c,$$
$$\|y_x^{(N)}\|_{L^{5/2}(0,T;L^\infty(S^1))} \le c\|y^{(N)}\|^{1/5}_{L^\infty(0,T;L^1(S^1))}\|y^{(N)}\|^{4/5}_{L^2(0,T;H^2(S^1))} \le c.$$

This implies the first bound in (2.8). The second bound follows from the first one and (2.7):

$$\begin{aligned}
&\|e^{y^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(S^1))} \\
&\le c\left(\|e^{y^{(N)}}\|_{L^{5/2}(0,T;L^1(S^1))} + \|(e^{y^{(N)}})_x\|_{L^{5/2}(0,T;L^1(S^1))}\right) \\
&\le c\|e^{y^{(N)}}\|_{L^{5/2}(0,T;L^1(S^1))} + c\|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(S^1))}\|y_x^{(N)}\|_{L^{5/2}(0,T;L^\infty(S^1))} \\
&\le c.
\end{aligned}$$

The lemma is proved. ∎

We also need an estimate for the discrete time derivative. We introduce the shift operator σ_N by $(\sigma_N(y^{(N)}))(x,t) = y_{k-1}(x)$ for $x \in S^1$, $t \in (t_{k-1}, t_k]$.

Lemma 2.4. *The following estimate holds:*

$$\|e^{y^{(N)}} - e^{\sigma_N(y^{(N)})}\|_{L^{10/9}(0,T;H^{-2}(0,1))} \le c\tau, \tag{2.9}$$

where $c>0$ does not depend on τ and N.

Proof. From (2.2) and Hölder's inequality we obtain

$$\begin{aligned}\frac{1}{\tau}\|e^{y^{(N)}}-e^{\sigma_N(y^{(N)})}\|_{L^{10/9}(0,T;H^{-2}(S^1))}&\le \|e^{y^{(N)}}y_{xx}^{(N)}\|_{L^{10/9}(0,T;L^2(S^1))}\\ &\le \|e^{y^{(N)}}\|_{L^{5/2}(0,T;L^\infty(S^1))}\|y_{xx}^{(N)}\|_{L^2(0,T;L^2(S^1))},\end{aligned}$$

and the right-hand side is uniformly bounded by (2.7) and (2.8) since $W^{1,1}(0,1)\hookrightarrow L^\infty(0,1)$. ∎

Now we are able to prove Theorem 2.1, i.e. to perform the limit $\tau\to 0$ in (2.2). From estimate (2.7) the existence of a subsequence of $y^{(N)}$ (not relabeled) follows such that, as $N\to\infty$ or, equivalently, $\tau\to 0$,

$$y^{(N)}\rightharpoonup y \quad \text{weakly in } L^2(0,T;H^2(S^1)). \tag{2.10}$$

Since the embedding $W^{1,1}(S^1)\subset L^2(S^1)$ is compact it follows from the second bound in (2.8) and from (2.9) by an application of Aubin's lemma [17, Thm. 5] that, up to the extraction of a subsequence, $e^{y^{(N)}}\to g$ strongly in $L^{5/2}(0,T;L^2(S^1))$.

We claim that $g=e^y$. For this, let z be a smooth function. Since $e^{y^{(N)}}\to g$ strongly in $L^2(0,T;L^2(S^1))$ and $y^{(N)}\rightharpoonup y$ weakly in $L^2(0,T;L^2(S^1))$, we can pass to the limit $N\to\infty$ in

$$0\le\int_0^T\int_{S^1}(e^{y^{(N)}}-e^z)(y^{(N)}-z)dxdt$$

to obtain the inequality

$$0\le\int_0^T\int_{S^1}(g-e^z)(y-z)dxdt.$$

The monotonicity of $x\mapsto e^x$ finally yields $g=e^y$.

Noticing that the uniform estimate (2.9) implies, for a subsequence,

$$\frac{1}{\tau}\left(e^{y^{(N)}}-e^{\sigma_N(y^{(N)})}\right)\rightharpoonup (e^y)_t \quad \text{weakly in } L^{10/9}(0,T;H^{-2}(S^1)), \tag{2.11}$$

we can pass to the limit $\tau\to 0$ in (2.2), using the convergence results (2.10)-(2.11), which concludes the proof of Theorem 2.1.

3. Regularity of solutions

THEOREM 3.1. *Let $u_0\in H^1(S^1)$ satisfy the assumptions of Theorem 2.1. Then the solution constructed in Theorem 2.1 satisfies the regularity properties*

$$\sqrt{u}\in L^2(0,T;H^3(S^1))\cap L^\infty(0,T;H^1(S^1)). \tag{3.1}$$

The theorem is an immediate consequence of the following lemma and the convergence properties shown in the previous section.

LEMMA 3.2. *Let $y_k\in C^\infty(S^1)$ be the solution of* (2.2) *constructed in Theorem 2.1 and set $u_k=e^{y_k}$, $k=1,\dots,N$. Then*

$$\frac{1}{\tau}\int_{S^1}\left(|(\sqrt{u_k})_x|^2-|(\sqrt{u_{k-1}})_x|^2\right)dx+\mu\int_{S^1}|(\sqrt{u_k})_{xxx}|^2dx\le 0,$$

where $\mu=(103+\sqrt{241})/72=1.646169\ldots$ *is the largest root of* $36x^2-103x+72=0$.

Proof. The proof is based on the algorithmic entropy construction method recently developed in [12]. The main idea of this method is to reformulate the task of proving entropy dissipation employing integration by parts as a decision problem for polynomial systems.

The functions $u_k=e^{y_k}$ and $u_{k-1}=e^{y_{k-1}}$ are strictly positive and smooth and satisfy

$$\frac{1}{\tau}(u_k-u_{k-1})+(u_k(\log u_k)_{xx})_{xx}=0 \quad \text{in } S^1. \tag{3.2}$$

We multiply this equation by $u_k^{-1/2}(u_k^{1/2})_{xx}$, integrate over S^1, and integrate by parts the second term:

$$\begin{aligned} 0=&\frac{1}{\tau}\int_{S^1} u_k^{-1/2}(u_k-u_{k-1})(u_k^{1/2})_{xx}dx \\ &-\int_{S^1}\big(u_k(\log u_k)_{xx}\big)_x\big(u_k^{-1/2}(u_k^{1/2})_{xx}\big)_x dx=: I_1-I_2. \end{aligned}$$

Integrating by parts once, an elementary computation shows that

$$\begin{aligned} I_1&=-\frac{1}{\tau}\int_{S^1}\big(|(u_k^{1/2})_x|^2-|(u_{k-1}^{1/2})_x|^2\big)dx-\frac{1}{4\tau}\int_{S^1}u_{k-1}|(\log u_k-\log u_{k-1})_x|^2dx \\ &\le -\frac{1}{\tau}\int_{S^1}\big(|(\sqrt{u_k})_x|^2-|(\sqrt{u_{k-1}})_x|^2\big)dx. \end{aligned}$$

We claim that the integral I_2 can be estimated as

$$I_2\ge\mu\int_{S^1}|(\sqrt{u_k})_{xxx}|^2dx, \tag{3.3}$$

where μ is as in the statement of the lemma. We write, omitting the index k in the following,

$$\begin{aligned} I_2=\int_{S^1}u\Big[&\frac{1}{2}\Big(\frac{u_x}{u}\Big)^6-2\Big(\frac{u_x}{u}\Big)^4\frac{u_{xx}}{u}+\Big(\frac{u_x}{u}\Big)^3\frac{u_{xxx}}{u}+2\Big(\frac{u_x}{u}\Big)^2\Big(\frac{u_{xx}}{u}\Big)^2 \\ &-2\frac{u_x}{u}\frac{u_{xx}}{u}\frac{u_{xxx}}{u}+\frac{1}{2}\Big(\frac{u_{xxx}}{u}\Big)^2\Big]dx. \end{aligned}$$

Moreover, we have

$$\begin{aligned} J=\int_{S^1}(\sqrt{u})^2_{xxx}dx=\int_{S^1}u\Big[&\frac{9}{64}\Big(\frac{u_x}{u}\Big)^6-\frac{9}{16}\Big(\frac{u_x}{u}\Big)^4\frac{u_{xx}}{u}+\frac{3}{8}\Big(\frac{u_x}{u}\Big)^3\frac{u_{xxx}}{u} \\ &+\frac{9}{16}\Big(\frac{u_x}{u}\Big)^2\Big(\frac{u_{xx}}{u}\Big)^2-\frac{3}{4}\frac{u_x}{u}\frac{u_{xx}}{u}\frac{u_{xxx}}{u}+\frac{1}{4}\Big(\frac{u_{xxx}}{u}\Big)^2\Big]dx. \end{aligned}$$

In order to show that inequality (3.3) holds, we need to perform suitable integrations by parts. It turns out that only the following integration by parts is useful:

$$\begin{aligned} 0&=\int_{S^1}\Big(\frac{u_x^5}{u^4}\Big)_x dx=\int_{S^1}u\Big[-4\Big(\frac{u_x}{u}\Big)^6+5\Big(\frac{u_x}{u}\Big)^4\frac{u_{xx}}{u}\Big]dx=J_1, \\ 0&=\int_{S^1}\Big(\frac{u_x^3u_{xx}}{u^3}\Big)_x dx=\int_{S^1}u\Big[-3\Big(\frac{u_x}{u}\Big)^4\frac{u_{xx}}{u}+\Big(\frac{u_x}{u}\Big)^3\frac{u_{xxx}}{u}+3\Big(\frac{u_x}{u}\Big)^2\Big(\frac{u_{xx}}{u}\Big)^2\Big]dx=J_2. \end{aligned}$$

Then we can write the integral as $I_2 = I_2 + c_1 J_1 + c_2 J_2$ for arbitrary constants c_1, $c_2 \in \mathbb{R}$. We wish to find $\mu > 0$ and c_1, $c_2 \in \mathbb{R}$ such that $I_2 + c_1 J_1 + c_2 J_2 \geq \mu J$. For this task we identify the derivative $\partial_x^j u/u$ with the real variable ξ_j and deal with the polynomials

$$\begin{aligned}
P(\xi) &= \frac{1}{2}\xi_1^6 - 2\xi_1^4\xi_2 + \xi_1^3\xi_3 + 2\xi_1^2\xi_2^2 - 2\xi_1\xi_2\xi_3 + \frac{1}{2}\xi_3^2, \quad \text{which corresponds to } I_2,\\
E(\xi) &= \frac{9}{64}\xi_1^6 - \frac{9}{16}\xi_1^4\xi_2 + \frac{3}{8}\xi_1^3\xi_3 + \frac{9}{16}\xi_1^2\xi_2^2 - \frac{3}{4}\xi_1\xi_2\xi_3 + \frac{1}{4}\xi_3^2, \quad \text{which corresponds to } J,\\
T_1(\xi) &= -4\xi_1^6 + 5\xi_1^4\xi_2, \quad \text{which corresponds to } J_1,\\
T_2(\xi) &= -3\xi_1^4\xi_2 + \xi_1^3\xi_3 + 3\xi_1^2\xi_2^2, \quad \text{which corresponds to } J_2.
\end{aligned}$$

Thus we wish to find constants $\mu > 0$ and c_1, $c_2 \in \mathbb{R}$ such that

$$(P - \mu E + c_1 T_1 + c_2 T_2)(\xi) \geq 0 \quad \text{for all } \xi = (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{R}^3,$$

which corresponds to a pointwise estimate of the integrand of I_2. The determination of all parameters such that the above inequality is true is called a quantifier elimination problem. In this situation it can be explicitly solved.

The above inequality is equivalent to

$$a_1\xi_1^6 + a_2\xi_1^4\xi_2 + a_3\xi_1^3\xi_3 + a_4\xi_1^2\xi_2^2 + a_5\xi_1\xi_2\xi_3 + a_6\xi_3^2 \geq 0 \quad \text{for all } \xi_1, \xi_2, \xi_3 \in \mathbb{R},$$

where

$$a_1 = \frac{1}{2} - \frac{9}{64}\mu - 4c_1, \qquad a_2 = -2 + \frac{9}{16}\mu + 5c_1 - 3c_2, \qquad a_3 = 1 - \frac{3}{8}\mu + c_2, \tag{3.4}$$

$$a_4 = 2 - \frac{9}{16}\mu + 3c_2, \qquad a_5 = -2 + \frac{3}{4}\mu, \qquad a_6 = \frac{1}{2} - \frac{1}{4}\mu. \tag{3.5}$$

Now, we recall (a slight generalization of) Lemma 12 of [12]:

LEMMA 3.3. [12] *Let the real polynomial*

$$P(\xi_1, \xi_2, \xi_3) = a_1\xi_1^6 + a_2\xi_1^4\xi_2 + a_3\xi_1^3\xi_3 + a_4\xi_1^2\xi_2^2 + a_5\xi_1\xi_2\xi_3 + a_6\xi_3^2$$

be given and let $a_6 > 0$. Then the statement

$$P(\xi_1, \xi_2, \xi_3) \geq 0 \quad \textit{for all } \xi_1, \xi_2, \xi_3 \in \mathbb{R}$$

is equivalent to

$$\textit{either} \quad \text{(i)} \quad 4a_4a_6 - a_5^2 = 2a_2a_6 - a_3a_5 = 0 \quad \textit{and} \quad 4a_1a_6 - a_3^2 \geq 0, \tag{3.6}$$

$$\textit{or} \quad \text{(ii)} \quad 4a_4a_6 - a_5^2 > 0 \quad \textit{and} \quad 4a_1a_4a_6 - a_1a_5^2 - a_2^2a_6 - a_3^2a_4 + a_2a_3a_5 \geq 0. \tag{3.7}$$

In both cases, the condition $a_6 > 0$, which is equivalent to $\mu < 2$, has to be satisfied. The first two conditions of case (i) give

$$c_2 = \frac{\mu}{24(2-\mu)}, \quad c_1 = \frac{\mu(3\mu - 8)}{240(2-\mu)^2},$$

and the third one is equivalent to

$$-36\mu^2 + 103\mu - 72 \geq 0.$$

This polynomial has two real roots,

$$\frac{103-\sqrt{241}}{72}=1.214942\dots \quad \text{and} \quad \frac{103+\sqrt{241}}{72}=1.646169\dots$$

Thus we obtain the requirement $\mu\leq(103+\sqrt{241})/72$ in case (i).

It can be seen that case (ii) gives a stronger condition on μ; we leave the details to the reader. □

REMARK 3.4. *From Theorem 3.1 it follows that*

$$u\in L^\infty(S^1\times(0,\infty))\cap H^1(0,T;H^{-1}(S^1)) \quad \textit{for all } T>0.$$

Indeed, we know that $u^{(N)}\in L^\infty(0,T;H^1(S^1))\cap L^2(0,T;\,H^3(S^1))$. *Thus, the first property is a consequence of the embedding* $H^1(S^1)\hookrightarrow L^\infty(S^1)$ *in one space dimension. Furthermore, we obtain from* (3.2)

$$\begin{aligned}\frac{1}{\tau}\big(u^{(N)}-\sigma_N(u^{(N)})\big)=&-2\big(\sqrt{u^{(N)}}(\sqrt{u^{(N)}})_{xxx}\\&-(\sqrt{u^{(N)}})_x(\sqrt{u^{(N)}})_{xx}\big)_x\in L^2(0,T;H^{-1}(S^1)).\end{aligned}$$

Then the limit $N\to\infty$ *shows the claim.*

4. Optimal logarithmic Sobolev inequality on S^1

The main goal of this section is the proof of a logarithmic Sobolev inequality for periodic functions. The following theorem is due to Weissler and Rothaus (see [9, 16, 18]). We give a simple proof using the entropy–entropy production method. Recall that S^1 is parametrized by $0\leq x\leq L$.

THEOREM 4.1. *Let* $\mathcal{H}_1=\{u\in H^1(S^1):u_x\not\equiv 0 \text{ a.e.}\}$ *and* $\|u\|^2_{L^2(S^1)}=\int_{S^1}u^2dx/L$. *Then*

$$\inf_{u\in\mathcal{H}_1}\frac{\int_{S^1}u_x^2dx}{\int_{S^1}u^2\log(u^2/\|u\|^2_{L^2(S^1)})dx}=\frac{2\pi^2}{L^2}. \tag{4.1}$$

We recall that the optimal constant in the usual Poincaré inequality is $L/2\pi$, i.e.

$$\inf_{v\in\mathcal{H}_1}\frac{\int_{S^1}v_x^2dx}{\int_{S^1}(v-\bar v)^2dx}=\frac{4\pi^2}{L^2}, \tag{4.2}$$

where $\bar v=\int_{S^1}vdx/L$.

Proof. Let I denote the value of the infimum in (4.1). First we prove that

$$I\leq\frac{1}{2}\inf_{v\in\mathcal{H}_1}\frac{\int_{S^1}v_x^2dx}{\int_{S^1}(v-\bar v)^2dx}, \tag{4.3}$$

since then the upper bound $I\leq 2\pi^2/L^2$ follows directly from (4.2). Let $v\in\mathcal{H}_1$ and set $u=u_\varepsilon=1+\varepsilon(v-\bar v)$. Without loss of generality, we may replace $v-\bar v$ by v such that $\int_{S^1}vdx=0$. Then $u^2=1+2\varepsilon v+\varepsilon^2v^2$ and the expansion $\log(1+x)=x-x^2/2+O(x^3)$

for $x \to 0$ yields

$$\begin{aligned}
\int_{S^1} u^2 \log(u^2) dx &= \int_{S^1} (1+2\varepsilon v+\varepsilon^2 v^2)\log(1+2\varepsilon v+\varepsilon^2 v^2) dx \\
&= 3\varepsilon^2 \int_{S^1} v^2 dx + O(\varepsilon^3), \\
\int_{S^1} u^2 dx \log\left(\frac{1}{L}\int_{S^1} u^2 dx\right) &= \int_{S^1} (1+\varepsilon^2 v^2) dx \log\left(\frac{1}{L}\int_{S^1} (1+\varepsilon^2 v^2) dx\right) \\
&= \varepsilon^2 \int_{S^1} v^2 dx + O(\varepsilon^4) \quad (\varepsilon \to 0).
\end{aligned}$$

Taking the difference of the two expansions gives

$$\int_{S^1} u^2 \log\left(\frac{u^2}{\int_{S^1} u^2 dx/L}\right) dx = 2\varepsilon^2 \int_{S^1} v^2 dx + O(\varepsilon^3).$$

Therefore, using $\int_{S^1} u_x^2 dx = \varepsilon^2 \int_{S^1} v_x^2 dx$,

$$\frac{\int_{S^1} u_x^2 dx}{\int_{S^1} u^2 \log(u^2/\|u\|^2_{L^2(S^1)}) dx} = \frac{1}{2}\frac{\int_{S^1} v_x^2 dx}{\int_{S^1} v^2 dx} + O(\varepsilon).$$

In the limit $\varepsilon \to 0$ we obtain (4.3).

In order to prove the lower bound for the infimum we use the entropy–entropy production method. For this we consider the heat equation

$$v_t = v_{xx} \quad \text{in } S^1 \times (0,\infty), \quad v(\cdot,0) = u^2 \quad \text{in } S^1$$

for some function $u \in H^1(S^1)$. We assume for simplicity that $\|u\|^2_{L^2(S^1)} = \int_{S^1} u^2 dx/L = 1$. Then

$$\frac{d}{dt}\int_{S^1} v \log v dx = -4 \int_{S^1} w_x^2 dx,$$

where the function $w := \sqrt{v}$ solves the equation $w_t = w_{xx} + w_x^2/w$. Now, the time derivative of

$$f(t) = \int_{S^1} w_x^2 dx - \frac{2\pi^2}{L^2}\int_{S^1} w^2 \log(w^2) dx$$

equals

$$f'(t) = -2\int_{S^1}\left(w_{xx}^2 + \frac{w_x^4}{3w^2} - \frac{4\pi^2}{L^2} w_x^2\right) dx \le -\frac{2}{3}\int_{S^1} \frac{w_x^4}{w^2} dx \le 0,$$

where we have used the Poincaré inequality

$$\int_{S^1} w_x^2 dx \le \frac{L^2}{4\pi^2}\int_{S^1} w_{xx}^2 dx. \tag{4.4}$$

This shows that $f(t)$ is non-increasing and moreover, for any $u \in H^1(S^1)$,

$$\int_{S^1} u_x^2 dx - \frac{2\pi^2}{L^2}\int_{S^1} u^2 \log\left(\frac{u^2}{\|u\|^2_{L^2(S^1)}}\right) dx = f(0) \ge f(t).$$

As the solution $v(\cdot,t)$ of the above heat equation and hence $w(\cdot,t)$ converges to zero in appropriate Sobolev norms as $t\to+\infty$, we conclude that $f(t)\to 0$ as $t\to+\infty$. This implies $I\geq 2\pi^2/L^2$. □

REMARK 4.2. Similar results as in Theorem 4.1 can be obtained for the so-called convex Sobolev inequalities. Let $\sigma(v)=(v^p-\bar{v}^p)/(p-1)$, where $\bar{v}=\int_{S^1} v dx/L$ for $1<p\leq 2$. We claim that

$$\inf_{v\in\mathcal{H}_1}\frac{\int_{S^1}\sigma''(v)v_x^2dx}{\int_{S^1}\sigma(v)dx}=\frac{8\pi^2}{L^2}.$$

As in the logarithmic case, the lower bound is achieved by an expansion around 1 and the usual Poincaré inequality. On the other hand, let v be a solution of the heat equation. Then

$$\frac{d}{dt}\int_{S^1}\sigma(v)dx=-\frac{4}{p}\int_{S^1}w_x^2dx$$

where $w=v^{p/2}$ solves

$$w_t=w_{xx}+\left(\frac{2}{p}-1\right)\frac{w_x^2}{w}, \tag{4.5}$$

and, using (4.4),

$$\begin{aligned}\frac{d}{dt}\int_{S^1}\left(w_x^2-\frac{2\pi^2p}{L^2}\sigma(v)\right)dx&=-2\int_{S^1}\left(w_{xx}^2-\frac{4\pi^2}{L^2}w_x^2+\left(\frac{2}{p}-1\right)\frac{w_x^4}{3w^2}\right)dx\\&\leq-\frac{2}{3}\left(\frac{2}{p}-1\right)\int_{S^1}\frac{w_x^4}{w^2}dx\leq 0.\end{aligned}$$

This proves the upper bound

$$\frac{p}{4}\int_{S^1}\sigma''(v)v_x^2dx=\int_{S^1}w_x^2dx\geq\frac{2\pi^2p}{L^2}\int_{S^1}\sigma(v)dx.$$

With the notation $v=u^{2/p}$ this result takes the more familiar form

$$\frac{1}{p-1}\left[\int_{S^1}u^2dx-L\left(\frac{1}{L}\int_{S^1}u^{2/p}dx\right)^p\right]\leq\frac{L^2}{2\pi^2p}\int_{S^1}u_x^2dx\quad\text{for all } u\in H^1(S^1). \tag{4.6}$$

The logarithmic case corresponds to the limit $p\to 1$ whereas the case $p=2$ gives the usual Poincaré inequality.

We may notice that the method gives more than what is stated in Theorem 4.1 since there is an integral remainder term. Namely, for any $p\in[1,2]$, for any $v\in H^1(S^1)$, we have

$$\frac{p}{4}\int_{S^1}\sigma''(v)v_x^2dx+\mathcal{R}[v]\geq\frac{2\pi^2p}{L^2}\int_{S^1}\sigma(v)dx$$

with

$$\mathcal{R}[v]=2\int_0^\infty\int_{S^1}\left(w_{xx}^2-\frac{4\pi^2}{L^2}w_x^2+\left(\frac{2}{p}-1\right)\frac{w_x^4}{3w^2}\right)dx\,dt,$$

where $w=w(x,t)$ is the solution to (4.5) with initial datum $u_0^{p/2}$. Inequality (4.6) can also be improved with an integral remainder term for any $p\in[1,2]$, where in the limit case $p=1$, one has to take $\sigma(v)=v\log(v/\bar{v})$. As a consequence, the only optimal functions in (4.1) or in (4.6) are the constants.

COROLLARY 4.3. *Let $n\in\mathbb{N}$, $n>0$ and let $\mathcal{H}_n=\{u\in H^n(S^1):u_x\not\equiv 0$ a.e.$\}$. Then*

$$\inf_{u\in\mathcal{H}_n}\frac{\int_{S^1}\left|u^{(n)}\right|^2dx}{\int_{S^1}u^2\log(u^2/\|u\|^2_{L^2(S^1)})dx}=\frac{1}{2}\left(\frac{2\pi}{L}\right)^{2n}. \tag{4.7}$$

Proof. We obtain a lower bound by applying successively Theorem 4.1 and the Poincaré inequality:

$$\int_{S^1}u^2\log\left(\frac{u^2}{\|u\|^2_{L^2(S^1)}}\right)dx\leq\frac{L^2}{2\pi^2}\int_{S^1}u_x^2dx\leq 2\left(\frac{L}{2\pi}\right)^{2n}\int_{S^1}\left|u^{(n)}\right|^2dx.$$

The upper bound is achieved as in the proof of Theorem 4.1 by expanding the quotient for $u=1+\varepsilon v$ with $\int_{S^1}v\,dx=0$ in powers of ε,

$$\frac{\int_{S^1}\left|u^{(n)}\right|^2dx}{\int_{S^1}u^2\log(u^2/\|u\|^2_{L^2(S^1)})dx}=\frac{1}{2}\frac{\int_{S^1}\left|v^{(n)}\right|^2dx}{\int_{S^1}v^2dx}+O(\varepsilon)\quad(\varepsilon\to 0),$$

and using the Poincaré inequality

$$\inf_{u\in\mathcal{H}_n}\frac{\int_{S^1}\left|v^{(n)}\right|^2dx}{\int_{S^1}|v-\bar{v}|^2dx}=\left(\frac{2\pi}{L}\right)^{2n}.$$

The best constant $\omega=(2\pi/L)^{2n}$ in such an inequality is easily recovered by looking for the smallest positive value of ω for which there exists a nontrivial periodic solution of $(-1)^nv^{(2n)}+\omega v=0$. ∎

5. Exponential time decay of the solutions

We show the exponential time decay of the solutions of (1.1). Our main result is contained in the following theorem.

THEOREM 5.1. *Assume that u_0 is a nonnegative measurable function such that $\int_{S^1}(u_0-\log u_0)dx$ and $\int_{S^1}u_0\log u_0dx$ are finite. Let u be the weak solution of* (1.1) *constructed in Theorem 2.1 and set $\bar{u}=\int_{S^1}u_0(x)dx/L$. Then*

$$\int_{S^1}u(\cdot,t)\log\left(\frac{u(\cdot,t)}{\bar{u}}\right)dx\leq e^{-M_1t}\int_{S^1}u_0\log\left(\frac{u_0}{\bar{u}}\right)dx,\quad\textit{where } M_1=\frac{32\pi^4}{L^4}.$$

Moreover, if in addition $\sqrt{u_0}\in H^1(S^1)$, then

$$\int_{S^1}\left((\sqrt{u})_x(\cdot,t)\right)^2dx\leq e^{-M_2t}\int_{S^1}(\sqrt{u_0})_x^2dx,\quad\textit{where } M_2=\frac{16\mu\pi^4}{L^4}$$

where $\mu=(103+\sqrt{214})/72=1.646169\ldots$ is the largest root of $36x^2-103x+72=0$.

Proof. Since we do not have enough regularity of the solutions of (1.1) in order to manipulate the differential equation directly, we need to regularize the equation first. For this we consider the semi-discrete problem

$$\frac{1}{\tau}(u_k - u_{k-1}) + (u_k(\log u_k)_{xx})_{xx} = 0 \quad \text{in } S^1 \tag{5.1}$$

as in the proof of Theorem 2.1. The solution $u_k \in C^\infty(S^1)$ of this problem for given u_{k-1} is strictly positive and we can use $\log u_k$ as a test function in the weak formulation of (5.1). In order to simplify the presentation we set $u := u_k$ and $z := u_{k-1}$. Then we obtain as in [15]

$$\frac{1}{\tau}\int_{S^1}(u\log u - z\log z)dx + \int_{S^1} u\left|(\log u)_{xx}\right|^2 dx \le 0. \tag{5.2}$$

From integration by parts it follows

$$\int_{S^1}\frac{u_x^2 u_{xx}}{u^2}dx = \frac{2}{3}\int_{S^1}\frac{u_x^4}{u^3}dx.$$

This identity gives

$$\begin{aligned}\int_{S^1} u\left|(\log u)_{xx}\right|^2 dx &= \int_{S^1}\left(\frac{u_{xx}^2}{u} + \frac{u_x^4}{u^3} - 2\frac{u_{xx}u_x^2}{u^2}\right)dx = \int_{S^1}\left(\frac{u_{xx}^2}{u} - \frac{1}{3}\frac{u_x^4}{u^3}\right)dx \\ &= \int_{S^1}\left(\frac{u_{xx}^2}{u} + \frac{1}{3}\frac{u_x^4}{u^3} - \frac{u_{xx}u_x^2}{u^2}\right)dx = 4\int_{S^1}|(\sqrt{u})_{xx}|^2 dx + \frac{1}{12}\int_{S^1}\frac{u_x^4}{u^3}dx.\end{aligned}$$

Thus, (5.2) becomes

$$\frac{1}{\tau}\int_{S^1}\left(u\log\left(\frac{u}{\bar u}\right) - z\log\left(\frac{z}{\bar u}\right)\right)dx + 4\int_{S^1}\left|(\sqrt{u})_{xx}\right|^2 dx \le 0. \tag{5.3}$$

Now we use Corollary 4.3 with $n=2$:

$$\int_{S^1} u\log\left(\frac{u}{\bar u}\right)dx \le \frac{L^4}{8\pi^4}\int_{S^1}\left|(\sqrt{u})_{xx}\right|^2 dx.$$

From this inequality and (5.3) we conclude

$$\frac{1}{\tau}\int_{S^1}\left(u\log\left(\frac{u}{\bar u}\right) - z\log\left(\frac{z}{\bar u}\right)\right)dx + \frac{32\pi^4}{L^4}\int_{S^1} u\log\left(\frac{u}{\bar u}\right)dx \le 0.$$

This is a difference inequality for the sequence

$$E_k := \int_{S^1} u_k\log\left(\frac{u_k}{\bar u}\right)dx,$$

yielding

$$(1+\tau M_1)E_k \le E_{k-1} \quad \text{or} \quad E_k \le E_0(1+\tau M_1)^{-k},$$

where M_1 is as in the statement of the theorem. For $t \in ((k-1)\tau, k\tau]$ we obtain further

$$E_k \le E_0(1+\tau M_1)^{-t/\tau}.$$

Now the proof goes exactly as in [15]. Indeed, the functions $u_k(x)$ converge a.e. to $u(x,t)$ and $(1+\tau M_1)^{-t/\tau}\to e^{-M_1 t}$ as $\tau\to 0$. This implies the first assertion. The second one is obtained similarly employing Lemma 3.2. □

REMARK 5.2. In the H^1-norm we obtain the following decay estimate if $\sqrt{u_0}\in H^1(S^1)$:

$$\|u(\cdot,t)-\bar{u}\|^2_{H^1(S^1)}\le Ce^{-M_2 t}, \tag{5.4}$$

where

$$M_2=\frac{16\mu\pi^4}{L^4}\quad\text{and}\quad C=4\Big(1+\frac{L^2}{4\pi^2}\Big)\left[\frac{\sqrt{L}}{2}\|(\sqrt{u_0})_x\|_{L^2(S^1)}+\sqrt{\bar{u}}\right]^2\|(\sqrt{u_0})_x\|^2_{L^2(S^1)}.$$

The decay rate is slightly worse than that in [5] which equals $32\pi^4/L^4$, but we do not need the strong condition $\|(\log u_0)_x\|_{L^2(S^1)}<12$ which is assumed in [5].

The proof of (5.4) uses the inequality

$$\|g-\bar{g}\|_{L^\infty(S^1)}\le\frac{\sqrt{L}}{2}\|g_x\|_{L^2(S^1)}\quad\text{for all } g\in H^1(S^1)$$

and the uniform bound

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(S^1)}&\le\|\sqrt{u(\cdot,t)}-\sqrt{\bar{u}}\|_{L^\infty(S^1)}+\sqrt{\bar{u}}\le\frac{\sqrt{L}}{2}\|(\sqrt{u})_x(\cdot,t)\|_{L^2(S^1)}+\sqrt{\bar{u}}\\&\le\frac{\sqrt{L}}{2}\|(\sqrt{u_0})_x\|_{L^2(S^1)}+\sqrt{\bar{u}}.\end{aligned}$$

By Poincaré's inequality, we have

$$\begin{aligned}\|u(\cdot,t)-\bar{u}\|^2_{H^1(S^1)}&\le\Big(1+\frac{L^2}{4\pi^2}\Big)\int_{S^1}\big(2\sqrt{u}(\sqrt{u})_x\big)^2(\cdot,t)dx\\&\le 4\Big(1+\frac{L^2}{4\pi^2}\Big)\|\sqrt{u(\cdot,t)}\|^2_{L^\infty(S^1)}\|(\sqrt{u})_x(\cdot,t)\|^2_{L^2(S^1)}\\&\le 4\Big(1+\frac{L^2}{4\pi^2}\Big)\Big(\frac{\sqrt{L}}{2}\|(\sqrt{u_0})_x\|_{L^2(S^1)}+\sqrt{\bar{u}}\Big)^2\|(\sqrt{u_0})_x\|^2_{L^2(S^1)}e^{-M_2 t}.\end{aligned}$$

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