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TRANSPORT IN SEMICONDUCTORS AT SATURATED VELOCITIES*

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Abstract. A model for the transport of electrons in a semiconductor is considered where the electrons travel with saturation speed in the direction of the driving force computed self consistently from the Poisson equation. Since the velocity is discontinuous at zeroes of the driving force, an interpretation of the model in the distributional sense is not necessarily possible. For a spatially one-dimensional model existence of distributional solutions is shown by passing to the limit in a regularized problem corresponding to a scaled drift-diffusion model with a velocity saturation assumption on the mobility. Several explicit solutions of the limiting problem are computed and compared to the results of numerical computations.

Key words. semiconductors, drift-diffusion model, velocity saturation

AMS subject classifications. 35L67, 35L80, 78A35

1. Introduction

We investigate initial-boundary value problems for the system

$$\frac{\partial^2\Phi}{\partial x^2} = n - C, \tag{1.1}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}\left(n\,\mathrm{sign}\frac{\partial\Phi}{\partial x}\right) = 0, \tag{1.2}$$

for the unknown functions $\Phi(x,t)$, $n(x,t)$, with $0<x<1$, $t>0$. The function $C(x)>0$ is considered given. The system (1.1), (1.2) can be interpreted as a model for the one-dimensional flow of electrons (density n) in a semiconductor crystal with built-in, positively charged background ions (density C) under the action of an electric field $-\frac{\partial\Phi}{\partial x}$. The electrostatic potential Φ satisfies the Poisson equation (1.1), and the field dependent drift velocity of the electrons is given by $\mathrm{sign}\frac{\partial\Phi}{\partial x}$.

Of course, some interpretation of the equation (1.2) is necessary since the drift velocity is in general discontinuous even for smooth electric fields. In [8] a theory for transport equations $\frac{\partial n}{\partial t} + \frac{\partial(nv)}{\partial x} = 0$ with given non-smooth coefficients $v(x,t)$ has been developed (see also [2], [3]). Solutions are constructed in the form of time dependent measures transported along Filippov characteristics $x = X(t)$, defined by appropriately generalized [4] solutions of

$$\dot{X}(t) = v(X(t),t).$$

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In the neighbourhood of jumps of v with respect to x, three typical types of behaviour can be distinguished:

1. The characteristics pass through the jump from left to right or from right to left. In this case, the solution in the neighbourhood of the jump can be interpreted in the distributional sense. This is the typical situation for shock waves in nonlinear hyperbolic equations. Note that the Filippov characteristics (which have the interpretation of particle paths) are different from the standard characteristics in this case.
2. The characteristics move into the jump from both sides. In this case, concentration occurs at the jump. The solution n is a Delta distribution, and the definition of the flux nv is not obvious (see [7]).
3. The characteristics move out of the jump on both sides. Then the Filippov characteristics and, consequently, solutions of the transport equation exist, but are not unique.

These comments are only partially relevant for solutions of (1.1), (1.2) because of the self consistent nonlinear coupling. It is plausible that concentration is inhibited by the Coulomb interaction. We expect that only case 1 will occur.

Our analysis below shows that the nonlinear coupling generically leads to solutions where the field vanishes in subsets of $(0,1)\times(0,\infty)$ of positive measure. This raises the question which value the drift velocity takes in such regions. As a first step, the sign function will be considered as set valued with

$$\operatorname{sign} E = \begin{cases} \{1\}, & E>0, \\ \{-1\}, & E<0, \\ [-1,1], & E=0, \end{cases}$$

and (1.2) will be replaced by

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \qquad v \in \operatorname{sign}\frac{\partial \Phi}{\partial x}. \tag{1.3}$$

If $\partial\Phi/\partial x = 0$ in an open domain, then $n = C$ and, consequently, $\partial(vC)/\partial x = 0$ holds there. In the examples in section 4 this provides sufficient information for the computation of v.

Also the evaluation of the product nv can be avoided by rewriting (1.3) even further:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left|\frac{\partial \Phi}{\partial x}\right| + vC\right) = 0, \qquad v \in \operatorname{sign}\frac{\partial \Phi}{\partial x}. \tag{1.4}$$

The formal equivalence of (1.1), (1.4) to (1.1), (1.2) is obvious.

In section 2 a regularized version of (1.1), (1.2) is derived by scaling a version of the semiconductor drift-diffusion model with a velocity saturation assumption on the (field dependent) mobility. Existence, uniqueness and smoothness of solutions of appropriate initial-boundary value problems has been shown in previous work (see [6] for references).

In section 3, uniform estimates in terms of the regularization parameter are derived and the limiting procedure is carried out proving convergence to solutions of (1.1), (1.4).

Finally, four typical weak solutions of the limiting problem in the form (1.1), (1.3) are computed explicitly in section 4. As an illustration they are compared with numerical solutions of the regularized problem.

2. Scaling. The regularized problem

In the drift-diffusion model (see, e.g., [6])

$$\varepsilon_s \frac{\partial^2 \Phi}{\partial x^2} = q(n-C), \tag{2.1}$$

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial n}{\partial x} - n v_\infty v\left(\frac{1}{E_{ref}}\frac{\partial \Phi}{\partial x}\right)\right), \tag{2.2}$$

the parameters ε_s, q, D, v_∞, E_{ref} denote the semiconductor permittivity, the elementary charge, the diffusivity of the electrons, the saturation velocity and, respectively, a reference field strength. The dimensionless function v is assumed to be smooth and to satisfy

$$v'(s) > 0, \qquad \lim_{s\to\pm\infty} v(s) = \pm 1. \tag{2.3}$$

A typical model is (see [1] for a derivation from a kinetic model)

$$v(s) = \frac{s}{\sqrt{1+s^2}}.$$

Assuming the equations to be posed on a space interval of length L, we scale length by L, time by L/v_∞, the potential by $L^2 E_{ref} v_\infty / D$, and the densities n and C by $\varepsilon_s E_{ref} v_\infty/(qD)$. The dimensionless parameter

$$\varepsilon = \frac{D}{L v_\infty}$$

measures the strength of the diffusion term in relation to the convection term in (2.2). The scaled version of (2.1), (2.2) reads

$$\frac{\partial^2 \Phi}{\partial x^2} = n - C, \tag{2.4}$$

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\left(\varepsilon\frac{\partial n}{\partial x} - n v\left(\frac{1}{\varepsilon}\frac{\partial \Phi}{\partial x}\right)\right), \tag{2.5}$$

where, for notational simplicity, the same symbols as in (2.1), (2.2) have been used for the scaled quantities. We shall be interested in situations where ε is small and $C(x)$ takes moderate (positive) values. Obviously, (1.1), (1.2) is the formal limit of (2.4), (2.5) as $\varepsilon \to 0$.

The system (2.4), (2.5) will be considered for $0 < x < 1$ and $t > 0$, subject to the initial condition

$$n(x,0) = n_I(x), \tag{2.6}$$

and boundary conditions

$$n(0,t) = n_l(t), \quad n(1,t) = n_r(t), \qquad \Phi(0,t) = 0, \quad \Phi(1,t) = U(t), \tag{2.7}$$

where the data satisfy

$$
\begin{aligned}
&n_I, C \in L^\infty((0,1)), \qquad n_l, n_r \in W^{1,1}_{loc}([0,\infty)),\\
&n_I, n_l, n_r \geq 0, \qquad U \in H^1_{loc}([0,\infty)).
\end{aligned} \tag{2.8}
$$

Global existence and uniqueness of a smooth solution of (2.4)–(2.7) with $n \geq 0$ is by now a standard result of the theory of the drift-diffusion model [6]. In the following section the limit $\varepsilon \to 0$ will be carried out.

3. The limiting problem

We start by deriving uniform estimates.

LEMMA 3.1. *Assume (2.3) and (2.8) and let (n,Φ) be the solution of (2.4)–(2.7). Then n and, consequently, $\frac{\partial^2 \Phi}{\partial x^2}$ are bounded in $L^\infty_{loc}([0,1]\times[0,\infty))$ uniformly in terms of $\varepsilon > 0$.*

Proof. Expanding the derivatives in (2.5) and using the Poisson equation gives

$$
\frac{\partial n}{\partial t} = \varepsilon \frac{\partial^2 n}{\partial x^2} - \frac{\partial n}{\partial x} v - \frac{n v'}{\varepsilon}(n - C).
$$

With (2.3) and (2.8), an application of the maximum principle immediately implies

$$
0 \leq n(x,t) \leq \max\{\sup_{(0,1)} n_I,\ \sup_{(0,1)} C,\ \sup_{(0,t)} n_l,\ \sup_{(0,t)} n_r\}.
$$

This proves the claim for n. The boundedness of $\frac{\partial^2 \Phi}{\partial x^2}$ is a trivial consequence of the Poisson equation. ∎

LEMMA 3.2. *With the assumptions of Lemma 3.1, $\varepsilon \frac{\partial n}{\partial x}$ and $\frac{\partial^2 \Phi}{\partial x \partial t}$ are bounded in $L^2_{loc}([0,1]\times[0,\infty))$ uniformly for small $\varepsilon > 0$.*

Proof. Multiplication of (2.5) by $n - n_D = n - n_l + x(n_l - n_r)$ and integration by parts gives

$$
\begin{aligned}
\frac{1}{2}\int_0^1 (n-n_D)^2 dx\Big|_{t=0}^T = &-\varepsilon \int_0^T \int_0^1 \left(\frac{\partial n}{\partial x}\right)^2 dx\,dt + \int_0^T \int_0^1 n v \frac{\partial n}{\partial x} dx\,dt\\
&- \int_0^T \int_0^1 (n-n_D)\frac{\partial n_D}{\partial t} dx\,dt + \varepsilon \int_0^T (n_r - n_l)^2 dt\\
&- \int_0^T (n_r - n_l)\int_0^1 n v\, dx\,dt.
\end{aligned} \tag{3.1}
$$

With $nv\frac{\partial n}{\partial x} \leq \frac{\varepsilon}{2}\left(\frac{\partial n}{\partial x}\right)^2 + \frac{1}{2\varepsilon}(nv)^2$, the first two terms of the right hand side can be estimated from above by

$$
-\frac{\varepsilon}{2}\int_0^T \int_0^1 \left(\frac{\partial n}{\partial x}\right)^2 dx\,dt + \frac{1}{2\varepsilon}\int_0^T \int_0^1 (nv)^2 dx\,dt.
$$

Since n, v, and (by assumption 2.8) also the remaining terms on the right hand side of (3.1) are uniformly bounded, this completes the proof of the uniform boundedness of $\varepsilon\frac{\partial n}{\partial x}$. Now we take the derivative of the Poisson equation (2.4) with respect to t, substitute for $\frac{\partial n}{\partial t}$ from (2.5) and integrate with respect to x. The result is the displacement current relation

$$\frac{\partial^2\Phi}{\partial x\partial t}=\varepsilon\frac{\partial n}{\partial x}-nv+\int_0^1 nv\,dx+U'+\varepsilon(n_l-n_r).$$

The lemma now follows from our previous estimates and the assumptions (2.8) on the data. □

By the previous two results, the spatial derivative of the electric field $\frac{\partial\Phi}{\partial x}$ is bounded in L^∞ and the time derivative is bounded in L^2. The following anisotropic generalization of the Morrey inequality provides information on the smoothness of $\frac{\partial\Phi}{\partial x}$.

LEMMA 3.3. *([5]) Let $\Omega\subset\mathbb{R}^n$ be an interval and let*

$$\frac{\partial u}{\partial x_j}\in L^{p_j}(\Omega),\quad 1<p_j\le\infty,\quad j=1,\dots,n,\qquad \textit{with } \alpha:=\sum_{j=1}^n\frac{1}{p_j}<1.$$

Then

$$\frac{|u(x)-u(y)|}{|x-y|^\beta}\le\frac{2}{\beta}\sum_{j=1}^n\left\|\frac{\partial u}{\partial x_j}\right\|_{L^{p_j}(\Omega)}\qquad\forall x,y\in\Omega,$$

holds with

$$\beta=\frac{1-\alpha}{1-\alpha+n/\min_j p_j}.$$

COROLLARY 3.1. *With the assumptions of Lemma 3.1, $\frac{\partial\Phi}{\partial x}$ is bounded in $C^{0,1/3}_{loc}([0,1]\times[0,\infty))$ uniformly for small $\varepsilon>0$.*

Now we turn to the limit $\varepsilon\to 0$. By the boundedness of n (Lemma 3.1) and of v (by $|v|<1$), we have convergence of n and v to n_0 and, respectively, v_0 in $L^\infty((0,1)\times(0,T))$ weak* for $T<\infty$ (for a subsequence). Also Corollary 3.1 implies uniform convergence of $\frac{\partial\Phi}{\partial x}$ to $\frac{\partial\Phi_0}{\partial x}$ in $[0,1]\times[0,T]$ (again for a subsequence). Obviously we can pass to the limit in the Poisson equation (2.4).

The weak formulation of the continuity equation (2.5) can be written as

$$\begin{aligned}&-\int_0^1\psi(t=0)n_I\,dx-\int_0^\infty\int_0^1 n\frac{\partial\psi}{\partial t}\,dx\,dt\\&=\int_0^\infty\int_0^1\left[\varepsilon n\frac{\partial^2\psi}{\partial x^2}-\varepsilon V\left(\frac{1}{\varepsilon}\frac{\partial\Phi}{\partial x}\right)\frac{\partial^2\psi}{\partial x^2}+v\left(\frac{1}{\varepsilon}\frac{\partial\Phi}{\partial x}\right)C\frac{\partial\psi}{\partial x}\right]dx\,dt,\end{aligned}\tag{3.2}$$

with $\psi\in C_0^\infty((0,1)\times[0,\infty))$ and any primitive V of v. It is easily shown that $\varepsilon V(z/\varepsilon)$ converges to $|z|$ uniformly in bounded z-intervals. This implies uniform convergence of $\varepsilon V\left(\frac{1}{\varepsilon}\frac{\partial\Phi}{\partial x}\right)$ to $\left|\frac{\partial\Phi_0}{\partial x}\right|$ in $\mathrm{supp}(\psi)$. Also, $v\left(\frac{1}{\varepsilon}\frac{\partial\Phi}{\partial x}\right)$ converges to $\mathrm{sign}\left(\frac{\partial\Phi_0}{\partial x}\right)$ pointwise in $\mathrm{supp}\left(\frac{\partial\Phi_0}{\partial x}\right)$ and, thus, v_0 is an element of (the set valued) $\mathrm{sign}\left(\frac{\partial\Phi_0}{\partial x}\right)$.

As a consequence of these observations we can pass to the limit in (3.2), showing that (n_0,Φ_0,v_0) is a weak solution of the system (1.1), (1.4) subject to the initial conditions (2.6). Also, by the uniform convergence, the limiting potential satisfies the boundary conditions for Φ in (2.7). We collect our results in the following theorem.

THEOREM 3.4. *Assume (2.3) and (2.8) and let* (n,Φ) *be the solution of (2.4)–(2.7). Then, as* $\varepsilon\to 0$*, restricting to subsequences,*

$$\begin{aligned} n &\to n_0 \quad \text{in } L^\infty_{loc}([0,1]\times[0,\infty)) \text{ weak *},\\ v\left(\frac{1}{\varepsilon}\frac{\partial\Phi}{\partial x}\right) &\to v_0 \quad \text{in } L^\infty_{loc}([0,1]\times[0,\infty)) \text{ weak *},\\ \Phi &\to \Phi_0 \quad \text{in } C^1_{loc}([0,1]\times[0,\infty)), \end{aligned}$$

where (n_0,v_0,Φ_0) *solve (1.1), (1.4) (in the sense of distributions), (2.6),* $\Phi_0(0,t)=0$*, and* $\Phi_0(1,t)=U(t)$*.*

Our estimates do not allow to pass to the limit in the boundary conditions (2.7) for the density. This was to be expected since the limiting transport equation is hyperbolic. The limiting problem becomes formally well posed for inflow boundary conditions of the form

$$\begin{aligned} n_0(0,t) &= n_l(t), \qquad \text{for } v_0(0,t)>0,\\ n_0(1,t) &= n_r(t), \qquad \text{for } v_0(1,t)<0. \end{aligned}$$

For $v_0(0,t)\le 0$ ($v_0(1,t)\ge 0$), no boundary condition can be prescribed at $x=0$ ($x=1$).

4. Some Typical Solutions

In this section we compute solutions of the problem

$$\frac{\partial^2\Phi}{\partial x^2} = n-1, \tag{4.1}$$

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv)=0, \qquad v\in \operatorname{sign}\left(\frac{\partial\Phi}{\partial x}\right), \tag{4.2}$$

for $x\in(0,1)$, $t>0$, subject to the initial conditions

$$n(x,0)=n_I=\text{const}, \tag{4.3}$$

and to the boundary conditions

$$\Phi(0,t)=0, \qquad \Phi(1,t)=U>0, \tag{4.4}$$

$$n(0,t)=0, \qquad \text{for } v(0,t)>0, \tag{4.5}$$

$$n(1,t)=1, \qquad \text{for } v(1,t)<0. \tag{4.6}$$

Uniqueness of solutions is only a conjecture and we do not make any claims in that direction. Consequently, some assumptions about the structure of the solution are introduced without justification in the following, although in some cases heuristic arguments are available. Support for our results will also be provided by comparisons with numerical results for the regularized problem (2.4)–(2.7). These are computed using a fractional time step approach where (4.1), (4.4) as a problem for Φ and (4.2), (4.5), (4.6) as a problem for n, are solved consecutively. An explicit finite difference method is used to solve (4.2), with upwinding for the convection terms, 10^4 equidistant grid points in the spatial direction, regularization parameter $\varepsilon = 10^{-4}$, and time steps according to the CFL limit $\Delta t = \Delta x$.

A possible physical interpretation of the problem is that the x-interval $[0,1]$ represents the n-region of a *pn*-diode with the *pn*-junction located at $x=0$ and a contact at $x=1$. In this interpretation the positivity of the applied voltage U means reverse bias, where current flow through the diode is expected to be blocked in a steady state. For $U < \frac{1}{2}$, a steady state solution is given by

$$n_s(x) = \begin{cases} 0\,, \text{ for } 0 < x < \sqrt{2U}\,, \\ 1\,, \text{ for } \sqrt{2U} < x < 1\,, \end{cases}$$
$$\Phi_s(x) = \begin{cases} x\sqrt{2U} - x^2/2\,, & \text{for } 0 < x < \sqrt{2U}\,, \\ U\,, & \text{for } \sqrt{2U} < x < 1\,, \end{cases} \tag{4.7}$$
$$v_s(x) = \begin{cases} 1\,, \text{ for } 0 < x < \sqrt{2U}\,, \\ 0\,, \text{ for } \sqrt{2U} < x < 1\,. \end{cases}$$

The value of v_s for $x > \sqrt{2U}$ is chosen by making the flux $n_s v_s$ constant ($n_s v_s = 0$, as expected). The occurrence of the depletion region $(0, \sqrt{2U})$ is a well known phenomenon in semiconductor physics. For $U > \frac{1}{2}$, the depletion region covers the whole interval $[0,1]$. In physically more accurate models (e.g., the bipolar drift-diffusion model over the full length of the diode), the approximative boundary condition $n_s(0) = 0$ breaks down in this case and the diode looses its ability to block the current, a phenomenon called *punch through* in the semiconductor literature (see [6]).

The explicit solution

$$\Phi(x,t) = Ux + \int_0^1 g(x,\xi)(n(\xi,t) - 1)\,d\xi \tag{4.8}$$

of the Poisson problem (4.1), (4.4), with the Green's function

$$g(x,\xi) = \begin{cases} x(\xi - 1)\,, \text{ for } x < \xi\,, \\ \xi(x-1)\,, \text{ for } x > \xi\,, \end{cases}$$

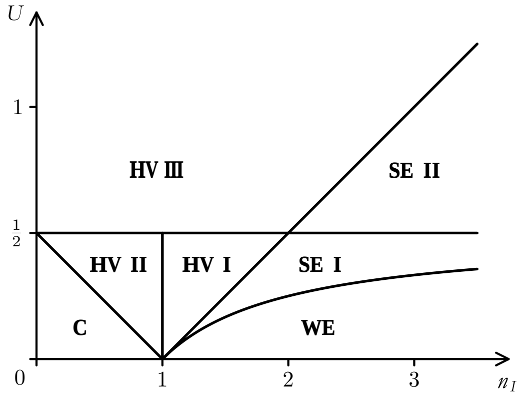
will be used in the following. The only information we need to extract from this formula is the sign of the electric field.

Initially (at $t=0$), three generic cases occur. For $2U > |n_I - 1|$ (the *high voltage case*), the initial potential $\Phi(x,0)$ is strictly monotonically increasing (implying $v(x,0) = 1$). On the other hand, for $0 < 2U < |n_I - 1|$, the initial field $\frac{\partial \Phi}{\partial x}(x,0)$ changes sign at

$$x = x_0 = \frac{1}{2} + \frac{U}{1 - n_I} \in (0,1)\,. \tag{4.9}$$

Depending on the convexity properties of the initial potential, we distinguish between the *compressive case* ($2U < 1 - n_I$) and the *expansive case* ($2U < n_I - 1$).

In terms of the qualitative behaviour of solutions of the problem (4.1)–(4.6), we arrive at a further subdivision of the parameter region defined by $U > 0$ and $n_I > 0$ into seven subregions, as shown in the following picture:



Legend:

C - compressive
WE - weakly expansive
SE - strongly expansive
HV - high voltage

diagram.01

We shall not discuss HV II, HV III, and SE II in the following. In these cases the solution is qualitatively similar to HV I and SE I, respectively. Apart from that, the solution for $U > \frac{1}{2}$ (cases HV III and SE II) is of limited physical interest, as mentioned above.

The remaining four cases are discussed in the following sections:

4.1. The compressive case: $n_I < 1$, $2U < 1 - n_I$. In this case the initial potential is concave with

$$v(x,0) = \operatorname{sign}(x_0 - x).$$

Initially, the characteristics of (4.2) are pointing inwards at the boundaries and both conditions (4.5) and (4.6) are effective. Close to $x = 0$, $t = 0$, the density is given by

$$n(x,t) = \begin{cases} 0, & \text{for } 0 < x < t, \\ n_I, & \text{for } 0 < t < x, \end{cases}$$

and similarly close to $x = 1$, $t = 0$. Less clear is what happens close to $x = x_0$. Assuming a tendency to charge neutrality, we construct a solution of the form

$$n(x,t) = \begin{cases} 0, & \text{for } 0 < x < t, \\ n_I, & \text{for } t < x < x_1(t), \\ 1, & \text{for } x_1(t) < x < x_2(t), \\ n_I, & \text{for } x_2(t) < x < 1 - t, \\ 1, & \text{for } 1 - t < x < 1, \end{cases} \tag{4.10}$$

with $x_1(0) = x_2(0) = x_0$. This representation is valid for $t < t_0$, where at $t = t_0$ at least one of the above subregions vanishes: $t_0 = x_1(t_0)$ or $x_1(t_0) = x_2(t_0)$ or $x_2(t_0) = 1 - t_0$.

The maximum of the potential at $x = x_0$ is assumed to develop into a plateau between $x_1(t)$ and $x_2(t)$:

$$\frac{\partial \Phi}{\partial x}(x,t) = 0 \qquad \text{for } x_1(t) \le x \le x_2(t). \tag{4.11}$$

After substitution of (4.10) into (4.8), (4.11) gives

$$t^2 - 2U + (1 - n_I)(x_1^2 - (1 - x_2)^2) = 0. \tag{4.12}$$

Since in the plateau region the field vanishes, the values of v are not determined. However, from $n=1$ and from the transport equation (4.2) we obtain $v(x,t)=\bar{v}(t)$ for $x_1(t)<x<x_2(t)$. Looking for a weak solution of (4.2), the Rankine-Hugoniot jump relations at $x=x_1(t)$ and at $x=x_2(t)$,

$$\dot{x}_1(n_I-1)=n_I-\bar{v}, \qquad \dot{x}_2(1-n_I)=\bar{v}+n_I, \tag{4.13}$$

have to be satisfied (in general: $\dot{x}_j[n]=[nv]$, where $[\cdot]$ denotes the jump of a quantity across $x=x_j$). Elimination of $\bar{v}$ from this system and subsequent integration with respect to time gives $(1-n_I)(x_2-x_1)=2n_It$. From this equation and from (4.12) we compute x_1 and x_2:

$$x_1(t)=\frac{1}{2}+\frac{U-t^2/2}{1-n_I-2n_It}-\frac{n_It}{1-n_I},$$
$$x_2(t)=\frac{1}{2}+\frac{U-t^2/2}{1-n_I-2n_It}+\frac{n_It}{1-n_I}.$$

The velocity in the plateau region is given by

$$\bar{v}(t)=\frac{n_I(x_1(t)+x_2(t)-1)-t}{x_1(t)+1-x_2(t)}.$$

For the condition $v\in\operatorname{sign}\frac{\partial\Phi}{\partial x}$, it remains to show that $|\bar{v}|\le 1$ holds. Using $t\le x_1$ and $x_2\le 1$, which holds in the domain of validity of the ansatz (4.10), we easily show

$$-1<-\max\{n_I,\ 1-n_I\}\le\bar{v}(t)\le n_I<1.$$

Finally, straightforward estimates give

$$\dot{x}_1<\bar{v},1, \qquad \dot{x}_2>\bar{v},-1,$$

and, thus, that the Filippov characteristics move through the discontinuity at $x=x_1(t)$ from left to right and at $x=x_2(t)$ from right to left.

As the next step we check, when the solution of the form (4.10) ceases to exist. It is easily shown that for $8U=(1-n_I)^2$ both $x_1(t)=t$ and $x_2(t)=1-t$ happen at the same time. We shall consider the situation

$$8U>(1-n_I)^2,$$

where a $t_0>0$ exists with $x_2(t_0)=1-t_0$ and $x_1(t_0)>t_0$. Note that at $t=t_0$, the plateau of the potential stretches from $x=x_1(t_0)$ to $x=1$. For $t>t_0$ the ansatz (4.10) is replaced by

$$n(x,t)=\begin{cases}0, & \text{for } 0<x<t,\\ n_I, & \text{for } t<x<x_1(t),\\ 1, & \text{for } x_1(t)<x<1.\end{cases} \tag{4.14}$$

Now $x_1(t)$ can be computed from the requirement $\frac{\partial\Phi}{\partial x}(x_1(t),t)=0$:

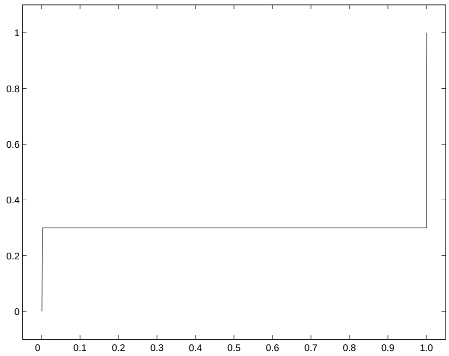
$$x_1(t)=\sqrt{\frac{2U-n_It^2}{1-n_I}},$$

and $v(x,t)=\tilde{v}(t)$, $x_1(t)<x<1$, from the Rankine-Hugoniot condition at $x=x_1(t)$:

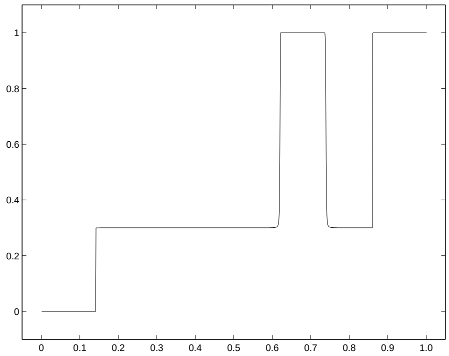
$$\tilde{v}(t)=n_I\left(1-\frac{t}{x_1(t)}\right).$$

For $t_1=\sqrt{2U}$, $x_1(t_1)=t_1$ and $\tilde{v}(t_1)=0$ hold. The region where n takes the initial value n_I vanishes, and the solution for $t>t_1$ is equal to the steady state (4.7). Thus, the steady state is reached in finite time.

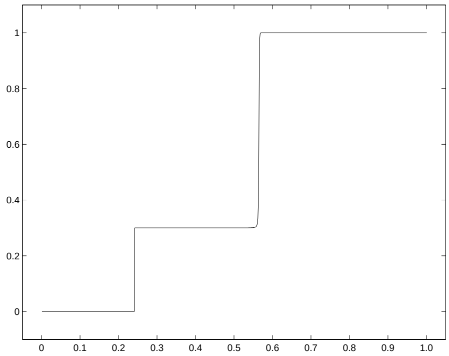
Finally, we present the results of numerical simulations for this case (with $n_I=0.3$ and $U=0.12$) in the four qualitatively different stages of the evolution process:



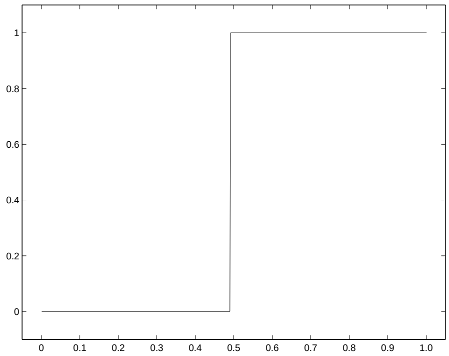
t=0.00 (initial)



t=0.12



t=0.25



t=0.49 (steady)

4.2. The high voltage case: $1<n_I<2$, $n_I-1<2U<1$. Now an initial density above the equilibrium value 1 is assumed. The assumptions on the voltage imply a strictly monotone initial potential and the existence of a steady state solution of the form (4.7). By $\frac{\partial\Phi}{\partial x}(x,0)>0$, the density for small times is given by

$$n(x,t)=\begin{cases}0, & \text{for } 0<x<t,\\ n_I, & \text{for } t<x<1.\end{cases}$$

The strict positivity of the field holds until

$$t=t_0=\frac{n_I-1+\sqrt{2Un_I-n_I+1}}{n_I}<1,$$

whence

$$\min_{x\in[0,1]}\frac{\partial\Phi}{\partial x}(x,t_0)=\frac{\partial\Phi}{\partial x}(t_0,t_0)=0.$$

For $t > t_0$, we construct a solution of the form

$$n(x,t) = \begin{cases} 0, & \text{for } 0 < x < x_1(t), \\ 1, & \text{for } x_1(t) < x < x_2(t), \\ n_I, & \text{for } x_2(t) < x < 1, \end{cases}$$

with

$$\frac{\partial \Phi}{\partial x}(x,t) = 0, \quad v(x,t) = \bar{v}(t), \qquad \text{for } x_1(t) < x < x_2(t).$$

From this ansatz and from the Rankine-Hugoniot jump conditions at $x = x_1$ and at $x = x_2$, the equations

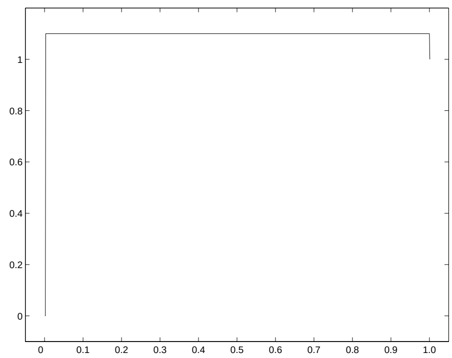
$$2U = x_1^2 + (n_I - 1)(1 - x_2)^2, \qquad x_1 + x_2(n_I - 1) = n_I t,$$

determining $x_1(t)$ and $x_2(t)$, can be derived. For

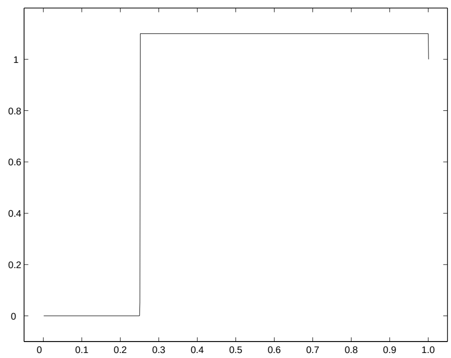
$$t = t_1 = \frac{n_I - 1 + \sqrt{2U}}{n_I},$$

$x_2(t_1) = 1$ and $x_1(t_1) = \sqrt{2U}$ hold and, thus, the steady state (4.7) is again reached in finite time.

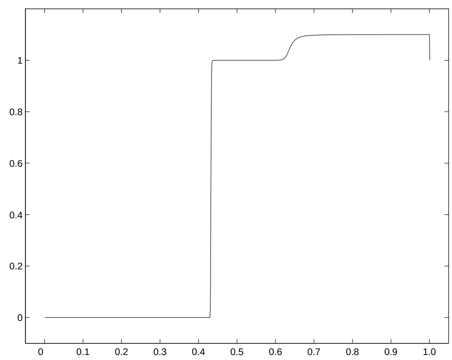
Again, we include results of our numerical simulations for this case (with the data $n_I = 1.1$ and $U = 0.1$) in the four qualitatively different stages of the evolution process:



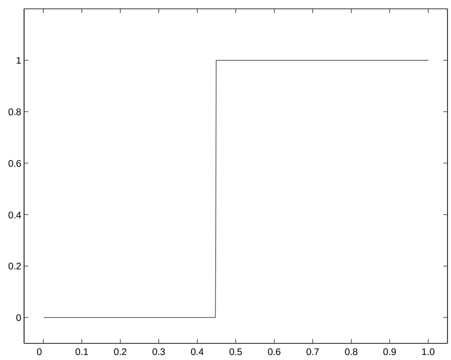
t=0.00 (initial)



t=0.25



t=0.45



t=0.50 (steady)

4.3. The strongly expansive case: $n_I > 1$**,** $n_I - 1 < 2Un_I < (n_I - 1)n_I$**.** Now (by the last inequality) the initial field vanishes at $x = x_0$ (see (4.9)). We construct a solution where the density is not piecewise constant as in the preceding cases, but (for small times) given by

$$n(x,t) = \begin{cases} n_I\,, & \text{for } 0 < x < x_1(t)\,, \\ \tilde{n}(x - t - x_0)\,, & \text{for } x_1(t) < x < x_0 + t\,, \\ n_I\,, & \text{for } x_0 + t < x < 1\,. \end{cases}$$

The potential remains convex with $\frac{\partial \Phi}{\partial x}(x_1(t), t) = 0$. With the ansatz for the density, this equation can be written as

$$U + \frac{n_I - 1}{2}\left(x_1^2 - (1 - x_0 - t)^2\right) + \int_{s(t)}^{0} (\eta + t + x_0 - 1)(\tilde{n}(\eta) - 1)\,d\eta = 0\,,$$

with $s(t) = x_1(t) - x_0 - t$. Introducing s as a new variable (instead of the time t), differentiation with respect to s gives

$$x_1(n_I + 1) - (1 - x_0 - t)(n_I - 1) + \frac{2n_I(\tilde{n}(s) - 1)}{n_I - \tilde{n}(s)} + \int_0^s (\tilde{n}(\eta) - 1)\,d\eta = 0\,, \qquad (4.15)$$

where the Rankine-Hugoniot condition at $x = x_1(t)$, $\dot{x}_1(n_I - \tilde{n}(s)) = -n_I - \tilde{n}(s)$, has been used for the computation of

$$\frac{dx_1}{ds} = \frac{\dot{x}_1}{\dot{s}} = \frac{\dot{x}_1}{\dot{x}_1 - 1} = \frac{n_I + \tilde{n}}{2n_I} \quad \text{and}$$
$$\frac{dt}{ds} = -\frac{n_I - \tilde{n}}{2n_I}\,.$$

With $s = 0$ we obtain

$$\tilde{n}(0) = n_I \frac{n_I + 1 - 2x_0 n_I}{3n_I - 1 - 2x_0 n_I}\,, \qquad (4.16)$$

lying between 1 and n_I. A further differentiation of (4.15) with respect to s leads to a differential equation for $\tilde{n}$:

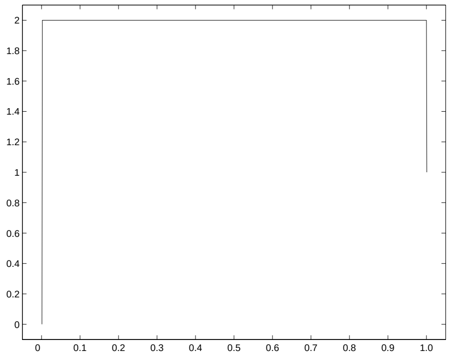
$$\tilde{n}' = -\frac{\tilde{n}(n_I - \tilde{n})^2}{n_I(n_I - 1)}\,. \qquad (4.17)$$

The initial value problem (4.16), (4.17) has to be solved for $s < 0$. Then $x_1(t)$ can be determined from

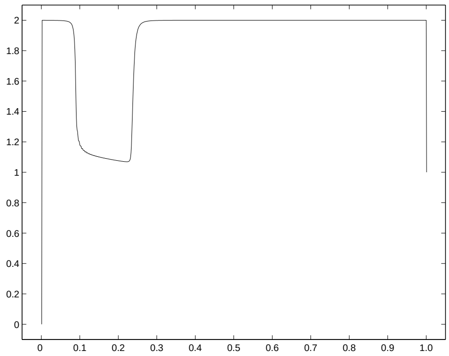
$$\dot{x}_1 = -\frac{n_I + \tilde{n}(x_1 - x_0 - t)}{n_I - \tilde{n}(x_1 - x_0 - t)}\,, \qquad x_1(0) = x_0\,.$$

Here we stop the explicit computations for this case. The further qualitative behaviour of the solution is as follows: Eventually, the location $x = x_1(t)$ of the minimum of the potential reaches the left boundary $x = 0$. After that, the potential is monotonically increasing (as a function of x), and the solution qualitatively behaves like in the preceding high voltage case.

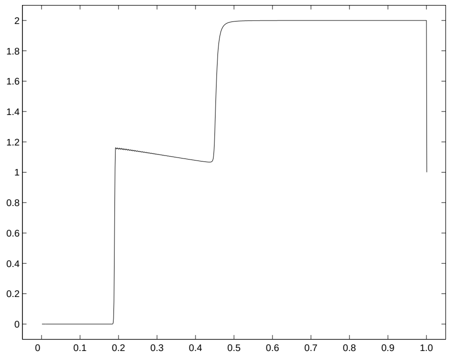
Finally, we present the results of numerical simulations for this case (with $n_I = 2.0$ and $U = 0.3$) in the six qualitatively different stages of the evolution process:



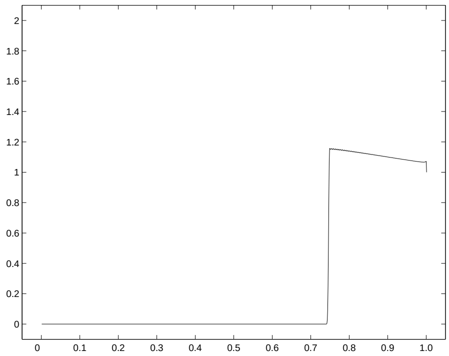
t=0.000 (initial)



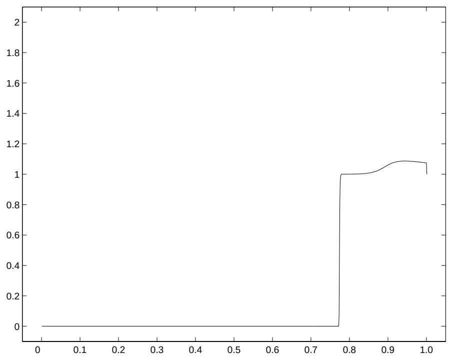
t=0.035



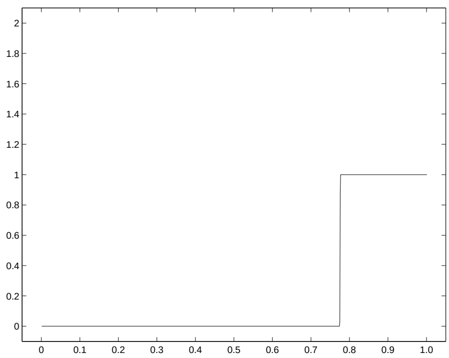
t=0.250



t=0.807



t=0.851



t=0.866 (steady)

4.4. The weakly expansive case: $0 < 2Un_I < n_I - 1$**.** Again, the initial field vanishes at $x = x_0$. For small times, the density now has the form

$$n(x,t) = \begin{cases} n_I\,, & \text{for } 0 < x < x_1(t)\,, \\ 1, & \text{for } x_1(t) < x < x_2(t)\,, \\ n_I\,, & \text{for } x_2(t) < x < 1\,, \end{cases}$$

with a plateau region for the potential:

$$\frac{\partial \Phi}{\partial x}(x,t) = 0, \quad v(x,t) = \bar{v}(t), \qquad \text{for } x_1(t) < x < x_2(t)\,.$$

Similarly to the compressive case, the jump locations and the velocity in the plateau region can be computed from the jump conditions:

$$
\begin{aligned}
x_1(t) &= \frac{1}{2} - \frac{U}{n_I - 1 - 2n_I t} - \frac{n_I t}{n_I - 1}, \\
x_2(t) &= \frac{1}{2} - \frac{U}{n_I - 1 - 2n_I t} + \frac{n_I t}{n_I - 1}, \\
\bar{v}(t) &= \frac{2n_I U(n_I - 1)}{(n_I - 1 - 2n_I t)^2}.
\end{aligned}
$$

The inequalities characterizing the weakly expansive case guarantee $0 < \bar{v}(0) < 1$. The above solution breaks down at

$$
t = t_0 = \frac{n_I - 1}{2n_I}\left(1 - \sqrt{\frac{2n_I U}{n_I - 1}}\right),
$$

whence $\bar{v}(t_0) = 1$ holds. For $t > t_0$, a unique minimum of the potential develops, and a solution of the form

$$
n(x,t) = \begin{cases} n_I, & \text{for } 0 < x < x_1(t), \\ \tilde{n}(x - x_1(t_0) - t + t_0), & \text{for } x_1(t) < x < x_1(t_0) + t - t_0, \\ 1, & \text{for } x_1(t_0) + t - t_0 < x < x_2(t_0) + t - t_0, \\ n_I, & \text{for } x_2(t_0) + t - t_0 < x < 1, \end{cases}
$$

can be constructed, with

$$
\frac{\partial \Phi}{\partial x}(x_1(t), t) = 0.
$$

The construction of $\tilde{n}$ and the qualitative behaviour after $t = t_0$ is similar to the strongly expansive case.

The results of numerical simulations for this case (with $n_I = 1.3$ and $U = 0.08$) in the six different stages of the evolution process are shown in the next page.

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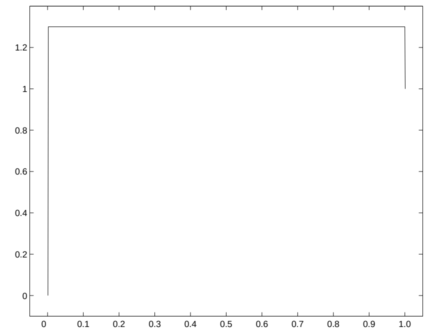
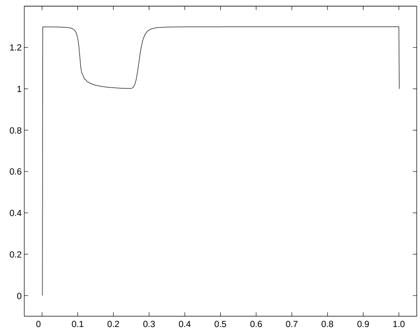
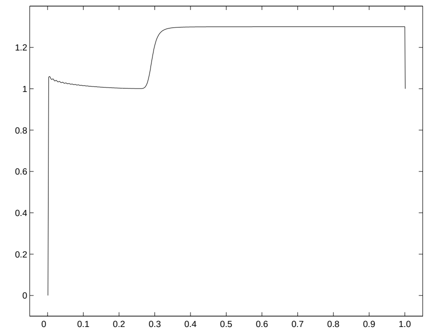


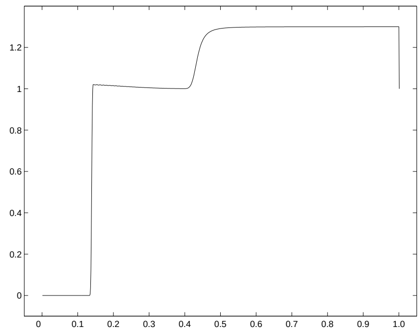
t=0.000 (initial)



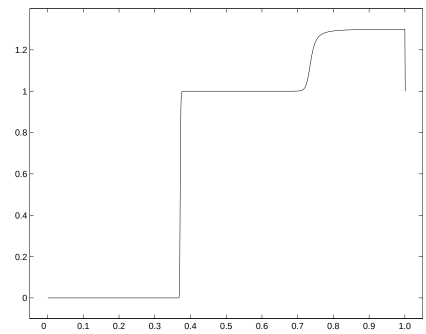
t=0.020



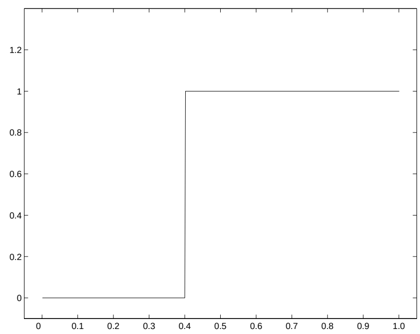
t=0.035



t=0.175



t=0.425



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