COMM. MATH. SCI.
Vol. 2, No. 1, pp. 35–52

STABILITY OF SOLITARY WAVES IN HIGHER-ORDER SOBOLEV SPACES*

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Abstract. The orbital stability of solitary waves has generally been established in Sobolev classes of relatively low order, such as H^1. It is shown here that at least for solitary-wave solutions of certain model equations, a sharp form of orbital stability is valid in L^2-based Sobolev classes of arbitrarily high order. Our theory includes the classical Korteweg-de Vries equation, the Benjamin-Ono equation and the cubic, nonlinear Schrödinger equation.

1. Introduction

Many equations for the description of wave motion that feature both nonlinearity and dispersion possess particular traveling-wave solutions called solitary waves. It has turned out that these special solutions often play a fundamental role in the long time behavior of quite general disturbances. In consequence of this, and because the issue is interesting in its own right, the stability of solitary waves to small perturbations has attracted considerable attention in the last three decades.

The mathematically exact theory for the stability of solitary waves began with Benjamin's theory [8] for the Korteweg-de Vries equation (see also [11]). In subsequent works, Benjamin's original conception was refined and extended in many ways. The existing theory is satisfactory both as regards its general conclusions about solitary waves and the range of its applicability, though it must be acknowledged that significant and difficult issues remain open (e.g. issues of asymptotic stability investigated by Pego and Weinstein [25] in particular cases and stability of solitary-wave solutions of complex systems like the Boussinesq systems or the full Euler equations for the propagation of surface water waves).

It is our purpose here to extend the existing theory in a direction not previously considered, and which we now explain in the context of Korteweg-de Vries-type equations. The evolution equations we have in mind take the form

$$u_t + f(u)_x - Lu_x = 0 \tag{1.1}$$

where $u(x,t)$ is a real-valued function of two real variables, $f\colon R \longrightarrow R$ is a smooth function (usually a polynomial), L is a Fourier-multiplier operator defined by

$$\widehat{Lv}(\zeta) = \alpha(\zeta)\hat{v}(\zeta) \tag{1.2}$$

for a non-negative, even dispersion symbol α, subscripts connote partial differentiation and the circumflex denotes the Fourier transform with respect to the spatial variable x. When an equation in the class depicted in (1.1) is interpreted as a model of physical phenomena, x is typically proportional to distance in the direction of propagation, t is proportional to time and u is often a displacement or a velocity. In the context of (1.1), a solitary wave is a traveling-wave solution of the form $\phi(x-ct)$ where c is a

*Received: October 21, 2003; accepted (in revised version): January 7, 2004. Communicated by Paul Milewski.

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fixed positive constant and ϕ is usually an even function tending to zero at $\pm\infty$. (For Schrödinger-type equations, the definition is slightly different as we will see later.)

Stability here is referred to the initial-value problem. Thus, in the context of (1.1), one imagines being provided with an initial wave profile, say at $t = 0$,

$$u(x, 0) = \psi(x), \tag{1.3}$$

for $x \in R$, and then inquiring into the subsequent evolution using (1.1). This presumes that the initial-value problem (1.1)-(1.2) is a well-posed problem so that a unique solution $u(x, t)$ departs from ψ under the influence of (1.1).

The solitary wave $\phi = \phi_c$ is said to be orbitally stable in a Banach space X with norm $\|.\|_X$ if whenever $\epsilon > 0$ is specified, there is a corresponding $\delta > 0$ such that

$$\|\phi - u(\cdot, 0)\|_X \leq \delta \tag{1.4}$$

implies

$$\inf_{y \in R} \|u(\cdot + y, t) - \phi\|_X \leq \epsilon, \tag{1.5}$$

for all $t \geq 0$. This result is interpreted to say that, if at some time, say $t = 0$, a solution of (1.1) is close to ϕ relative to X, then it remains close in shape for all subsequent (and previous) time. Of course, it is possible that in some contexts the stipulation (1.3) might need strengthening, say to

$$\|\phi - u(\cdot, 0)\|_Y \leq \delta, \tag{1.6}$$

where $Y \subset X$ is a smaller space with a stronger norm, though one would generally prefer that stability in X subsist on the data lying close to ϕ in X and nothing more. These general ruminations about orbital stability can be found in [13], and as shown in this reference, it is the case that in many circumstances where stability is obtained as just outlined, there is in fact a smooth function $\gamma : R \longrightarrow R$ such that

$$\|u(\cdot, t) - \phi(\cdot + \gamma(t))\|_X \leq \epsilon \tag{1.7}$$

for all $t \in R$ and

$$|\gamma'(t) + c| \leq \epsilon \tag{1.8}$$

for all t, where c is the phase speed of the solitary wave whose stability is in question. This latter result may be interpreted as saying that the bulk of the wave motion emanating from the perturbed solitary wave propagates at a speed very near to the original phase velocity c. This is consistent with, but not necessarily equivalent to, asymptotic stability.

As just outlined, the theory accords well with what is observed in real situations, and with the outcome of numerical stimulations of equations of the form depicted in (1.1). One aspect does not fit well with what is observed in computer approximations and comprises a limitation of the theory. This point is explained next.

The space X for which the conclusion (1.4) holds is usually dictated by the Hamiltonian for the equation. In the case of (1.1), the functional

$$H(u) = \int_{-\infty}^{\infty} \left(\frac{1}{2} uLu + F(u)\right) dx$$

is a Hamiltonian for (1.1), where $F' = f$ and $F(0) = 0$, say. That is, (1.1) is formally equivalent to

$$\frac{\partial u}{\partial t} = J\nabla H(u),$$

where the skew-adjoint operator J is simply $-\partial_x$ and ∇H is the gradient of the functional H. (That is to say, $\nabla H(u)$ is the Gateaux derivative of H at u in the direction v,

$$H'(u)v = (\nabla H(u), v)$$

where the inner product is that of $L^2(R)$ in this case.) Stability in the sense defined by (1.4)-(1.5) or (1.6)-(1.5) is naturally referred by the existing theory to the so-called energy space

$$\|g\|_X = \left\{\int_{-\infty}^{\infty} (1 + \alpha(k))|\hat{g}(u)|^2 dk\right\}^{\frac{1}{2}}.$$

It is our purpose here to establish orbital stability as indicated above for smaller spaces X whose norms are much stronger. What this means practically is that not only does the bulk of what emanates from the perturbed solitary wave stay close in shape and propagation speed to the original solitary wave, but emerging residual oscillations must also be very small and not only in the energy norm. An example of what our theory denies is easily displayed. (In general, for $s \geq 0$, $H^s(R)$ is the subspace of square integrable functions whose s^{th} generalized derivative is also square integrable.) Suppose initial data ψ lies near to a solitary-wave solution ϕ_c in the strong sense of a small space like $H^k(R)$ for k large, say in the context of the KdV-equation itself ((1.1) with $f(u) = u^2/2$ and $Lu = -\partial_x^2 u$). The energy space in this case is $H^1(R)$. The currently existing theory would not preclude, for some positive time t, that the solution u corresponding to initial data ψ might have the form

$$u(x,t) = \phi_c(x - \theta) + \rho(x)$$

for some $\theta \in R$, where ρ has the form

$$\rho(x) = \epsilon\mu(x)\cos(\epsilon^{\alpha} x)$$

with $\alpha < 0$, say $\alpha = -1/k$. For $\epsilon << 1$, ρ is indeed small in $H^1(R)$, but features very significant high frequency oscillations that, for example, never appear in numerical simulations when ψ is as described above. In particular, the $H^k(R)$-norm is not generally small, no matter how small is ϵ. The present theory precludes such behavior if ψ does not initially feature these sort of oscillations.

It is worth remarking that at least in the case of the Benjamin-Ono equation, the issue has extra interest for the following reason. The existing stability theory for the Benjamin-Ono solitary waves (see [10]) adduces stability in the sense of (1.4)-(1.5) where $X = H^{\frac{1}{2}}(R)$. However, it must additionally be assumed that $\psi \in H^s(R)$ where $s \geq 3/2$. (Well-posedness for the case $s = 3/2$ follows from the subsequent work of Ponce [24] whilst well-posedness for $s > 3/2$ was in the original paper [10] based on Kato's theory; see Kato [19] and Tom [26].) Because of the recent results of Molinet, Tzvetkov and Saut [23], it appears that a well-posedness theory for the Benjamin-Ono equation in very weak spaces is troublesome, and so having a stability theory in higher order Sobolev classes allows one to match it to the well-posedness results.

2. Notation and Well-Posedness Results

For $1 \leq p < +\infty$, $L^p = L^p(R)$ is the usual Banach space of classes of real-valued, Lebesgue measurable functions $f : R \longrightarrow R$ that are p^{th}-power integrable whereas $L^\infty = L^\infty(R)$ is the measurable, essentially bounded functions. The case $p = 2$ already appeared in the Introduction. The norm of $f \in L^p$ is denoted $|f|_p$. For $s \geq 0$, the Sobolev space H^s is the subspace of functions $f \in L^2$ such that

$$\|f\|_s^2 = \int_\infty^\infty (1 + \zeta^2)^s |\hat{f}(\zeta)|^2 d\zeta < +\infty.$$

The space H^s is a Hilbert space with this norm. The cases $s = 1$ and $s = k$ were featured in our previous commentary.

Logically, prior to a discussion of stability as formulated above in terms of pertubations of the initial data should be a theory for the initial-valued problem itself. This is a subject that has attracted considerable attention and it is not our purpose to provide a survey of the theory. The results outlined below suffice for the stability theory developed here. More subtle results are available in some cases but these do not concern us here.

For the initial-value problems for the KdV-equation

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, & \text{for } x \in R \text{ and } t \geq 0, \\ u(x,0) = \psi(x), \end{cases} \tag{2.1}$$

and the mKdV-equation

$$\begin{cases} u_t + u^2 u_x + u_{xxx} = 0, & \text{for } x \in R \text{ and } t \geq 0, \\ u(x,0) = \psi(x), \end{cases} \tag{2.2}$$

the following theory suffices (see [19]).

THEOREM 2.1. *Let $s > 3/2$. For each $\psi \in H^s$, there exist a $T > 0$ depending only on $\|\psi\|_{H^s}$ and a unique solution u to the KdV-equation (respectively, the mKdV-equation) such that $u \in C([0,T]; H^s)$. Moreover, for any fixed $T > 0$, if U_T is the mapping which associates to ψ the solution u on the interval $[0,T]$, then U_T is locally Lipschitz continuous from H^s into $C([0,T]; H^s)$.*

For the Benjamin-Ono equation

$$\begin{cases} u_t + uu_x + H(u_{xx}) = 0, & \text{for } x \in R \text{ and } t \geq 0, \\ u(x,0) = \psi(x), \end{cases} \tag{2.3}$$

where H denotes the Hilbert transform defined by the principle value integral

$$H(u(x)) = \frac{1}{\pi} P.V. \int \frac{u(y)}{x - y} dy,$$

the following theory suffices (see [2]).

THEOREM 2.2. *Let $s > 3/2$. For each $\psi \in H^s$, there exist a $T > 0$ depending only on $\|\psi\|_{H^s}$, and a unique solution u of (2.3) such that $u \in C^k([0,T]; H^{s-2k}(R))$ for all $k \in N$ with $s - 2k \geq -1$. For any fixed $T > 0$, let U_T be the mapping which associcates to ψ the solution u on the interval $[0,T]$. Then U_T is continuous from H^s into $C^k([0,T]; H^{s-2k}(R))$, for the same range of k.*

Virtually identical theory holds for the Intermediate Long Wave equation

$$\begin{cases} u_t + uu_x + \frac{1}{\delta}u_x + T_\delta(u_{xx}) = 0, & \text{for } x \in R \text{ and } t \geq 0, \\ u(x,0) = \psi(x), \end{cases} \tag{2.4}$$

where the symbol α_δ of $L = T_\delta \partial_x$ (see (1.1)-(1.2)) has the form

$$\alpha_\delta(\zeta) = 2\pi coth(2\pi\delta\zeta) - \frac{1}{\delta} \tag{2.5}$$

(see, again, [2]).

For the cubic, nonlinear Schrödinger equation

$$\begin{cases} iu_t + u_{xx} - 2\sigma|u|^2 u = 0, & \text{for } x, t \in R, \\ u(x,0) = \psi(x), \end{cases} \tag{2.6}$$

where $\sigma = \pm 1$, we only need the following result (see [20]). Here, the spaces feature complex-valued functions.

Theorem 2.3. *Let $s \geq 1$. For each $\psi \in H^s$, there exist a $T > 0$, depending only on $\|\psi\|_{H^s}$ and a unique solution u to the NLS equation (2.4) such that $u \in C([0,T]; H^s)$. For any fixed $T > 0$, let U_T be the mapping which associates to ψ the solution u on the interval $[0,T]$. Then U_T is continous from H^s into $C([0,T]; H^s)$.*

3. Korteweg-de Vries Equation

Considered here are the classical Korteweg-de Vries equation (KdV)

$$u_t + u_{xxx} + uu_x = 0$$

and the modified Korteweg-de Vries equation (mKdV)

$$u_t + u_{xxx} + u^2 u_x = 0.$$

These equations possess infinitely many integral invariants I_i $(i = 1, 2, ...)$ [22]. Reproduced below are the first six invariants for the KdV and the mKdV equations:

$$I_1(u) = \int u dx,$$
$$I_2(u) = \int u^2 dx,$$

and for the KdV-equation

$$I_3(u) = \int \left(u_x^2 - \frac{1}{3}u^3\right) dx,$$
$$I_4(u) = \int \left(\frac{9}{5}u_{xx}^2 - 3uu_x^2 + \frac{1}{4}u^4\right) dx,$$
$$I_5(u) = \int \left(\frac{1}{5}u^5 - 6u^2u_x^2 + \frac{36}{5}uu_{xx}^2 - \frac{108}{35}u_{xxx}^2\right) dx,$$
$$I_6(u) = \int \left(\frac{1}{6}u^6 - 10u^3u_x^2 + 18u^2u_{xx}^2 - 5u_x^4 - \frac{108}{7}uu_{xxx}^2 + \frac{120}{7}u_{xx}^3 + \frac{36}{7}u_{xxxx}^2\right) dx,$$

whereas for the mKdV-equation,

$$
\begin{aligned}
I_3(u) &= \int \left(\frac{3}{2}u_x^2 - \frac{1}{4}u^4\right)dx,\\
I_4(u) &= \int \left(\frac{1}{6}u^6 - 5u^2u_x^2 + 3u_{xx}^2\right)dx,\\
I_5(u) &= \int \left(\frac{1}{8}u^8 - \frac{21}{2}u^4u_x^2 + \frac{63}{5}u^2u_{xx}^2 - \frac{63}{10}u_x^4 - \frac{27}{5}u_{xxx}^2\right)dx,\\
I_6(u) &= \int \left(\frac{1}{10}u^{10} - 18u^6u_x^2 + \frac{162}{5}u^4u_{xx}^2 - \frac{342}{5}u^2u_x^4 - \frac{972}{35}u^2u_{xxx}^2 + \frac{432}{7}uu_{xx}^3 + \right.\\
&\qquad\qquad \left. \frac{5508}{35}u_x^2u_{xx}^2 + \frac{324}{35}u_{xxxx}^2\right)dx.
\end{aligned}
$$

These integral invariants play a central role in our stability argument.

For the KdV-equation, the general result in this direction may be expressed as follows (see [22]). There is a pair of sequences of polynomials $\{P_j\}_{j\geq 2}$ and $\{Q_j\}_{j\geq 2}$,

$$P_j = P_j(y_0, \cdots, y_{j-2}),$$

$$Q_j = Q_j(z_0, \cdots, z_{j+2}),$$

such that if $u = u(x,t)$ is a C^{k+2}-solution of KdV, then

$$\frac{\partial}{\partial t}\left\{P_j(u, u_x, \cdots, \partial_x^{j-2}u)\right\} = \frac{\partial}{\partial x}\left\{Q_j(u, u_x, \cdots, \partial_x^{j+2}u)\right\}. \tag{3.1}$$

Moreover, $P_j(y_0, \cdots, y_m)$ is exactly a linear combination of monomials $y_0^{r_0} \cdots y_m^{r_m}$ for which

$$\sum_{i=0}^{m}\left(1 + \frac{i}{2}\right)r_i = j. \tag{3.2}$$

Similarly, Q_j is comprised of monomials $z_0^{s_0} \cdots z_k^{s_k}$ such that

$$\sum_{i=0}^{k}\left(1 + \frac{i}{2}\right)s_i = j + 2. \tag{3.3}$$

There is likewise a pair of sequences of polynomials $\{\bar{P}_j\}$ and $\{\bar{Q}_j\}$, (see again [22]), such that $\bar{P}_j(y_0, \cdots, y_m)$ is a linear combination of monomials $y_0^{r_0} \cdots y_m^{r_m}$ for which

$$1 + \frac{1}{2}\sum_{i=0}^{m}\left(1 + i\right)r_i = j \tag{3.4}$$

whereas $\bar{Q}_j$ is comprised of monomials $z_0^{s_0} \cdots z_k^{s_k}$ such that

$$1 + \frac{1}{2}\sum_{i=0}^{k}\left(1 + i\right)s_i = j + 2 \tag{3.5}$$

and $\bar{P}_j$ and $\bar{Q}_j$ satisfy

$$\frac{\partial}{\partial t}\left\{\bar{P}_j(u, u_x, \cdots, \partial_x^{j-2}u)\right\} = \frac{\partial}{\partial x}\left\{\bar{Q}_j(u, u_x, \cdots, \partial_x^{j+2}u)\right\} \tag{3.6}$$

whenever $u = u(x,t)$ is a C^{j+2}-solution of the m-KdV equation.

If the solution u of the KdV-equation (mKdV respectively) is not only sufficiently smooth, but also, along with its first few derivatives, decays to zero at $\pm\infty$ rapidly enough, then (3.1) ((3.6), respectively) implies the integral

$$I_j(u) = \int P_j(u, u_x, \cdots, \partial_x^{j-2}u)dx$$

is time invariant. Indeed, because of the continuous dependence of u on its initial value ψ in H^{j-2}, the density of H^{j+3} in H^{j-2} and the fact that $Q_j(f, f_x, \cdots, \partial_x^{j+2}f) \longrightarrow 0$ as $x \longrightarrow \pm\infty$ for $f \in H^{j+3}$, it follows that $I_j(u)$ is time independent for solutions $u \in C([0,T), H^{j-2})$ of the KdV-equation (mKdV-equation respectively).

For $j \geq 2$, the invariant functional I_j for the KdV (and mKdV) has the form

$$I_j(u) = \int c_j(\partial_x^{j-2}u)^2 + \cdots dx \tag{3.7}$$

where the signs are organized so that $c_j > 0$.

As is well known (see [12]), these invariants imply global bounds on solutions. Thus if $u \in C([0,\infty), H^{j-2})$ solves the KdV-equation (mKdV-equation), then

$$\sup_{t\geq 0} \|u(\cdot,t)\|_{j-2} \leq K_j, \tag{3.8}$$

where $K_j = K_j(\psi)$ depends only on the H^{j-2}-norm of the initial data ψ.

Another point which follows immediately upon consideration of the individual monomials making up P_j, is that if u, v are both H^{j-2}-solutions of the KdV-equation (mKdV-equation) with initial data ψ and $\tilde{\psi}$ respectively, then

$$|I_j(u) - I_j(v)| \leq L_j\|u - v\|_{j-2}, \tag{3.9}$$

where L_j is a constant depending on $\|\psi\|_{j-2}$ and $\|\tilde{\psi}\|_{j-2}$. Indeed, (3.9) holds for any functions u, v in $C([0,T]; H^{j-2})$ where the constant L_j depends on $\|u\|_{C([0,T];H^{j-2})}$ and $\|v\|_{C([0,T];H^{j-2})}$. However, if u, v solve the KdV-equation (mKdV-equation) then L_j depends only on $\|\psi\|_{j-2}$ and $\|\tilde{\psi}\|_{j-2}$ on account of (3.8), and so may be taken to be time-independent. Of course, $\|u - v\|_{j-2}$ need not be time-independent!

THEOREM 3.1. *Let $n \geq 1$ be an integer. The solitary-wave solutions ϕ_c, $c > 0$ of the KdV-equation and the mKdV-equation are stable in H^n.*

Remark 1. The solitary-wave solution of the generalized KdV-equation

$$u_t + u_{xxx} + u^p u_x = 0$$

has the form $\phi_c(x-ct)$ where ϕ_c satisfies the ordinary differential equation

$$\partial^2_{\zeta\zeta}\phi_c + c\phi_c - \frac{1}{p+1}\phi_c^{p+1} = 0. \tag{3.10}$$

Up to translation of the independent variable, there is only one solution of (3.10) that decays to zero as $\zeta \longrightarrow \pm\infty$ for $p=1$ and only two (ϕ_c and $-\phi_c$) when $p=2$. These have the exact form

$$\phi_c(\zeta) = \alpha sech^{\frac{2}{p}}(\beta\zeta)$$

where $\alpha = \frac{1}{2}c(p+1)(p+2)\frac{1}{p}$ and $\beta = \frac{1}{2}p\sqrt{c}$. Indeed, these solutions define travelling-waves of the generalized KdV-equation for all integers p.

Remark 2. For the KdV-equation, nonlinear stability of ϕ_c in H^1 was proved by Benjamin [8] and Bona [11], and for the mKdV-equation, the stability result in H^1 is a consequence of the work in [3], [14], and [27].

Proof. The argument precedes by an induction wherein stability is shown sequentially to hold in $X = H^n$ for integer values $n = 1, 2, \cdots$.

According to the theory developed in [13] or [25], given a speed $c > 0$ and $\epsilon_1 > 0$, there is a $\delta_1 = \delta_1(\epsilon_1, c)$ such that if $\|\psi - \phi_c\|_1 \leq \delta_1$, then there is a C^1-mapping $\gamma : R \longrightarrow R$ (γ depends on c and ψ) such that

i) $\gamma(0) = 0$,

ii) if u is the solution of the KdV-equation with initial value ψ, then

$$\inf_{y\in R} \|u(\cdot,t) - \phi_c(\cdot + y)\|_1 \leq \|u(\cdot,t) - \phi_c(\cdot + \gamma(t))\|_1 < \epsilon_1$$

for all $t \geq 0$, and

iii) $|\gamma'(t) + c| \leq c_1\epsilon_1$, for all $t \geq 0$, when c_1 is a constant that depends only on c. In particular, orbital stability in the sense specified in the Introduction is known in H^1.

Attention is now given to stability in H^2. Let $c > 0$ be fixed and let $h(x,t) = u(x,t) - \phi_c(x + \gamma(t))$. Stability in H^2 follows if we can show that given $\epsilon_2 > 0$, there is a $\delta_2 > 0$ such that $\|h(\cdot,t)\|_2 \leq \epsilon_2$ for all t, provided $\|h(\cdot,0)\|_2 \leq \delta_2$. Write ϕ for ϕ_c for ease of reading. The difference

$$\Delta I_4(u) = I_4(u(\cdot,t)) - I_4(\phi(\cdot + \gamma(t)))$$

is central to our argument in favor of H^2-stability. A little calculation and an integration by parts show that

$$\Delta I_4(u) = \int\left\{\frac{9}{5}h_{xx}^2 + \frac{18}{5}h\phi_{xxxx} - 6\phi\phi_x h_x - 3\phi h_x^2 - 3h\phi_x^2 - 6hh_x\phi_x - 3hh_x^2 + \frac{1}{4}h^4 + h^3\phi \right.$$
$$\left. + \frac{3}{2}h^2\phi^2 + h\phi^3\right\}dx. \tag{3.11}$$

Since $I_4(u(\cdot,t))$ is time independent, it follows that $\Delta I_4(u) = \Delta I_4(\psi)$ depends only on the initial data ψ. Evaluating (3.11) at $t = 0$ and making straightforward estimates, it is determined that

$$\Delta I_4(u) \leq c_0\delta_2 + c_1\delta_2^4, \tag{3.12}$$

where δ_2 is any upper bound for $\|\psi - \phi\|_2$ and c_0, c_1 are constants depending only on ϕ, and hence only on c.

Because of the H^1-stability result enunciated above, for any $\epsilon_1 > 0$, there is a $\delta_1 > 0$ such that if $\|\psi - \phi\|_1 \le \delta_1$, then there is a C^1-function γ such that

$$\|u(\cdot,t) - \phi(\cdot + \gamma(t))\|_1 = \|h(\cdot,t)\|_1 \le \epsilon_1 \tag{3.13}$$

for all t.

Suppose now that $\|\psi - \phi\|_2 \le \delta_1$ which certainly implies $\|\psi - \phi\|_1 \le \delta_1$. Using (3.11) and (3.13), the quantity $\Delta I_4(u)$ may be bounded below as follows;

$$\Delta I_4(u) \ge \frac{9}{5}|h_{xx}|_2^2 - \epsilon_1 D_0 - \epsilon_1^4 D_1 \tag{3.14}$$

where D_0, D_1 are constants depending only on c.

Combining (3.12) and (3.14), there obtains the inequality

$$\frac{9}{5}|h_{xx}(.,t)|_2^2 \le \epsilon_1 D_0 + \epsilon_1^4 D_1 + \delta_2 c_0 + \delta_2^4 c_1$$

holding for all t. In consequence, it is deduced that

$$\|h(\cdot,t)\|_2^2 \le \epsilon_1 M_0 + \delta_2 M_1$$

where M_0, M_1 are smooth functions of ϵ_1, δ_2, and c, and in particular are bounded on bounded sets.

It remains simply to choose ϵ_1 so that $\epsilon_1 M_0 < \epsilon_2^2/4$. This implies the existence of a $\delta_1 > 0$ as in (3.13). Then choose $\delta_2 \le \delta_1$ small enough that $\delta_2 M_1 \le \epsilon_2^2/4$ also. The stability conclusion then follows.

Notice that we proved something a little stronger than just orbital stability in H^2. It was deduced in fact that if $\|\psi - \phi\|_2 < \delta_2$, then for all t,

$$\|h(\cdot,t)\|_2 = \|u(\cdot,t) - \phi(\cdot + \gamma(t))\|_2 \le \epsilon_2$$

where $\gamma(t)$ is the same smooth function appearing in the H^1-stability result of [13].

Proceed inductively, supposing that for all $j < k$, stability holds in H^j in the stronger sense that, given an $\epsilon_j > 0$, there is a $\delta_j > 0$ such that if $\|\psi - \phi\|_j \le \delta_j$, then $\|h(\cdot,t)\|_j \le \epsilon_j$ for all t. Presuming that $\psi \in H^k$, the stability in H^k is established by using the invariant functional I_{k+2}.

Fix an $\epsilon_k > 0$. As in the case $k = 2$, define

$$\Delta I_{k+2}(u) = I_{k+2}(u(\cdot,t)) - I_{k+2}(\phi(\cdot + \gamma(t))). \tag{3.15}$$

This quantity is time independent if u is the solution of the KdV-equation with initial value ψ, and thus depends only upon ψ. An upper bound for $\Delta I_{k+2}(u)$ is easily determined in terms of an upper bound δ_k for $\|\psi - \phi\|_k$ by evaluating (3.15) at $t = 0$, *viz.*

$$\Delta I_{k+2}(u) \leq c_{k+2}\delta_k^2 + c'_{k+2}\delta_k^{k+2}. \tag{3.16}$$

For any positive value ϵ_{k-1}, there is a $\delta_{k-1} > 0$ for which $\|\psi - \phi\|_{k-1} \leq \delta_{k-1}$ implies

$$\|h(\cdot,t)\|_{k-1} = \|u(\cdot,t) - \phi(\cdot + \gamma(t))\|_{k-1} \leq \epsilon_{k-1}. \tag{3.17}$$

for all t. A direct calculation of $\Delta I_{k+2}(u)$ in terms of h and ϕ reveals that

$$\Delta I_{k+2}(u) = c_{k+2}\int (\partial_x^k h)^2 + \text{lower-order terms}, \tag{3.18}$$

where $c_{k+2} > 0$. Because of (3.17), it is straightforward, using the general form of I_j described in (3.1)-(3.3), to ascertain that

$$|\text{lower-order terms}| \leq N_k\epsilon_{k-1}^2 + N'_k\epsilon_{k-1}^{k+2} \tag{3.19}$$

where N_k and N'_k are constants depending only on ϕ and so only on c. Combining (3.16), (3.18) and (3.19), there appears the inequality

$$|\partial_x^k h|_2^2 \leq c_k\delta_k^2 + c'_k\delta_k^{k+2} + N_k\epsilon_{k-1}^2 + N'_k\epsilon_{k-1}^{k+2}. \tag{3.20}$$

The desired result now follows by first choosing ϵ_{k-1} small enough and then choosing δ_k accordingly.

An argument for the higher-order stability of solitary-wave solutions of the mKdV-equation follows the same pattern, though making use of (3.4)-(3.7) rather than (3.1)-(3.3). $\square$

4. Benjamin-Ono and Intermediate Long-Wave Equations

We turn now to consideration of the Benjamin-Ono equation

$$u_t + uu_x + H(u_{xx}) = 0,$$

$x, t \in R$, where H denotes the Hilbert transform defined by the principle-value integral

$$H(u(x)) = \frac{1}{\pi}P.V.\int \frac{u(y)}{x-y}dy.$$

This equation is in the form (1.1) where $f(u) = u^2/2$ and the dispersion symbol $\alpha(\zeta) = 2\pi|\zeta|$. Solitary-wave solutions of the Benjamin-Ono equation satisfy the equation

$$-c\phi'_c + \phi_c\phi'_c + H\phi''_c = 0. \tag{4.1}$$

For any wave speed $c > 0$, (4.1) has an exact solution of the form

$$\phi_c(y) = \frac{4c}{1 + c^2y^2}, \tag{4.2}$$

which was found by Benjamin [9]. Up to translations in the independent variable, these solutions are unique, as shown by Amick and Toland (see [6], [7], and also [5]).

It is known that these solitary waves are stable in $H^{\frac{1}{2}}$ (see [10]). Like the KdV- and mKdV-equations, the Benjamin-Ono-equation possesses infinitely many integral invariants (see [15] and [21]). The following are the first six invariants in the form given in [15]:

$$
\begin{aligned}
I_{-1}(u) &= \int u dx,\\
I_0(u) &= \int \frac{1}{2}u^2 dx,\\
I_1(u) &= -\int\left\{\frac{1}{3}u^3 + uH(u_x)\right\}dx,\\
I_2(u) &= \int\left\{\frac{1}{4}u^4 + \frac{3}{2}u^2H(u_x) + 2u_x^2\right\}dx,\\
I_3(u) &= \int\left\{-\frac{1}{5}u^5 - \left(\frac{4}{3}u^3H(u_x) + u^2H(uu_x)\right) - \left(2uH(u_x)^2 + 6uu_x^2\right)\right.\\
&\qquad\qquad\qquad \left. +4uH(u_{xxx})\right\}dx,\\
I_4(u) &= \int\left\{\frac{1}{6}u^6 + \left(\frac{5}{4}u^4H(u_x) + \frac{5}{3}u^3H(uu_x)\right) + \frac{5}{2}\left(5u^2u_x^2 + u^2H(u_x)^2\right.\right.\\
&\qquad \left.\left. +2uH(u_x)H(uu_x)\right) - 10\left(u_x^2H(u_x) + 2uu_{xx}H(u_x)\right) + 8u_{xx}^2\right\}dx.
\end{aligned}
$$

In general, the integral invariants $I_n(u)$, $n = 0, 1, \cdots$, of the Benjamin-Ono-equation can be written in the form (see [2])

$$
I_n(u) = \int (-1)^n \frac{u^{n+2}}{n+2}dx + \sum_{m=1}^{n-1}\int P_{n+2-m,m}(u)dx + i(n)c_n\int u\frac{\partial^n}{\partial x^n}A_n(u)dx
$$

where c_n is a positive constant,

$$
A_n(u) = \begin{cases} u & \text{if } n \text{ is even,}\\ H(u) & \text{if } n \text{ is odd}\end{cases} \tag{4.3}
$$

and

$$
i(n) = \begin{cases} (-1)^p & \text{if } n = 2p,\\ (-1)^{p+1} & \text{if } n = 2p+1.\end{cases} \tag{4.4}
$$

The polynomial $P_{j,k}(u)$ denotes the sum of all terms which are homogenenous of degree j in u and which involve exactly k derivatives in x.

Using these invariants, one can demonstrate stability of solitary waves in higher-order Sobolev spaces. Here is a result analogous to Theorem 3.1.

THEOREM 4.1. *The solitary-wave solutions ϕ_c, $c > 0$, of the Benjamin-Ono equation are stable in $H^{\frac{n}{2}}$ for any positive integer n.*

Remark. For $n = 1$ and $n = 2$, it must be additionally assumed that the perturbed solitary wave ψ lies in H^s for some $s \geq 3/2$ in order that a continuous dependence theory be available. Of course, it is only presumed that $\phi - \psi$ is small in $H^{\frac{n}{2}}$ despite the stronger regularity presumption on ϕ.

Proof. We first give a direct proof for H^1, which is the case $n = 2$. As before, define

$$h(x,t) = u(x,t) - \phi_c(x + \gamma(t)),$$

where u is the solution of the Benjamin-Ono equation and γ is the C^1-function guaranteed by [13]. Consider the difference

$$\begin{aligned}
&I_2(u(\cdot,t)) - I_2(\phi(\cdot + \gamma(t))) = I_2(h(\cdot,t) + \phi(\cdot + \gamma(t))) - I_2(\phi(\cdot + \gamma(t))) \\
&\quad = \int \left\{ \frac{1}{4}(h+\phi)^4 + \frac{3}{2}(h+\phi)^2 H(h_x + \phi_x) + 2(h_x + \phi_x)^2 \right\} dx \\
&\qquad\qquad - \int \left\{ \frac{1}{4}\phi^4 + \frac{3}{2}\phi^2 H(\phi_x) + 2(\phi_x)^2 \right\} dx \\
&\quad = \int \left\{ \frac{1}{4}h^4 + h^3\phi \frac{3}{2}h^2\phi^2 + h\phi^3 \frac{3}{2}h^2 H(h_x) + \frac{3}{2}h^2 H(\phi_x) + 3h\phi H(h_x) + 3h\phi H(\phi_x) \right. \\
&\qquad\qquad \left. + \frac{3}{2}\phi^2 H(h_x) + 2h_x^2 + 4h_x\phi_x \right\} dx
\end{aligned}$$

$$\begin{aligned}
&\geq 2\|h\|_1^2 - 2|h|_2^2 - 4|h|_2\|\phi\|_2 - c_0|h|_2\|\phi\|_1^2 - c_0|\phi|_\infty\|\phi\|_1|h|_2 - c_0|\phi|_\infty|h|_2\|h\|_1 - \\
&c_0\|\phi\|_1|h|_\infty|h|_2 - c_0|h|_\infty^{\frac{1}{2}}\|h\|_{\frac{1}{2}}^{\frac{3}{2}}\|h\|_1 - |\phi|_6^3|h|_2 - c_0|\phi|_4^2|h|_\infty|h|_2 - c_0|\phi|_\infty|h|_\infty|h|_2^2 - |h|_\infty\|h\|_{\frac{1}{2}}^3 \\
&\geq c_1\|h\|_1^2 - c_2\|h\|_{\frac{1}{2}} - c_3\|h\|_{\frac{1}{2}}^2 - c_4\|h\|_{\frac{1}{2}}^4 - c_5\|h\|_{\frac{1}{2}}^6 = c_1\|h\|_1^2 - c(\|h\|_{\frac{1}{2}})\|h\|_{\frac{1}{2}}
\end{aligned}$$

where c_i $(i = 1,2,3,4,5)$ depend only on $\|\phi\|_1$ and hence only on c. In the above estimates, use was made of the Sobolev embeddings $H^1 \longrightarrow L^\infty$ and $H^{\frac{1}{2}} \longrightarrow L^3$. Combining a similarly derived upper bound for $I_2(u(0)) - I_2(\phi)$, it follows that

$$\begin{aligned}
c_1\|u(\cdot,t)\|_1^2 &\leq I_2(u(\cdot,0)) - I_2(\phi) + c_2\|h\|_{\frac{1}{2}} + c_3\|h\|_{\frac{1}{2}}^2 + c_4\|h\|_{\frac{1}{2}}^4 + c_5\|h\|_{\frac{1}{2}}^6 \\
&\leq c_1\|h(\cdot,0)\|_1^2 + c(\delta_1)\delta_1 + c(\epsilon_1)\epsilon_1,
\end{aligned}$$

because $\|h(\cdot,0)\|_{\frac{1}{2}} \leq \delta_1$ and $\|h(\cdot,t)\|_{\frac{1}{2}} \leq \epsilon_1$ for all t.

For any $\epsilon_2 > 0$, choose $\epsilon_1 > 0$ such that $c(\epsilon_1)\epsilon_1 < \epsilon_2^2/8$ and $\epsilon_1 < \epsilon_2/2$, where $c(\epsilon_1)\epsilon_1 = c_2\epsilon_1 + c_3\epsilon_1^2 + c_4\epsilon_1^4 + c_5\epsilon_1^6$. Choose $\delta_2 > 0$ such that $\delta_2 < \delta_1$ and $c_1\delta_2^2 + c(\delta_2)\delta_2 < \epsilon_2^2/8$. With these choices, if $\|h(\cdot,0)\|_1 < \delta_2$, then

$$\|h(\cdot,t)\|_1^2 \leq c(\epsilon_1)\epsilon_1 + c(\delta_1)\delta_1 + c_1\delta_1^2 \leq \frac{1}{4}\epsilon_2^2,$$

and thus $\|h(\cdot,t)\|_1 < \epsilon_2$ for all $t \geq 0$.

For $n \geq 3$ we proceed inductively as in Section 3, using the explicit form for invariants spelled out in (4.3) and (4.4). We pass over the details. This completes the proof of Theorem 4.1. □

The intermediate long-wave-equation

$$u_t + uu_x + \frac{1}{\delta}u_x + T(u_{xx}) = 0 \tag{4.5}$$

where

$$Tu(x) = -\frac{1}{2\delta}P.V.\int_{-\infty}^{\infty} coth(\frac{\pi(x-y)}{2\delta})u(y)dx \tag{4.6}$$

also possesses an infinite sequence of invariants which are in involution (see [2]). The first few of these invariants are

$$\begin{aligned}
I_{-1}(u) &= \int udx,\\
I_0(u) &= \int \frac{1}{2}u^2dx,\\
I_1(u) &= -\int\left\{\frac{1}{3}u^3 - uT(u_x) + \frac{1}{\delta}u^2\right\}dx,\\
I_2(u) &= \int\left\{\frac{1}{4}u^4 + \frac{3}{2}u^2T(u_x) + \frac{1}{2}u_x^2 + \frac{3}{2}[T(u)_x]^2 + \frac{1}{\delta}[\frac{3}{2}u^3 + \frac{9}{2}uT(u_x)] + \frac{3}{2\delta^2}u^2\right\}dx.
\end{aligned}$$

As the intermediate long-wave-equation (4.5) enjoys properties similar to those of the Benjamin-Ono equation, one obtains for the Cauchy problem exactly the same results (see [2]). The stability of solitary-wave solutions of (4.5) in $H^{\frac{1}{2}}$ was already established in [1]. Consequently, the stability of solitary-wave solutions of (4.5) in higher-order Sobolev spaces is similarly derived. The proofs parallel those given for the Benjamin-Ono-equation and hence are omitted. Here is the precise statement.

THEOREM 4.2. *The solitary-wave solutions ϕ_c, $c > 0$, of the intermediate long-wave equation are stable in $H^{\frac{n}{2}}$ for any positive integer n.*

5. Cubic Nonlinear Schrödinger Equation

Attention is now given to the cubic nonlinear Schrödinger equation (NLS from now on), namely

$$iu_t + u_{xx} - 2\sigma|u|^2u = 0 \tag{5.1}$$

for $x, t \in R$, where $\sigma = \pm 1$. Naturally, $u = u(x,t)$ is complex-valued in this case. The case $\sigma = -1$ is the so-called focussing NLS equation and it supports travelling-wave solutions u of the form

$$u(x,t) = e^{i\omega t}\phi_{\omega,\theta}(x - \theta t) \tag{5.2}$$

where $(\omega, \theta) \in S^1 \times R$. An important special case arises when $\theta = 0$ and $\omega = \Omega > 0$. These are standing-wave solutions $u(x,t) = e^{i\Omega t}\phi_\Omega(x)$, often referred to as "bound states". Of special interest in many physical situations governed approximately by the cubic nonlinear Schrödinger equation are the so-called "ground states" that minimize energy subject to fixed charge. These wave forms ϕ_Ω, which are the analog of the solitary waves for the KdV- and BBM-equations, are positive, real-valued, radially symmetric, and rapidly decreasing to zero at infinity. The ground states were shown by Cazenave and Lions (see [16] and [18]) to be orbitally stable in the following sense. For any $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that if $\psi \in H^1(R)$ and $\|\psi - \phi_\Omega\|_{H^1} \leq \delta$, then there are maps μ, γ: $R \longrightarrow R$ such that if u is the solution of (5.1) with initial data ψ, then

$$\|u(\cdot, t) - e^{i\mu(t)}\phi_\Omega(x - \gamma(t))\|_{H^1} \leq \epsilon$$

for all t. Bona and Soyeur [13] later broadened the results to include traveling waves and provided a more detailed view of the functions μ and γ. One key to their calculation is the operator T_θ: $H^1(R) \longrightarrow H^1(R)$ defined by

$$(T_\theta u)(x) = exp\left(i\frac{\theta x}{2}\right)u(x)$$

for $u \in H^1(R)$ and $\theta \in R$. We here paraphrase their results and refer the readers to [13] for detailed analysis.

LEMMA 5.1. *Let $(\omega, \theta) \in S^1 \times R$ be such that $\Omega = \omega + \frac{1}{4}|\theta|^2 > 0$. If ϕ_Ω is a bound state of (5.1), then $\Phi_{\omega,\theta} = T_\theta\phi_\Omega$ is a traveling-wave solution of (5.1).*

THEOREM 5.2. *For any $(\omega, \theta) \in S^1 \times R$ such that $\omega + \frac{1}{4}|\theta|^2 = \Omega > 0$, define the traveling wave $\Phi_{\omega,\theta} = T_\theta\phi_\Omega$. The traveling-wave solution $v(x,t) = e^{i\omega t}\Phi_{\omega,\theta}(x - \theta t)$ is orbitally stable in the sense that for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that when $\|\psi(\cdot) - \Phi_{\omega,\theta}(\cdot, 0)\|_{H^1} \leq \delta$, then there are C^1 mappings $p : R \longrightarrow R$ and $q : R \longrightarrow R$ for which the solution u of (5.1) emanating from the initial data ψ satisfies*

$$\|u(\cdot, t) - e^{ip(t)}\Phi_{\omega,\theta}(\cdot - q(t))\|_{H^1} \leq \epsilon \tag{5.3}$$

for all t. Moreover, p and q are close to ω and θ in the sense that

$$p'(t) = \omega + O(\epsilon),$$

$$q'(t) = \theta + O(\epsilon) \tag{5.4}$$

as $\epsilon \longrightarrow 0$, uniformly in t.

Just like the evolution equations studied in the earlier sections, the cubic NLS-equation possesses infinitely many integral invariants I_i, $i = 1, 2 \cdots$. Reproduced here are the first six invariants for this equation:

$$\begin{aligned}
I_1 &= \int |u| dx, \\
I_2 &= \int |u|^2 dx, \\
I_3 &= \int \left\{ u\bar{u}_x - \bar{u}u_x \right\} dx, \\
I_4 &= \int \left\{ |u_x|^2 - \sigma|u|^4 \right\} dx, \\
I_5 &= \int \left\{ \sigma u_x\bar{u}_{xx} - \sigma\bar{u}_x u_{xx} + \frac{3}{2}u^2(\bar{u^2})_x - \frac{3}{2}\bar{u^2}(u^2)_x \right\} dx,
\end{aligned}$$

$$I_6 = \int \left\{ |u_{xx}|^2 + 4\sigma|u|^2|u_x|^2 - \sigma|u|^2\bar{u}u_{xx} - \sigma|u|^2u\bar{u}_{xx} + 2|u|^6 \right\} dx.$$

The general form for calculating the infinitely many conservation laws for the NLS-equation can be found in [4]. As appears immediately from an examination of the generating function for these conservation laws given in formula (1.6.12) of [4], the even numbered ones are potentially useful as regards stability of solitary waves. Indeed, just as for the equations appearing in the preceding sections, there is a sequence $\{P_{2j}\}_{j\geq 1}$ of polynomials with P_{2j} a function of j variables, *viz.*

$$P_{2j} = P_{2j}(y_0, \cdots, y_{j-1}),$$

such that if u is a solution of (5.1) that lies at least in $C(0,T;H^{j-1}(R))$, then

$$I_{2j} = \int_{-\infty}^{\infty} P_{2j}(u, u_x, \cdots \partial_x^{j-1}u)dx = \text{constant}. \tag{5.5}$$

(This result relies upon the well-posedness theory for (5.1) in the Sobolev spaces $H^s(R)$, see, for example, [17].)

As follows from an analysis of the structure of these polynomials, $P_{2j}(y_0, \cdots, y_{j-1})$ is exactly a linear combination of monomials $z_0^{r_0} \cdots z_m^{r_m}$ where z_k is y_k or $\bar{y}_k$ and

$$\sum_{i=0}^{m}(1+i)r_i = 2j. \tag{5.6}$$

Thus, for $j \geq 1$, the invariant I_{2j} has the form

$$I_{2j}(u) = \int c_{2j}|\partial_x^{j-1}u|^2dx + \int d_{2j}|u|^{2j} + \sum_{m=1}^{2j-1}\int P_{2j-m,m}(u)dx \tag{5.7}$$

where c_{2j}, d_{2j} are constants and $P_{j,k}(u)$ denotes the sum of all terms which are homogeneous of degree j in u and which involve exactly k derivatives in x. In particular, it is known that $c_{2j} \neq 0$, and hence we may take it to be positive.

Theorem 5.3. *The solitary-wave solutions $v(x,t) = e^{i\omega t}\Phi_{\omega,\theta}(x-\theta t)$ of the cubic nonlinear Schrödinger equation are stable in H^n, for any positive integer n.*

Proof. Orbital stability in the sense specified in Theorem 5.2 for H^1 is already known (see [16], [18] and [13]). Attention is first given to stability in H^2.

As before, let $h(x,t) = u(x,t) - e^{ip(t)}\Phi_{\omega,\theta}(x-q(t))$ where p and q are C^1-functions that provide stability in H^1, as outlined above and established in [13]. Stability in H^2 follows if we can show that, given $\epsilon_2 > 0$, there is a $\delta_2 > 0$ such that $\|h(\cdot,t)\|_2 \leq \epsilon_2$ for all t provided $\|h(\cdot,0)\|_2 \leq \delta_2$. Write ϕ for $e^{ip(t)}\Phi_{\omega,\theta}$ for ease of reading. Because of the H^1-stability result, for any $\epsilon_1 > 0$ there is a $\delta_1 > 0$ such that if $\|h(\cdot,0)\|_1 \leq \delta_1$, then $\|h(\cdot,t)\|_1 \leq \epsilon_1$, for all t.

Define $\Delta I_6(u) = I_6(u(\cdot,t)) - I_6(\phi(\cdot + q(t)))$. A calculation reveals that

$$\Delta I_6(u) = \int(\phi_{xx}+h_{xx})(\bar{\phi}_{xx}+\bar{h}_{xx}) - 4\int(\phi+h)(\bar{\phi}+\bar{h})(\phi_x+h_x)(\bar{\phi}_x+\bar{h}_x) - \int|\phi_{xx}|^2 +$$

$$4\int|\phi|^2|\phi_x|^2+\int(\phi+h)(\bar\phi^2+2\bar\phi\bar h+\bar h^2)(\phi_{xx}+h_{xx})-\int|\phi|^2\bar\phi\phi_{xx}-\int|\phi|^2\phi\bar\phi_{xx}-$$
$$2\int|\phi|^6+\int(\phi^2+2\phi h+h^2)(\bar\phi+\bar h)(\bar\phi_{xx}+\bar h_{xx})+2\int(\phi+h)^3(\bar\phi+\bar h)^3. \qquad (5.8)$$

The following inequalities come to our aid in analysing ΔI_6:

(a) $\displaystyle\int(\phi_{xx}+h_{xx})(\bar\phi_{xx}+\bar h_{xx})-\int|\phi_{xx}|^2\le\int|h_{xx}|^2+a_1\int|h_x|^2.$

(b) $\displaystyle\left|-4\int(\phi+h)(\bar\phi+\bar h)(\phi_x+h_x)(\bar\phi_x+\bar h_x)+4\int|\phi|^2|\phi_x|^2\right|\le b_1\|h\|_{L^2}$

$$+b_2\|h_x\|_{L^2}+b_3\|h\|_{L^2}^2+b_4\|h_x\|_{L^2}^2+b_5\|h_x\|_{L^2}^3+b_6\|h_x\|_{L^2}^4.$$

(c) $\displaystyle\left|\int(\phi+h)(\bar\phi^2+2\bar\phi\bar h+\bar h^2)(\phi_{xx}+h_{xx})-\int|\phi|^2\bar\phi\phi_{xx}\right|\le c_1\|h\|_{L^2}+c_2\|h_x\|_{L^2}$

$$+c_3\|h\|_{L^2}^2+c_4\|h_x\|_{L^2}^2+c_5\|h_x\|_{L^2}^3+c_6\|h\|_{L^2}^4+c_7\|h_x\|_{L^2}^4.$$

(d) $\displaystyle\left|2\int(\phi+h)^3(\bar\phi+\bar h)^3-2\int|\phi|^6\right|\le d_1\|h\|_{L^2}+d_2\|h\|_{L^2}^2+d_3\|h_x\|_{L^2}^2$

$$+d_4\|h\|_{L^2}^4+d_5\|h_x\|_{L^2}^4+d_6\|h_x\|_{L^2}^6.$$

(Here a_i, b_i, c_i, d_i denote various constants that depend only on ϕ.) Combining (a)-(d) with (5.9), there obtains

$$\Delta I_6(u)\le C_0\delta_2+C_1\delta_2^6 \qquad (5.8)$$

where δ_2 is any upper bound for $\|\psi-\phi\|_2$ and C_0,C_1 are constants depending only on norms of ϕ, and hence only on ω and θ.

Attention is now turned to an effective lower bound for $\Delta I_6(u)$. To this end, notice that

$$\int(\phi_{xx}+h_{xx})(\bar\phi_{xx}+\bar h_{xx})-\int|\phi_{xx}|^2\ge\int|h_{xx}|^2-c_1\int|h_x|^2\ge\int|h_{xx}|^2-c_1\epsilon_1^2 \quad (5.9)$$

according to (5.7). Combining (5.8), (5.10) and (b)-(d), there obtains the lower bound

$$\Delta I_6(u)\ge|h_{xx}|_2^2-\epsilon_1D_0-\epsilon_1^6D_1 \qquad (5.10)$$

where D_0 and D_1 are constants depending only on ω and θ. Using (5.9) and (5.11), there obtains the inequality

$$|h_{xx}(\cdot,t)|_2^2\le\epsilon_1D_0+\epsilon_1^6D_1+\delta_2C_0+\delta_2^6C_1$$

holding for all t. In consequence, it is deduced that

$$\|h(\cdot,t)\|_2^2\le\epsilon_1M_0+\delta_2M_1$$

where M_0,M_1 are smooth functions of $\epsilon_1,\delta_2,\omega,\theta$, and in particular are bounded on bounded sets.

It remains simply to choose ϵ_1 so that $\epsilon_1 M_0 < \epsilon_2^2/2$. This implies the existence of a $\delta_1 > 0$ for which $\|u(\cdot,t) - \phi(\cdot + q(t))\|_1 = \|h(\cdot,t)\|_1 \le \epsilon_1$, provided $\|\phi - \psi\|_1 \le \delta_1$. Then choose $\delta_2 \le \delta_1$ small enough that $\delta_2 M_1 \le \epsilon_2^2/2$ also. The stability conclusion then follows.

We proceed inductively, supposing that for all $j < k$, stability holds in H^j in the stronger sense that, given an $\epsilon_j > 0$, there is a $\delta_j > 0$ such that if $\|\psi - \phi\|_j \le \delta_j$ then $\|h(\cdot,t)\|_j \le \epsilon_j$ for all t. Presuming that $\psi \in H^k$, the stability in H^k is established by using the invariant functional I_{2k+2}.

Fix an $\epsilon_k > 0$. As in the case $k = 2$, define

$$\Delta I_{2k+2}(u) = I_{2k+2}(u(\cdot,t)) - I_{2k+2}(\phi(\cdot + q(t))) \tag{5.11}$$

where $q(t)$ is as before, a C^1-function that provides stability in H^1.

The upper bound for $\Delta I_{2k+2}(u)$ is calculated in term of an upper bound δ_k for $\|\psi - \phi\|_k$ by evaluating (5.12) at $t = 0$, *viz.*

$$\Delta I_{2k+2}(u) = \Delta I_{2k+2}(\psi) \le c_k \delta_k^2 + c_k' \delta_k^{2k+2}. \tag{5.12}$$

For any positive value ϵ_{k-1}, there is a $\delta_{k-1} > 0$ for which $\|\psi - \phi\|_{k-1} \le \delta_{k-1}$ implies

$$\|u(\cdot,t) - \phi(\cdot + q(t))\|_{k-1} \le \epsilon_{k-1} \tag{5.13}$$

for all t. Using (5.14), a direct calculation of $\Delta I_{2k+2}(u)$ in terms of h and ϕ yields a lower bound of the form

$$\Delta I_{2k+2}(u) \ge c_k |\partial_x^k h|_2^2 - \epsilon_{k-1} N_k - \epsilon_{k-1}^{k+2} N_k' \tag{5.14}$$

where $c_k > 0$, and N_k, N_k' depend only on ω and θ.

Stability in H^k follows from (5.13) and (5.15) just as in the H^2 case. ∎

Acknowledgements. The authors are grateful for partial support from the USA's National Science Foundation. Jean-Claude Saut made helpful comments at an early stage of the development for which we are also grateful.

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