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STRONG SOLUTION TO A KIND OF CROSS DIFFUSION PARABOLIC SYSTEM*

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Abstract. A kind of strongly coupled cross diffusion parabolic system, which can be used as the energy transport model in semiconductor science, is studied in this paper. The existence and uniqueness of strong solution to the initial boundary value problem are obtained, under the condition that the initial data are a small perturbation of an isothermal stationary solution. Based on this result, an application of Newton iteration on solving this problem is also given.

Key words. Cross diffusion parabolic system, energy transport model, strong solution, Newton iteration.

AMS subject classifications. 35M10, 35K55, 35K65, 78A35.

1. Introduction

In the last decades, more and more strongly coupled parabolic systems with (degenerate or non-degenerate) cross diffusion terms were derived from applied science, for instance, chemotaxis phenomenon in biomathematics, generalized drift diffusion and energy transport model in semiconductor science, separation of granular material, etc. In real applications, due to more information included, such kinds of cross diffusion models describe the phenomena more clearly than the classical weakly coupled diffusion systems. But very few theoretical results have been obtained up to now.

In the present paper, we consider the following cross diffusion parabolic system.

$$\left\{\begin{array}{l} n_t - \mathrm{div} J_1(\nabla n, \nabla E, n, E, \nabla V) = 0 \\ E_t - \mathrm{div} J_2(\nabla n, \nabla E, n, E, \nabla V) = -\nabla V \cdot J_1(\nabla n, \nabla E, n, E, \nabla V) + R(n, E) \\ \triangle V = n - C(x) \end{array}\right. \tag{1.1}$$

where $J_i(r, s, n, E, \phi)$, $i = 1, 2$, are smooth functions and are linear functions on r and s. In real applications, n and E may denote some kinds of densities, J_1 and J_2 the flux densities of such densities, and $R(n, E)$ relaxation term. It is well known that if the system is not weakly coupled, no general theory as the results in [13] can be used directly and one must find another way to deal with this system. In fact, the structure is completely different from the weakly coupled case so that the usual method including the maximum principle and the regularity theory can not be used.

We will study a special case of (1.1), which could describe the conservation of energy in semiconductor simulation. For more details of the energy transport model, we refer to the references [11, 1, 8, 12, 2, 15]. This system is a degenerate quasi-linear

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strongly coupled parabolic system with principal part in divergence form. Obviously, it is important to make investigation on it both theoretically and numerically.

Up to now, only partial results are available in the literature concerning the wellposedness of the energy transport model. We summarize some of the available results. For the stationary case, we refer to references, [5, 9, 4]. For the transient case, the first results on the existence of a weak solution and the large time behavior to a more general parabolic system were obtained by P. Degond, S. Génieys and A. Jüngel [6]. Furthermore, A. Jüngel established in [7] the regularity and uniqueness when the coefficient matrix depends merely on space variable x. However, in both [6] and [7] it is required that the diffusion matrix is uniformly positive definite, while the situation in the commonly used energy transport models does not satisfy this requirement. So it would be very interesting to study directly the energy transport model, i.e. system (1.1) with

$$\Phi = \nabla V \tag{1.2}$$
$$J_1 = -\big(\nabla(nT^{\frac{1}{2}-\beta}) - nT^{-\frac{1}{2}-\beta}\Phi\big) \tag{1.3}$$
$$J_2 = -(2-\beta)\big(\nabla(nT^{\frac{3}{2}-\beta}) - nT^{\frac{1}{2}-\beta}\Phi\big) \tag{1.4}$$
$$R = r_0\frac{n(T_0-T)}{T^{\frac{1}{2}-\beta}} \tag{1.5}$$
$$E = (2-\beta)nT, \tag{1.6}$$

which can be rewritten into the following form

$$n_t + \mathrm{div}J_1 = 0 \tag{1.7}$$
$$E_t + \mathrm{div}J_2 = J_1\cdot\Phi + R \tag{1.8}$$
$$\lambda^2\mathrm{div}\Phi = n - C(x), \quad \Phi = \nabla V, \tag{1.9}$$

where the unknowns n and T are the electric density and the electron temperature respectively, V is the electrostatic potential, E the density of the internal energy, $R(n,T)$ the energy relaxation term, J_1 the carrier flux density, J_2 the energy flux density, or heat flux and $C(x)$ the doping profile which represents the background of the device. There are also some physical constants such as, T_0 the lattice temperature, λ the scaled Debye length. $\beta \in [0, 1/2]$. The case with $\beta = 1/2$ is called *Chen* model, [3], and $\beta = 0$ is *Lyumkis* model, [14]. For convenience, we suppose $T_0 = 1$ which has been used in numerical simulation.

In general, this system will be considered on $Q_\tau = \Omega\times[0,\tau)$. The problem will be discussed with the following boundary conditions and initial conditions in this paper

$$J_1\cdot\gamma|_{\partial\Omega} = J_2\cdot\gamma|_{\partial\Omega} = \Phi\cdot\gamma|_{\partial\Omega} = 0, \tag{1.10}$$
$$n(x,0) = n_I, \quad T(x,0) = T_I \tag{1.11}$$

where the boundary conditions are equivalent to the homogeneous Neumann boundary conditions when we consider positive solution n and T,

$$\nabla n\cdot\gamma|_{\partial\Omega_N} = \nabla T\cdot\gamma|_{\partial\Omega_N} = \Phi\cdot\gamma|_{\partial\Omega_N} = 0, \tag{1.12}$$

We make the following assumptions

(I) $C(x) \in L_\infty(\Omega)$, $0 < \underline{C} \le C(x) \le \overline{C}$;

(II) Ω is a bounded domain in $\mathbb{R}^N$, $\partial\Omega \in C^{0,1}$, $N \le 3$.

In section 2, we will discuss the existence of a typical isothermal state first, and then prove the existence and uniqueness of a kind of strong solution when the initial state is near this isothermal state. In section 3, we give a scheme by Newton iteration to calculate this solution. At the same time, we obtain that the speed of convergence of this scheme is quadratic.

2. The Existence and Uniqueness of Strong Solution

As mentioned in the introduction we first look for a special stationary isothermal state $(\mathcal{N}, T_0 = 1, \Psi)$ with $\mathcal{N} > 0$, namely, to solve

$$\begin{aligned} &\nabla\mathcal{N} - \mathcal{N}\Psi = 0, \\ &\mathrm{div}\Psi = \mathcal{N} - C(x), \Psi = \nabla\mathcal{V}, \end{aligned} \tag{2.1}$$

with boundary condition

$$\Psi \cdot \gamma|_{\partial\Omega} = 0. \tag{2.2}$$

Theorem 2.1. *Suppose that (I) and (II) hold, and $\overline{C} \leq 2/(1-2\beta)$. Then (2.1) (2.2) has a $W_p^2(\Omega)$, $1 < p < \infty$, solution $(\mathcal{N}, \mathcal{V})$ which satisfies that*

$$\underline{C} \leq \mathcal{N} \leq \overline{C}, \qquad \text{in } \Omega. \tag{2.3}$$

In the theorem, we have used the notation $W_p^m(\Omega) = \{u \in L_p(\Omega) : Du, ..., D^m u \in L_p(\Omega)\}$.

Proof. It is easy to see that (2.1)(2.2) is equivalent to

$$\begin{cases} \triangle\mathcal{V} = \theta e^{\mathcal{V}} - C(x), \\ \nabla\mathcal{V} \cdot \gamma|_{\partial\Omega} = 0, \end{cases} \tag{2.4}$$

where $\theta > 0$ is an arbitrary fixed constant. The existence and uniqueness of $W_2^1(\Omega)$ weak solution of (2.4) can be easily obtained by monotone operator theory, similar to Lemma 3.2 in [4]. In the following we will get the estimates (2.3).

Let $c_1 = \ln(\overline{C}/\theta)$, $c_2 = \ln(\underline{C}/\theta)$. Using $(\mathcal{V} - c_1)^+$ as a test function in (2.4), it holds that

$$\begin{aligned} \int_\Omega |\nabla(\mathcal{V} - c_1)^+|^2 &= -\int_{\Omega_{c_1}} \theta e^{\mathcal{V}}(\mathcal{V} - c_1)^+ + \int_{\Omega_{c_1}} C(x)(\mathcal{V} - c_1)^+ \\ &\leq -\int_{\Omega_{c_1}} (\theta e^{c_1} - \overline{C})(\mathcal{V} - c_1)^+ = 0, \end{aligned} \tag{2.5}$$

where $\Omega_{c_1} = \{\mathcal{V} - c_1 > 0\}$. Thus it follows that

$$\mathcal{V} \leq c_1, \qquad \mathcal{N} = \theta e^{\mathcal{V}} \leq \overline{C}.$$

Similarly, using $(\mathcal{V} - c_2)^-$ as a test function, we have

$$\mathcal{V} \geq c_2, \qquad \mathcal{N} = \theta e^{\mathcal{V}} \geq \underline{C}.$$

Thus by regularity theory on elliptic equations, with the help of $\theta e^{\mathcal{V}} - C(x) \in L_\infty(\Omega)$, we know that $\mathcal{V} \in W_p^2(\Omega)$, $\forall 1 < p < \infty$, namely $\Psi \in W_p^1(\Omega)$.

Remark 2.2. *If $C(x) \in W_2^2(\Omega)$, then by regularity theory of elliptic equations, we have $\mathcal{N}, \mathcal{V} \in W_2^4(\Omega)$.*

We turn to the dynamic system (1.7)-(1.9) next.

We introduce the following notations

$$\mathcal{B}_1 := \{u \in W_2^r(\Omega) : \nabla u \cdot \gamma|_{\partial\Omega} = 0\}, \tag{2.6}$$

$$\mathcal{B}_2 := \{u \in W_2^{r,\frac{r}{2}}(Q_\tau) : \nabla u \cdot \gamma|_{\partial\Omega} = 0, u|_{t=0} = 0\}, \tag{2.7}$$

$$\mathcal{B}_3 := \{\Phi = \nabla V, V \in W_2^{r,\frac{r}{2}}(Q_\tau) : \Phi \cdot \gamma|_{\partial\Omega} = 0\}, \tag{2.8}$$

$$\mathcal{B}_4 := \{u \in W_2^{r-2,\frac{r-2}{2}}(Q_\tau)\}, \tag{2.9}$$

where $r = 2$ if $N \leq 2$ and $r = 4$ if $N = 3$, $W_2^{2,1}(Q_\tau) = \{u \in L_2(Q_\tau) : Du, D^2u, u_t \in L_2(Q_\tau)\}$, $W_2^{4,2}(Q_\tau) = \{u \in L_2(Q_\tau) : D^\alpha u, u_t, D^2u_t, u_{tt} \in L_2(Q_\tau), \alpha = 1, ..., 4\}$

THEOREM 2.3. *Suppose (I) (II) hold, $C(x) \in W_2^{r-2}(\Omega)$, $\overline{C} \leq 2/(1-2\beta)$, $n_I, T_I \in \mathcal{B}_1(\Omega)$, then for any fixed $\tau > 0$, there exist $\delta > 0$ and $\eta(\delta) > 0$ such that when*

$$\|n_I - \mathcal{N}\|_{\mathcal{B}_1} + \|T_I - 1\|_{\mathcal{B}_1} < \delta, \tag{2.10}$$

there exists a unique solution (n, T, Φ) such that $(n - n_I, T - 1, \Phi) \in (\mathcal{B}_2)^2 \times \mathcal{B}_3$ for (1.7)-(1.9) and satisfying

$$\|n - \mathcal{N}\|_{\mathcal{B}_2} + \|T - 1\|_{\mathcal{B}_2} + \|\Phi - \Psi\|_{\mathcal{B}_3} < \eta(\delta), \tag{2.11}$$

with $\eta(\delta) \to 0$, $\delta \to 0$.

REMARK 2.4. *The assumption $\overline{C} \leq 2/(1-2\beta)$ is reasonable since in the commonly used Chen model, $\beta = 1/2$, which means $\overline{C}$ is an arbitrary positive constant.*

Proof. We will use implicit function theorem to get the existence and uniqueness of the solution. At first we write the equations in (1.7)-(1.9) explicitly in the following form

$$G_i(n, T, \Phi) = 0, \quad i = 1, 2, 3 \tag{2.12}$$

with

$$\begin{aligned} G_1(n, T, \Phi) &= n_t - T^{\frac{1}{2}-\beta}\triangle n - (\frac{1}{2} - \beta)T^{-\frac{1}{2}-\beta}n\triangle T \\ &- (1-2\beta)T^{-\frac{1}{2}-\beta}\nabla n \cdot \nabla T + (\frac{1}{4} - \beta^2)T^{-\frac{3}{2}-\beta}n|\nabla T|^2 + T^{-\frac{1}{2}-\beta}\nabla n \cdot \Phi \\ &- (\frac{1}{2} + \beta)T^{-\frac{3}{2}-\beta}n\nabla T \cdot \Phi + nT^{-\frac{1}{2}-\beta}(n - C) \end{aligned} \tag{2.13}$$

$$\begin{aligned} G_2(n, T, \Phi) &= T_t - T^{\frac{1}{2}-\beta}\triangle T - (1-2\beta)T^{-\frac{1}{2}-\beta}|\nabla T|^2 \\ &- 2T^{\frac{1}{2}-\beta}n^{-1}\nabla n \cdot \nabla T + T^{-\frac{1}{2}-\beta}n^{-1}\nabla n \cdot \Phi + (\frac{3}{2} - \beta)T^{-\frac{1}{2}-\beta}\nabla T \cdot \Phi \\ &- \frac{1}{2-\beta}T^{-\frac{1}{2}-\beta}|\Phi|^2 - T^{-\frac{1}{2}+\beta}(1 - T) \end{aligned} \tag{2.14}$$

$$G_3(n, T, \Phi) = \operatorname{div}\Phi - n + C(x) \tag{2.15}$$

Introducing a mapping

$$\mathcal{F} : \mathcal{U} \to (\mathcal{B}_4)^3 \tag{2.16}$$

with an open subset of $(\mathcal{B}_1)^2 \times ((\mathcal{B}_2)^2 \times \mathcal{B}_3)$, that is

$$\begin{aligned}\mathcal{U} = \{&((n_I, T_I), (P, Q, V)) \in (\mathcal{B}_1)^2 \times ((\mathcal{B}_2)^2 \times \mathcal{B}_3) : \\ &\exists \overline{D} \geq \underline{D} > 0 \text{ such that } \underline{D} < P + n_I, Q + T_I < \overline{D}, \\ &\text{and } \|(n_I, T_I)\|_{(\mathcal{B}_1)^2}, \|(P, Q, \Phi)\|_{(\mathcal{B}_2)^2 \times \mathcal{B}_3} < \overline{D}\}\end{aligned} \tag{2.17}$$

in the following sense

$$G_i(P + n_I, Q + T_I, \Phi) = f_i, \quad i = 1, 2, 3. \tag{2.18}$$

The continuity and continuous differentiability of $\mathcal{F}$ can be obtained by Sobolev embedding theorem in the Appendix and Hölder inequality. In fact, we deal with it in two cases, $N \leq 2$ and $N = 3$. When $N \leq 2$, if $u \in W_2^{2,1}(Q_\tau)$, we have $Du \in L^4(Q_\tau)$ and $\|Du\|_{L^4(Q_\tau)} \leq C\|u\|_{W_2^{2,1}(Q_\tau)}$ by Sobolev embedding theorem. When $N = 3$, if $u \in W_2^{4,2}(Q_\tau)$ which means $u_t, D^2u \in W_2^{2,1}(Q_\tau)$, we have $u \in W_8^{2,1}(Q_\tau)$, thus $\|Du\|_{L_\infty(Q_\tau)} \leq C\|u\|_{W_8^{2,1}(Q_\tau)} \leq C\|u\|_{W_2^{4,2}(Q_\tau)}$ by Sobolev embedding. Then in any case $\mathcal{F}$ is continuously differentiable in $\mathcal{U}$ by Hölder inequality and the above estimates. Since the proofs involved are fundamental calculations, we will only give some examples on calculating certain typical terms in the definition of $\mathcal{F}$. When $N \leq 2$, we have

$$\begin{aligned}&\|T_1^{-\frac{1}{2}-\beta}\nabla n_1 \cdot \nabla T_1 - T_2^{-\frac{1}{2}-\beta}\nabla n_2 \cdot \nabla T_2\|_{L_2(Q_\tau)} \\ &\quad\leq \quad C(\overline{D}, \underline{D})\|T_1 - T_2\|_{L_\infty(Q_\tau)}\|\nabla n_1\|_{L_4(Q_\tau)}\|\nabla T_1\|_{L_4(Q_\tau)} \\ &\qquad\qquad + C(\underline{D})\|\nabla(n_1 - n_2)\|_{L_4(Q_\tau)}\|\nabla T_1\|_{L_4(Q_\tau)} \\ &\qquad\qquad + C(\underline{D})\|\nabla n_2\|_{L_4(Q_\tau)}\|\nabla(T_1 - T_2)\|_{L_4(Q_\tau)} \\ &\quad\leq \quad C(\|n_1 - n_2\|_{W_2^{2,1}(Q_\tau)} + \|T_1 - T_2\|_{W_2^{2,1}(Q_\tau)})\end{aligned} \tag{2.19}$$

while when $N = 3$,

$$\begin{aligned}&\|T_1^{-\frac{1}{2}-\beta}\nabla n_1 \cdot \nabla T_1 - T_2^{-\frac{1}{2}-\beta}\nabla n_2 \cdot \nabla T_2\|_{W_2^{2,1}(Q_\tau)} \\ &\quad\leq \quad \|(T_1^{-\frac{1}{2}-\beta} - T_2^{-\frac{1}{2}-\beta})\nabla n_1 \cdot \nabla T_1\|_{W_2^{2,1}(Q_\tau)} \\ &\qquad\quad + \|T_2^{-\frac{1}{2}-\beta}\nabla(n_1 - n_2) \cdot \nabla T_1\|_{W_2^{2,1}(Q_\tau)} \\ &\qquad\quad + \|T_2^{-\frac{1}{2}-\beta}\nabla n_2 \cdot \nabla(T_1 - T_2)\|_{W_2^{2,1}(Q_\tau)} \\ &\quad\leq \quad C(\overline{D}, \underline{D})\|T_1 - T_2\|_{W_2^{4,2}(Q_\tau)}\|n_1\|_{W_2^{4,2}(Q_\tau)}\|T_1\|_{W_2^{4,2}(Q_\tau)} \\ &\qquad\quad + C(\underline{D})\|n_1 - n_2\|_{W_2^{4,2}(Q_\tau)}\|T_1\|_{W_2^{4,2}(Q_\tau)} \\ &\qquad\quad + C(\underline{D})\|n_2\|_{W_2^{4,2}(Q_\tau)}\|T_1 - T_2\|_{W_2^{4,2}(Q_\tau)} \\ &\quad\leq \quad C(\|n_1 - n_2\|_{W_2^{4,2}(Q_\tau)} + \|T_1 - T_2\|_{W_2^{4,2}(Q_\tau)})\end{aligned} \tag{2.20}$$

In (2.19) and (2.20), for the simplicity of notations, we have used $n_i = P_i + n_I$ and $T_i = Q_i + T_I$.

It is obvious that the isothermal state is a fixed point of $\mathcal{F}$, i.e., $\mathcal{F}(X_0) = 0$ with $X_0 = ((\mathcal{N}, 1), (0, 0, \Psi))$.

Then it is remained to prove that the Frèchet derivative $\mathcal{F}_2'(X_0)$ is invertible, which means to solve the following linear system uniquely for any $f_i \in \mathcal{B}_4$, $i = 1, 2, 3$,

$$P_t - \triangle P - (\frac{1}{2} - \beta)\mathcal{N}\triangle Q + \mathcal{L}_1(P, Q, \Phi) = f_1 \tag{2.21}$$

$$Q_t - \triangle Q + \mathcal{L}_2(P, Q, \Phi) = f_2 \tag{2.22}$$

$$\operatorname{div}\Phi - P = f_3, \Phi = \nabla V, \tag{2.23}$$

where $\mathcal{L}_i$, $i = 1, 2$ are linear operators,

$$\mathcal{L}_i(P, Q, \Phi) = A_{i1}\nabla P + A_{i2}\nabla Q + A_{i3}P + A_{i4}Q + A_{i5}\Phi. \tag{2.24}$$

A_{ij} is functions only depend on $\mathcal{N}$, $\nabla\mathcal{N}$ and Ψ. By estimates in Theorem 2.1, we have $\|A_{ij}\|_{L_\infty(Q_\tau)} \le K$. Here and here after, we denote K the constants which are independent of the solution (P, Q, V) to the problem (2.21)-(2.23).

At first we will prove that problem (2.21)-(2.23) has a unique solution in $W_2^{2,1}(Q_\tau)\times W_2^{2,1}(Q_\tau) \times L_2(0, \tau; W_2^1(\Omega))$ with $P, Q|_{t=0} = 0$, $\nabla P \cdot \gamma|_{\partial\Omega} = \nabla Q \cdot \gamma|_{\partial\Omega} = \Phi \cdot \gamma|_{\partial\Omega} = 0$ when $f_i \in L_2(Q_\tau)$, $i = 1, 2, 3$. The main point lies in the following a priori estimate.

Since the difference of any two solutions V_1 and V_2 to (2.23) with $\Phi\cdot\gamma|_{\partial\Omega} = 0$ is a constant, independent of the space variable x, there exists a $\bar{V}$ such that $\int_\Omega \bar{V}\,dx = 0$ and $\nabla V = \nabla\bar{V} = \Phi$. Multiplying (2.23) by $\bar{V}$ and integrating it over Ω, we have, with the help of Poincarè inequality,

$$\|\Phi(\cdot, t)\|^2_{L_2(\Omega)} \le K(\|P(\cdot, t)\|^2_{L_2(\Omega)} + \|f_3(\cdot, t)\|^2_{L_2(\Omega)}).$$

Integrating the above over $[0, \tau]$, it follows

$$\|\Phi\|_{L_2(Q_\tau)} \le K(\|P\|_{L_2(Q_\tau)} + \|f_3\|_{L_2(Q_\tau)}). \tag{2.25}$$

Multiplying (2.21) by P and integrating it over Ω, we have, due to $\nabla P\cdot\gamma|_{\partial\Omega} = 0$ and $\|A_{ij}\|_{L_\infty(Q_\tau)} \le K$, that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega P^2 + \int_\Omega(|\nabla P|^2 + (\frac{1}{2} - \beta)\mathcal{N}\nabla Q\cdot\nabla P) \le \frac{1}{8}\int_\Omega(|\nabla P|^2 + |\nabla Q|^2) + K\int_\Omega(P^2 + Q^2) + K.$$

Multiplying (2.22) by Q and integrating it over Ω, we have, with the help of $\nabla Q\cdot\gamma|_{\partial\Omega} = 0$ and $\|A_{ij}\|_{L_\infty(Q_\tau)} \le K$, that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega Q^2 + \int_\Omega |\nabla Q|^2 \le \frac{1}{8}\int_\Omega(|\nabla P|^2 + |\nabla Q|^2) + K\int_\Omega(P^2 + Q^2) + K.$$

Combining the two inequalities above, it follows that

$$\frac{d}{dt}\int_\Omega(P^2 + Q^2) + \int_\Omega(|\nabla P|^2 + (\frac{1}{2} - \beta)\mathcal{N}\nabla Q\cdot\nabla P + |\nabla Q|^2) \le K\int_\Omega(P^2 + Q^2) + K. \tag{2.26}$$

With the assumption $\overline{C} \le 2/(1 - 2\beta)$, we have

$$\frac{d}{dt}\int_\Omega(P^2 + Q^2) + \int_\Omega(|\nabla P|^2 + |\nabla Q|^2) \le K\int_\Omega(P^2 + Q^2) + K. \tag{2.27}$$

By Gronwall inequality, we have

$$\sup_{0\le t\le\tau}\int_\Omega(P^2 + Q^2) + \int_0^\tau\int_\Omega(|\nabla P|^2 + |\nabla Q|^2) \le K. \tag{2.28}$$

In order to obtain the second order estimate, we need to take gradients on (2.21) and (2.22) with respect to the space variable. It should be pointed out that it is not necessary to use the third order differentiability because we can use difference quotient instead.

Taking gradient to (2.21) and (2.22) with respect to the space variable, multiplying them by ∇P and ∇Q separately, integrating over Ω, and using $\nabla P \cdot \gamma|_{\partial\Omega} = \nabla Q \cdot \gamma|_{\partial\Omega} = 0$ and $\|A_{ij}\|_{L_\infty(Q_\tau)} \le K$ whenever it is necessary, we have,

$$\sup_{0\le t\le\tau}\int_\Omega(|\nabla P|^2+|\nabla Q|^2)(t)+\int_0^\tau\int_\Omega(|\triangle P|^2+|\triangle Q|^2)\le K. \tag{2.29}$$

Combining (2.28) (2.29) with (2.21) and (2.22), we get the L_2 estimates on P_t and Q_t. Thus for any $W_2^{2,1}$ solution,

$$\|P\|_{W_2^{2,1}(Q_\tau)}, \|Q\|_{W_2^{2,1}(Q_\tau)} \le K. \tag{2.30}$$

Next we will use continuity method to prove the $W_2^{2,1}$ solvability of (2.21)-(2.23) with homogeneous Neumann boundary conditions.

We consider the following family of systems, $\xi \in [0,1]$,

$$\begin{aligned}(1-\xi)(P_t^\xi-\triangle P^\xi)+\xi[P_t-\triangle P-(\frac{1}{2}-\beta)\mathcal{N}\triangle Q+\mathcal{L}_1(P,Q,\Phi)]&=f_1\\ (1-\xi)(Q_t^\xi-\triangle Q^\xi)+\xi[Q_t-\triangle Q+\mathcal{L}_2(P,Q,\Phi)]&=f_2\\ \mathrm{div}\Phi^\xi=P^\xi+f_3,\ \Phi^\xi&=\nabla V^\xi,\\ P^\xi|_{t=0}=Q^\xi|_{t=0}&=0,\\ \nabla P^\xi\cdot\gamma|_{\partial\Omega}=\nabla Q^\xi\cdot\gamma|_{\partial\Omega}=\Phi^\xi\cdot\gamma|_{\partial\Omega}&=0.\end{aligned}$$

When $\xi = 0$, it is known that the above problem has a solution $P^0, Q^0 \in W_2^{2,1}(Q_\tau)$, $\Phi^0 \in L_2(0,\tau;W_2^1(\Omega))$. Similar to the estimate (2.30) we can get

$$\|P^\xi\|_{W_2^{2,1}(Q_\tau)}, \|Q^\xi\|_{W_2^{2,1}(Q_\tau)}, \|\Phi^\xi\|_{L_2(0,\tau;W_2^1(\Omega))} \le K,$$

where K is independent of ξ. Thus from continuity method we know that there exists a solution, when $\xi = 1$, $P, Q \in W_2^{2,1}(Q_\tau)$, $\Phi \in L_2(0,\tau;W_2^1(\Omega))$. The uniqueness of this solution can be obtained directly from (2.30).

Then in the case of $N \le 2$, we have finished the proof. For $N = 3$, we need more regularity of P, Q, Φ, i.e. $P, Q \in W_2^{4,2}(Q_\tau)$, with the help of n_I, $T_I \in W_2^4(\Omega)$ and $f_i \in W_2^{2,1}(Q_\tau)$. These could be obtained by the following procedures. Differentiating (2.21)-(2.23) with respect to t, or with respect to t and x for higher order estimates, we can obtain the desired a priori estimates with the help of homogeneous Neumann boundary condition and the system itself. We omit the details here. ∎

3. The Application of Newton Iteration

In the previous section, we have obtained the existence of a solution to the energy transport model with homogeneous Neumann boundary condition in a neighborhood of the isothermal state. Now in this section, we will simulate this solution by Newton Iteration for any fixed initial data n_I, T_I.

Let

$$F(P,Q,\Phi) = \mathcal{F}((n_I,T_I),(P,Q,\Phi)). \tag{3.1}$$

By theorem 2.3, we have a $X^* = (P^*, Q^*, \Phi^*) \in \mathcal{O}_\eta(\mathcal{N} - n_I, 1 - T_I, \Psi)$ such that $F(X^*) = 0$, where we use the notation $\mathcal{O}_\eta(\mathcal{N} - n_I, 1 - T_I, \Psi)$ to represent an η neighborhood of $(\mathcal{N} - n_I, 1 - T_I, \Psi)$ in $(\mathcal{B}_2)^2 \times \mathcal{B}_3$. Then there exists a $\delta^* > 0$ such that $\mathcal{O}_{\delta^*}(X^*) \subset \mathcal{O}_\eta(\mathcal{N} - n_I, 1 - T_I, \Psi)$. It follows from theorem 2.3 that $0 < \underline{D} \leq P + n_I, Q + T_I \leq \overline{D}$, $\|(P + n_I, Q + T_I, \Phi)\|_{(\mathcal{B}_2)^2 \times \mathcal{B}_3}$ for any $(P, Q, \Phi) \in \mathcal{O}_{\delta^*}(X^*)$.

For the numerical simulation of X^*, from which we can get the solution $n = P^* + n_I$, $T = Q^* + T_I$ and $\Phi = \Phi^*$, we employ the Newton iteration in the following steps:

1, Choose $X_0 \in \mathcal{O}_{\delta^*}(X^*)$;

2, For $n = 0, 1, ...$ set $X_{n+1} = X_n - (F'(X_n))^{-1} F(X_n)$.

In the above, we use the abbreviations $X_n = (P_n, Q_n, \Phi_n)$ for $n = 0, 1, ...$ To ensure that this iteration is well defined and convergent, we need to verify the required properties of mapping F, which will be stated in the following theorem.

Theorem 3.1. *The operator $F : (\mathcal{B}_2)^2 \times \mathcal{B}_3 \to (\mathcal{B}_4)^3$ is Frèchet differentiable in $\mathcal{O}_{\delta^*}(X^*)$. When $r = 4$ in the definition of space $\mathcal{B}_i$, $i = 1, ..., 4$, the Frèchet derivative $F'(X)$ is Lipschitz continuous in $\mathcal{O}_{\delta^*}(X^*)$, i.e.*

$$\sup_{\|X\|_{(\mathcal{B}_2)^2 \times \mathcal{B}_3} \leq 1} \|F'(X_1)(X) - F'(X_2)(X)\|_{(\mathcal{B}_4)^3} \leq L\|X_1 - X_2\|_{(\mathcal{B}_2)^2 \times \mathcal{B}_3}, \tag{3.2}$$

for all $X_1, X_2 \in \mathcal{O}_{\delta^*}(X^*)$, where L depends on $\overline{C}$, $\underline{C}$, η and the norm of isothermal state $\|X_0\|_{(\mathcal{B}_2)^2 \times \mathcal{B}_3}$. Furthermore, as a continuous linear operator, $(F'(X))^{-1} : \mathcal{B}_4 \to (\mathcal{B}_2)^2 \times \mathcal{B}_3$ exists for all $X \in \mathcal{O}_{\delta^*}(X^*)$.

Proof. By definition, the exact form of operator F is from (2.13)-(2.15)

$$F(P, Q, \Phi) = G_i(P + n_I, Q + T_I, \Phi) \tag{3.3}$$

with fixed initial data n_I and T_I. The continuous differentiability of F can be obtained by the same method as the discussion in (2.19) and (2.20) where we have proved the same property of $\mathcal{F}$. The derivative of F at $\hat{X} = (\hat{P}, \hat{Q}, \hat{\Phi})$ is

$$\begin{aligned} & F'(\hat{X})(P, Q, \Phi) \\ = & \left[\begin{array}{l} P_t - (\hat{Q} + T_I)^{\frac{1}{2} - \beta} \triangle P - (\frac{1}{2} - \beta)(\hat{Q} + T_I)^{-\frac{1}{2} - \beta}(\hat{P} + n_I) \triangle Q + L_1(P, Q, \Phi) \\ Q_t - (\hat{Q} + T_I)^{\frac{1}{2} - \beta} \triangle Q + L_2(P, Q, \Phi) \\ \mathrm{div}\Phi - P, \end{array} \right] \end{aligned} \tag{3.4}$$

where L_i, $i = 1, 2$ are linear operators,

$$\begin{aligned} L_1(P, Q, \Phi) = & B_{11} \nabla P + B_{12} \nabla Q + B_{13} P + B_{14} Q + B_{15} \Phi \\ & - (\frac{1}{2} - \beta)(\hat{Q} + T_I)^{-\frac{1}{2} - \beta} \triangle(\hat{Q} + T_I) P \\ & - (\frac{1}{2} - \beta)(\hat{Q} + T_I)^{-\frac{1}{2} - \beta} \triangle(\hat{P} + n_I) Q \\ & + (\frac{1}{4} - \beta^2)(\hat{Q} + T_I)^{-\frac{3}{2} - \beta}(\hat{P} + n_I) \triangle(\hat{Q} + T_I) Q. \end{aligned} \tag{3.5}$$

$$\begin{aligned} L_2(P, Q, \Phi) = & B_{21} \nabla P + B_{22} \nabla Q + B_{23} P + B_{24} Q + B_{25} \Phi \\ & - (\frac{1}{2} - \beta)(\hat{Q} + T_I)^{-\frac{1}{2} - \beta} \triangle(\hat{Q} + T_I) Q \end{aligned} \tag{3.6}$$

and B_{ij} are functions which only depend on $\nabla(\hat{P} + n_I), \hat{P} + n_I$, $\nabla(\hat{Q} + n_I), \hat{Q} + n_I$ and $\hat{\Phi}$.

Next we will prove that $F'(X)$ is Lipschitz continuous in $\mathcal{O}_{\delta^*}(X^*)$ when $r = 4$ in the definition of $\mathcal{B}_i$. $\forall X_1 = (P_1, Q_1, \Phi_1), X_2 = (P_2, Q_2, \Phi_2) \in \mathcal{O}_{\delta^*}(X^*)$, we need to estimate the following quantity,

$$\|F'(X_1)(P, Q, \Phi) - F'(X_2)(P, Q, \Phi)\|_{(W_2^{2,1})^3}.$$

In fact, among the terms in $F'(X_1)(P, Q, \Phi) - F'(X_2)(P, Q, \Phi)$, such terms like $\|B_{ij}(X_1)P - B_{ij}(X_2)P\|_{W_2^{2,1}(Q_\tau)}$ can be estimated similarly to (2.20). The others can be controlled by the following method, (we use notations $n_i = P_i + n_I$ and $T_i = Q_i + T_I$),

$$\|(T_1^{\frac{1}{2}-\beta} - T_2^{\frac{1}{2}-\beta})\triangle P\|_{W_2^{2,1}(Q_\tau)} \leq C(\overline{D}, \underline{D})\|T_1 - T_2\|_{W_2^{4,2}(Q_\tau)}\|P\|_{W_2^{4,2}(Q_\tau)}$$

$$\begin{aligned} &\|(T_1^{-\frac{3}{2}-\beta} n_1 \triangle T_1 - T_2^{-\frac{3}{2}-\beta} n_2 \triangle T_2)Q\|_{W_2^{2,1}(Q_\tau)} \\ \leq\ & C(\overline{D}, \underline{D})(\|T_1 - T_2\|_{W_2^{4,2}(Q_\tau)} + \|n_1 - n_2\|_{W_2^{4,2}(Q_\tau)})\|Q\|_{W_2^{4,2}(Q_\tau)} \end{aligned}$$

where we only give an example of them. Thus we have proved (3.2).

The last step is to get the existence of $(F'(X))^{-1}$, which can be obtained by a priori estimates and continuity method as the proof of the invertibility of $\mathcal{F}'(X_0)$ in Theorem 2.3, with only a little difference in the a priori estimates. In Theorem 2.3, A_{ij} depend only on the isothermal state, which makes the estimates easily obtained. Now in the present case, the coefficients of linear operator $(F'(X))^{-1}$ are more complicated. They include not only the similar terms as A_{ij}, i.e. B_{ij}, which have estimates $\|B_{ij}\|_{L_\infty(Q_\tau)} \leq C$ by Sobolev embedding, but also some other terms whose coefficients are not L_∞ bounded, i.e. the last three terms in (3.5) and the last term in (3.6). To deal with such terms, we need to use integration by parts with homogeneous Neumann boundary condition, and Hölder inequality. We omit the details.
□

Hence, the Newton iteration is well-defined and the above theorem ensures the convergence.

THEOREM 3.2. *There exists a constant $\hat{\delta} \leq \delta^*$ such that the iterative sequence $X_{n+1} = X_n - (F'(X_n))^{-1}F(X_n)$ converges to the solution X^* for every initial value $X_0 \in \mathcal{O}_{\hat{\delta}}(X^*)$. Furthermore, there is a suitable constant M, depending on δ^* and L, such that*

$$\begin{aligned} &\|X_n - X^*\|_{(\mathcal{B}_2)^2\times\mathcal{B}_3} \leq M\|X_n - X^*\|^2_{(\mathcal{B}_2)^2\times\mathcal{B}_3} \\ &\|X_n - X^*\|_{(\mathcal{B}_2)^2\times\mathcal{B}_3} \leq (M\hat{\delta})^{2^n}/M. \end{aligned}$$

REMARK 3.3. *The step $X_{n+1} = X_n - (F'(X_n))^{-1}F(X_n)$ can be easily obtained numerically by finite difference schemes or finite element schemes on linear parabolic systems.*

4. Appendix We state the following Sobolev embedding theorem for $W_p^{2,1}(Q_\tau)$ in [10], which have been used in the proof of the present paper.

Theorem. If Ω satisfies the uniform cone condition, $u \in W_p^{2,1}(Q_\tau)$, $p \geq 1$, then

$$W_p^{2,1}(Q_\tau) \hookrightarrow \begin{cases} C^{k,\frac{k}{2}}(Q_\tau), & k = 2 - \dfrac{N+2}{p}, \quad p > \dfrac{N+2}{2}, \\ L_s(Q_\tau), & 1 \leq s < \infty, \quad p = \dfrac{N+2}{2}, \\ L_s(Q_\tau), & s \leq \dfrac{(N+2)p}{N+2-2p}, \quad p < \dfrac{N+2}{2}. \end{cases} \tag{4.1}$$

$$\left.\begin{array}{l} \|Du\|_{L_\infty(Q_\tau)} \\ \|Du\|_{L_s(Q_\tau)} \\ \|Du\|_{L_s(Q_\tau)} \end{array}\right\} \leq C\|u\|_{W_p^{2,1}(Q_\tau)}, \quad \begin{array}{rl} & p > N+2, \\ 1 \leq s < \infty, & p = N+2, \\ s \leq \dfrac{(N+2)p}{N+2-p}, & p < N+2. \end{array} \tag{4.2}$$

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