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UNIQUENESS OF BV ENTROPY SOLUTIONS FOR HIGH DIMENSIONAL QUASILINEAR PARABOLIC EQUATIONS WITH ARBITRARY DEGENERACY*

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Abstract. This paper deals with the Cauchy problem for the quasilinear parabolic equations with arbitrary degeneracy. We give a new and natural definition of BV entropy solution. The uniqueness of the natural BV entropy solutions is obtained. In order to prove the uniqueness, the discontinuity conditions for these solutions are established. Also, we construct an example of the natural BV entropy solution.

Key words. Arbitrary degeneracy, BV entropy solution, Discontinuity condition, Uniqueness.

1. Introduction

In this paper we are concerned with the uniqueness of BV entropy solutions of the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta A(u) + \nabla \vec{B}(u), \qquad (x,t) \in Q_T, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $Q_T \equiv \mathbb{R}^N \times (0,T)$ and

$$A(u) = \int_0^u a(s)ds, \ a(s) \geq 0, \ \vec{B}(u) = \int_0^u \vec{b}(s)ds,$$

with $a(s), \vec{b}(s)$ appropriately smooth.

Since $a(s)$ is allowed to have zero points, we call them points of degeneracy of (1.1), the equation (1.1) does not admit classical solutions in general. For equations with arbitrary degeneracy, namely, with the set $E = \{s; a(s) = 0\}$ including interior points, the solutions of (1.1) even might be discontinuous.

Equations with degeneracy arise from a wide variety of diffusive processes in nature. They are suggested as mathematical models of physical problems in many fields such as the flow through a porous medium and sedimentation consolidation processes of flocculated suspensions, see e.g. [1], [2], [5], [15] and the references cited therein.

During the past fifty years, such equations have been paid much attention by many mathematicians. One of the main problems for such equations is the uniqueness of solutions. For equations (1.1) with arbitrary degeneracy, the set of degenerate points $E = \{s; a(s) = 0\}$ may include interior points. Suppose $E \supset [c,d] (c < d)$. Then for $u \in [c,d]$, (1.1) turns out to be the first order quasilinear conservation law

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial B_i(u)}{\partial x_i} \tag{1.3}$$

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whose solutions, as is well-known, might have discontinuity, even if the initial value is smooth enough. This means that shock waves appear in the solution in finite time (see [8]). Since equations with arbitrary degeneracy are hyperbolic–parabolic mixed type, it is natural to discuss solutions in a suitable class of discontinuous functions. It was Vol'pert and Hudjaev(see [11]), who first devoted to the uniqueness of solutions of equations (1.1) with arbitrary degeneracy. The BV space is selected in [11]. They defined generalized solutions for (1.1) as follows.

DEFINITION 1.1. *A function $u \in BV(Q_T) \cap L^\infty(Q_T)$ is said to be a BV entropy solution of (1.1) on Q_T, if $\dfrac{\partial A(u)}{\partial x_i} \in L^2_{loc}(Q_T)$, $i = 1, 2, \cdots, N$, and for any $0 \le \varphi \in C_0^\infty(Q_T)$, $k \in \mathbb{R}$, the following integral inequality holds*

$$\iint_{Q_T} sgn\,(u(x,t)-k)\left[(u-k)\frac{\partial \varphi}{\partial t} - (\vec{B}\,(u) - \vec{B}\,(k))\nabla\varphi - \nabla A(u)\nabla\varphi\right] dxdt \ge 0. \tag{1.4}$$

The existence of BV entropy solutions is proved in [11]. The authors of [11] also obtained the uniqueness of BV solutions under the conditions $\sqrt{A'(u)}\frac{\partial u}{\partial x_j} \in L^\infty(Q_T)$ $(j = 1, 2, \cdots, N)$. However there are very few solutions satisfying this condition, even for the porous medium equation. It is easy to verify that the typical Barenblatt solutions are excluded from this class. The uniqueness of BV solutions to the Cauchy problem (1.1) was completed by Wu and Yin [14] in one dimensional case. The proof is based on the analysis of the discontinuity condition and a deep study of the properties of functions in BV and BV_x.

Recently, we noticed that for high dimensional case, overcoming some technical difficulties, Vol'pert proved in [10] the uniqueness of BV entropy solutions under the following assumption

$$\frac{\partial A(u)}{\partial x_i} \in BV_{x_i}(Q_T) \cap L^\infty(Q_T), \quad i = 1, 2, \cdots, N. \tag{1.5}$$

In 1999, Carrillo[3] discussed the uniqueness in a large class of solutions for equations with arbitrary degeneracy. However, in the proof of uniqueness no information about the discontinuity solutions is supplied.

Clearly, in the definition of Vol'pert and Hujaev above, the solutions and the test functions are taken in different function spaces and are not allowed to be in the same function space. In this paper, we introduce a new class of generalized solutions, called strong BV entropy solutions. The idea of the new definition is motivated by the fact that, in many cases, in defining weak solutions of partial differential equations, the test functions are chosen in the same or nearly the same function space as the solutions. Moreover, the solutions of equations with arbitrary degeneracy might be discontinuous and the discontinuity surface has lower dimensional Hausdorff measure zero. If the test functions are taken in $C_0^\infty(Q_T)$ functions space, then the integral inequality (1.3) can not give precise information of solutions in the discontinuity surface. So the definition we introduced here seems more natural.

DEFINITION 1.2. *A function $u \in BV(Q_T) \bigcap L^\infty(Q_T)$ is called a natural BV entropy solution of (1.1), if $\overline{\dfrac{\partial A(u)}{\partial x_i}}$ is integrable on Q_T with respect to the measure $\dfrac{\partial \varphi}{\partial x_i}$ for* *any $0 \le \varphi \in BV_c(Q_T)$ and for any $k \in \mathbb{R}$ the following inequality holds*

$$\iint_{Q_T} \frac{1}{2}(sgn(u^+ - k) + sgn(u^- - k)) \left[(\overline{u} - k)\frac{\partial \varphi}{\partial t} - \overline{\frac{\partial A(u)}{\partial x_i}} \frac{\partial \varphi}{\partial x_i} \right] \tag{1.6}$$

$$- \iint_{Q_T} \frac{1}{2}(sgn(u^+ - k) + sgn(u^- - k))(\overline{B_i(u)} - B_i(k))\frac{\partial \varphi}{\partial x_i} \ge 0.$$

If in addition, for any $h(x) \in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{t \to 0^+} \int_{\mathbb{R}^N} u(x,t)h(x)dx = \int_{\mathbb{R}^N} u_0(x)h(x)dx, \tag{1.7}$$

then u is said to be a natural BV entropy solution of the Cauchy problem (1.1)–(1.2).

It should be pointed out that our definition of natural BV entropy solution is quite different from the Vol'pert and Hujaev's definition. In Definition 1.1, since the test functions $\varphi \in C_0^\infty(Q_T)$, the integrals in (1.4) are Lebesgue integrals. However, in Definition 1.2, the test functions only belongs to $BV_c(Q_T)$, so the integrals in (1.6) are non-Lebesgue integrals. In [10], in order to obtain the uniqueness of BV entropy solutions, the author assumed that (1.5) is valid, namely, each second order derivatives of $A(u)$ is a Radon measure. Unfortunately, in high dimensional case, it seems impossible to prove that each $\frac{\partial^2 A(u)}{\partial {x_i}^2} (i = 1, 2, \cdots, N)$ is a Radon measure, although one can easily do that for $\Delta A(u)$.

Notice that for $\varphi \in BV_c(Q_T)$, all terms on the left of (1.6) are integrals with respect to the measure $\frac{\partial \varphi}{\partial x_i}$ which is zero on any subset of Q_T with $N-$dimensional Hausdorff measure zero. The difference of the measure for the integrals make our proof quite different from the one dimensional case(see the deduction of discontinuity conditions in [15], pp. 308). Since for $u \in BV(Q_T) \cap L^\infty(Q_T)$, $u^+, u^-, \bar{u}$ and $\overline{B(u)}$ exist on Q_T except a possible set of $N-$ dimensional Hausdorff measure zero, all integrals in (1.6) exist. The main result of this paper is the following theorem.

Theorem 1.3. *Let u_1 and u_2 be strong BV entropy solutions of the Cauchy problem for (1.1) with initial data u_{10} and u_{20} respectively. If for almost all $(t, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in (0,T) \times \mathbb{R}^{N-1}$, as a function of x_i, the left and right-limit of $\frac{\partial A(u(t,x_i',\cdot))}{\partial x_i}$ exist and*

$$\frac{\partial A(u_k)}{\partial x_i} \in L^\infty(Q_T), \quad i = 1, 2, \cdots, N, \quad k = 1, 2. \tag{1.8}$$

Then for almost all $t \in (0,T)$,

$$\int_{\mathbb{R}^N} |\bar{u}_1(x,t) - \bar{u}_2(x,t)|\omega_\lambda(x)dx \le e^{K_\lambda t} \int_{\mathbb{R}^N} |u_{01}(x) - u_{02}(x)|\omega_\lambda(x)dx, \tag{1.9}$$

where $\lambda > 0, K_\lambda$ is a constant depending only on λ and the bound of u_1 and u_2, and

$$\omega_\lambda(x) = \exp\{-\lambda\sqrt{1+x^2}\}.$$

Remark 1.1. *The above result can be extended without any essential difficulty to more general equations with arbitrary degeneracy of the form*

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}\left(a^{ij}(t,x,u)\frac{\partial u}{\partial x_j}\right) + \frac{\partial}{\partial x_i}b^i(t,x,u) + c(t,x,u),$$

where $a^{ij} = a^{ji}$ *and*

$$a^{ij}(x,t,u)\xi_i\xi_j \geq 0, \ \ \forall \xi = (\xi_i, \xi_2, \cdots, \xi_N) \in \mathbb{R}^N.$$

In this paper, we denote by $BV(Q_T)$ the set of all functions of locally bounded variation, namely, a subset of $L^1_{loc}(Q_T)$, in which the weak derivatives of each function are Radon measures on Q_T. A little general class, denoted by $BV_{x_i}(Q_T)$, is another subset of $L^1_{loc}(Q_T)$, in which only the derivative in x_i of each function is required to be a Radon measure on Q_T. Clearly

$$BV(Q_T) \subset BV_{x_i}(Q_T).$$

We need series of fine properties of BV functions and $BV_{x_i}(i = 1, \cdots, N)$ functions in the sequel, whose proofs can be found in [6], [9], [11], [12] and [14] respectively.

2. Discontinuity conditions

In this section we deduce the discontinuity conditions for BV entropy solutions of the equation (1.1). For high dimensional case, we can't deduce the discontinuity condition so directly from the measure equality (1.1) as in [14] for one dimensional case. The Steklov mean value operator is used to overcome this difficulty.

THEOREM 2.1. *Let u be a strong BV entropy solution of the equation (1.1). Then H–almost everywhere on Γ_u^*,*

$$(u^+ - u^-)\gamma_t - \sum_{i=1}^{N} \left[B_i(u^+) - B_i(u^-)\right] \gamma_{x_i} - \sum_{i=1}^{N}(w_i^r - w_i^l)|\gamma_{x_i}| = 0, \tag{2.1}$$

$$A'(s) = 0, \qquad \forall s \in [u_*, u^*], \tag{2.2}$$

where w_i^r and w_i^l denote the right approximate limit and the left approximate limit of $w_i(x_i, x_i', t) = \dfrac{\partial A(u(x_i, x_i', t))}{\partial x_i}$ as a function of x_i respectively, $u_* = \min\{u^-, u^+\}$, $u^* = \max\{u^-, u^+\}$.

Proof. Taking $k > \max\{\|u^+\|_{L^\infty(Q_T)}, \|u^-\|_{L^\infty(Q_T)}\}$ and $k < \min\{-\|u^+\|_{L^\infty(Q_T)}, -\|u^-\|_{L^\infty(Q_T)}\}$ respectively, we get from the inequality (1.6) that

$$\iint_{Q_T} \overline{w_i}\frac{\partial \psi}{\partial x_i} = \iint_{Q_T} \bar{u}\frac{\partial \psi}{\partial t} - \iint_{Q_T} \overline{B_i(u)}\frac{\partial \psi}{\partial x_i}, \tag{2.3}$$

for any $\psi \in BV_c(Q_T)$. Here and below we always adopt the summation convention on repeated indices.

Define the operators

$$\begin{aligned} S_h^t: \ \ & BV(Q_T) \to BV(Q_T), \quad f \to S_h^t f, \\ S_{h_i}^i: \ \ & BV(Q_T) \to BV(Q_T), \quad f \to S_{h_i}^i f, \end{aligned}$$

where

$$S_h^t f(t,x) \equiv \frac{1}{h}\int_t^{t+h} f(\tau, x)d\tau, \quad S_{h_i}^i f(t,x) \equiv \frac{1}{h_i}\int_{x_i}^{x_i+h_i} f(\xi_i, x_i', t)d\xi_i$$

for $i = 1, 2, \cdots, N$.

Clearly

$$\lim_{h\to 0^+} S_h^t f(t,\cdot) = f^r(t,\cdot),\ \lim_{h_i\to 0^+} S_{h_i}^i f(x_i,\cdot) = f^r(x_i,\cdot),$$
$$\lim_{h\to 0^-} S_h^t f(t,\cdot) = f^l(t,\cdot),\ \lim_{h_i\to 0^-} S_{h_i}^i f(x_i,\cdot) = f^l(x_i,\cdot).$$

Replacing ψ by $S_{h_N}^N \cdots S_{h_1}^1 S_h^t \psi$ in (2.3), we have

$$\begin{aligned}&\iint_{Q_T} \overline{w_i}\frac{\partial S_{h_N}^N \cdots S_{h_1}^1 S_l^t \psi}{\partial x_i}\\ =&\iint_{Q_T} \overline{u}\frac{\partial S_{h_N}^N \cdots S_{h_1}^1 S_h^t \psi}{\partial t} - \iint_{Q_T} \overline{B_i(u)}\frac{\partial S_{h_N}^N \cdots S_{h_1}^1 S_h^t \psi}{\partial x_i}.\end{aligned}$$

Noting the commutativity of the operator $S_h^t(S_{h_i}^i)$ with the differential operator $\frac{\partial}{\partial t}(\frac{\partial}{\partial x_i})$ $(i = 1, 2, \cdots, N)$, we have

$$\begin{aligned}&\iint_{Q_T} \overline{w_i} S_{h_i}^i \left(\frac{\partial S_{h_N}^N \cdots S_{h_1}^1 S_h^t \psi}{\partial x_i}\right)\\ =&\iint_{Q_T} \overline{u} S_h^t \left(\frac{\partial S_{h_N}^N \cdots S_{h_1}^1 \psi}{\partial t}\right) - \iint_{Q_T} \overline{B_i(u)} S_{h_i}^i \left(\frac{\partial S_{h_N}^N \cdots S_{h_1}^1 S_h^t \psi}{\partial x_i}\right).\end{aligned}$$

Clearly, we have

$$\iint_{Q_T} f(x,t) S_{h_i}^i g(x,t)dtdx = \iint_{Q_T} S_{-h_i}^i f(x,t) g(x,t) dtdx$$

for any $f, g \in BV_c(Q_T)$. Thus

$$\begin{aligned}&\iint_{Q_T} S_{-h_N}^N \cdots S_{-h_1}^1 S_{-h}^t S_{-h_i}^i \overline{w_i}\frac{\partial \psi}{\partial x_i}\\ =&\iint_{Q_T} S_{-h_N}^N \cdots S_{-h_1}^1 S_{-h}^t \overline{u}\frac{\partial \psi}{\partial t} - \iint_{Q_T} S_{-h_N}^N \cdots S_{-h_1}^1 S_{-h}^t S_{-h_i}^i \overline{B_i(u)}\frac{\partial \psi}{\partial x_i}. \qquad (2.4)\end{aligned}$$

Letting $h, h_1, \cdots, h_N \to 0^+, 0^-$ in turn in (2.4) and combining the deduced relations, we get for any $\psi \in BV_c(Q_T)$

$$\begin{aligned}&\iint_{Q_T} (w_i^r - w_i^l)\frac{\partial \psi}{\partial x_i}\\ =&\iint_{Q_T} (u^r - u^l)\frac{\partial \psi}{\partial t} - \iint_{Q_T} (B_i(u^r) - B_i(u^l))\frac{\partial \psi}{\partial x_i}, \qquad (2.5)\end{aligned}$$

Set

$$\begin{aligned}&D^+ = \{(x,t):\ u^+(x,t) > 0, u^-(x,t) > 0\},\\ &D^- = \{(x,t):\ u^+(x,t) < 0, u^-(x,t) < 0\},\\ &D^{x,+} = \{t:\ u_*(t,x) > 0\},\ D^{x,-} = \{t:\ u^*(t,x) < 0\},\\ &D^{x_i',t,+} = \{x_i:\ u_*(x_i,x_i',t) > 0\},\ D^{x_i',t,-} = \{x_i:\ u^*(x_i,x_i',t) < 0\}.\end{aligned}$$

From the properties of BV functions(see [4]), the set $F^+ = \{(x,t) \in Q_T :\ u > 0\}$ has finite perimeter, namely, $\chi_{F^+} \in BV(Q_T)$. Clearly, $\chi_{D^+} = \chi_{F^+}$ *a.e.* in Q_T. For any $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \chi_{D^+} \frac{\partial \varphi}{\partial x_i} dxdt = \iint_{Q_T} \chi_{F^+} \frac{\partial \varphi}{\partial x_i} dxdt = - \iint_{Q_T} \varphi \frac{\partial \chi_{F^+}}{\partial x_i},$$

then, by the definition of BV functions, $\chi_{D^+} \in BV(Q_T)$ and thus $\chi_{D^+}\varphi \in BV_c(Q_T)$. Similarly, we have $\chi_{D^-}\varphi \ \in BV_c(Q_T)$. Taking $\psi = \chi_{D^+}\varphi$ in (2.5), we have

$$\begin{aligned} &\iint_{Q_T} (w_i^r - w_i^l) \frac{\partial \varphi \chi_{D^+}}{\partial x_i} + \iint_{Q_T} (w_i^r - w_i^l) \chi_{D^+} \frac{\partial \varphi}{\partial x_i} dtdx \\ =& \iint_{Q_T} (u^r - u^l) \varphi \frac{\partial \chi_{D^+}}{\partial t} + \iint_{Q_T} (u^r - u^l) \chi_{D^+} \frac{\partial \varphi}{\partial t} dtdx \\ &- \iint_{Q_T} (B_i(u^r) - B_i(u^l)) \chi_{D^+} \frac{\partial \varphi}{\partial x_i} dtdx \\ &- \iint_{Q_T} (B_i(u^r) - B_i(u^l)) \varphi \frac{\partial \chi_{D^+}}{\partial x_i}. \end{aligned}$$

Noticing that an integrable function(with respect to Lebesgue measure) is approximately continuous almost everywhere, we deduce that $u^r(t,\cdot) = u^l(t,\cdot)$ *a.e.* in $(0,T)$ for fixed $x \in \mathbb{R}^N$, $B_i(u^r(x_i,\cdot)) = B_i(u^l(x_i,\cdot))$ and $w_i^r(x_i,\cdot) = w_i^l(x_i,\cdot)$ *a.e.* in $\mathbb{R}$ for fixed $(t,x_i') \in \mathbb{R}^{N-1} \times (0,T)$ and hence

$$\begin{aligned} &\iint_{Q_T} (u^r - u^l) \chi_{D^+} \frac{\partial \varphi}{\partial t} dtdx = \iint_{Q_T} (w_i^r - w_i^l) \chi_{D^+} \frac{\partial \varphi}{\partial x_i} dtdx \\ &\quad = \iint_{Q_T} (B_i(u^r) - B_i(u^l)) \chi_{D^+} \frac{\partial \varphi}{\partial x_i} dtdx = 0. \end{aligned}$$

Since the functions $u_*(\cdot,x_i',t)$ and $u^*(\cdot,x_i',t)$ are lower semi-continuous and upper semi-continuous respectively, we see that the sets $D^{t,x_i',+}$ and $D^{t,x_i',-}$ can be expressed as the unions of at most countable open intervals. It is easily seen that the measure $\dfrac{\partial \chi_{D^+}(t,x_i',\cdot)}{\partial x_i}$ and $\dfrac{\partial \chi_{D^-}(t,x_i',\cdot)}{\partial x_i}$ concentrate on the endpoints of the component open intervals of the sets $D^{x_i',t,+}$ and $D^{x_i',t,-}$. By Lemma 3.7.4 in [15], we have

$$\begin{aligned} &\iint_{Q_T} (u^r - u^l) \varphi \frac{\partial \chi_{D^+}}{\partial t} \\ =& \iint_{\mathbb{R}^N} dx \int_0^T (u^r - u^l) \varphi \frac{\partial \chi_{D^{+,x}}(\cdot)}{\partial t} \\ =& \iint_{\mathbb{R}^N} dx \sum_{t \in D_r^{x,+} \setminus D_l^{x,+}} (u^r - u^l)\varphi - \iint_{\mathbb{R}^N} dx \sum_{t \in D_l^{x,+} \setminus D_r^{x,+}} (u^r - u^l)\varphi \\ =& \int_{\Gamma_{\chi_{D^+}}} \varphi (u^+ - u^-) \gamma_t dH^N. \end{aligned}$$

Similarly, we can deduce that

$$\iint_{Q_T} (B_i(u^r) - B_i(u^l)) \varphi \frac{\partial \chi_{D^+}}{\partial x_i} = \int_{\Gamma_{\chi_{D^+}}} \varphi (B_i(u^+)) - (B_i(u^-)) \gamma_{x_i} dH^N$$

and

$$\iint_{Q_T}(w_i^r - w_i^l)\varphi\frac{\partial\chi_{D^+}}{\partial x_i} = \int_{\Gamma_{\chi_{D^+}}}\varphi(w_i^r - w_i^l)|\gamma_{x_i}|dH^N.$$

Thus

$$\int_{\Gamma_{\chi_{D^+}}}\varphi[(u^+ - u^-)\gamma_t - (B_i(u^+) - B_i(u^-))\gamma_{x_i} - (w_i^r - w_i^l)|\gamma_{x_i}|]dH^N = 0. \quad (2.6)$$

Moreover, taking $\psi = \chi_{D^-}\varphi$ in (2.6), we can derive in a similar fashion that

$$\int_{\Gamma_{\chi_{D^-}}}\varphi[(u^+ - u^-)\gamma_t - (B_i(u^+) - B_i(u^-))\gamma_{x_i} - (w_i^r - w_i^l)|\gamma_{x_i}|]dH^N = 0. \quad (2.7)$$

Combining (2.6) and (2.7), we have

$$\int_{\Gamma_u}\varphi[(u^+ - u^-)\gamma_t - (B_i(u^+) - B_i(u^-))\gamma_{x_i} - (w_i^r - w_i^l)|\gamma_{x_i}|]dH^N = 0. \quad (2.8)$$

For any bounded and measurable subset S of Γ_u^*, similar to Lemma 3.7.6 in [15], we can select a sequence $\{\varphi_j\} \subset C_0^\infty(Q_T)$ such that

$$|\varphi_j(x,t)| \leq 1, \lim_{j\to\infty}\|\varphi_j - \chi_S\|_X = 0 \text{ in } X = L^1\left(Q_T, \sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial u}{\partial t}\right|\right).$$

From this and Corollary 2.1 it follows that(precisely along a subsequence)

$$\lim_{j\to\infty}\varphi_j(x,t) = \chi_S(x,t)$$

H−almost everywhere on Γ_u^*. Then φ by replacing φ_j in (2.8), letting $j \to \infty$ and using the dominated convergence theorem, we obtain

$$\int_S\varphi[(u^+ - u^-)\gamma_t - (B_i(u^+) - B_i(u^-))\gamma_{x_i} - (w_i^r - w_i^l)|\gamma_{x_i}|]dH^N = 0,$$

which implies (2.1) by the arbitrariness of S.

(2.2) can be deduced similar to that of the proof of (4.8) in [15](Chp3, pp 308) and we omit the details. The theorem is proved.

3. Proof of Theorem 1.1

Let u_1 and u_2 be two BV entropy solutions of the equation (1.1) with the initial data u_{01} and u_{02} respectively. Denote

$$z = u_1 - u_2,\ \beta_i = \int_0^1 b_i(\theta u_1 + (1-\theta)u_2)d\theta,$$
$$\sigma = \frac{1}{2}(\text{sgn}z^+ + \text{sgn}z^-),\ w_i = \frac{\partial A(u_1)}{\partial x_i} - \frac{\partial A(u_2)}{\partial x_i}.$$

For any $\varphi \in C_0^\infty(Q_T)$, $\varphi \geq 0$, define

$$J(u_1, u_2, \varphi) = \iint_{Q_T}\sigma\left(\bar{z}\frac{\partial\varphi}{\partial t} - \overline{\beta_i z}\frac{\partial\varphi}{\partial x_i} - \overline{w_i}\frac{\partial\varphi}{\partial x_i}\right)dxdt.$$

All efforts we make in the following is to prove

$$J(u_1,u_2,\varphi)\geq 0, \qquad \forall 0\leq\varphi\in C_0^\infty(Q_T). \tag{3.1}$$

Once this is done, the proof of Theorem 1.1 can be completed in the same way just as in [14].

Clearly, $J(u_1,u_2,\varphi)$ can be rewritten as

$$\begin{aligned}&J(u_1,u_2,\varphi)\\ =&\iint_{Q_T}\sigma\left(\frac{\partial z\varphi}{\partial t}-\frac{\partial\beta_i z\varphi}{\partial x_i}\right)-\iint_{Q_T}\sigma\varphi\left(\frac{\partial z}{\partial t}-\frac{\partial\beta_i z}{\partial x_i}\right)-\iint_{Q_T}\sigma\overline{w_i}\frac{\partial\varphi}{\partial x_i}dxdt.\end{aligned}$$

In order to obtain the crucial inequality (3.1), we need the following

LEMMA 3.1. *For any $\varphi\geq 0,\varphi\in C_0^\infty(Q_T)$,*

$$\begin{aligned}&J(u_1,u_2,\varphi)\\ \geq&-\int_{\Gamma_z}\varphi(sgnz^+-sgnz^-)(\bar z\gamma_t-\overline{\beta_i z}\gamma_{x_i}-\widetilde{w_i}\gamma_{x_i})dH_N\\ &+\int_{\Gamma_0}\varphi(|w_i^l sgnz^r|+|w_i^r sgnz^l|)\gamma_{x_i}dH_N,\end{aligned} \tag{3.2}$$

where

$$\Gamma_0=\{(x,t)\in\Gamma_z:\ z^+(x,t)z^-(x,t)=0\}.$$

Proof. By the theory of BV functions(see [11]),

$$\iint_{Q_T}\sigma\left(\frac{\partial z\varphi}{\partial t}-\frac{\partial\beta_i z\varphi}{\partial x_i}\right)=-\int_{\Gamma_z}\varphi(\mathrm{sgn}z^+-\mathrm{sgn}z^-)(\bar z\gamma_t-\overline{\beta_i z}\gamma_{x_i})dH_N. \tag{3.3}$$

Set

$$\begin{aligned}\Gamma_+&=\{(x,t)\in\Gamma_z:\ z^+(x,t)z^-(x,t)>0\},\\ \Gamma_-&=\{(x,t)\in\Gamma_z:\ z^+(x,t)z^-(x,t)<0\},\\ E^+&=\{(x,t):\ z^+(x,t)>0,z^-(x,t)>0\},\\ E^-&=\{(x,t):\ z^+(x,t)<0,z^-(x,t)<0\}.\end{aligned}$$

According to the definition of σ, we have

$$\begin{aligned}&\iint_{Q_T}\sigma\varphi\left(\frac{\partial z}{\partial t}-\frac{\partial\beta_i z}{\partial x_i}\right)\\ =&\iint_{\Gamma_0}\sigma\varphi\left(\frac{\partial z}{\partial t}-\frac{\partial\beta_i z}{\partial x_i}\right)+\iint_{Q_T}(\chi_{E^+}-\chi_{E^-})\varphi\left(\frac{\partial z}{\partial t}-\frac{\partial\beta_i z}{\partial x_i}\right),\end{aligned} \tag{3.4}$$

Noticing mes $\Gamma_0=0$ and $\chi_{E^+}=\overline{\chi_{E^+}},\chi_{E^-}=\overline{\chi_{E^-}}$ *a.e.* in Q_T, we have

$$\begin{aligned}&\iint_{Q_T}\sigma\overline{w_i}\frac{\partial\varphi}{\partial x_i}dxdt\\ =&\iint_{Q_T}\overline{w_i}\ \overline{\chi_{E^+}}\frac{\partial\varphi}{\partial x_i}dxdt-\iint_{Q_T}\overline{w_i}\ \overline{\chi_{E^-}}\frac{\partial\varphi}{\partial x_i}dxdt.\end{aligned}$$

Obviously,

$$\iint_{Q_T} \overline{w_i}\frac{\partial \psi}{\partial x_i} = \iint_{Q_T} \overline{z}\frac{\partial \psi}{\partial t} - \iint_{Q_T} \overline{\beta_i z}\frac{\partial \psi}{\partial x_i}, \qquad \forall\ \psi \in BV_c(Q_T). \tag{3.5}$$

From this and the product rule for BV functions, it follows that

$$\begin{aligned}\iint_{Q_T} \sigma\overline{w_i}\frac{\partial \varphi}{\partial x_i}dxdt \quad &= -\iint_{Q_T} \overline{\chi_{E^+}}\varphi\left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i}\right) - \iint_{Q_T} \overline{w_i}\varphi\frac{\partial \chi_{E^+}}{\partial x_i} \\ &\quad + \iint_{Q_T} \overline{\chi_{E^-}}\varphi\left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i}\right) + \iint_{Q_T} \overline{w_i}\varphi\frac{\partial \chi_{E^-}}{\partial x_i},\end{aligned}$$

and hence

$$\begin{aligned}&\iint_{Q_T} \sigma\varphi\left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i}\right) + \iint_{Q_T} \sigma\overline{w_i}\frac{\partial \varphi}{\partial x_i}dxdt \\ =& \iint_{\Gamma_0} \sigma\varphi\left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i}\right) + \iint_{Q_T} (\chi_{E^+} - \overline{\chi_{E^+}})\varphi\left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i}\right) \\ &- \iint_{Q_T} (\chi_{E^-} - \overline{\chi_{E^-}})\varphi\left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i}\right) \\ &- \iint_{Q_T} \overline{w_i}\varphi\frac{\partial \chi_{E^+}}{\partial x_i} + \iint_{Q_T} \overline{w_i}\varphi\frac{\partial \chi_{E^-}}{\partial x_i}.\end{aligned} \tag{3.6}$$

Now we treat the terms on the right hand side of (3.6) respectively. Let

$$z_*(x,t) = \min\{z^+(x,t), z^-(x,t)\},\ z^*(x,t) = \max\{z^+(x,t), z^-(x,t)\}.$$

Then $E^+ = \{(x,t) \in Q_T : z_* > 0\},\ E^- = \{(x,t) \in Q_T : z^* < 0\}$.

For any $i = 1, 2, \cdots, N$, define

$$E_i^{t,x_i',+} = \{x_i : z_*(t, x_i', x_i) > 0\}, \quad E_i^{t,x_i',-} = \{x_i : z^*(t, x_i', x_i) < 0\}.$$

Since the functions $z_*(\cdot, x_i', t)$ and $z^*(\cdot, x_i', t)$ are lower and super semi–continuous respectively, we see that the sets $E_i^{t,x_i',+}$ and $E_i^{t,x_i',-}$ can be expressed as the unions of at most countable open intervals. Denote by $E_r^{t,x_i',+}, E_r^{t,x_i',-}$ and $E_l^{t,x_i',+}, E_l^{t,x_i',-}$ the right endpoints and the left endpoints respectively. We decompose these sets as follows:

$$\begin{aligned}E_r^{t,x_i',+} &= E_{r,+}^{t,x_i',+} \cup E_{r,0}^{t,x_i',+}, \qquad & E_l^{t,x_i',+} &= E_{l,+}^{t,x_i',+} \cup E_{l,0}^{t,x_i',+}, \\ E_r^{t,x_i',-} &= E_{r,-}^{t,x_i',-} \cup E_{r,0}^{t,x_i',-}, \qquad & E_l^{t,x_i',-} &= E_{l,-}^{t,x_i',-} \cup E_{l,0}^{t,x_i',-},\end{aligned}$$

where

$$\begin{aligned}E_{r,+}^{t,x_i',+} &= \{x_i \in E_r^{t,+};\ z^l(t,x) > 0\}, \qquad & E_{r,0}^{t,x_i',+} &= \{x_i \in E_r^{t,+};\ z^l(t,x) = 0\}, \\ E_{l,+}^{t,x_i',+} &= \{x_i \in E_l^{t,+};\ z^r(t,x) > 0\}, \qquad & E_{l,0}^{t,x_i',+} &= \{x_i \in E_l^{t,+};\ z^r(t,x) = 0\}, \\ E_{r,-}^{t,x_i',-} &= \{x_i \in E_r^{t,-};\ z^l(t,x) < 0\}, \qquad & E_{r,0}^{t,x_i',-} &= \{x_i \in E_r^{t,-};\ z^l(t,x) = 0\}, \\ E_{l,+}^{t,x_i',-} &= \{x_i \in E_l^{t,-};\ z^r(t,x) < 0\}, \qquad & E_{l,0}^{t,x_i',-} &= \{x_i \in E_l^{t,-};\ z^r(t,x) = 0\}.\end{aligned}$$

We introduce the following notations

$$\Gamma_0 = \{(x,t) \in \Gamma_z;\ z^+(x,t)z^-(x,t) = 0\}, \qquad \Gamma_0^{t,x_i'} = \{x_i;\ (x,t) \in \Gamma_0\},$$
$$\Gamma_+ = \{(x,t) \in \Gamma_z;\ z^+(x,t)z^-(x,t) > 0\}, \qquad \Gamma_+^{t,x_i'} = \{x_i;\ (x,t) \in \Gamma_+\},$$

$$\Gamma_{r,+}^{t,x_i'} = \{x_i \in \Gamma_0^{t,x_i'};\ z^r(x,t) > 0\}, \qquad \Gamma_{r,-}^{t,x_i'} = \{x_i \in \Gamma_0^{t,x_i'};\ z^r(x,t) < 0\},$$
$$\Gamma_{l,+}^{t,x_i'} = \{x_i \in \Gamma_0^{t,x_i'};\ z^l(x,t) > 0\}, \qquad \Gamma_{l,-}^{t,x_i'} = \{x_i \in \Gamma_0^{t,x_i'};\ z^l(x,t) < 0\}.$$

Similar to the proof of Lemma 3.4.1 in [15], we have, for almost all $(t, x_i') \in (0,T) \times \mathbb{R}^{N-1}$,

$$\Gamma_0^{t,x_i'} = \Gamma_{r,+}^{t,x_i'} \cup \Gamma_{r,-}^{t,x_i'} \cup \Gamma_{l,+}^{t,x_i'} \cup \Gamma_{l,-}^{t,x_i'}$$

and

$$\Gamma_{r,+}^{t,x_i'} = E_{l,+}^{t,x_i',+} \backslash E_{r,-}^{t,x_i',-}, \qquad \Gamma_{r,-}^{t,x_i'} = E_{l,-}^{t,x_i',-} \backslash E_{r,+}^{t,x_i',+}$$
$$\Gamma_{l,+}^{t,x_i'} = E_{r,+}^{t,x_i',+} \backslash E_{l,-}^{t,x_i',-}, \qquad \Gamma_{l,-}^{t,x_i'} = E_{r,-}^{t,x_i',-} \backslash E_{l,+}^{t,x_i',+}.$$

Using Lemma 3.7.5 in [15], we have

$$\begin{aligned}&\iint_{\Gamma_0} \sigma\varphi \left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i} \right) \\ = &\int_{\Gamma_0} \sigma\varphi[(z^+ - z^-)\gamma_t - ((\beta_i z)^+ - (\beta_i z)^-)\gamma_{x_i}] dH_N.\end{aligned}$$

According to the theory of BV functions(see [9]), the set of non-regular points of χ_{E^+}, χ_{E^-} are N-dimensional Hausdorff measure zero. Since the measures $\frac{\partial z}{\partial t}$ and $\frac{\partial \beta_i z}{\partial x_i}$ are zero in any set of N-dimensional Hausdorff measure zero and $\Gamma_{\chi_{E^+}} \subset \Gamma_z$, we have

$$\begin{aligned}&\iint_{Q_T} (\chi_{E^-} - \overline{\chi_{E^-}})\varphi \left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i} \right) \\ =&\int_{\Gamma_{\chi_{E^-}}} (\chi_{E^-} - \overline{\chi_{E^-}})\varphi[(z^+ - z^-)\gamma_t - ((\beta_i z)^+ - (\beta_i z)^-)\gamma_{x_i}] dH_N. \qquad (3.7)\end{aligned}$$

By virtue of the discontinuity conditions (2.1) and the relations (3.6)-(3.7) we deduce that

$$\begin{aligned}&\iint_{\Gamma_0} \sigma\varphi \left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i} \right) + \iint_{Q_T} (\chi_{E^+} - \overline{\chi_{E^+}})\varphi \left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i} \right) \\ &\quad - \iint_{Q_T} (\chi_{E^-} - \overline{\chi_{E^-}})\varphi \left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i} \right) \\ =&\iint_{(0,T)\times\mathbb{R}^{N-1}} \frac{1}{2} \sum_{x_i \in \Gamma_0^{t,x_i'}} (\text{sgn} z^+ + \text{sgn} z^-)\varphi(w_i^r - w_i^l) dt dx_i' \\ &\quad + \iint_{(0,T)\times\mathbb{R}^{N-1}} \sum_{x_i \in \Gamma_{\chi_{E^+}}^{t,x_i'}} (\chi_{E^+} - \overline{\chi_{E^+}})\varphi(w_i^r - w_i^l) dt dx_i'\end{aligned}$$

$$- \iint_{(0,T)\times\mathbb{R}^{N-1}} \sum_{x_i\in\Gamma^{t,x_i'}_{\chi_{E^-}}} (\chi_{E^-} - \overline{\chi_{E^-}})\varphi(w_i^r - w_i^l)dtdx_i'$$

From the definition of $\Gamma_0^{t,x_i'}$, $\Gamma_{r,+}^{t,x_i'}$, $\Gamma_{r,-}^{t,x_i'}$, $\Gamma_{l,+}^{t,x_i'}$ and $\Gamma_{l,-}^{t,x_i'}$, we have

$$\begin{aligned}
&\iint_{(0,T)\times\mathbb{R}^{N-1}} \frac{1}{2} \sum_{x_i\in\Gamma_0^{t,x_i'}} (\mathrm{sgn}z^+ + \mathrm{sgn}z^-)\varphi(w_i^r - w_i^l)dtdx_i' \\
&\quad = \iint_{(0,T)\times\mathbb{R}^{N-1}} \kappa_i^1(t,x_i')dtdx_i',
\end{aligned}$$

where

$$\begin{aligned}
\kappa_i^1(t,x_i') &= \frac{1}{2} \sum_{x_i\in\Gamma_0^{t,x_i'}} (\mathrm{sgn}z^+ + \mathrm{sgn}z^-)\varphi(w_i^r - w_i^l) \\
&= \frac{1}{2} \sum_{x_i\in\Gamma_{r,+}^{t,x_i'}} \varphi(w_i^r - w_i^l) - \frac{1}{2} \sum_{x_i\in\Gamma_{r,-}^{t,x_i'}} \varphi(w_i^r - w_i^l) \\
&\quad + \frac{1}{2} \sum_{x_i\in\Gamma_{l,+}^{t,x_i'}} \varphi(w_i^r - w_i^l) - \frac{1}{2} \sum_{x_i\in\Gamma_{l,-}^{t,x_i'}} \varphi(w_i^r - w_i^l)
\end{aligned} \tag{3.8}$$

for almost all $(t,x_i') \in (0,T)\times\mathbb{R}^{N-1}$.

From the theory of BV space(see [9]), we have

$$\begin{aligned}
&\sum_{x_i\in\Gamma^{t,x_i'}_{\chi_{E^+}}} (\chi_{E^+} - \overline{\chi_{E^+}})\varphi(w_i^r - w_i^l) \\
&= -\frac{1}{2} \sum_{x_i\in E_r^{t,x_i',+}\setminus E_l^{t,x_i',+}} \varphi(w_i^r - w_i^l) - \frac{1}{2} \sum_{x_i\in E_l^{t,x_i',+}\setminus E_r^{t,x_i',+}} \varphi(w_i^r - w_i^l) \\
&\qquad - \sum_{x_i\in E_l^{t,x_i',+}\cap E_r^{t,x_i',+}} \varphi(w_i^r - w_i^l),
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\sum_{x_i\in\Gamma^{t,x_i'}_{\chi_{E^-}}} (\chi_{E^-} - \overline{\chi_{E^-}})\varphi(w_i^r - w_i^l) \\
&= -\frac{1}{2} \sum_{x_i\in E_r^{t,x_i',-}\setminus E_l^{t,x_i',-}} \varphi(w_i^r - w_i^l) - \frac{1}{2} \sum_{x_i\in E_l^{t,x_i',-}\setminus E_r^{t,x_i',-}} \varphi(w_i^r - w_i^l) \\
&\qquad - \sum_{x_i\in E_l^{t,x_i',-}\cap E_r^{t,x_i',-}} \varphi(w_i^r - w_i^l),
\end{aligned} \tag{3.10}$$

for almost for $(t,x_i') \in (0,T)\times\mathbb{R}^{N-1}$.

By Lemma 3.7.8 in [15], we have

$$\iint_{Q_T} \overline{w_i}\varphi\frac{\partial\chi_{E^+}}{\partial x_i} = \int_{\mathbb{R}^{N-1}\times(0,T)} dtdx_i' \int_{R^1} \widetilde{w_i}\varphi\frac{\partial\chi_{E^+}(t,x_i',\cdot)}{\partial x_i}$$

$$\iint_{Q_T} \overline{w_i}\varphi \frac{\partial \chi_{E^-}}{\partial x_i} = \int_{\mathbb{R}^{N-1}\times(0,T)} dt dx_i' \int_{R^1} \widetilde{w_i}\varphi \frac{\partial \chi_{E^+}(t,x_i',\cdot)}{\partial x_i}.$$

Noticing the sets $E_i^{t,x_i',+}$ and $E_i^{t,x_i',-}$ can be expressed as the unions of at most countable open intervals and

$$\frac{\partial \chi_{E^+}(t,x_i',\cdot)}{\partial x_i} = \frac{\partial \chi_{E^{t,x_i',+}}(x_i)}{\partial x_i}, \quad \frac{\partial \chi_{E^-}(t,x_i',\cdot)}{\partial x_i} = \frac{\partial \chi_{E^{t,x_i',-}}(x_i)}{\partial x_i},$$

one can see from the theory of BV functions that measures $\frac{\partial \chi_{E^+}(t,x_i',\cdot)}{\partial x_i}$ and $\frac{\partial \chi_{E^-}(t,x_i',\cdot)}{\partial x_i}$ concentrate on the end points of the open intervals of $E^{t,x_i',+}$ and $E^{t,x_i',-}$ respectively. Thus, for almost all $(t,x_i') \in (0,T)\times\mathbb{R}^{N-1}$,

$$\int_{R^1} \widetilde{w_i}\varphi \frac{\partial \chi_{E^+}(t,x_i',\cdot)}{\partial x_i} = \sum_{x_i \in E_l^{t,x_i',+}\setminus E_r^{t,x_i',+}} \widetilde{w_i}\varphi - \sum_{x_i \in E_r^{t,x_i',+}\setminus E_l^{t,x_i',+}} \widetilde{w_i}\varphi,$$

and

$$\int_{R^1} \widetilde{w_i}\varphi \frac{\partial \chi_{E^-}(t,x_i',\cdot)}{\partial x_i} = \sum_{x_i \in E_l^{t,x_i',-}\setminus E_r^{t,x_i',-}} \widetilde{w_i}\varphi - \sum_{x_i \in E_r^{t,x_i',-}\setminus E_l^{t,x_i',-}} \widetilde{w_i}\varphi,$$

and hence

$$\iint_{Q_T} \overline{w_i}\varphi \frac{\partial \chi_{E^+}}{\partial x_i} - \iint_{Q_T} \overline{w_i}\varphi \frac{\partial \chi_{E^-}}{\partial x_i} = \iint_{(0,T)\times\mathbb{R}^{N-1}} \lambda_i(t,x_i') dt dx_i', \tag{3.11}$$

where

$$\begin{aligned}\lambda_i(t,x_i') = & \sum_{x_i \in E_l^{t,x_i',+}\setminus E_r^{t,x_i',+}} \widetilde{w_i}\varphi - \sum_{x_i \in E_r^{t,x_i',+}\setminus E_l^{t,x_i',+}} \widetilde{w_i}\varphi \\ & - \sum_{x_i \in E_l^{t,x_i',-}\setminus E_r^{t,x_i',-}} \widetilde{w_i}\varphi + \sum_{x_i \in E_r^{t,x_i',-}\setminus E_l^{t,x_i',-}} \widetilde{w_i}\varphi,\end{aligned}$$

for almost all $(t,x_i') \in (0,T)\times\mathbb{R}^{N-1}$.

Now, combining (3.9), (3.10) and (3.11), we can obtain

$$\begin{aligned}&\iint_{Q_T} \sigma\varphi\left(\frac{\partial z}{\partial t} - \frac{\partial \beta_i z}{\partial x_i}\right) + \iint_{Q_T} \sigma\overline{w_i}\frac{\partial \varphi}{\partial x_i} dx dt \\ &= -\int_{\Gamma_z} \varphi(\mathrm{sgn} z^+ - \mathrm{sgn} z^-)\widetilde{w_i}\gamma_{x_i}) dH_N - \iint_{(0,T)\times\mathbb{R}^{N-1}} \kappa_i(t,x_i') dt dx_i',)\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}\kappa_i(t,x_i') = & -\sum_{x_i \in \Gamma_{l,+}^{t,x_i'} \cup\ E_{r,0}^{t,x_i',+}} \varphi w_i^l + \sum_{x_i \in \Gamma_{r,+}^{t,x_i'} \cup\ E_{l,0}^{t,x_i',+}} \varphi w_i^r \\ & + \sum_{x_i \in \Gamma_{l,-}^{t,x_i'} \cup\ E_{r,0}^{t,x_i',-}} \varphi w_i^l - \sum_{x_i \in \Gamma_{r,-}^{t,x_i'} \cup\ E_{l,0}^{t,x_i',-}} \varphi w_i^r.\end{aligned}$$

So far, we have proved that

$$J(u_1,u_2,\varphi) = -\int_{\Gamma_z}\varphi(\mathrm{sgn} z^+ - \mathrm{sgn} z^-)(\bar z\gamma_t - \overline{\beta_i z}\gamma_{x_i} - \widetilde{w_i}\gamma_{x_i})dH_N + \iint_{(0,T)\times\mathbb{R}^{N-1}}\kappa_i(t,x_i')dtdx_i'.$$

Similar to the proof of the inequality (3.22) in Lemma 3.4.1 in [15] (see pp. 319–320), we can obtain

$$\iint_{(0,T)\times\mathbb{R}^{N-1}}\kappa_i(t,x_i')dtdx_i' \geq \int_{\Gamma_0}\varphi(|w_i^l \mathrm{sgn} z^r| + |w_i^r \mathrm{sgn} z^l|)\gamma_{x_i}dH_N. \tag{3.13}$$

Thus complete the proof of Lemma 3.1.

Using Lemma 3.1, we can prove Theorem 1.1 by the same method as in [15] (see pp 321–323).

4. An Example

In this section, we give an example of the natural BV entropy solution. Consider the quasilinear equations of the form

$$\frac{\partial u}{\partial t} = \Delta A(u) \tag{4.1}$$

where

$$A'(s) = 0 \quad \text{if } s<0; \quad A'(s)>0 \quad \text{if } s>0,$$

subject to radial initial value condition

$$u(x,0) = u_0(x) = u_0(|x|)\begin{cases} >0, & 0<|x|<1, \\ <0, & |x|>1 \end{cases} \tag{4.2}$$

with

$$\lim_{|x|\to 1^-} u_0(x) = 0, \quad \lim_{|x|\to 1^+} u_0(x) < 0.$$

Denote $r=|x|$. Suppose $\frac{\partial^2 A(u_0)}{\partial r^2}$ is a regular measure. By virtue of [7], the problem (4.1)–(4.2) admits a radial BV entropy solution $u(r,t)$, namely, $u(x,t) = u(r,t)$. It is easy to verify that a radial solution $u(r,t)$ of the equation (4.1) satisfies

$$\frac{\partial r^{N-1}u}{\partial t} = \frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial A(u)}{\partial r}\right), \qquad (r,t)\in Q_T, \tag{4.3}$$

where $Q_T = (0,\infty)\times(0,T)$.

Since we have prescribed an initial value with a discontinuous point $r=1$, the corresponding BV solution $u(r,t)$ would have a curve of discontinuity $r=\lambda(t)$ with $\lambda'(t)>0$ emerging from $(0,1)$. Using the discontinuity conditions, we can obtain the relations which satisfied by $\lambda(t)$

$$u(t,r) = u_0(r) \ \text{ if } \ r>\lambda(t), \quad u^+\big|_{r=\lambda(t)} < 0\ , \quad u^-\big|_{r=\lambda(t)} = 0$$

and

$$u_0(\lambda(t))\lambda'(t) - \left(\frac{\partial A(u)}{\partial r}\right)^-\Bigg|_{r=\lambda(t)} = 0.$$

Thus $(u(t,r),\lambda)$ is a solution of (4.3) with boundary and initial conditions

$$u\Big|_{r=\lambda(t)} = 0\ , \tag{4.4}$$

$$\frac{\partial A(u)}{\partial r}\bigg|_{r=\lambda(t)} = u_0(\lambda(t))\lambda'(t), \tag{4.5}$$

$$u(r,0) = u_0(r) \qquad r > 0. \tag{4.6}$$

Define

$$w(t,r) = \begin{cases} u(t,r), & \text{if } \ r < \lambda(t), \\ u_0(r), & \text{if } \ r > \lambda(t). \end{cases}$$

We will prove that if $(u(t,r),\lambda(t))$ is a BV entropy solution of the problem (4.3)–(4.6), then $w(t,r)$ is a BV solution of the problem (4.3),(4.6) with $r = \lambda(t)$ as the line of discontinuity.

By a solution of the problem (4.3)–(4.6), we mean a pair of functions $(u(t,r),\lambda(t))$ such that

$$r^{N-1}u \in L^\infty(G) \cap C(\overline{G}\backslash\{(0,1)\}) \cap C^{1,2}(G), \quad G = \{(t,r):\ 0 < t < T, r < \lambda(t)\}.$$

$$\iint_D \left|\frac{\partial r^{N-1}u}{\partial t}\right| dtdr,\ \ \iint_D \left|\frac{\partial r^{N-1}u}{\partial r}\right| dtdr < +\infty,$$

where D denotes an universal bounded domain in $(0,T)\times(0,+\infty)$ with $\overline{D} \subset \overline{G} - \{t = 0\}$.

$$\frac{\partial r^{N-1}A(u)}{\partial r} \in C(\overline{G}\backslash\{(0,1)\}),\ \lambda(t) \in C[0,T] \cap C^1(0,T],\ \lambda'(t) > 0$$

and (4.3), (4.4), (4.5), (4.6) are satisfied.

Now we prove that $w(t,r)$ satisfies the following integral inequality

$$\begin{aligned} J(w,k,\varphi) \equiv &\iint_{Q_T} \frac{1}{2}(\text{sgn}(u^+ - k) + \text{sgn}(u^- - k))r^{N-1}(\bar{u} - k)\frac{\partial\varphi}{\partial t} \\ &- \iint_{Q_T} \frac{1}{2}(\text{sgn}(u^+ - k) + \text{sgn}(u^- - k))r^{N-1}\overline{\frac{\partial A(u)}{\partial r}}\frac{\partial\varphi}{\partial r} \ge 0, \end{aligned} \tag{4.7}$$

where $k \in \mathbb{R}$, $\varphi \in BV_c(Q_T), \varphi \ge 0$ and the integral on Q_T being respect to the measure $\dfrac{\partial\varphi}{\partial t}$ and $\dfrac{\partial\varphi}{\partial r}$.

From the initial condition (4.2) and the definition of the free boundary problem, we can divide $J(w,k,\varphi)$ into three parts:

$$\begin{aligned} J(w,k,\varphi) = &\iint_{r>\lambda(t)} r^{N-1}\text{sgn}(u_0 - k)(u_0 - k)\frac{\partial\varphi}{\partial t} \\ &+ \iint_{r<\lambda(t)} r^{N-1}\text{sgn}(u - k)\left[(u - k)\frac{\partial\varphi}{\partial t} - \frac{\partial A(u)}{\partial r}\frac{\partial\varphi}{\partial r}\right] \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&+\iint_{r=\lambda(t)}\frac{1}{2}(\operatorname{sgn}(u^+-k)+\operatorname{sgn}(u^--k))r^{N-1}\\
&\qquad\cdot\left[(\bar{u}-k)\frac{\partial\varphi}{\partial t}+\overline{\frac{\partial A(u)}{\partial r}}\frac{\partial\varphi}{\partial r}\right]\\
&\equiv J_1(u_0,k,\varphi)+J_2(u,k,\varphi)+J_3(u,k,\varphi).
\end{aligned}
$$

We will treat J_1, J_2 and J_3 in (4.8) respectively.

Let $j_\varepsilon(s)$ be the kernel of a mollifier in one dimension, namely, $j_\varepsilon(s)=\frac{1}{\varepsilon}j\left(\frac{s}{\varepsilon}\right)$, $0\le j(s)\in C_0^\infty(\mathbb{R})$, $\operatorname{supp}j(s)\in[-1,1]$, and $\int_{-\infty}^{+\infty}j(s)ds=1$. Define

$$
\varphi_\varepsilon(t,r)=\int_{\mathbb{R}}j_\varepsilon(r-\rho)j_\varepsilon(t-s)\varphi(s,\rho)dsd\rho.
$$

Since $\varphi\in BV_c(Q_T)$, it follows that $\varphi_\varepsilon\in C_0^\infty(Q_T)$ and $\varphi_\varepsilon\to\varphi$ in $BV(Q_T)$. Thus

$$
\begin{aligned}
J_1(u_0,k,\varphi)&=\lim_{\varepsilon\to0}\iint_{r>\lambda(t)}r^{N-1}\operatorname{sgn}(u_0-k)(u_0-k)\frac{\partial\varphi_\varepsilon}{\partial t}dt\\
&=\lim_{\varepsilon\to0}\int_0^T r^{N-1}|u_0-k|\lambda'(t)\varphi_\varepsilon\Big|_{r=\lambda(t)}dt. \qquad (4.9)
\end{aligned}
$$

For any $\eta>0$, define $H_\eta(s)=\frac{s}{\sqrt{s^2+\eta}}$, $h_\eta(s)=H_\eta'(s)$. It is easy to verify that $H_\eta(s)$ is strictly increasing with respect to s for fixed $\eta>0$. Moreover,

$$
\lim_{\eta\to0}H_\eta(s)=\operatorname{sgn}s,\qquad \lim_{\eta\to0}sh_\eta(s)=0.
$$

Obviously,

$$
J_2(u,k,\varphi)=\lim_{\eta\to0^+}J_{2,\eta}(u,k,\varphi) \qquad (4.10)
$$

where

$$
J_{2,\eta}(u,k,\varphi)=\iint_{r<\lambda(t)}r^{N-1}H_\eta(u-k)\left[(u-k)\frac{\partial\varphi}{\partial t}-\frac{\partial A(u)}{\partial r}\frac{\partial\varphi}{\partial r}\right].
$$

From (4.3), we have

$$
\begin{aligned}
J_{2,\eta}(u,k,\varphi)&=\lim_{\varepsilon\to0}\iint_{r<\lambda(t)}r^{N-1}H_\eta(u-k)\left[(u-k)\frac{\partial\varphi_\varepsilon}{\partial t}-\frac{\partial A(u)}{\partial r}\frac{\partial\varphi_\varepsilon}{\partial r}\right]dtdr\\
&\ge-\lim_{\varepsilon\to0}\int_0^T\lambda(t)^{N-1}\lambda'(t)H_\eta(k)[k-u_0(\lambda(t))]\varphi_\varepsilon(t,\lambda(t))dt\\
&\quad-\lim_{\varepsilon\to0}\iint_{r<\lambda(t)}r^{N-1}h_\eta(u-k)\frac{\partial u}{\partial t}(u-k)\varphi_\varepsilon dtdr
\end{aligned}
$$

For any $\delta>0$, denote

$$
G_\delta=\{(t,r)\in G; r<\lambda(t)-\delta\}.
$$

Since $u\in C^1(G_\delta)$, we get by letting $\eta\to0^+$ that

$$
\iint_{r<\lambda(t)-\delta}r^{N-1}h_\eta(u-k)\frac{\partial u}{\partial t}(u-k)\varphi_\varepsilon dtdr\to0
$$

uniformly for ε. Moreover,

$$\begin{aligned}
&\left|\iint_{\lambda(t)-\delta<r<\lambda(t)} r^{N-1}h_\eta(u-k)\frac{\partial u}{\partial t}(u-k)\varphi_\varepsilon dtdr\right| \\
&\leq \iint_{\lambda(t)-\delta<r<\lambda(t)} r^{N-1}\left|\frac{\partial u}{\partial t}\right|\varphi_\varepsilon dtdr \to 0 \quad \text{ is uniformly for } \varepsilon, \ \text{ as } \delta \to 0.
\end{aligned}$$

Thus

$$\iint_{r<\lambda(t)} r^{N-1}h_\eta(u-k)\frac{\partial u}{\partial t}(u-k)\varphi_\varepsilon dtdr \to 0 \qquad \text{is uniformly for } \ \varepsilon, \ \text{ as } \eta \to 0,$$

namely,

$$\lim_{\eta\to 0} J_{2,\eta}(u,k,\varphi) \geq \int_0^T \lambda(t)^{N-1}\lambda'(t)\mathrm{sgn}(k)[u_0(\lambda(t)-k]\varphi_\varepsilon(t,\lambda(t))dt. \tag{4.11}$$

is uniformly for ε.

Combining (4.10)–(4.11), we obtain

$$\begin{aligned}
&J_1(w,k,\varphi)+J_2(w,k,\varphi) \\
&\geq \lim_{\varepsilon\to 0}\int_0^T \lambda(t)^{N-1}\lambda'(t)[|u_0-k|-|k|+\mathrm{sgn}(k)u_0(\lambda(t))]\varphi_\varepsilon\,\Big|_{r=\lambda(t)}\, dt \geq 0.
\end{aligned} \tag{4.12}$$

Denote

$$\sigma_k = \frac{1}{2}(\mathrm{sgn}(u^+-k)+\mathrm{sgn}(u^--k)).$$

We have

$$\begin{aligned}
J_3(w,k,\varphi) &= \lim_{\varepsilon\to 0}\iint_{r=\lambda(t)} \sigma_k r^{N-1}(\bar{u}-k)\frac{\partial\varphi_\varepsilon}{\partial t} \\
&\quad + \lim_{\varepsilon\to 0}\iint_{r=\lambda(t)} \sigma_k r^{N-1}\overline{\frac{\partial A(u)}{\partial r}}\frac{\partial\varphi_\varepsilon}{\partial r}.
\end{aligned}$$

Using the product rule for BV functions, we have

$$\begin{aligned}
J_3(w,k,\varphi) &= \lim_{\varepsilon\to 0}\iint_{r=\lambda(t)} \sigma_k\frac{\partial r^{N-1}u\varphi_\varepsilon}{\partial t} \\
&\quad + \lim_{\varepsilon\to 0}\iint_{r=\lambda(t)} \sigma_k\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial A(u)}{\partial r}\varphi_\varepsilon\right) \\
&\quad + \lim_{\varepsilon\to 0}\iint_{r=\lambda(t)} \sigma_k\varphi_\varepsilon\left[\frac{\partial r^{N-1}u}{\partial t}-\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial A(u)}{\partial r}\right)\right].
\end{aligned}$$

It gives from the measure equality (4.3) that

$$\begin{aligned}
J_3(w,k,\varphi) &= \lim_{\varepsilon\to 0}\iint_{r=\lambda(t)} \sigma_k\varphi_\varepsilon\frac{\partial r^{N-1}u}{\partial t} \\
&\quad + \lim_{\varepsilon\to 0}\iint_{r=\lambda(t)} \sigma_k\varphi_\varepsilon\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial A(u)}{\partial r}\right).
\end{aligned} \tag{4.13}$$

By virtue of formula which transforms the double integral of BV functions into a curve integral(see [9]), we get

$$\iint_{r=\lambda(t)} \sigma_k \varphi_\varepsilon \frac{\partial r^{N-1} u}{\partial t} = \int_{r=\lambda(t)} \sigma_k \varphi_\varepsilon r^{N-1} [u^r(r,t) - u^l(r,t)] |\gamma_t| dH$$

and

$$\begin{aligned} &\iint_{r=\lambda(t)} \sigma_k \varphi_\varepsilon \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial A(u)}{\partial r} \right) \\ =&\int_{r=\lambda(t)} \sigma_k \varphi_\varepsilon r^{N-1} \left[\left(\frac{\partial A(u)}{\partial r} \right)^r - \left(\frac{\partial A(u)}{\partial r} \right)^l \right] |\gamma_r| dH. \end{aligned}$$

Applying the theory of BV functions again, we have for any $v \in L^\infty(Q_T) \cap BV(Q_T)$,

$$v^+(t,r) - v^-(t,r) = [v^r(t,r) - v^l(t,r)] \mathrm{sgn} \gamma_r$$

hold $H-$almost every on Γ_v^*. Thus

$$\begin{aligned} &\iint_{r=\lambda(t)} \sigma_k \varphi_\varepsilon \frac{\partial r^{N-1} u}{\partial t} - \iint_{r=\lambda(t)} \sigma_k \varphi_\varepsilon \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial A(u)}{\partial r} \right) \\ =&\int_{r=\lambda(t)} \sigma_k \varphi_\varepsilon r^{N-1} \Bigg[(u^+(r,t) - u^-(r,t)) \gamma_t \\ &\qquad - \left(\left(\frac{\partial A(u)}{\partial r} \right)^+ - \left(\frac{\partial A(u)}{\partial r} \right)^- \right) \gamma_r \Bigg] dH. \end{aligned}$$

Using the jump conditions gives

$$\iint_{r=\lambda(t)} \sigma_k \varphi_\varepsilon \frac{\partial r^{N-1} u}{\partial t} - \iint_{r=\lambda(t)} \sigma_k \varphi_\varepsilon \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial A(u)}{\partial r} \right) = 0.$$

From (4.13), we have

$$J_3(w,k,\varphi) = 0. \tag{4.14}$$

Combining (4.8), (4.12) and (4.14) we derive (4.7) immediately.

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