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An infinite-dimensional phenomenon in finite-dimensional metric topology

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We show that there are homotopy equivalences $h : N \to M$ between closed manifolds which are induced by cell-like maps $p : N \to X$ and $q : M \to X$ but which are not homotopic to homeomorphisms. The phenomenon is based on the construction of cell-like maps that kill certain $\mathbb{L}$-classes. The image space in these constructions is necessarily infinite-dimensional. In dimension > 5 we classify all such homotopy equivalences. As an application, we show that such homotopy equivalences are realized by deformations of Riemannian manifolds in Gromov-Hausdorff space preserving a contractibility function.

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1. Introduction

The authors were led to the questions studied in this paper by two different routes. The first route was via a quest to understand precompact subsets of Riemannian manifolds in Gromov-Hausdorff space, while the second arose via our efforts to understand cell-like maps, homology manifolds, and topological resolutions. Connecting these problems to each other led to a new functorial subgroup of the structure group of topological surgery theory and to examples casting light on both of these problems.

Beginning with the first question, recall that the Gromov-Hausdorff metric is a complete metric on the isomorphism classes of compact metric spaces. The Gromov-Hausdorff distance from a metric space X to the one-point metric space P is diam(X)/2, so Gromov-Hausdorff closeness imposes little connection between the topologies of compact metric spaces.

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However, if one assumes a uniform local contractibility condition, then much more structure is preserved. Let $\rho : [0, R) \to [0, \infty)$ be a function with $\rho(0) = 0$ and $\rho(t) \geq t$, such that ρ is continuous at 0. Following Borsuk [10] and Gromov [36], we say that X is $LGC(\rho)$ if every ball of radius $r < R$ in X is nullhomotopic in the concentric ball of radius $\rho(r)$. This is a generalization of the idea of injectivity radius for Riemannian manifolds. Sufficiently Gromov-Hausdorff close n-dimensional $LGC(\rho)$ spaces are homotopy equivalent – and there are explicit estimates on the required degree of closeness in terms of n and ρ. See [10] and especially the theorem on p. 392 of [62].

A theorem of Chapman and Ferry [17] implies that if M is a closed n-manifold with a fixed topological metric, $n \geq 5$,[1] with contractibility function ρ then there is an $\epsilon > 0$ such that any $LGC(\rho)$ n-manifold within ϵ of M in Gromov-Hausdorff space is homeomorphic to it. A natural question is whether we can make this relationship depend solely on ρ and n? If the answer were yes, one would obtain a straightforward explanation of the following result of Ferry, [27].

Theorem. For every n and contractibility function ρ, precompact collections of closed $LGC(\rho)$ Riemannian manifolds in Gromov-Hausdorff space contain only finitely many homeomorphism types. (Here again, dimensions 3 and 4 are due to Perelman [61] and Freedman-Quinn [34].)

If the ϵ above depended only on n and ρ, we could cover a given precompact space by finitely many ϵ-balls and each of these balls would contain a unique homeomorphism type, giving us our finiteness result. This strategy is correct with respect to homotopy types, as mentioned above, and for simple homotopy types and rational Pontrjagin classes, as shown in [27], but it fails for homeomorphism types. It can happen that for certain precompact collections of closed Riemannian manifolds with contractibility function ρ, there are limit points X with the property that every ϵ neighborhood of X contains manifolds of different homeomorphism types. This only happens when the limit points in question are infinite-dimensional spaces with finite cohomological dimension.

The most straightforward way to detect this phenomenon is via the symbol of the signature operator on a Riemannian manifold. This lies in $KO_*(M)$. (By work of Sullivan and Teleman, this makes sense for topological manifolds, except in dimension 4.) We will show that for $n \geq 6$, there exist ρ

[1] This theorem is also true in the remaining dimensions < 5 by work of Freedman-Quinn and Perelman.

and arbitrarily close n-manifolds in a suitably chosen precompact subset of $LGC(\rho)$ whose symbols can differ by any given odd torsion element of $KO_*(M)$.

Definition.

1. We will say M *deforms to* N with contractibility function ρ if there are paths M_t and N_t, $0 \leq t < 1$, $M = M_0$ and $N = N_0$, in a precompact subset of Gromov-Hausdorff space consisting of manifolds with contractibility function ρ such that the Gromov-Hausdorff distance between M_t and N_t goes to zero as t approaches 1. Note that this definition is symmetric in M and N.
2. If we allow ρ to vary, we obtain an equivalence relation called *deformation equivalence* where M is *deformation equivalent* to N if there is a closed manifold P such that M deforms to P and P deforms to N. Theorem 2.7 implies that in this case M also deforms to N with some contractibility function $\bar{\rho}$, so deformation equivalence is an equivalence relation.

The notion of deformation in part (1) of the definition above arose naturally in differential geometry. See [36], [37], [38], [39].

Theorem 1. If M^m, $m \geq 6$, is a closed simply connected manifold such that $\pi_2(M)$ is finite, then there are manifolds $\bar{M}$ which are deformation equivalent to M in some precompact collection of $LGC(\rho)$-manifolds for some ρ if and only if $KO_m(M)$ has odd torsion. Indeed, for each odd torsion class τ in $KO_m(M)$ there is a unique homotopy equivalence $f : N \to M$ which is realized by a deformation and whose signature operator differs from that of M by τ. This gives many examples – for instance between S^3-bundles over S^4. See Proposition 2.16.

For the general non-simply-connected-or-π_2-not-finite situation, there are secondary invariants that arise in the problem. These invariants are related to η invariants, except that the familiar Atiyah-Patodi-Singer η invariants usually give rise to torsion-free invariants, and the generalization we need must contain torsion information. We shall give a complete analysis of the deformation problem for dimensions ≥ 6 in Theorem 2.7.[2] We mention here some consequences and examples:

[2] Our construction requires us to embed a certain 3-dimensional metric space into M. In dimensions ≥ 7, this is similar to the technique used in [21] to study large Riemannian manifolds and disprove a variant of the Novikov Conjecture. In dimension 6, a good deal of extra care is required to ensure embeddibility.

Definition. If M is a closed n-manifold, a *homotopy structure* on M is a homotopy equivalence $f : N \to M$ from another closed n-manifold to M. Homotopy structures f and $f' : N' \to M$ are *equivalent* if there is a homeomorphism $\Phi : N' \to N$ so that $f \circ \Phi$ is homotopic to f'. The set of homotopy structures on M is denoted by $\mathcal{S}(M)$. $\mathcal{S}(M)$ is an abelian group, the abelian group structure being obtained geometrically, through Siebenmann Periodicity, or algebraically through work of Ranicki.

Theorem 2. For any closed n-manifold M, the set of homotopy structures $f : M' \to M$ that are obtainable by deformations in some precompact subset of $LGC(\rho)$ manifolds in Gromov-Hausdorff space defines a subset $S^{CE}(M)$ that is an odd torsion subgroup of the structure group $S(M)$.

This notation will be explained below.

Theorem 3. If the Farrell-Jones conjecture is true for Γ and $\underline{E}\Gamma$ is equivariantly finite, then $S^{CE}(M)$ is finite for any M with fundamental group Γ. In particular, if M has word hyperbolic fundamental group, or has fundamental group that is a lattice is a semisimple Lie group, then $S^{CE}(M)$ is finite.

Theorem 3 depends on the work of Farrell and Jones [26], Bartels and Lück [6], and Kammeyer-Lück-Rüping [43] on the Farrell-Jones conjecture[3]. See the discussion following Corollary 2.8.

Theorem 4. There is a closed M such that $S^{CE}(M)$ is infinite.

This theorem is in sharp contrast with Ferry's theorem mentioned above. The resolution of this tension is that for any given ρ only a finite subset (no reason to believe it is a subgroup!) of S(M) occurs. By varying ρ we obtain this plenitude of deformations.

The fundamental group involved in Theorem 4 is linear, being one of the subgroups of right angled Artin groups, studied in [9] and [51]. The invariant that detects infinitely many homeomorphism types is based on a modification of the theory of higher rho invariants of [74].

The Borel conjecture is currently unresolved in its full generality, so the following corollary to our analysis is especially gratifying.

Theorem 5. If M^n, $n \geq 6$, is closed aspherical then $S^{CE}(M) = 0$.

[3] The Borel Conjecture is the torsion-free version of the Farrell-Jones conjecture. Originally, the Borel Conjecture proposed that closed aspherical manifolds with isomorphic fundamental groups should be homeomorphic. It has since been generalized to Farrell-Jones, which includes groups with torsion.

We now turn to the second source of motivation, which is the direction from which our proofs develop. The theorem of Chapman and Ferry mentioned earlier implies that the limit points of manifolds in Gromov-Hausdorff space which are limits of more than one topological type are not manifolds. In this case, it turns out that the limit points are infinite-dimensional homology manifolds with finite cohomological dimension. The possibility of infinite dimensional limit points in a precompact subset of Gromov-Hausdorff space was established by T. Moore in [57], based on work of the first author [18] and R. D. Edwards [72].

Definition.

(i) A compact subset X of an n-manifold M^n is said to be *cell-like* if for every open neighborhood U of X in M^n, the inclusion $X \to U$ is nullhomotopic. This is a topological property of X [48] and is the Čech analogue of "contractible". The space $\sin(1/x)$-with-a-bar is an example of a cell-like set which is not contractible.

(ii) A map $f : Y \to Z$ between compact metric spaces is *cell-like or CE* if for each $z \in Z$, $f^{-1}(z)$ is cell-like. The empty set is not considered to be cell-like, so cell-like maps must be surjective.

Cell-like maps with domain a compact manifold or finite polyhedron are weak homotopy equivalences over every open subset of the range [47, 49]. That is, if $c : M \to X$ is cell-like, then for every open $U \subset X$, $c|_{c^{-1}(U)} : c^{-1}(U) \to U$ is a weak homotopy equivalence. The Vietoris-Begle Theorem implies that the range space of such a cell-like map always has finite cohomological dimension. If the range has finite covering dimension, then c is a homotopy equivalence over every open set.

When X is infinite-dimensional, the range X need not have the homotopy type of a CW complex and f need not be a homotopy equivalence.

The first example of this sort was given by J. Taylor in [69], exploiting maps discovered by J. F. Adams [2] that go from an iterated suspension of a Moore space to the Moore space. These maps are zero 0 on reduced homology, yet induce an isomorphism on (nontrivial) complex K-theory.

Taylor's examples have infinite-dimensional domain and range. Cell-like maps with finite dimensional domain and infinite dimensional range were constructed by the first author in [18], using a result of Edwards [72]. The resulting cell-like images have paradoxical properties. If $f : M \to X$ is a CE map with infinite-dimensional range, the classical Vietoris-Begle Theorem shows that X nevertheless has finite cohomological dimension and that X

can contain no finite dimensional subsets of dimension exceeding the dimension of M. The map f induces isomorphisms on any connective homology theory [30] but need not induce isomorphisms on periodic K and L theories. We will have particular interest in cell-like maps $f : M^n \to X$ such that the induced map $f_\# : KO_n(M) \to KO_n(X)$ has kernel.

Section 2 reviews information about cell-like maps and classical surgery. This section also contains statements of our main results and sets the stage for the work to follow. It also contains detailed calculations for several classes of manifolds. Section 3 contains the details of the construction of useful cell-like maps from manifolds to compact metric spaces. The main theorems are proved in sections 4 and 5, using controlled surgery over these cell-like images. Finally in section 6 this is related to $LGC(\rho)$ subsets of Gromov-Hausdorff space as well other natural geometric questions (such as the existence of a topological injectivity function for deformations). The proof of this connection depends on our main theorem that constructs CE maps and, therefore, deformations. At the end, we discuss a modification of the higher ρ-invariants that contains enough torsion information to give the examples in Theorem 4.

Our work leaves open the following question:

Question 1.1. Can nonhomeomorphic Riemannian manifolds M and M' be deformed to each other in a precompact subset of Gromov-Hausdorff space, respecting a contractibility function as above, while maintaining an upper bound on volume? Greene and Petersen [35] have shown that this cannot happen in the presence of an upper bound on volume for certain contractibility functions.

2. Surgery and cell-like maps

We begin by formulating a useful lifting property of cell-like maps:

LIFTING PROPERTY: Let $f : M \to X$ be a cell-like map with M an absolute neighborhood retract.[4] Given a space W with $\dim W < \infty$, $\epsilon > 0$, a closed subset $A \subset W$, a map $g : W \to X$, and a map $h : A \to M$ with $f \circ h = g|_A$, there is a map $\bar{h} : W \to M$ extending h such that g is

[4]If M is compact metric with finite covering dimension, M is an absolute neighborhood retract $\equiv$ ANR if and only if it is locally contractible.

ϵ-homotopic to $f \circ \bar{h}$ rel A:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, hook] & M \arrow[d, "f"] \\ W \arrow[ur, "\bar{h}"] \arrow[r, "g"] & X. \end{tikzcd}$$

See [47, 49] for details. Note that the upper triangle is strictly commutative while the lower triangle is ϵ homotopy commutative.

Definition 2.1. A homotopy equivalence $f : N \to M$ between closed manifolds is *realized by cell-like maps* if there exist a space X and cell-like maps $c_1 : N \to X$, $c_2 : M \to X$ so that the diagram

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr, "CE"'] & & M \arrow[dl, "CE"] \\ & X & \end{tikzcd}$$

homotopy commutes. We will also say that f *factors through cell-like maps* and we will call closed manifolds N and M satisfying this property *CE-related*.

In view of the lifting property, every pair of cell-like maps $c_1 : N \to X$, $c_2 : M \to X$ induces a homotopy equivalence $f : N \to M$. Relative lifting implies that the induced homotopy equivalence is unique up to homotopy. If $\dim X < \infty$ and $n \geq 5$, Quinn's uniqueness of resolutions theorem asserts that this homotopy equivalence is homotopic to a homeomorphism. See [63], Prop 3.2.3. In case the range space X is infinite dimensional the uniqueness of resolutions need not hold and the induced map f need not be homotopic to a homeomorphism.

Two simple homotopy equivalences of manifolds $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ are called *equivalent* if there is a homeomorphism $h : N_1 \to N_2$ such that $f_2 \circ h$ is homotopic to f_1. We recall that the set $\mathcal{S}^s(M)$ of equivalence classes of simple homotopy equivalences $f : N \to M$ is called the set of *simple topological structures* on M. The structure set $\mathcal{S}^s(M)$ is functorial and has an abelian group structure defined either by Siebenmann periodicity [46] or by algebraic surgery theory [64]. Ranicki's theory gives the induced homomorphism formula for topological structures [65]:

Proposition 2.2. *Let M^n be a closed topological n-manifold, $n \geq 5$ and let $h : M \to N$ be a simple homotopy equivalence, $[h] \in \mathcal{S}^s(N)$. Then the isomorphism $h_* : \mathcal{S}^s(M) \to \mathcal{S}^s(N)$ is defined by the formula*

$$h_*([f]) = [h \circ f] - [h].$$

The structure set $\mathcal{S}^h$ is defined similarly, using homotopy equivalences and replacing the relation of homeomorphism by h-cobordism. The next proposition shows that the homotopy equivalences arising most naturally in this paper are simple. We will omit the decorations unless we explicitly wish to study $\mathcal{S}^h$. Similarly, $\mathbb{L}$ will be an abbreviation for $\mathbb{L}^s$.

Proposition 2.3. *A homotopy equivalence $f : M \to N$ that factors through cell-like maps is a simple homotopy equivalence.*

Proof. Let $p : M \to X$ and $q : N \to X$ be cell-like maps such that f is a homotopy lift of p with respect q. Theorem D of [29] states that there is a simple homotopy equivalence $g : M \to N$ such that p is homotopic to $q \circ g$. As noted above, f is homotopic to g. This implies the equality of the Whitehead torsions: $\tau(f) = \tau(g) = 0$. Hence f is a simple homotopy equivalence. □

We denote the subset of structures realized by cell-like maps by $\mathcal{S}^{CE}(M) \subset \mathcal{S}(M)$.

Theorem 2.4. *Let M^n be a closed simply connected topological n-manifold with finite $\pi_2(M)$, $n > 5$. Then $\mathcal{S}^{CE}(M)$ is the odd torsion subgroup of $\mathcal{S}(M)$.*

The proof of Theorem 2.4 follows Corollary 4.6.

Remark 2.5. On page 531 of [50], Lacher asks whether two closed manifolds that admit CE maps to the same space X must be homeomorphic. The theorem above shows that the answer to his question is "no" when X is allowed to be infinite-dimensional. See Corollary 2.15 below for an example.

We recall the Sullivan-Wall surgery exact sequence [70] for closed orientable high-dimensional topological manifolds:

$$(1) \qquad \cdots \longrightarrow L_{n+1}(\mathbb{Z}\pi_1(M)) \dashrightarrow \mathcal{S}(M) \xrightarrow{\eta} [M, \mathrm{G/TOP}] \xrightarrow{\theta} L_n(\mathbb{Z}\pi_1(M))$$

The map η is called the *normal invariant* and the homomorphism θ is called the *surgery obstruction*. The Sullivan-Wall surgery exact sequence was extended by Quinn and Ranicki[5] to the functorial exact sequence of abelian groups below:

$$\cdots \to L_{n+1}(\mathbb{Z}\pi_1(M)) \to \mathcal{S}_n(M) \xrightarrow{\eta'} H_n(M;\mathbb{L}) \xrightarrow{\theta'} L_n(\mathbb{Z}\pi_1(M)) \to \cdots \tag{2}$$

where $H_n(M;\mathbb{L}) = H^0(M;\mathbb{L}) = [M, \mathrm{G/TOP}\times\mathbb{Z}]$, $\mathcal{S}(M) \subset \mathcal{S}_n(M)$, and $\eta'|_{\mathcal{S}(M)} = \eta$. The homomorphism θ' is called the *assembly map* for M. This sequence is defined and functorial when M is a finite polyhedron. This was extended to arbitrary CW complexes in [76]. See also [74], page 89. We write $L_n = L_n(\mathbb{Z})$ and recall that $L_n = \mathbb{Z}$ if $n = 4k$, $L_n = \mathbb{Z}/2$ if $n = 4k+2$, and $L_n = 0$ for odd n.

In general, Ranicki's algebraic surgery functor gives us a long exact sequence

$$\cdots \to \mathcal{S}_n(P, Q) \to H_n(P, Q;\mathbb{L}) \to L_n(\mathbb{Z}\pi_1 P, \mathbb{Z}\pi_1 Q) \to \cdots$$

for any CW pair (P, Q). If P happens to be a compact n-dimensional manifold with nonempty boundary Q, then $\mathcal{S}_n(P)$ is the usual rel boundary structure set.[6]

In the closed connected case, $\mathcal{S}_n(P)$ differs from the usual geometrically defined structure set by at most a $\mathbb{Z}$. We also have a long exact sequence

$$\cdots \to \mathcal{S}_{n+1}(P, Q) \to \mathcal{S}_n(Q) \to \mathcal{S}_n(P) \to \mathcal{S}_n(P, Q) \to \cdots$$

where for an n-dimensional manifold with nonempty boundary $(P, \partial P)$, $\mathcal{S}_n(P, \partial P)$ is the *not* rel boundary structure set.

All of these sequences are 4-periodic. If $Q \to P$ induces an isomorphism on π_1, then $\mathcal{S}_k(P, Q) \cong H_k(P, Q;\mathbb{L})$ because the Wall groups $L_*(\mathbb{Z}\pi_1 P, \mathbb{Z}\pi_1 Q)$ are zero by Wall's $\pi - \pi$ Theorem. See Corollary 3.1.1 [70] . Composing this isomorphism with the boundary map in Ranicki's exact sequence, we

[5] Our notation differs from Ranicki's in that we've shifted the index on the structure set by one and omitted a bar over $\mathcal{S}$.

[6] In particular, our notation for a manifold with boundary $(M, \partial M)$ has $\mathcal{S}_n(M, \partial M)$ denoting structures on the pair $(M, \partial M)$ and does not indicate structures on M rel ∂M. Restriction gives a natural boundary map $\mathcal{S}_n(M, \partial M) \to \mathcal{S}_{n-1}(\partial M)$.

have a homomorphism $\partial' : H_{k+1}(P, Q; \mathbb{L}) \to \mathcal{S}_k(Q)$. For a closed connected n-manifold there is a split monomorphism

$$0 \longrightarrow \mathcal{S}(M) \xrightarrow{\ i\ } \mathcal{S}_n(M) \longrightarrow \mathbb{Z}\,. \tag{3}$$

To state the main theorem for closed connected non-simply connected manifolds we need the following.

Definition 2.6. If K is a CW complex, let $P_2(K)$ be the CW complex obtained from K by attaching cells in dimensions 4 and higher to kill the homotopy groups of K in dimensions 3 and above. Thus, $K \subset P_2(K)$, $\pi_i(P_2(K)) = 0$ for $i \geq 3$, and $P_2(K) - K$ consists of cells of dimension ≥ 4. Note that $P_2(K)$ will not, in general, be a finite complex. The space $P_2(K)$ is called *the second stage of the Postnikov tower of* K.

Let M be a closed n-manifold. We denote by

$$\delta : H_{n+1}(P_2(M), M; \mathbb{L}) \to \mathcal{S}(M)$$

the composition:

$$H_{n+1}(P_2(M), M; \mathbb{L}) \cong \mathcal{S}_{n+1}(P_2(M), M) \stackrel{\partial}{\to} \mathcal{S}_n(M) \stackrel{p}{\to} \mathcal{S}(M),$$

where p is any splitting of i.

Let $\phi : A \to B$ be a homomorphism of abelian groups. By $\phi^T : \mathrm{T}(A) \to \mathrm{T}(B)$ we denote the restriction $\phi|_{\mathrm{T}(A)}$ of ϕ to the torsion subgroups and by $\phi_{[q]} : A_{[q]} \to B_{[q]}$ we denote the localization of ϕ away from q.

Here is our main theorem for non-simply connected manifolds.

Theorem 2.7. *Let M^n be a closed topological n-manifold, $n > 5$. Then*

$$\mathcal{S}^{CE}(M) = \mathrm{im}(\delta^T_{[2]}).$$

In particular, $\mathcal{S}^{CE}(M)$ is a subgroup of the odd torsion of $\mathcal{S}(M)$.[7,8]

[7]Using results of [11] and [3], one sees that replacing $P_2(M)$ by $P_k(M)$, $k \geq 2$, would not not change $\mathrm{im}(\delta^T_{[2]})$.

[8]An important step in the proof of Theorem 2.7 consists of showing that if an odd torsion element $\alpha \in H_n(M; \mathbb{L})$ dies under the inclusion $M \to P_2(M)$, then there is a CE map $f : M \to X$ such that $f_*(\alpha) = 0$ in Steenrod $\mathbb{L}$-homology. Since 2-local $\mathbb{L}$-homology is Eilenberg-MacLane, see Remark 4.36 of [52] and Proposition 6.8, the Vietoris-Begle theorem says that 2-torsion cannot be killed by cell-like maps.

For the remainder of Section 2 we will derive consequences of Theorem 2.7, which is proven following Proposition 5.4. Since torsion elements of $\mathcal{S}_n(M)$ lie in the kernel of the map $\mathcal{S}_n(M) \to \mathbb{Z}$, ∂ maps $T(\mathcal{S}_{n+1}(P_2(M), M))$ into $T(\mathcal{S}(M)) \equiv T(\mathcal{S}_n(M))$, so $\mathrm{im}(\delta^T_{[2]})$ is independent of the choice of the splitting p. Since the study of $\mathcal{S}^{CE}(M)$ reduces to an analysis of odd torsion, we can invert 2 in most of our applications. This allows us to omit decorations on L-groups and structure sets.

Corollary 2.8. *If $L_{n+1}(\pi_1(M))$ has finitely generated odd torsion, then $\mathcal{S}^{CE}_n(M)$ is finite.*

Proof. Examination of the surgery exact sequence

$$H_{n+1}(M;\mathbb{L}) \longrightarrow L_{n+1}(\pi_1(M)) \longrightarrow \mathcal{S}_n(M) \longrightarrow H_n(M;\mathbb{L})$$

together with the observation that the $\mathbb{L}$-homology terms are finitely generated, shows that the odd torsion subgroup of $\mathcal{S}_n(M)$ is finite. □

This implies Theorem 3. The Farrell-Jones conjecture for $L(\Gamma) \otimes \mathbb{Z}[\frac{1}{2}]$ only makes use of the equivariant homotopy theory of $\underline{E}\Gamma$ = the classifying space for proper Γ-actions. For a lattice, $K\backslash G/\Gamma$ is finite (by the Borel-Serre compactification) and similarly the Rips complex is a suitable space when Γ is hyperbolic [58]. In these cases, the Farrell-Jones conjecture is affirmed (even integrally) in [43], [6] and [7].

Corollary 2.9. *Let $f_* : \mathcal{S}(M) \to \mathcal{S}(N)$ be the induced homomorphism for a continuous map $f : M \to N$ between two closed n-manifolds, $n > 5$. Then $f_*(\mathcal{S}^{CE}(M)) \subset \mathcal{S}^{CE}(N)$.*

Proof. We have a commuting diagram

$$\begin{tikzcd} \mathcal{S}_{n+1}(P_2(M), M) \arrow[r] \arrow[d, "f_*"] & \mathcal{S}(M) \arrow[d, "f_*"] \\ \mathcal{S}_{n+1}(P_2(N), N) \arrow[r] & \mathcal{S}(N) \end{tikzcd}$$

from which the result follows immediately. □

Corollary 2.10. *Let $n \geq 6$ and let $f : N \to M$ be a homotopy equivalence between closed n-manifolds that is realized by cell-like maps. Then f preserves rational Pontrjagin classes.*

Proof. This is Remark 1.7 of [28]. Rationally, the group of normal invariants is isomorphic to 4-periodic rational cohomology. Under this isomorphism, a normal invariant corresponds to the difference of the L-polynomials. Since the L-polynomials agree, the rational Pontrjagin classes also agree. □

Corollary 2.11. *Being CE-related is an equivalence relation on closed n-manifolds, $n > 5$.*

Proof. We prove transitivity. Let M_1 be CE-related to M_2 and M_2 CE-related to M_3. Let $h_1 : M_1 \to M_2$ and $h_2 : M_2 \to M_3$ be corresponding homotopy equivalences. It suffices to show that the composition $h_2 \circ h_1$ is realized by cell-like maps. In view of Corollary 2.9 we have $(h_2)_*([h_1]) \in \mathcal{S}^{CE}(M_3)$ and hence by the formula for the induced homomorphism (Proposition 2.2) we obtain that $[h_2 \circ h_1] = [h_2] + (h_2)_*([h_1]) \in \mathcal{S}^{CE}(M_3)$. □

We will refer to a manifold that admits a nontrivial deformation as being "malleable". Manifolds which are not malleable are "immutable". In special cases, it is not hard to understand the map $H_{n+1}(P_2(M), M; \mathbb{L}) \to \mathcal{S}(M)$ well enough to get concrete "immutability" and "malleability" results. We begin with two typical immutability statements:

Corollary 2.12. *If M^n is a closed manifold with $n \geq 6$ and either*

(i) M is aspherical, or
(ii) M is homotopy equivalent to a complex projective space, or
(iii) M is homotopy equivalent to a lens space,

then any homotopy equivalence $f : N \to M$ that factors through cell-like maps is homotopic to a homeomorphism.

Proof. If M is aspherical, then $M = P_2(M)$ and $H_{n+1}(P_2(M), M; \mathbb{L}) = 0$, so structures in the image of $H_{n+1}(P_2(M), M; \mathbb{L}) = 0$ are trivial.

If M is homotopy equivalent to $\mathbb{C}P^k$, then $P_2(M) = \mathbb{C}P^\infty$. But

$$H_{n+1}(\mathbb{C}P^\infty, \mathbb{C}P^k; \mathbb{L}) = \lim_{\ell \to \infty} H_{n+1}(\mathbb{C}P^\ell, \mathbb{C}P^k; \mathbb{L})$$

which has no odd torsion, so no nontrivial element of $\mathcal{S}(M)$ can be the image of an odd torsion element. See Lemma 2.13 below.

If M is homotopy equivalent to a $2k-1$-dimensional lens space, then $P_2(M)$ is an infinite dimensional lens space, constructed by attaching one cell in each dimension $2k$ and above to M. It's straightforward to write down the chain complex for $C_*(P_2(M), M)$ and compute the integral homology. A quick calculation using the Atiyah-Hirzebruch spectral sequence shows that $H_{2k}(P_2(M), M; \mathbb{L}) = \mathbb{Z}$, so M is immutable. $\square$

Lemma 2.13. *If (K, L) is a CW pair and $H_*(K, L; \mathbb{Z})$ has no odd torsion, then $H_*(K, L; \mathbb{L})$ has no odd torsion.*

Proof. For CW pairs, $H_n(K, L; \mathbb{L}) \otimes \mathbb{Q} \cong \bigoplus_k H_{n-4k}(K, L; \mathbb{Q})$. Comparing this to the Atiyah-Hirzebruch spectral sequence gives the result, since there can be no nonzero differentials between terms of the form $H_p(K, L; L_{4k})$ on the E_2-page. $\square$

Corollary 2.14. *It follows that all simply connected manifolds with finite π_2 and no odd torsion in homology are immutable in the sense of Corollary 2.12.*

Here is a simple example of malleability.

Corollary 2.15. *There are closed nonhomeomorphic 6-dimensional manifolds M and N which are CE-related.*

Proof. Let $p \geq 5$ be a prime number. By general position, the Moore complex $P = S^1 \cup_p B^2$ can be PL-embedded in $\mathbb{R}^6$. Suspending embeds $P' = S^2 \cup_p B^3$ into $\mathbb{R}^7$. Let W be a regular neighborhood of P' in $\mathbb{R}^7$ and let $\partial W = M$. The manifold M is stably parallelizable because it is a closed codimension one submanifold of euclidean space $\mathbb{R}^7$. Clearly, M is simply connected. By Lefschetz duality, $H_2(W, M) = H^5(W) = H^5(P') = 0$ and $H_3(W, M) = H^4(W) = H^4(P') = 0$. The exact sequence of the pair (W, M) implies $H_2(M) = \mathbb{Z}/p$.

By the Atiyah-Hirzebruch spectral sequence $H_2(M; \mathbb{L}) \cong \mathbb{Z}/p$. Choose a nontrivial p-torsion element $\alpha \in H_6(M; \mathbb{L}) \cong H_2(M; \mathbb{L})$.

The Sullivan-Wall and the Quinn-Ranicki exact sequences form a commutative diagram

$$\begin{tikzcd} & \mathcal{S}(M) \arrow[r, "\eta"] \arrow[d, "\subset"] & {[M; G/TOP]} \arrow[r] \arrow[d, "D"] & \mathbb{Z}/2 \arrow[d, "="] \\ 0 \arrow[r] & \mathcal{S}_6(M) \arrow[r, "\eta'"] & H_6(M, \mathbb{L}) \arrow[r] & \mathbb{Z}/2. \end{tikzcd}$$

Let $\beta = \bar{\eta}'^{-1}(\alpha)$. Since β is a torsion element, $\beta \in \mathcal{S}(M) \subset \mathcal{S}_6(M)$. Thus, by Theorem 2.4, β defines a homotopy equivalence $f : N \to M$ that belongs to $\mathcal{S}^{CE}(M)$. It remains to show that N is not homeomorphic to M.

We show that N has a nontrivial topological stable normal bundle. Since $D\eta(\beta) = \alpha \neq 0$, we have $\eta(\beta) = [\gamma] \neq 0$ for some map $\gamma : M \to G/TOP$. The class $[\gamma]$ represents the difference between topological stable normal bundles on M and N which are defined by two lifts $\nu_M : M \to BTOP$ and $\sigma : M \to BTOP$ of the Spivak map $M \to BG$ with respect to the fibration $p : BTOP \to BG$. Here ν_M denotes a classifying map for the topological stable normal bundle on M. Note that $\nu_N = \sigma \circ f$. Thus, the lifts ν_M and σ are not fiberwise homotopic. We need to show that ν_M and σ are not homotopic in $BTOP$.

Since the stable normal bundle of M is trivial, the map $\nu_M : M \to BTOP$ is nullhomotopic. Note that the map σ is homotopic to $i \circ \gamma$ where $i : G/TOP \to BTOP$ is the inclusion of the fiber into the total space of the fibration p. We recall that the groups $\pi_i(BG) = \pi_{i-1}(G) = \pi^s_{i-1}$ are 2 and 3 torsion for $i \leq 8$. The homotopy exact sequence of the fibration p implies that after inverting 2 and 3 the inclusion i is an 8-equivalence. Therefore, the map $i \circ \gamma$ is not nullhomotopic.

Thus, ν_N is not nullhomotopic, the topological stable normal bundle of N is nontrivial and, hence, N is not homeomorphic to M. $\square$

Thanks to Diarmuid Crowley for pointing out malleable examples in which both M and N are smooth 3-sphere bundles over S^4.

Proposition 2.16. *There are nonhomeomorphic S^3-bundles over S^4 which are equivalent under deformation.*

Proof. $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$, so 3-sphere bundles over S^4 are classified by pairs of integers (m, n) corresponding to elements $m\sigma + n\rho \in \pi_3(SO(4))$ with respect to generators σ, ρ introduced by James and Whitehead and described in [16]. If $M_{m,n}$ is the sphere bundle corresponding to $m\sigma + n\rho$, we have $H^4(M_{m,n}) \cong \mathbb{Z}/n$ and the only other nonvanishing cohomology groups are $H^0(M_{m,n}) \cong H^7(M_{m,n}) \cong \mathbb{Z}$.

The paper [16] gives a complete classification of these manifolds up to homotopy equivalence, homeomorphism, and diffeomorphism and includes a computation of normal invariants of homotopy equivalences between nonhomeomorphic manifolds, allowing a complete classification of these manifolds up to deformation. The classification is somewhat lengthy to write down, however, so we content ourselves with a simple example. There are homotopy equivalences $f_p : M_{0,p} \to M_{12,p}$ for all integers p. According to Proposition 2.1 of [16], the normal invariant of f_p is 1, which implies that $M_{0,p}$ deforms to $M_{12,p}$. In case $p \equiv 3 \mod 4$ is a prime, Corollary 1.3 of [16] shows that $M_{0,p}$ and $M_{12,p}$ are not homeomorphic.

We remind the reader that this means that there exist a contractibility function $\rho : [0, R) \to [0, \infty)$ and precompact families of Riemannian metrics $M_{0,p,t}$ and $M_{12,p,t}$ with $0 < t \leq 1$ such that ρ is a contractibility function for each of these metrics and such that $\lim_{t\to 0} d_{GH}(M_{0,p,t}, M_{12,p,t}) = 0$. As t approaches 1, $M_{0,p,t}$ and $M_{12,p,t}$ are homotopy equivalent with control going to zero. Crossing with $\mathbb{CP}^2$, gives examples with a topological injectivity function.[9] $\square$

Proposition 2.17. *For the M of Corollary 2.15 we have $\mathcal{S}^{CE}(M) \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$.*

Proof. $H_7(M; \mathbb{L}_\bullet) \cong [M; \mathrm{G/TOP}]$ and $\bar{H}_7(M; \mathbb{L}) \cong [M, *; \mathrm{G/TOP} \times \mathbb{Z}, *]$, so

$$\mathcal{S}(M) = \mathcal{S}^{CE}(M) = H_7(M; \mathbb{L}_\bullet) \cong \bar{H}_7(M; \mathbb{L}) \cong \mathbb{Z}/p \oplus \mathbb{Z}/p. \qquad \square$$

Crossing with a sphere produces further examples of malleability.

Corollary 2.18. *(i.) If $f : M' \to M$ is a simple-homotopy equivalence between closed n-manifolds with odd order normal invariant in $H_n(M; \mathbb{L})$, then $\mathrm{id}_{S^k} \times f : S^k \times M' \to S^k \times M$ factors through cell-like maps, $k \geq 3$, $n + k \geq 7$.*

(ii.) If $f : M' \to M$ is a homotopy equivalence, not necessarily simple, with odd order normal invariant in $H_n(M; \mathbb{L})$ and k is even, then $\mathrm{id}_{S^k} \times f$ is h-cobordant to a map that factors through cell-like maps, $k \geq 3$, $n + k \geq 7$.

(iii.) If $f : M' \to M$ is a homotopy equivalence, not necessarily simple, with odd order normal invariant in $H_n(M; \mathbb{L})$ and k is odd, then $\mathrm{id}_{S^k} \times f : S^k \times M' \to S^k \times M$ factors through cell-like maps, $k \geq 3$, $n + k \geq 7$.

Proof. (i) If $f : M' \to M$ is a simple-homotopy equivalence, then the normal invariant of $[f]$ is a homotopy class of maps $\eta(f) : M \to \mathrm{G/TOP}$. The normal invariants of $\mathrm{id}_{S^k} \times f$ and $\mathrm{id}_{B^{k+1}} \times f$ are the composition of projection onto M with $\eta(f)$, so the normal invariants of $\mathrm{id}_{S^k} \times f$ and $\mathrm{id}_{B^{k+1}} \times f$ have odd

[9] See Definition 6.1.

order. Pushing forward to $\mathbb{L}$ and dualizing, we have a diagram

$$\begin{tikzcd}
H_{n+k+1}(B^{k+1} \times M, S^k \times M; \mathbb{L}) \arrow[r, "\cong"] & \mathcal{S}^s_{n+k+1}(B^{k+1} \times M, S^k \times M) \arrow[r, "\partial"] \arrow[d] & \mathcal{S}^s_{n+k}(S^k \times M) \\
& \mathcal{S}^s_{n+k+1}(P_2(B^{k+1} \times M), S^k \times M) \arrow[ur, "\partial"] & \\
H_{n+k+1}(P_2(S^k \times M), S^k \times M; \mathbb{L}) \arrow[r, "\cong"] & \mathcal{S}^s_{n+k+1}(P_2(S^k \times M), S^k \times M) \arrow[u, "\cong"'] \arrow[uur, dashed, "\partial"'] &
\end{tikzcd}$$

where both horizontal isomorphisms come from the $\pi - \pi$ theorem and we have used the inclusion-induced homotopy equivalences $P_2(M) \cong P_2(S^k \times M) \cong P_2(B^{k+1} \times M)$ when $k \geq 3$. This shows that $[\mathrm{id}_{S^k} \times f]$ is in the image of $H_{n+k+1}(P_2(S^k \times M), S^k \times M; \mathbb{L})$ via the composition of the dashed arrow with the bottom horizontal arrow and that it comes from an odd order element, namely, the image of $\eta(\mathrm{id}_{B^{k+1}} \times f)$ in $\mathcal{S}_{n+k+1}(P_2(B^{k+1} \times M), S^k \times M)$.

(ii) Consider the diagram above with $\mathcal{S}^s$ replaced by $\mathcal{S}^h$. The homotopy equivalence $f : M' \to M$ satisfies a symmetry $\tau(f) = (-1)^{n-1}\tau(f)^*$ which can be seen in the PL case by computing torsions using triangulations and dual triangulations. This shows that $2\tau(f) = (-1)^{n-1}(\tau(f)^* + (-1)^{n-1}\tau(f))$. Torsions of the form $\tau + (-1)^{n-1}\tau$ can be varied away by including into an h-cobordism of torsion τ and retracting to the other end. After crossing with S^k, $\tau(\mathrm{id}_{S^k} \times f) = 2\tau(f)$ so $[\mathrm{id}_{S^k} \times f]$ therefore lies in the image of $\mathcal{S}^s_{n+k}(S^k \times M)$ in $\mathcal{S}^h_{n+k}(S^k \times M)$, which is to say that $\mathrm{id}_{S^k} \times f$ is h-cobordant to a simple homotopy equivalence. The result follows as in case (i).

(iii) The product formula for Whitehead torsion implies that $\tau(\mathrm{id}_{S^k} \times f) = 0$ and argument in (i) applies as above. □

We recall that by definition a fake lens space of order p is the orbit space of a free action of $\mathbb{Z}/p$ on a sphere. Since simple homotopy equivalent lens spaces are diffeomorphic, the actions giving rise to the fake lens spaces L' below are topologically nonlinear. Explicit constructions of fake lens spaces as quotients of Brieskorn spheres are studied in [59].

Corollary 2.19. *There exist a 5-dimensional lens space L and a fake lens space L' such that $L' \times S^3$ and $L \times S^3$ are CE-related and $L' \times S^3$ and $L \times S^3$ are not homeomorphic.*

Proof. Let L be the lens space $L_{11}(1,1,3)$ in the notation of [53], p. 403. The first Pontrjagin class of this manifold is zero. Thus, $H_i(pt;\mathbb{L}_\bullet) = L_i$ for $i > 0$ and $H_i(pt;\mathbb{L}_\bullet) = 0$ for $i \leq 0$. In the Atiyah-Hirzebruch spectral sequence for the lens space L the term $E^2_{1,4} = H_1(L; H_4(pt;\mathbb{L}_\bullet)) = H_1(L) = \mathbb{Z}/11$ survives to $E^\infty_{1,4}$ and hence to $H_5(L;\mathbb{L}_\bullet)$. By Theorem 10.1 of [40], $L_5(\mathbb{Z}[\mathbb{Z}/11]) = 0$, so this homology class comes from the structure set $\mathcal{S}(L)$. Thus there is a simple homotopy equivalence $f : L' \to L$ with nontrivial normal invariant of order 11. It follows from Proposition 2.18 that $L \times S^3$ and $L' \times S^3$ are CE-related. Theorem 6.3 will show that they deform to each other.

By Corollary 2.18 the simple homotopy equivalence $f \times 1_{S^3} : L' \times S^3 \to L \times S^3$ also has nontrivial normal invariant of order 11, so by the argument of Corollary 2.15 the manifolds $L \times S^3$ and $L' \times S^3$ cannot be homeomorphic, since the first Pontrjagin class is topologically invariant and zero for L and nonzero for L'. $\square$

Remark 2.20. According to [53], $L_{11}(1,1,4)$ and $L_{11}(1,6,4)$ are homotopy equivalent. The first Pontrjagin class of $L_{11}(1,1,4)$ is zero and the first Pontrjagin class of $L_{11}(1,6,4)$ is three, so they are not homeomorphic and, therefore, not simple-homotopy equivalent. The normal invariant of the homotopy equivalence has odd order, so crossing with S^3 produces nonhomeomorphic lens spaces crossed spheres that deform to each other.

In contrast, if L and L' are as in Corollary 2.19, then the induced homotopy equivalence between $L' \times S^2$ and $L \times S^2$ is not realized by a deformation. $P_2(L \times S^2) = L^\infty \times \mathbb{CP}^\infty$, where L^∞ is an infinite lens space. The Künneth Theorem for ordinary homology together with the Atiyah-Hirzebruch spectral sequence shows that the normal invariant of f does not go to zero in $H_7(P_2(L \times S^2);\mathbb{L})$, so the structure $[f]$ cannot lie in the image of $H_8(P_2(L \times S^2), L \times S^2;\mathbb{L})$.

Manifolds with finite fundamental group.

We continue to derive further consequences of the as yet unproven Theorem 2.7.

Proposition 2.21. *Let M be a manifold with finite fundamental group such that $\pi_2(M)$ is finite. Then for $n > 5$,*

$$S^{CE}(M^n) \cong \mathrm{OddTorsion}(\mathrm{Ker}(KO_n(M) \to KO_n(K(\pi_1 M, 1)))).$$

Proof. The homotopy fiber of $P_2(M) \to K(\pi_1 M, 1) = B\pi$ is $K(\pi_2(M), 2)$. By [77], Theorem 2, $P_2(M) \to B\pi$ induces an isomorphism on $\mathbb{L} \wedge M(p)$

homology, so we can use $K(\pi_1 M, 1)$ and $P_2(M)$ interchangeably in our calculations. Also, $\mathcal{S}(M) \cong \mathcal{S}(M, *)$.

To begin, we have a commuting diagram of surgery exact sequences below:

$$\begin{tikzcd}
 & & & H_{n+1}(P_2(M), *; \mathbb{L}) \arrow[d] & \\
 & & \mathcal{S}_{n+1}(P_2(M), M) \arrow[r, "\cong"] \arrow[d] & H_{n+1}(P_2(M), M; \mathbb{L}) \arrow[d] & \\
H_{n+1}(M, *; \mathbb{L}) \arrow[r] \arrow[d] & L_{n+1}(\pi_1 M, e) \arrow[r] \arrow[d] & \mathcal{S}_n(M, *) \arrow[r] \arrow[d] & H_n(M, *; \mathbb{L}) \arrow[r] \arrow[d] & L_n(\pi_1 M, e) \arrow[d] \\
H_{n+1}(P_2(M), *; \mathbb{L}) \arrow[r] & L_{n+1}(\pi_1 M, e) \arrow[r] & \mathcal{S}_n(P_2(M), *) \arrow[r] & H_n(P_2(M), *; \mathbb{L}) \arrow[r] & L_n(\pi_1 M, e)
\end{tikzcd}$$

By the $\pi - \pi$ theorem, the top horizontal arrow is an isomorphism. Since we are interested in odd primary behavior, we can invert 2, which replaces $\mathbb{L}$-homology by KO-homology and $P_2(M)$ by $B\pi$, where $\pi = \pi_1(M)$. By a transfer argument, see [1], the reduced KO-homology of $B\pi$ is torsion, so the map $L_{n+1} \to \mathcal{S}_n(B\pi, *)$ is a rational isomorphism. This gives us the diagram below at odd primes:

$$\begin{tikzcd}
 & & & KO_{n+1}(B\pi, *) \arrow[d] & \\
 & & \mathcal{S}_{n+1}(B\pi, M) \arrow[r, "\cong"] \arrow[d] & KO_{n+1}(B\pi, M) \arrow[d, "\cong\otimes\mathbb{Q}"] \arrow[dl, "\delta"'] & \\
KO_{n+1}(M, *) \arrow[r] \arrow[d] & L_{n+1}(\pi, e) \arrow[r] \arrow[d, "\cong"] & \mathcal{S}_n(M, *) \arrow[r] \arrow[d] & KO_n(M, *) \arrow[r] \arrow[d] & L_n(\pi, e) \arrow[d] \\
KO_{n+1}(B\pi, *) \arrow[r] & L_{n+1}(\pi, e) \arrow[r, "\cong\otimes\mathbb{Q}"] & \mathcal{S}_n(B\pi, *) \arrow[r] & KO_n(B\pi, *) \arrow[r] & L_n(\pi, e).
\end{tikzcd}$$

In [71], page 2, Wall shows that for π finite, $L_n(\pi)$ is the direct sum of a free abelian group and a 2-torsion group. We draw two conclusions:

1. Conclusion 1: Let α be an odd torsion element in the kernel of $\mathcal{S}_n(M, *) \to KO_n(M, *)$. α is the image of $\alpha' \in L_{n+1}(\pi, e)$, which is impossible since α' must have infinite order and also go to zero in $\mathcal{S}_n(B\pi, *)$. Thus, $\mathcal{S}_n(M, *) \to KO_n(M, *)$ is a monomorphism on odd torsion.
2. Conclusion 2: Suppose that β is an odd torsion element in the kernel of $KO_n(M, *) \to KO_n(B\pi, *)$. β is the image of $\beta' \in KO_{n+1}(B\pi, M)$

and must be odd torsion since $KO_{n+1}(B\pi, M) \to KO_n(M, *)$ is a rational isomorphism.

This shows that $\mathcal{S}^{CE}(M, *)$ maps isomorphically onto the odd torsion in the kernel of $KO_n(M, *) \to KO_n(B\pi, *)$, completing the proof of the proposition. □

Remark 2.22. A related result holds for manifolds with abelian fundamental group $\pi = \mathbb{Z}^k \oplus A$. Splitting off infinite cyclic factors using Shaneson's thesis shows that the Wall groups of finitely generated abelian groups are sums of free abelian groups and finite 2-groups. $B\pi = T^k \times BA$ and an easy spectral sequence argument shows that the groups $KO_*(T^k, B\pi)$ are torsion, where $B\pi \to T^k$ is the projection. The result is a diagram (see below) with the same formal properties as the second diagram in the proof of Proposition 2.21. The short exact sequence

$$\to \mathcal{S}_{*+1}(T^k, M) \to \mathcal{S}_*(M) \to \mathcal{S}_*(T^k) \to$$

shows that $\mathcal{S}_{*+1}(T^k, M) \cong \mathcal{S}_*(M)$. Comparing the long exact KO-homology sequences of (T^k, M) and $(T^k, B\pi)$ shows that the odd KO-homology in the kernel $KO_{n+1}(T^k, M) \to KO_{n+1}(T^k, B\pi)$ is isomorphic to $\mathcal{S}^{CE}(M)$.

$$\begin{tikzcd}
 & & & KO_{n+1}(T^k, B\pi) \arrow[d] & \\
 & & \mathcal{S}_{n+1}(B\pi, M) \arrow[r, "\cong"] \arrow[d] & KO_{n+1}(B\pi, M) \arrow[d] \arrow[dl, "\delta"'] & \\
KO_{n+2}(T^k, M) \arrow[r] \arrow[d] & L_{n+2}(\mathbb{Z}^k, \pi) \arrow[r] \arrow[d, "\cong"] & \mathcal{S}_{n+1}(T^k, M) \arrow[r] \arrow[d] & KO_{n+1}(T^k, M) \arrow[r] \arrow[d] & L_n(\mathbb{Z}^k, \pi) \arrow[d] \\
KO_{n+1}(T^k, B\pi) \arrow[r] & L_{n+1}(\mathbb{Z}^k, \pi) \arrow[r, "\cong\otimes\mathbb{Q}"] & \mathcal{S}_n(T^k, B\pi) \arrow[r] & KO_n(T^k, B\pi) \arrow[r] & L_n(\mathbb{Z}^k, \pi)
\end{tikzcd}$$

□

Proposition 2.23. *Let M^n be a closed manifold, $n \geq 7$, $\pi_1(M) = \pi$ and with $\pi_2(M)$ finite. If π has split injective assembly map away from 2, then $\mathcal{S}^{CE}(M)$ is isomorphic to the odd torsion subgroup of $H_{n+1}(B\pi, M; \mathbb{L})$.*

Proof. Consider the diagram below (away from 2).

$$\begin{tikzcd}
 & & \mathcal{S}_{n+1}(B\pi, M) \arrow[r, "\cong"] \arrow[d] & H_{n+1}(B\pi, M; \mathbb{L}) \arrow[d] \arrow[dl, "\delta"'] \\
H_{n+1}(M; \mathbb{L}) \arrow[r] \arrow[d] & L_{n+1}(\pi) \arrow[r, "w"] \arrow[d, "id"'] & \mathcal{S}(M) \arrow[r] \arrow[d, "q"'] & H_n(M; \mathbb{L}) \arrow[d] \\
H_{n+1}(B\pi; \mathbb{L}) \arrow[r, shift right, "A"'] & L_{n+1}(\pi) \arrow[l, shift right, "j"'] \arrow[r, shift right] & \mathcal{S}_n(B\pi) \arrow[l, shift right, "s"'] \arrow[r, "0"] & H_n(B\pi; \mathbb{L})
\end{tikzcd}$$

The assembly map A is a split monomorphism, so there are splittings j and s, as shown. $w \circ s$ is a splitting of q, so δ maps $H_{n+1}(B\pi, M; \mathbb{L})$ isomorphically onto a direct summand of $\mathcal{S}(M)$. The result now follows from Theorem 2.7. A great many torsion-free groups satisfy this version of the Novikov Conjecture. See [41, 67]. □

Remark 2.24. One can unify some of the calculations we have given in this section when the C^*-algebra assembly map is known to be split injective and one consequently has "a refined normal invariant" $\mathcal{S}(M) \to KO^{\pi}_{n+1}(\underline{\mathrm{E}}\pi, \widetilde{M})[1/2]$ analogous to the map in the preceding proposition given by the projection $\mathcal{S}(M) \to H_{n+1}(B\pi, M; \mathbb{L})$ (under an $\mathbb{L}$-theory integral Novikov hypothesis), e.g. groups that admit uniform embeddings in Hilbert space, see [67]. In that case, $\mathcal{S}^{CE}(M)$ is isomorphic to the the image of the odd torsion of $KO_{n+1}(B\pi, M) \cong KO^{\pi}_{n+1}(E\pi, \widetilde{M})$ in $KO^{\pi}_{n+1}(\underline{\mathrm{E}}\pi, \widetilde{M})[1/2]$.

Spherical space forms.

We now give a proof of immutability valid for all spherical space forms. We begin by recalling that Conclusion 1 in the proof of Proposition 2.21 said the map $\mathcal{S}(M) \to H_n(M; \mathbb{L})$ is a monomorphism on odd torsion, so $\mathcal{S}^{CE}(M) \to H_n(M; \mathbb{L})$ is a monomorphism.

As above, for any X with free action of a group G with p-Sylow subgroup G_p, the map $X/G_p \to X/G$ is split surjective in any p-local homology theory, with a splitting induced by the transfer. In particular, the transfer $\tau : KO_n(X/G) \to KO_n(X/G_p)$ is split injective on p-torsion.

Now, let G be a finite group acting freely on S^n with quotient $M = S^n/G$ and let p be an odd prime. By Thm. 11.6 of [15], the p-Sylow subgroup G_p of G must be cyclic so, as Wall observes, $L = S^n/G_p$ has the homotopy type of a linear lens space. There is a transfer $\tau : \mathcal{S}^{CE}_n(M) \to \mathcal{S}^{CE}_n(L)$ described as follows: A CE map $M \to X$ induces an isomorphism on π_1 which induces a bijection between covering spaces of M and covering spaces of X. Let X_L be the covering space of X corresponding to $L \to M$. X is locally n-connected for all n, so covering space theory gives us a pullback diagram

$$\begin{tikzcd}
N_L \arrow[r] \arrow[d, "h.e."] & N \arrow[d, "h.e."] \\
L \arrow[r] \arrow[d, "CE"] & M \arrow[d, "CE"] \\
X_L \arrow[r] & X
\end{tikzcd}$$

and the vertical map on the left is CE because the pullback of a CE map is CE. The homotopy equivalence $N_L \to L$ is controlled over X_L because the tracks of the homotopies are the lifts of the tracks of the homotopies over X.

By Wall, [70], Chapter 14, the structure group of an odd lens space is torsion free. See also [74], pp 110-111. This implies that $S^{CE}(L)$ is trivial. The diagram below then shows that the p-torsion in $\mathcal{S}^{CE}(M)$ must be trivial.

$$\begin{tikzcd}
\mathcal{S}_n^{CE}(L) \arrow[r] & \mathcal{S}_n(L) \arrow[r] & H_n(L;\mathbb{L}) \\
\mathcal{S}_n^{CE}(M) \arrow[u, "\tau"] \arrow[r, "1-1"] \arrow[rr, bend right, "1-1"'] & \mathcal{S}_n(M) \arrow[u, "\tau"] \arrow[r] & H_n(M;\mathbb{L}) \arrow[u, "\tau", "1-1"']
\end{tikzcd}$$

Repeating for each odd p, it follows that $S^{CE}(M)$ is trivial.

Proposition 2.25 (Proof of Theorem 4)**.** *$\mathcal{S}^{CE}(M)$ can be infinite.*

Proof. Let $M(\mathbb{Z}/p, n)$ be a Moore space, p an odd prime. Triangulate $M(\mathbb{Z}/p, n)$ as a flag complex L and form the Bestvina-Brady group $\pi = H_L$ as in [9], [51]. For $n \geq 3$, $B\pi$ has free, finitely generated homology through dimension n and its homology in dimension $n+1$ contains an infinite sum of $\mathbb{Z}/p$'s. See Corollaries 8 and 9 in [51]. It also follows that $H_{n+2}(\pi)$ is finitely generated free abelian and all higher homology is zero, whence it follows immediately from the Atiyah-Hirzebruch spectral sequence that $H_n(\pi;\mathbb{L})$ contains an infinite sum of $\mathbb{Z}/p$'s. By periodicity, the same is true for $H_{n-4k}(\pi;\mathbb{L})$ for any k.

Let K be the 3-skeleton of $B\pi$, which is finite for $n \geq 3$. Embed K in $\mathbb{R}^{m+1}$, $m \geq 8$, and let M^m be the boundary of a regular neighborhood. Consider the diagram

$$\begin{tikzcd} \mathcal{S}_{m+1}(B\pi, M) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_{m+1}(B\pi, M; \mathbb{L}) \arrow[dl, "\delta"] \\ H_m(M; \mathbb{L}) & \end{tikzcd}$$

By the Borel Conjecture for Bestvina-Brady groups [6], the vertical arrow on the left is an isomorphism, so δ is an isomorphism. (Actually for the purposes of this argument, the much easier integral Novikov conjecture would suffice, see [14, 22, 42, 5]). Since $\pi_2(M) = 0$, we have $B\pi = P_2(M)$, so Theorem 2.7 tells us that $\mathcal{S}^{CE}(M)$ is the image of the odd torsion under the map δ. Let $m + 1 = n - 4k$, $k \geq 1$. By our construction, $H_{m+1}(B\pi; \mathbb{L})$ contains infinitely generated p torsion. Since $H_{m+1}(M; \mathbb{L})$ is finitely generated, $H_{m+1}(B\pi, M; \mathbb{L})$ contains infinitely generated p torsion and $\mathcal{S}(M)$ contains infinitely generated p torsion in the image of δ. $\square$

Each structure $[\alpha]$ above is represented by a homotopy equivalence $\alpha : M_\alpha \to M$.

Proposition 2.26. *There are infinitely many nonhomeomorphic manifolds $M_\alpha \in \mathcal{S}^{CE}(M)$ when $m \geq 8$.*

Proof. If $m \geq 8$, the manifold M has a handle decomposition with no handles in the middle dimension. As in [75], this allows us to define an absolute "higher ρ invariant" for our manifolds M_α in a quotient group of $L_{m+1}(\pi)$. Since the construction in [75] was rational and we are interested in torsion phenomena, we will review the construction.

Let N^m be a closed, oriented m-manifold such that the $\mathbb{Z}\pi_1 N$-chain complex $C_*(N)$ is chain-homotopy equivalent to chain complex of finitely generated projective $\mathbb{Z}\pi_1 N$-modules $\{P_i\}$ with $P_i = 0$, $i = [m/2]$. Following Hausmann, we call such manifolds *anti-simple*. Let $P_*^{<i}$ be the truncation of P_*. There is a chain retraction $P_* \to P_*^{<i}$ and $(P_*^{<i}, P_*)$ is a symmetric algebraic Poincaré pair. This is well-defined in that if Q_* is a chain complex of finitely generated projective modules chain-homotopy equivalent to P_* with $Q_i = 0$, then there is a chain-homotopy equivalence of pairs $(P_*^{<i}, P_*) \sim (Q_*^{<i}, Q_*)$. If the manifold N is the boundary of an oriented manifold W^{m+1} with a map to $B\pi_1 N$ extending $N \to B\pi_1 N$, the pair (W, N) gives us another symmetric algebraic Poincaré pair over $\mathbb{Z}\pi_1 N$ and we can paste the two together along

P_* to get a closed $(m+1)$-dimensional symmetric algebraic chain complex and, therefore, an element of $L^{m+1}(\pi_1(N))$. Two such coboundaries of N define an element ω of $\Omega_{m+1}(B\pi_1 N)$, so our element of $L^{m+1}(\pi_1(N))$ is well-defined up to the image of the map $\Omega_{m+1}(B\pi_1 N) \to L^{m+1}(\pi_1(N))$ that sends each element to its symmetric signature. The resulting element in $L^{m+1}(\pi_1(N))/\Omega_{m+1}(B\pi_1 N)$ is the higher ρ invariant of N.

Remark 2.27. One can define the higher ρ invariant without assuming explicitly that M^n bounds if one inverts the torsion present in Witt bordism in dimension n of $B\pi$. (This follows from the argument in [74].) Thus, for the Bestvina-Brady groups used here, in low dimensions one need not invert anything and one has a more general higher ρ invariant available to distinguish homotopy equivalent anti-simple manifolds with Bestvina-Brady fundamental groups.

Returning to our manifold M, let P_* be the $\mathbb{Z}\pi$-chain complex obtained by gluing together two copies of a handle decomposition of a regular neighborhood of K in $\mathbb{R}^m$. We have $P_i = 0$ for $3 < i < m-3$. Since the manifolds M_α are homotopy equivalent to M, they are also anti-simple. Since they are obtained from M by Wall realization, they are cobordant to M, so they bound and have higher ρ invariants. By construction, the higher ρ invariant of M_α differs from the higher ρ invariant of M by the image of α'' in $L^{m+1}(\pi)/\Omega_{m+1}(B\pi)$. Since $B\pi$ has finite $m+1$-skeleton, the Atiyah-Hirzebruch spectral sequence shows that $\Omega_{m+1}(B\pi)$ is finitely generated. Since the collection of α's in $L^{m+1}(\pi)$ is infinitely generated, there are infinitely many nonhomeomorphic M_α's. □

Next we show that, by itself, infinite odd torsion in the L-group does not suffice to produce infinitely many deformable manifolds.

Proposition 2.28. *There is a closed manifold M^n such that $L_{n+1}(\pi)$ has infinitely generated odd torsion but $\mathcal{S}^{CE}(M) = 0$.*

Proof. Let A be the universal finitely presented acyclic group of [8] and let Σ be a homology sphere with fundamental group A. Since $H_1(A) = H_2(A) = 0$, such a homology sphere exists in dimensions ≥ 5 by a well-known theorem of Kervaire [44]. The surgery exact sequence for Σ is

$$H_{n+1}(\Sigma;\mathbb{L}) \longrightarrow L_{n+1}(A) \longrightarrow \mathcal{S}(\Sigma) \longrightarrow H_n(\Sigma;\mathbb{L})$$

Inspection of this sequence gives us

$$\tilde{L}_{n+1}(A) \cong \mathcal{S}(\Sigma),$$

where $\tilde{L}_{n+1}(A) = L_{n+1}(A)/L_{n+1}(e)$. Now consider the commutative diagram of topological surgery exact sequences

$$\begin{tikzcd}
 & 0 \arrow[r] & \tilde{L}_{n+1}(A) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathcal{S}(\Sigma) \arrow[r] \arrow[d] & 0 \\
0 = \bar{H}_{n+1}(BA;\mathbb{L}) \arrow[rr] & & \tilde{L}_{n+1}(A) \arrow[r] & \mathcal{S}(BA) &
\end{tikzcd}$$

If an element of $\mathcal{S}(\Sigma)$ goes to 0 in $\mathcal{S}(BA)$, then it comes from an element of $\tilde{L}_{n+1}(A)$ which maps to a nonzero element of $\mathcal{S}(BA)$, yielding a contradiction. Since elements of $\mathcal{S}^{CE}(\Sigma)$ must die in $\mathcal{S}(BA)$, $\mathcal{S}^{CE}(\Sigma) = 0$. If $L_{n+1}(A)$ contains infinitely generated odd torsion, we are done. Otherwise, let π be a finitely presented group such that $L_{n+1}(\pi)$ has infinitely generated odd torsion. Note that in view of the Borel conjecture for Bestvina-Brady groups, the group π from Proposition 2.25 is such for an appropriate choice of n. We consider the amalgamated free product $\Gamma = A *_\pi (\pi \times \mathbb{Z}/2)$. Γ is $\mathbb{Z}[1/2]$-acyclic. Using the isomorphism (away from 2)[10]

$$L(\pi \times \mathbb{Z}/2) \cong L(\pi) \times L(\pi)$$

and Cappell's Mayer-Vietoris sequence [12], we have

$$0 \longrightarrow \tilde{L}_n(A) \oplus \tilde{L}_n(\pi) \longrightarrow \tilde{L}_n(\Gamma) \longrightarrow 0$$

after inverting 2. This shows that Γ has infinite odd torsion in L-theory with $H_1(\Gamma) = \mathbb{Z}/2$. Suspending once to $A *_\Gamma A$ kills H_2, so a $\mathbb{Z}[1/2]$ version of Kervaire's theorem produces a $\mathbb{Z}[1/2]$-homology sphere with fundamental group $A *_\Gamma A$. Cappells's theorem mod finitely generated odd torsion shows that the L-theory of $L_{n-1}(A *_\Gamma A)$ contains infinite odd torsion, and we can complete the argument as above. $\square$

3. Cell-like maps that kill $\mathbb{L}$-classes

Singular homology behaves badly for non-ANRs. This is illustrated in [4], where it is shown that an infinite compact wedge of S^2's has uncountable singular rational homology in infinitely many dimensions. Since we will be

[10]By [66], Prop. 4.4, $L_n(\mathbb{Z}\pi) \to L_n(\mathbb{Q}\pi)$ is an isomorphism modulo 8 torsion. Inverting 2, we have $L_n(\mathbb{Z}[\pi \times \mathbb{Z}/2]) \cong L_n(\mathbb{Q}[\pi \times \mathbb{Z}/2]) \cong L_n(\mathbb{Q}[\pi] \times \mathbb{Q}[\pi]) \cong L_n(\mathbb{Q}[\pi]) \times L_n(\mathbb{Q}[\pi]) \cong L_n(\mathbb{Z}[\pi]) \times L_n(\mathbb{Z}[\pi])$

dealing with compact non-ANR spaces, we use the Steenrod extension of a generalized homology theory h_*, [54], [27], [13], [24], which satisfies the usual Eilenberg-Steenrod axioms for (generalized) homology theories, together with the union axiom. For the Steenrod homology defined by a homology theory h_* we use the same notation h_* since they agree on CW complex pairs. As every generalized homology theory h_* has reduced and nonreduced versions, the same holds true for the Steenrod homology. For a nonreduced theory h_* we denote by $\bar{h}_*$ its reduced version.

Though the reduced homology $\bar{h}_*$ is defined for single spaces, still one can define it for pairs by setting $\bar{h}_*(X, A) = \bar{h}_*(X/A) = h_*(X, A)$. A reduced Steenrod homology theory is determined by two axioms:

1. (Exactness) Given any compact metrizable pair (B, A), there is a long exact sequence

$$\ldots \to \bar{h}_i(A) \to \bar{h}_i(B) \to \bar{h}_i(B/A) \to \bar{h}_{i-1}(A) \to \ldots$$

2. (Milnor's Additivity Axiom [55]) Given a countable collection X_i of pointed compact metric spaces and letting $\bigvee X_i \subset \prod X_i$ be the null wedge, we have an isomorphism

$$\bar{h}_*(\bigvee X_i) \cong \prod \bar{h}_*(X_i).$$

We emphasize that h is homotopy invariant. Every homology theory has a unique Steenrod extension satisfying these two axioms.

We use the notation $KO_*(X) = H_*(X; KO)$ for periodic KO-homology and we use $\overline{KO}_*$ to stand for reduced KO homology. We need the following facts [3], [11], [77].

Theorem 3.1. *If $p > 1$ is an integer and $n \geq 3$, $\overline{KO}_*(K(\pi, n); \mathbb{Z}/p) = 0$ for any group π. If π is torsion, $\overline{KO}_*(K(\pi, n)) = 0$, for $n = 2$.*

Let $M(p)$ denote the $\mathbb{Z}/p$ Moore spectrum. For odd p, we have a chain of homotopy equivalences of spectra

$$\overline{KO}_* \wedge M(p) \sim \overline{KO}_*[\frac{1}{2}] \wedge M(p) \sim \mathbb{L}[\frac{1}{2}] \wedge M(p) \sim \mathbb{L} \wedge M(p).$$

This implies the following:

Corollary 3.2. *Let p be odd, then for any finite group $\bar{H}_*(K(\pi, 2); \mathbb{L} \wedge M(p)) = 0$ where $\mathbb{L} \wedge M(p)$ is $\mathbb{L}$-theory with coefficients in $\mathbb{Z}/p$.*

We recall that for an extraordinary homology theory given by a spectrum $\mathbb{E}$ of CW complexes there are Universal Coefficient Formulas for coefficients $\mathbb{Z}/p$ and $\mathbb{Q}$:

(4)

$$0 \to H_n(K;\mathbb{E}) \otimes \mathbb{Z}/p \to H_n(K;\mathbb{E} \wedge M(p)) \to \mathrm{Tor}(H_{n-1}(K;\mathbb{E}), \mathbb{Z}/p) \to 0$$

and

$$H_n(K;\mathbb{E}_{(0)}) = H_n(K;\mathbb{E}) \otimes \mathbb{Q}.$$

Here $\mathrm{Tor}(H, \mathbb{Z}/p) = \{c \in H \mid pc = 0\}$ and $\mathbb{E}_{(0)}$ denotes the localization at 0. Every compact metric space X can be written as an inverse limit $X = \varprojlim\{K_i\}$ of finite polyhedra and any two such sequences are pro-equivalent. By $\check{H}_*(X;\mathbb{E}) = \varprojlim\{H_*(K_i,\mathbb{E})\}$ we denote the Čech $\mathbb{E}$-homology. The Steenrod homology [54], [27], [13], [24] $H_n(X;\mathbb{E})$ of X fits into the following exact sequence

$$0 \to \lim{}^1\{H_{n+1}(K_i;\mathbb{E})\} \to H_n(X;\mathbb{E}) \to \check{H}_n(X;\mathbb{E}) \to 0.$$

If $H_k(pt;\mathbb{E})$ is finitely generated for each k, the Mittag-Leffler condition holds with rational or finite coefficients, so we have

$$H_n(X;\mathbb{E} \wedge M(p)) = \check{H}_n(X;\mathbb{E} \wedge M(p)) \quad \text{and} \quad H_n(X;\mathbb{E}_{(0)}) = \check{H}_n(X;\mathbb{E}_{(0)}).$$

In the case of $\mathbb{Z}/p$-coefficients we obtain an exact sequence which is natural in X:

(5)

$$0 \to \varprojlim(H_n(K_i;\mathbb{E}) \otimes \mathbb{Z}/p) \to H_n(X;\mathbb{E} \wedge M(p)) \xrightarrow{\phi'} \mathrm{Tor}(\check{H}_{n-1}(X;\mathbb{E}), \mathbb{Z}/p).$$

Lemma 3.3. *Let M be a simply connected finite complex with finite $\pi_2(M)$. Then for every element $\gamma \in H_k(M;\mathbb{L})$ of odd order p there exists an odd torsion element $\alpha \in H_{k+1}(P_2(M), M;\mathbb{L})$ such that $\partial(\alpha) = \gamma$ where ∂ is the connecting homomorphism in the exact sequence of the pair $(P_2(M), M)$.*

Proof. Note that $P_2(M) = K(\pi_2(M), 2)$.

If $\pi_2(M) = 0$, the space $P_2(M)$ is contractible and the lemma is trivial.

If $\pi_2(M)$ is torsion, then by Corollary 3.2, $\bar{H}_*(P_2(M);\mathbb{L} \wedge M(p)) = 0$. Then by the Universal Coefficient diagram

(6)

$$\begin{tikzcd}
H_{k+2}(P_2(M),M;\mathbb{L}\wedge M(p)) \arrow[r, "epi"] \arrow[d, "\partial", "iso"'] & \mathrm{Tor}(H_{k+1}(P_2(M),M;\mathbb{L}),\,\mathbb{Z}/p) \arrow[r, "mono"] & H_{k+1}(P_2(M),M;\mathbb{L}) \arrow[d, "\partial"] \\
H_{k+1}(M;\mathbb{L}\wedge M(p)) \arrow[r, "epi"] & \mathrm{Tor}(H_k(M;\mathbb{L}),\,\mathbb{Z}/p) \arrow[r, "mono"] & H_k(M;\mathbb{L})
\end{tikzcd}$$

we obtain the required result. □

The following proposition is proven in [72] Appendix B.

Proposition 3.4. *Let E be a CW complex with trivial homotopy groups $\pi_i(E) = 0$, $i \geq k$ for some k, and let $q : X \to Y$ be a cell-like map between compacta. Then q induces a bijection of the homotopy classes $q^* : [Y, E] \to [X, E]$.*

Remark 3.5. If E does not satisfy this condition, then the conclusion does not hold, despite the map being an ordinary homology isomorphism with arbitrary coefficient systems. Indeed this remarkable possibility is precisely the basis of the examples given in [18] of infinite dimensional compacta with finite cohomological dimension. What follows is the systematic exploitation of this.

Let $q : M \to X$ be a cell-like map. According to Proposition 3.4 for every map $h : M \to P_2(M)$ there is a map $g : X \to P_2(M)$ such that $g \circ q$ is homotopic to h. In particular, there is an induced map $\tilde{g} : M_q \to M_h$ between their mapping cylinders, $\tilde{g}|_M = id_M$, $\tilde{g}|_X = g$. We apply this when h is the inclusion $j : M \subset P_2(M)$ and denote the induced map by $i : M_q \to M_j$. Denote by

$$i_* : H_*(M_q, M; \mathbb{L}) \to H_*(P_2(M), M; \mathbb{L})$$

the induced homomorphism for Steenrod $\mathbb{L}$-homology groups [27], [45].

Proposition 3.6. *Let M^n be a closed connected topological n-manifold, $n \geq 6$, let p be odd, and let $\beta \in H_*(P_2(M), M; \mathbb{L} \wedge M(p))$, then there exist a cell-like map $q : M \to X$ and an element $\widehat{\beta} \in H_*(M_q, M; \mathbb{L} \wedge M(p))$ such that $i_*(\widehat{\beta}) = \beta$.*

The proof of Proposition 3.6 will follow Lemma 3.12.

Proposition 3.7. *Let M^n be a closed connected topological n-manifold, $n \geq 6$. If $\alpha \in H_*(P_2(M), M; \mathbb{L})$ is an odd torsion element, then there exist a cell-like map $q : M \to X$ and an odd order element $\widehat{\alpha} \in H_*(M_q, M; \mathbb{L})$ such that $i_*(\widehat{\alpha}) = \alpha$.*

Proof. Let $\alpha \in H_k(P_2(M), M; \mathbb{L})$ be an element of order p where p is odd. Then by the universal coefficient formula, there is an epimorphism

$$\phi : H_{k+1}(P_2(M), M; \mathbb{L} \wedge M(p)) \to \mathrm{Tor}(H_k(P_2(M), M; \mathbb{L}), \mathbb{Z}/p).$$

Note that $\mathrm{Tor}(H, \mathbb{Z}/p) = \{c \in H \mid pc = 0\}$ so that there is an inclusion $\mathrm{Tor}(H, \mathbb{Z}/p) \subset H$ which is natural in H. Thus, $\alpha \in \mathrm{Tor}(H_k(P_2(M), M; \mathbb{L}), \mathbb{Z}/p)$. Hence, there is an element $\beta \in H_{k+1}(P_2(M), M; \mathbb{L} \wedge M(p))$ such that $\phi(\beta) = \alpha$. By Proposition 3.6 there exist a cell-like map $q : M \to X$ and an element $\widehat{\beta}$ such that $i_*(\widehat{\beta}) = \beta$. The commuting diagram of universal coefficient formulas gives (see Equation (4)):

$$\begin{tikzcd}
H_{k+1}(M_q, M; \mathbb{L} \wedge M(p)) \arrow[r, "\phi'"] \arrow[d, "i_*"] & \mathrm{Tor}(H_k(M_q, M; \mathbb{L}), \mathbb{Z}/p) \arrow[r, "\subset"] \arrow[d] & H_k(M_q, M; \mathbb{L}) \arrow[d, "i_*"] \\
H_{k+1}(P_2(M), M; \mathbb{L} \wedge M(p)) \arrow[r, "\phi"] & \mathrm{Tor}(H_k(P_2(M), M; \mathbb{L}), \mathbb{Z}/p) \arrow[r, "\subset"] & H_k(P_2(M), M; \mathbb{L})
\end{tikzcd}$$

which implies that $i_*(\hat{\alpha}) = \alpha$ where $\widehat{\alpha} = \phi'(\widehat{\beta})$ is an element of order p. □

Remark 3.8. By Proposition 3.4 a cell-like map induces a rational isomorphism on $\mathbb{L}$-homology. Therefore, $H_*(M_q, M; \mathbb{L})$ is a torsion group.

Theorem 3.9. *Let M^n be a closed simply connected topological n-manifold, $n \geq 6$, with $\pi_2(M)$ finite. Then for every odd torsion element $\gamma \in H_*(M; \mathbb{L})$ there exist X and a cell-like map $q : M \to X$ such that $q_*(\gamma) = 0$.*

Proof. By Lemma 3.3 there is an odd torsion element $\alpha \in H_*(P_2(M), M; \mathbb{L})$ such that $\partial(\alpha) = \gamma$. By Proposition 3.7 there exists a cell-like map $q : M \to X$ and an element $\widehat{\alpha} \in H_*(M_q, M; \mathbb{L})$ such that $i_*(\widehat{\alpha}) = \alpha$. Then the commutative diagram

$$\begin{tikzcd}
H_{*+1}(M_q, M; \mathbb{L}) \arrow[r] \arrow[d, "i_*"] & H_*(M; \mathbb{L}) \arrow[r, "q_*"] \arrow[d, "="] & H_*(X; \mathbb{L}) \arrow[d] \\
H_{*+1}(P_2(M), M; \mathbb{L}) \arrow[r] & H_*(M; \mathbb{L}) \arrow[r] & H_*(P_2(M); \mathbb{L})
\end{tikzcd}$$

implies that $q_*(\gamma) = 0$. □

Remark 3.10. Without the finiteness assumption on $\pi_2(M)$ one can show that q kills an element $\gamma \otimes 1_{\mathbb{Z}/p}$ with $\mathbb{Z}/p$ coefficients.

We recall that the *cohomological dimension* of a topological space X with respect to the coefficient group G is the following number:

$$\dim_G X = \max\{n \mid \check{H}^n(X, A; G) \neq 0 \text{ for some closed } A \subset X\}.$$

We recall that the existence of cell-like maps of manifolds that raise dimension to infinity follows from the following two theorems. First we apply Theorem D by the first author that produces examples of infinite dimensional compact metric spaces with finite integral cohomological dimension [18]:

Theorem D. *Let h_* be a generalized homology theory. Suppose that $\bar{h}_*(K(\mathbb{Z}, n)) = 0$. Then for any finite polyhedron P and a nonzero element $\alpha \in \bar{h}_*(P)$ there is a compact metric space Y and a map $f : Y \to P$ such that*

(i) $\dim_{\mathbb{Z}} Y \leq n$;
(ii) $\alpha \in \operatorname{im}\{f_* : \bar{h}_*(Y) \to \bar{h}_*(P)\}$.

Since Y is not a CW complex, $\bar{h}(Y)$ above is Steenrod homology. Theorem D when applied to $P = S^m$ with $m > n$ produces Y with $\dim_{\mathbb{Z}} Y \leq n$ and $\dim Y > n$. Alexandroff's theorem [18], [72] about coincidence of cohomological and covering dimensions of compacta when the latter is finite implies that $\dim Y = \infty$.

Then we apply the following Edwards Resolution Theorem [25] [72]:

Theorem E. *For any compact metric space Y with $\dim_{\mathbb{Z}} Y = n$ there is a n-dimensional compact metric space X and a cell-like map $f : X \to Y$.*

Any embedding of the above compactum $X \subset M$ in a manifold M defines a cell-like map $p : M \to Z$ extending f where p restricted to $M - X$ is one-to-one. We note that in the formulation of Theorem D in [18] the cohomology h^* were used instead of homology with the corresponding conclusion (ii) $f^*(\alpha) \neq 0$. The same proof works for homology.

We recall that proofs of both theorems deal with inverse sequences of finite polyhedra. In the proof of Theorem D we construct Y as the limit of an inverse sequence of polyhedra $\{P_i, g_i^{i+1}\}$ with $P_1 = P$ where a polyhedron

P_{i+1} is constructed by a certain modification of P_i. In the proof of Theorem E we present Y as the limit of an inverse sequence of polyhedra $\{K_i, p_i^{i+1}\}$ with mesh of triangulations tending to zero when $i \to \infty$. Then we form an inverse sequence $\{K_i^{(n)}; q_i^{i+1}\}$ of the n-skeletons where bonding maps are cellular approximations of the restrictions p_i^{i+1} to $K_i^{(n)}$. It was well known in 40s-50s that the limit space X is n-dimensional and it admits a natural UV^{n-1}-map $f : X \to Y$. The condition $\dim_{\mathbb{Z}} Y = n$ allows us to improve the map f to a cell-like map.

Relative versions of Theorems D and E were established by adjustments of proofs of Theorems D and E. A map of pairs $f : (X, L) \to (Y, L)$ is called *strict* if $f(X - L) = Y - L$ and $f|_L = id_L$.

The following theorem is taken from [20] (Theorem 7.2).

Theorem D1. *Let h_* be a generalized homology theory. Suppose that $\bar{h}_*(K(G, n))$ is 0. Then for any finite polyhedral pair (K, L) and any element $\alpha \in h_*(K, L)$ there is a compactum $Y \supset L$ and a strict map $f : (Y, L) \to (K, L)$ such that*

(i) $\dim_G(Y - L) \leq n$;
(ii) $\alpha \in \mathrm{im}\{f_* : h_*(Y, L) \to h_*(K, L)\}$.

For $G = \mathbb{Z}$ this is Theorem 3.3′ in [21].

Theorem E1. (Theorem 8.6 [20]) *For any pair of compact metric spaces (Y, Z) with $\dim_{\mathbb{Z}}(Y - Z) = n$ there is a compact metric space X containing Z with $\dim(X - Z) = n$ and a strict cell-like map $f : (X, Z) \to (Y, Z)$.*

Theorems D1 and E1 are proven by the same techniques as Theorems D and E. In the proof of Theorem D1 we consider a triangulation of $K - L$ with mesh of simplexes tending to zero when they are approaching to L and apply the construction from the proof of Theorem D. In the proof of Theorem E1 we consider a presentation of (Y, Z) as the limit of an inverse sequence $\{(K_i, Z)\}$ such that for each i the space $K_i - Z$ is a simplicial complex with mesh of simplexes tending to zero when they are approaching to Z and the bonding maps $p_i^{i+1} : (K_{i+1}, Z) \to (K_i, Z)$ restricted to Z being the identity. Simplicial approximations of the restriction of p_i^{i+1} to the n-skeleta $K_i^{(n)}$ define the maps $q_i^{i+1} : (K_{i+1} \cup Z, Z) \to (K_i \cup Z, Z)$. Then $X = \lim_{\leftarrow}\{K_i \cup Z, q_i^{i+1}\}$. A cell-like map $f : X \to Y$ is defined by the same reasoning as in the proof of Theorem E.

Lemma 3.11. *Let (Z, M) be a compact pair such that $\dim(Z - M) \leq n$ and let M be a manifold of dimension $2n+1$. Suppose there is a retraction $\rho : Z \to M$. Then ρ is homotopic $\operatorname{rel} M$ to a retraction $r : Z \to M$ with $r|_{(Z-M)}$ one-to-one.*

Proof. The condition $\dim X \leq n$ for a compact metric space X implies that every continuous map $\phi : X \to M$ to a $2n+1$-dimensional manifold can be approximated by an embedding. Moreover, the space of embeddings $\mathrm{Emb}(X, M)$ is a dense G_δ in the space of mappings $\mathrm{Map}(X, M)$. The same argument shows that under the condition $\dim(Z - M) \leq n$ the space of retraction-embeddings

$$\mathrm{RetEmb}(Z, M) = \{f : Z \to M \mid f|_M = id_M, f|_{Z-M} \text{ is one-to one}\}$$

is dense in the space of all retractions $\mathrm{Ret}(Z, M)$. □

The following lemma is a reformulation of Lemma 3.7 from [21].

Lemma 3.12. *Let Z be a compact and $r : Z \to M$ be a retraction with $r|_{(Z-M)}$ one-to-one. Let $g : (Z, M) \to (Y, M)$ be a cell-like map which is the identity over M. Then there is a cell-like map $q : M \to X$ and a map $r' : Y \to X$ such that the diagram*

$$\begin{array}{ccc} Z & \xrightarrow{r} & M \\ \downarrow g & & \downarrow q \\ Y & \xrightarrow{r'} & X \end{array}$$

commutes. □

Proof of Proposition 3.6. First we consider the case $\dim M \geq 7$. We consider the generalized homology theory $h_* = \mathbb{L} \wedge M(p)$, i.e., $\mathbb{L}$-theory with coefficients in $\mathbb{Z}/p$. Let $\beta \in h_{k+1}(P_2(M), M)$. There is a finite complex K, $M \subset K \subset P_2(M)$, and an element $\gamma \in h_*(K, M)$ such that γ is taken to β by the inclusion homomorphism.

By Theorem 3.1, $\bar{h}_*(K(\mathbb{Z}, 3)) = 0$. Then we apply Theorem D1 to (K, M) and γ to obtain a strict map $f : (Y, M) \to (K, M)$ satisfying the conditions $\dim_{\mathbb{Z}}(Y - M) \leq 3$ and $\gamma \in \mathrm{im}\{f_* : h(Y, M) \to h(K, M)\}$. Then we apply Theorem E1 to obtain a cell-like map $g : (Z, M) \to (Y, M)$ with $\dim(Z - M) \leq 3$.

Because $P_2(M) - M$ has no cells of dimension ≤ 3, there is a homotopy of $f \circ g$ rel M that sweeps $Z - M$ to M. Thus, $f \circ g$ is homotopic to a retraction

$\rho : Z \to M$. Since $\dim M \geq 7$, by Lemma 3.11, $f \circ g$ is homotopic rel M to a retraction $r : Z \to M$ which is one-to one on $Z - M$. By Lemma 3.12 there is a cell-like map $q : M \to X$ and a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "r"] \arrow[d, "g"] & M \arrow[d, "q"] \\ Y \arrow[r, "r'"] & X. \end{tikzcd}$$

By Proposition 3.4 there is a map $g' : X \to P_2(M)$ such that $g' \circ q$ is homotopic to the inclusion $M \subset P_2(M)$. Hence $f \circ g \sim r \sim g' \circ q \circ r = g' \circ r' \circ g$. Since g is cell-like, the map f is homotopic to $g' \circ r'$ by Proposition 3.4. Then there is a homotopy commutative diagram of the mapping cylinders

$$\begin{tikzcd} M_{j'} \arrow[r, "r'"] \arrow[d, "f"] & M_q \arrow[dl, "i"] \\ M_j & \end{tikzcd}$$

where $j : M \to P_2(M)$ and $j' : M \to Y$ are the embeddings. For Steenrod h_*-homology this gives us the following diagram:

$$\begin{tikzcd} h_*(Y, M) \arrow[r, "="] \arrow[d, "f_*"] & h_*(M_{j'}, M) \arrow[r, "r'_*"] \arrow[d] & h_*(M_q, M) \arrow[dl, "i_*"] \\ h_*(P_2(M), M) \arrow[r] & h_*(M_j, M) & \end{tikzcd}$$

By condition (ii) of Theorem D1 there is $\gamma' \in h_*(Y, M)$ such that $f_*(\gamma') = \gamma$. Then $i_*(\widehat{\beta}) = \beta$ where $\widehat{\beta} = r'_*(\gamma')$. $\square$

The rest of this section is devoted to the case of $n = 6$. To cover this case we extend Theorem E1 to the following.

Theorem E2. *Let (Y, L) be a pair of compacta such that*

$$\dim_{\mathbb{Z}_p}(Y - L) \leq 2 \quad \textit{and} \quad \dim_{\mathbb{Z}[\frac{1}{p}]}(Y - L) \leq 2.$$

Then there is a strict cell-like map $g : (Z, L) \to (Y, L)$ such that

$$\dim(Z - L) \leq 3, \quad \dim_{\mathbb{Z}/p}(Z - L) \leq 2, \text{ and } \dim_{\mathbb{Z}[\frac{1}{p}]}(Z - L) \leq 2.$$

We note that these conditions imply that $\dim(Z - L)^2 \leq 5$ [20].

Theorem E2 is a relative version of Theorem 3 from [19]. Its proof is a modification of the proof of Theorem 3 [19] which is performed in the same lines as the proof of Theorem E1 is obtained as a modification of the proof of Theorem E.

Lemma 3.11 can be accordingly modified by using results of [23], [68] about approximations by embeddings of maps $f : X \to M^{2n}$ of a compact metric space X with $\dim X^2 \leq 2n-1$ to a $2n$-dimensional manifold:

Lemma 3.13. *Let (Z, M) be a compact pair such that $\dim(Z-M)^2 \leq 2n-1$ and let M be a manifold of dimension $2m$. Suppose there is a retraction $\rho : Z \to M$. Then ρ is homotopic $\operatorname{rel} M$ to a retraction $r : Z \to M$ with $r|_{(Z-M)}$ one-to-one.*

Proof of Proposition 3.6, the case of dimension 6. In view of Corollary 3.2, $\bar{h}_*(K(G,2)) = 0$ for $h_* = \mathbb{L} \wedge M(p)$ and $G = \mathbb{Z}/p \oplus \mathbb{Z}[\frac{1}{p}]$. By Theorem D1 there is a strict map $f : (Y, M) \to (K, M)$ satisfying the conditions $\dim_G(Y - M) \leq 3$ and $\gamma \in \operatorname{im}\{f_* : h_*(Y, M) \to h_*(K, M)\}$. Then we apply Theorem E2 to obtain a cell-like map $g : (Z, M) \to (Y, M)$ with $\dim(Z - M) \leq 3$ and $\dim(Z - M)^2 \leq 5$. Since $P_2(M) - M$ has no cells of dimension ≤ 3, there is a homotopy of $f \circ g$ rel M that sweeps $Z - M$ to M. Thus, $f \circ g$ is homotopic to a retraction $\rho : Z \to M$. Since $\dim M = 6$, by Lemma 3.13, $f \circ g$ is homotopic rel M to a retraction $r : Z \to M$ which is one-to one on $Z - M$. The rest of the proof is the same as for $n > 6$.

4. Continuously controlled topology and cell-like maps of simply connected manifolds

We recall that a map of pairs $f : (Z, Y) \to (Z', Y)$ is strict if $(Z - Y) \subset Z' - Y$ and $f|_Y = id_Y$. A proper homotopy $f_t : Z \to Z'$ which is strict at each level is called *strict* if the homotopy $f_t : (Z, Y) \to (Z', Y)$ is continuous.

Let X be a locally compact space compactified to $\bar{X}$ by a compact *corona* $Y = \bar{X} - X$. A proper map $f : Z \to X$ is a *strict homotopy equivalence* if there is a proper map $g : X \to Z$ such that $g \circ f$ and $f \circ g$ are strictly homotopic to $id_{\bar{Z}}$ and $id_{\bar{X}}$ respectively where Z is given a compactification as above.

Definition 4.1.

(i) Let X be an open manifold and let Y be the compact corona of a compactification $\bar{X}$ of X. Two strict homotopy equivalences $f : W \to$

X and $f' : W' \to X$ are *equivalent* if there is a homeomorphism $h : W \to W'$ such that $f = f' \circ h$.

(ii) The set of the equivalence classes of strict homotopy equivalences of manifolds is called the set of *continuously controlled structures* on X at Y and it is denoted by $\mathcal{S}^{cc}(\bar{X}, Y)$.

We note that if $\tilde{X}$ is another compactification of X with compact corona Y' such that there is a continuous strict map $\phi : \bar{X} \to \tilde{X}$ which is the identity on X, then there is a map $\phi_* : \mathcal{S}^{cc}(\bar{X}, Y) \to \mathcal{S}^{cc}(\tilde{X}, Y')$.

Definition 4.2. A pair (X, Y) is said to be *locally 1-connected at* Y if for each $y \in Y$ and neighborhood U of y in X there is a smaller neighborhood V of y in X so that the inclusion-induced map $\pi_1(V - Y) \to \pi_1(U - Y)$ is zero.

Proposition 4.3. *Let X be a simply connected open manifold of dimension $n \geq 5$ compactified by a compact corona Y in such a way that the pair $(\bar{X}, Y)$ is locally 1-connected. Then there is a surgery exact sequence*

$$\cdots \to \bar{H}_n(Y; \mathbb{L}) \to \mathcal{S}^{cc}(\bar{X}, Y) \to [X, \mathrm{G/TOP}] \to \bar{H}_{n-1}(Y; \mathbb{L})$$

which is natural with respect to maps between coronas, as above. Here $H_*(-; \mathbb{L})$ is Steenrod $\mathbb{L}$-homology.

Proof. This sequence can be obtained by adjusting the bounded surgery theory of [32] to the continuously controlled case. It is presented on p. 313 of [60] in a form where the homology terms are Ranicki-Wall L-groups of the continuously controlled additive category $\mathcal{B}(\bar{X}, Y; \mathbb{Z})$. Theorem 2.4 of [60] states that these terms are in fact the Steenrod $\mathbb{L}$-homology groups of the corona.[11]

The naturality follows from the definition of the continuously controlled category. □

Definition 4.4. A subset X of a manifold M has *property* UV^1 if for every neighborhood U of X there is a neighborhood V of X contained in U so that $\pi_1(V) \to \pi_1(U)$ is trivial. A map $f : M \to Z$ is said to be UV^1 if each point-inverse $f^{-1}(z)$ is nonempty and UV^1 in M. See section 2 of [50] for details.

[11] As explained in [60], the proof is axiomatic and the axioms given on p. 315 of [60] are the usual ones that we listed in the previous section.

Let M be a closed simply connected n-manifold and let $q : M \to Y$ be a UV^1-map. Then the mapping cone C_q is a compactification of $M \times \mathbb{R}$ by $Y_+ = Y \sqcup pt$ which is locally 1-connected at Y_+. Since $(C_q - Y_+)$ is homotopy equivalent to M and $\bar{H}_*(Y_+; \mathbb{L}) = H_*(Y; \mathbb{L})$, the controlled surgery exact sequence becomes the following

$$\cdots \to H_{n+1}(Y; \mathbb{L}) \to \mathcal{S}^{cc}(C_q, Y_+) \to H_n(M; \mathbb{L}_\bullet) \to H_n(Y; \mathbb{L})$$

where $\mathbb{L}_\bullet$ is the connected cover of the spectrum $\mathbb{L}$. Note that $(\mathbb{L}_\bullet)_0 = \mathrm{G/TOP}$ and by Poincaré duality over $\mathbb{L}_\bullet$ [64], $[M, \mathrm{G/TOP}] = H^0(M, \mathbb{L}_\bullet) = H_n(M, \mathbb{L}_\bullet)$. Thus the n^{th} homotopy group $\mathcal{S}^{cc}_n(C_q, Y_+)$ of the fiber of the controlled assembly map of spectra $\mathbb{H}_*(M; \mathbb{L}) \to \mathbb{H}_*(Y; \mathbb{L})$ differs from $\mathcal{S}^{cc}(C_q, Y_+)$ by at most a copy of $\mathbb{Z}$.

Proposition 4.5. *Let M be a simply connected n-manifold, $n \geq 5$, and let $q : M \to X$ be a UV^1 map and let M_q be its mapping cylinder. Then there is a commutative diagram:*

$$\begin{tikzcd}
\mathcal{S}^{cc}(C_q, X_+) \arrow[r, "split-mono"] \arrow[d, "forget"] & \mathcal{S}^{cc}_{n+1}(C_q, X_+) \arrow[r, "\eta|"] & H_{n+1}(M_q, M; \mathbb{L}) \arrow[d, "\bar{\partial}"] \\
\mathcal{S}(M) \arrow[r, "split-mono"] & \mathcal{S}_n(M) \arrow[r, "\bar{\eta}"] & \bar{H}_n(M; \mathbb{L})
\end{tikzcd}$$

where $\eta|$ and $\bar{\eta}$ are isomorphisms.

Proof. We have two vertical fibration sequences of spectra on the right, leading to the diagram below at level n:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
 & \mathcal{S}^{cc}(C_q, X_+) \arrow[r] \arrow[d] & \mathcal{S}^{cc}_{n+1}(C_q, X_+) \arrow[r, "\eta|"] \arrow[d, "\eta"] & H_{n+1}(M_q, M; \mathbb{L}) \arrow[d, "\bar{\partial}"] \\
H_n(M; \mathbb{L}_\bullet) \cong & {[M \times (0,1), \mathrm{G/TOP}]} \arrow[r] \arrow[d, "q_*"] & H_n(M; \mathbb{L}) \arrow[r, "\cong"] \arrow[d, "q_*"] & H_n(M; \mathbb{L}) \arrow[d, "q_*"] \\
 & H_n(X; \mathbb{L}) \arrow[r, "\cong"] & H_n(X; \mathbb{L}) \arrow[r, "\cong"] \arrow[d] & H_n(X; \mathbb{L}) \arrow[d] \\
 & & {} & {}
\end{tikzcd}$$

from which we see that $\eta|$ is an equivalence. Using the exact sequence

$$0 \longrightarrow H_n(M;\mathbb{L}_\bullet) \longrightarrow H_n(M;\mathbb{L}) \longrightarrow H_n(M;\mathbb{L}/\mathbb{L}_\bullet) \cong \mathbb{Z}$$

one chases the diagram above to show first that the composition

$$\eta' : \mathcal{S}^{cc}(C_q, X_+) \to H_{n+1}(M_q, M;\mathbb{L})$$

is a monomorphism and then that its cokernel is a subgroup of $\mathbb{Z}$. From this $H_n(M;\mathbb{L}/\mathbb{L}_\bullet) \cong \mathbb{Z}$ follows from the observation that $H_n(M;\mathbb{L}/\mathbb{L}_\bullet)$ is rationally isomorphic to $\mathbb{Q}$ and that the only nonzero term in degree n on the E_2 page of the Atiyah-Hirzebruch spectral sequence computing $H_n(M;\mathbb{L}/\mathbb{L}_\bullet)$ is isomorphic to $\mathbb{Z}$.

If $[g] \in \mathcal{S}^{cc}(C_q, X_+)$ is a structure, then there is a manifold N compactified by X_+ so that $g : N \to M \times (0, 1)$ extends over X_+ by the identity and such that this extended map is a strict homotopy equivalence rel X_+.

If $q : M \to X$ is cell-like, then X is locally k-connected for all k, so there is a retraction from a neighborhood of X_+ in the compactification of N to X_+. The proof of the existence of mapping cylinder neighborhoods in [63] now shows that X_+ has a mapping cylinder neighborhood in the compactification of N. This splits N as $N' \times (0, 1)$ and gives a homotopy equivalence $N' \to M$. The thin h-cobordism theorem guarantees that this gives a well-defined forgetful map from $\mathcal{S}^{cc}(C_q, X_+) \to \mathcal{S}(M)$. The mapping cylinder projection provides a cell-like map $q_N : N \to X$.

We put $\bar{H}_n(M;\mathbb{L})$, rather than $H_n(M;\mathbb{L})$ in the lower right corner of this diagram because both the vertical and horizontal maps have images in $\bar{H}_n(M;\mathbb{L})$ and, as observed in Proposition 4.5, $\bar{\eta}$ is an isomorphism. □

The argument above gives us an important corollary.

Corollary 4.6. *Let $q : M \to X$ be a cell-like map of a simply connected closed manifold M. Then*

(1) $\mathcal{S}^{cc}(C_q, X_+)$ is generated by strict maps $f : (C_p, X) \to (C_q, X)$ where $p : N \to X$ is a cell-like map.
(2) The forget control map takes f to a homotopy equivalence $h : N \to M$ which factors through the cell-like maps q and p. □

Proof of Theorem 2.4. ($\mathrm{T}^{odd}(\mathcal{S}(M)) \subset \mathcal{S}^{CE}(M)$.)
Let α be an odd torsion element of $\mathcal{S}(M)$. Let $\gamma = \eta(\alpha) \in \bar{H}_n(M;\mathbb{L})$. By Theorem 3.9 there is a cell-like map $q : M \to X$ such that $q_*(\gamma) = 0$. Consider the diagram of Proposition 4.5. By Lemma 3.7 there is a torsion element $\widehat{\gamma} \in H_{n+1}(M_q, M;\mathbb{L})$ such that $\bar{\partial}(\widehat{\gamma}) = \gamma$. Let $\alpha' = \eta'^{-1}(\widehat{\gamma})$. Since α is the image of α' under the forgetful map, by Proposition 4.6 we have $\alpha \in \mathcal{S}^{CE}(M)$.

($\mathrm{T}^{odd}(\mathcal{S}(M)) \supset \mathcal{S}^{CE}(M)$.)
Suppose that $c : N \to X$ and $q : M \to X$ are cell-like maps and that $f : N \to M$ is a homotopy equivalence such that $q \circ f \simeq c$.

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr, "c"', "CE"] & & M \arrow[dl, "q", "CE"'] \\ & X & \end{tikzcd}$$

We consider the diagram of Proposition 4.5:

$$\begin{tikzcd} \mathcal{S}^{cc}(C_q, X_+) \arrow[r, "\eta'"] \arrow[d, "forget"] & H_{n+1}(M_q, M;\mathbb{L}) \arrow[d, "\bar{\partial}"] & \\ \mathcal{S}(M) \arrow[r, "\eta"] & \bar{H}_n(M;\mathbb{L}) \arrow[r] & \mathbb{Z} \end{tikzcd}$$

By the Vietoris-Begle theorem a cell-like map induces an isomorphism of ordinary cohomology or Steenrod homology with any coefficients (see Proposition 3.4). Therefore $H_n(M;\mathbb{L}) \to H_n(X;\mathbb{L})$ is an isomorphism rationally and the image of $H_{n+1}(M_q, M;\mathbb{L})$ in $H_n(M;\mathbb{L})$ is therefore a torsion group. Since $\mathbb{L}$ is an Eilenberg-MacLane spectrum at 2, $H_n(M;\mathbb{L}) \to H_n(X;\mathbb{L})$ is an isomorphism at 2 and hence the image of $H_{n+1}(M_q, M;\mathbb{L})$ in $H_n(M;\mathbb{L})$ is odd torsion. By Proposition 4.6 $[f]$ is the image of $[c] \in \mathcal{S}^{cc}(C_q, X_+)$ under the forgetful map. Then $[f] = \eta^{-1}\bar{\partial}(\gamma)$ is an odd torsion element where $\gamma = \bar{s}([c]) \in H_{n+1}(M_q, M;\mathbb{L})$. □

5. Continuous control near the corona

We move on to the nonsimply connected case. We will use germs of continuously controlled structures near infinity to recover the main diagram in the proof of Proposition 4.5. The computation of $\mathcal{S}^{CE}(M)$ in the nonsimply connected case is made more complicated because we no longer have the isomorphism $\mathcal{S}(M) \cong \bar{H}_n(M;\mathbb{L})$. We note, that by Proposition 3.4, if $q: M \to X$ is cell-like, then there are maps $M \xrightarrow{q} X \longrightarrow P_2(M)$ such that the composition is the inclusion, where $P_2(M)$ is the second stage of the Postnikov system of M. Elements of $H_n(M;\mathbb{L})$ which survive to $H_n(P_2(M);\mathbb{L})$ therefore cannot be in the kernel of $H_n(M;\mathbb{L}) \to H_n(X;\mathbb{L})$, so we are led to examine the boundary map $H_{n+1}(P_2(M), M;\mathbb{L}) \to H_n(M;\mathbb{L})$, leading to a proof of our main result.

Proposition 5.1. *Let (P,Q) be a CW pair with an inclusion isomorphism $\pi_1(Q) = \pi_1(P) = \pi$. Then the homomorphism $\partial' : H_{n+1}(P,Q;\mathbb{L}) \to \mathcal{S}_n(Q)$ defined in §2 coincides with the induced homomorphism on homotopy groups in the diagram below:*

$$\begin{tikzcd}
\mathbb{H}_{*+1}(P,Q;\mathbb{L}) \arrow[r] \arrow[d, "\partial'"] & \mathbb{H}_*(Q;\mathbb{L}) \arrow[r] \arrow[d] & \mathbb{H}_*(P;\mathbb{L}) \arrow[d, "A_P"] \\
\mathcal{S}_*(Q) \arrow[r] & \mathbb{H}_*(Q;\mathbb{L}) \arrow[r, "A_Q"] & \mathbb{L}_*(\mathbb{Z}\pi)
\end{tikzcd}$$

where A_P and A_Q are the assembly maps for P and Q.

Proof. The proof is a diagram chase. ☐

We recall the notation $\delta = p \circ \partial'$ where $p : \mathcal{S}_n(M) \to \mathcal{S}(M)$ is the projection.

To prove Theorem 2.7 we need the germ version of continuously controlled surgery theory constructed in [60].

Definition 5.2.

(i) Let N be an open manifold and let X be a compact corona of a compactification $\bar{N}$ of an end of N. A *strict homotopy equivalence near X* is a strict map $\bar{f} : \bar{W} \to \bar{N}$, where $\bar{W}$ is a compactification of an end of W by X and $\bar{f}|_X = id_X$, such that there are neighborhoods $\bar{U} \supset \bar{V}$ of X in $\bar{N}$ and $\bar{U}' \supset \bar{V}'$ of X in $\bar{W}$ such that $f(\bar{U}) \subset \bar{U}'$ and there is a strict map $\bar{g} : \bar{U}' \to \bar{U}$ with $\bar{g}|_X = id_X$ such that

  (a) $\bar{g} \circ \bar{f}|_{\bar{V}}$ is strict homotopic in $\bar{U}$ to $id_{\bar{V}}$.

  (b) $\bar{f} \circ \bar{g}|_{V'}$ is strict homotopic in $\bar{U}'$ to $id_{\bar{V}'}$.

(ii) Two strict homotopy equivalences near X, $\bar{f} : \bar{W} \to \bar{N}$ and $\bar{f}' : \bar{W}' \to \bar{N}$ are *equivalent* if there exist a neighborhood $\bar{V}$ of X in $\bar{W}$ and a strict map $h : \bar{V} \to \bar{W}'$, $h|_X = id_X$ which is an open imbedding and $\bar{f}' \circ h : \bar{V} \to \bar{N}$ is strict homotopic to $\bar{f}|_{\bar{V}}$.

(iii) The set of the equivalence classes of strict homotopy equivalences of manifolds near X is called the set of *germs of continuously controlled structures* on N at X and it is denoted as $\mathcal{S}^{cc}(\bar{N}, X)_\infty$.

One can define Top reductions near the boundary as germs of homotopy classes $[N, \mathrm{G/TOP}]_\infty$ of maps at X and the corresponding L-groups and form a surgery exact sequence. This was done in §15 of [32] in the case of bounded control and in §2 of [60] for continuous control. We are interested in the case from [60] where U is X and Z is empty. Thus, in our case of interest, $N = M \times (0,1)$ and the Top reductions of the Spivak bundle are just Top reductions of the Spivak bundle of M. Here, $\bar{N}$ is an open mapping cylinder[12] $\mathring{M}_q$ of a cell-like map $q : M \to X$ of a closed orientable manifold.

Proposition 5.3. *Let $q : M \to X$ be a cell-like map of a closed orientable n-manifold, then there is an exact sequence*

$$\cdots \to \bar{H}_{n+1}(X; \mathbb{L}) \to \mathcal{S}^{cc}(\mathring{M}_q, X)_\infty \to [M, \mathrm{G/TOP}] \to H_n(X; \mathbb{L}).$$

By Proposition 4.6, forget control defines a map $\phi : \mathcal{S}^{cc}(\mathring{M}_q, X)_\infty \to \mathcal{S}(M)$. Moreover, there is a commutative diagram:

$$\begin{tikzcd}
\mathcal{S}^{cc}(\mathring{M}_q, X)_\infty \arrow[r] \arrow[d, "\phi"] & {[M, \mathrm{G/TOP}]} \arrow[r] \arrow[d] & H_n(X; \mathbb{L}) \arrow[d, "A"] \\
\mathcal{S}(M) \arrow[r] & {[M, \mathrm{G/TOP}]} \arrow[r] & L_n(\mathbb{Z}\pi_1(M)).
\end{tikzcd}$$

Here A is the assembly map for X.

Proposition 5.4. *If $q : M \to X$ is a cell-like map of a closed connected n-manifold, then the forget control map $\phi : \mathcal{S}^{cc}(\mathring{M}_q, X)_\infty \to \mathcal{S}(M)$ factors as*

$$\mathcal{S}^{cc}(\mathring{M}_q, X)_\infty \xrightarrow{j} H_{n+1}(M_q, M; \mathbb{L}) \xrightarrow{i_*} H_{n+1}(P_2(M), M; \mathbb{L}) \xrightarrow{\delta} \mathcal{S}(M)$$

where j is a monomorphism with cokernel $\mathbb{Z}$ or 0.

Proof. Proposition 3.4 defines a map $g : X \to P_2(M)$ such that $g \circ q$ is homotopic to the inclusion $M \to P_2(M)$. We consider the diagram of (horizontal)

[12] This is the usual mapping cylinder of q with the domain copy of M stripped off.

fibrations of spectra

$$\begin{tikzcd}
\mathbb{H}_{*+1}(X, M; \mathbb{L}) \arrow[r] \arrow[d] & \mathbb{H}_*(M; \mathbb{L}) \arrow[r] \arrow[d, "="] & \mathbb{H}_*(X; \mathbb{L}) \arrow[d, "g_*"] \\
\mathbb{H}_{*+1}(P_2(M), M; \mathbb{L}) \arrow[r] \arrow[d] & \mathbb{H}_*(M; \mathbb{L}) \arrow[r] \arrow[d, "="] & \mathbb{H}_*(P_2(M); \mathbb{L}) \arrow[d, "A_E"] \\
\mathcal{S}_*(M) \arrow[r] & \mathbb{H}_*(M; \mathbb{L}) \arrow[r] & \mathbb{L}_*(\mathbb{Z}\pi).
\end{tikzcd}$$

In dimension n the homomorphism between homotopy groups of the fibers gives

$$H_{n+1}(M_q, M; \mathbb{L}) \xrightarrow{i_*} H_{n+1}(P_2(M), M; \mathbb{L}) \xrightarrow{\partial'} \mathcal{S}_n(M)$$

where $H_{n+1}(M_q, M; \mathbb{L})$ differs from $\mathcal{S}^{cc}(\mathring{M}_q, X)_\infty$ by a potential summand $\mathbb{Z}$. The proof of this is similar to the proof of Proposition 4.5, using the fibration sequence

$$\mathcal{S}^{cc}_{*+1}(\mathring{M}_q, X)_\infty \to \mathbb{H}_n(M;\, \mathbb{L}) \to \mathbb{H}_n(X;\, \mathbb{L})$$

in place of

$$\mathcal{S}^{cc}_{*+1}(C_q, X_+) \to \mathbb{H}_n(M;\, \mathbb{L}) \to \mathbb{H}_n(X;\, \mathbb{L})$$

The result then follows from Proposition 5.1. □

Proof of Theorem 2.7. ($\mathcal{S}^{CE}(M) \supset im(\delta^T_{[2]})$.) We are given an odd torsion element $\alpha \in H_{n+1}(P_2(M), M; \mathbb{L})$ with $\delta(\alpha) = [f] \in \mathcal{S}(M)$ where δ is the composition

$$H_{n+1}(P_2(M), M; \mathbb{L}) \cong \mathcal{S}_{n+1}(P_2(M), M) \to \mathcal{S}_n(M) \to \mathcal{S}(M).$$

By Proposition 3.7, there exist a cell-like map $q : M \to X$ and an odd torsion element $\widehat{\alpha} \in H_{n+1}(M_q, M; \mathbb{L}) \cong \mathcal{S}^{cc}_{n+1}(\mathring{M}_q, X)_\infty$ so that α is the image of $\widehat{\alpha}$ under the inclusion-induced map $i_* : H_{n+1}(M_q, M; \mathbb{L}) \to H_{n+1}(P_2(M), M; \mathbb{L})$. Since $\widehat{\alpha}$ has finite order and $j : \mathcal{S}^{cc}(\mathring{M}_q, X)_\infty \to \mathcal{S}^{cc}_{n+1}(\mathring{M}_q, X)_\infty$ is an isomorphism on torsion subgroups, $\widehat{\alpha} = j(\alpha')$, where $\alpha' \in \mathcal{S}^{cc}(\mathring{M}_q, X)_\infty$. By Propostion 5.4 $\phi(\alpha') = [f]$. Let $g : W \to M \times (0,1)$ be a representative for α'. As in the proof of Corollary 4.6 we may assume that $W = N \times (0,1)$ and $\bar{W} = M_p$, where $p : N \to X$ is cell-like. Thus, $[f] = \phi(\alpha')$ is realized by cell-like maps p and q.

($\mathcal{S}^{CE}(M) \subset im(\delta^T_{[2]})$). Suppose that $c : N \to X$ and $q : M \to X$ are cell-like maps and that $f : N \to M$ is a homotopy equivalence such that $q \circ f \simeq c$.

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr, "c"', "CE"] & & M \arrow[dl, "q", "CE"'] \\ & X & \end{tikzcd}$$

As above, there is an inclusion-induced map $p : X \to P_2(M)$ and the forgetful map $H_{n+1}(M_q, M; \mathbb{L}) \cong \mathcal{S}^{cc}_{\infty}(M_q, X) \to \mathcal{S}(M)$ factors through $H_{n+1}(P_2(M), M; \mathbb{L})$. It therefore suffices to show that the image of $H_{n+1}(M_q, M; \mathbb{L})$ in $H_{n+1}(P_2(M), M; \mathbb{L})$ is an odd torsion group. By the Vietoris-Begle theorem a cell-like map induces an isomorphism of ordinary cohomology or Steenrod homology with any coefficients (see Proposition 3.4). Therefore $H_*(M; \mathbb{L}) \to H_*(X; \mathbb{L})$ is an isomorphism rationally, and hence, the image of $H_*(M_q, M; \mathbb{L})$ in $H_*(P_2(M), M; \mathbb{L})$ is torsion. Since $\mathbb{L}$ is an Eilenberg-MacLane spectrum at 2, $H_*(M; \mathbb{L}) \to H_*(X; \mathbb{L})$ is an isomorphism at 2 and hence $H_*(P_2(M), M; \mathbb{L})$ is odd torsion. $\square$

6. Deforming Riemannian manifolds in Gromov-Hausdorff space

In this section, we apply the theory of CE equivalence, developed above, to study one parameter families of Riemannian manifolds in Gromov-Hausdorff space.

Definition 6.1.

(i) A continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(0) = 0$, continuous at 0, with $\rho(t) \geq t$ for all t is a *contractibility function* for a metric space X if there is $R > 0$ such that for each $x \in X$ and $t \leq R$, the t-ball $B_t(x)$ centered at x can be contracted to a point in the $\rho(t)$-ball $B_{\rho(t)}(x)$.

(ii) Similarly, if X is an n-manifold, ρ is a *topological injectivity function* for X if for each $x \in X$ and $t \leq R$ there is an open subset $U \subset X$ so that U is homeomorphic to $\mathbb{R}^n$ and $B_t(x) \subset U \subset B_{\rho(t)}(x)$.

Let $\rho = \rho_1 : [0, R) \to [0, \infty)$ be a contractibility function. The theorem of Petersen on page 392 of [62] shows that for every $\epsilon > 0$ there is a $\delta > 0$ so that if X and Y are compact n-dimensional metric spaces with contractibility function ρ such that $d_{GH}(X, Y) < \delta$, then X and Y are homotopy equivalent by maps and homotopies that move points by less than ϵ. Moreover, given ϵ, there is an explicit computation of the necessary δ.

Combining this with the results of Chapman-Ferry, Freedman-Quinn, and Perelman cited in the introduction, we see that if M is a closed topological n-manifold with a given metric d_M, and a contractibility function ρ, then there is a $\delta > 0$ such that any other topological n-manifold with contractibility function ρ and $d_{GH}(M, N) < \delta$ must be homeomorphic to M.

In this section, we show that the condition that M be stationary with a fixed metric is necessary: that there are families of nonhomeomorphic Riemannian manifolds with a common contractibility function that can be deformed arbitrarily close to each other in a precompact region of Gromov-Hausdorff space. We get a complete algebraic description of this behavior and produce many examples of nonhomeomorphic families of manifolds with common contractibility functions and/or topological injectivity functions that can be similarly pushed together.

Definition 6.2.

(i) If Z is a metric space, $X \subset Z$, and $\epsilon > 0$, $N_\epsilon(X) = \{z \in Z \,|\, d(z, X) < \epsilon\}$.

(ii) If X and Y are compact subsets of a metric space Z, the *Hausdorff distance* between X and Y is

$$d_H(X, Y) = \inf\{\epsilon > 0 \mid X \subset N_\epsilon(Y), Y \subset N_\epsilon(X)\}.$$

Here, X and Y are isometrically embedded in Z.

(iii) If X and Y are compact metric spaces, the *Gromov-Hausdorff distance* from X to Y is

$$d_{GH}(X, Y) = \inf_Z \{d_H(X, Y) \mid X, Y \subset Z\}.$$

(iv) Let $\mathcal{CM}$ be the set of isometry classes of compact metric spaces with the Gromov-Hausdorff metric.

(v) Let $\mathcal{M}^{man}(n, \rho)$ be the set of all $(X, d) \in \mathcal{CM}$ such that X is a topological n-manifold with (topological) metric d with contractibility function ρ.

It is well-known that $\mathcal{CM}$ is a complete metric space (see [37] or [62] for an exposition).

Theorem 6.3.

(i) *If $n \neq 3$ and $X \in \mathcal{CM}$ is in the closure of $\mathcal{M}^{man}(n,\rho)$, then there is an $\epsilon > 0$ so that there are only finitely many homeomorphism types of manifolds $M \in \mathcal{M}^{man}(n,\rho)$ with $d_GH(M,X) < \epsilon$. If $d_{GH}(M,X)$, $d_{GH}(M',X) < \epsilon$, then there exists a simple homotopy equivalence $h : M' \to M$ which preserves rational Pontryagin classes.*

(ii) *If $[f] \in \mathcal{S}^{CE}(M)$ with $f : N \to M$, M and N smooth, then there exist a contractibility function ρ and a compact metric space X such that every neighborhood of X in $\mathcal{CM}$ contains manifolds lying in $\mathcal{M}^{man}(n,\rho)$ and homeomorphic to both M and N.*

(iii) *There exist examples as in (ii) such that M and N are not homeomorphic.*

Proof. Part (i) is Theorem 2.10 of [27].

For part (ii), and let $q : M \to X$ and $p : N \to X$ be cell-like maps. By the main results of [31] and [57], there exist a contractibility function ρ, and sequences of Riemannian metrics $\{d_i^M\}$ and $\{d_i^N\}$ on M and N respectively lying in $\mathcal{M}^{man}(n,\rho)$ and converging in $\mathcal{CM}$ to (X,d) for some metric d.

For part (iii), let M and N be the manifolds from Corollary 2.15 or Proposition 2.16. □

Let $\overline{\mathcal{M}^{man}(n,\rho)}$ be the closure of $\mathcal{M}^{man}(n,\rho)$ in Gromov-Hausdorff space and let $\partial\mathcal{M}^{man}(n,\rho)$ be the boundary. For a compact metric space X, we will denote its isometry class by the same letter X.

Theorem 6.4. *Suppose that the isometry type of a metric space X belongs to $\partial\mathcal{M}^{man}(n,\rho)$. Then there is an $\epsilon > 0$ such every two manifolds $M, N \in B_\epsilon(X) \cap \mathcal{M}^{man}(n,\rho)$ are CE-related.*

Proof. The proof will follow Proposition 6.9. □

Definition 6.5. A map $f : M \to X$ has the *δ-lifting property in dimensions* $\leq k$ if for every PL pair (P,Q), $\dim P \leq k$ for every commutative diagram

$$\begin{tikzcd} Q \arrow[r, "g'"] \arrow[d] & M \arrow[d, "f"] \\ P \arrow[r, "g"] \arrow[ur, dashed, "\bar{g}"] & X \end{tikzcd}$$

there is a map $\bar{g} : P \to M$ extending g' such that $\mathrm{dist}(f \circ \bar{g}, g) < \delta$.

Proposition 6.6. *Let X be a locally k-connected space for $k > n$, then there exists $\delta > 0$ such that every map $f : Z \to X$ from a compact n-dimensional ANR with the δ-lifting property in dimensions $\leq n+1$ is a weak homotopy equivalence through dimension n (i.e., such that f is $n+1$-connected). Furthermore, f induces isomorphisms of Steenrod homology groups $f_* : H_i(M) \to H_i(X)$ for $i \leq n$.*

Proof. The weak homotopy equivalence in dimension n easy follows from the lifting property. This implies the result for singular homology. We note that the Steenrod homologies coincide with the singular homologies in the locally $n+1$-connected case. □

Proposition 6.7. *If $X \in \partial\mathcal{M}^{man}(n,\rho)$, then for every $\delta > 0$ there exists $\epsilon > 0$ such that for every $M \in \mathcal{M}^{man}(n,\rho)$ with $d_{GH}(M,X) < \epsilon$ there is a map $f : M \to X$ with the δ-lifting property in dimensions $\leq n+1$.*

Proof. The space X is locally k-connected for all finite k (see [29]). Then for small ϵ a map $f : M \to X$ can be constructed by induction by means of a small triangulation on M (if M does not admit a triangulation, one can embed M in euclidean space, take a fine triangulation of a neighborhood of M, and lift all of the simplices in the neighborhood that meet M). Given $\delta_0 > 0$, we may assume that $d(x, f(x)) < \delta_0$. Clearly, for a proper choice of δ_0 the map f will have the δ-lifting property. □

Proposition 6.8 ([52] Remark 4.36)**.** *For any CW complex K there is an isomorphism*

$$H_n(K;\mathbb{L}_{(2)}) \cong \bigoplus_i H_{n+4i}(K;\mathbb{Z}/2) \oplus H_{n+4i-2}(K;\mathbb{Z}/2)$$

which is natural with respect to maps $K \to L$.

Proposition 6.9. *If $X \in \partial\mathcal{M}^{man}(n,\rho)$, then there exists $\epsilon > 0$ such that for every $M \in \mathcal{M}^{man}(n,\rho)$ with $d_{GH}(M,X) < \epsilon$ there is a map $f : M \to X$ such that $f_* : H_*(M;\mathbb{L}_{(2)}) \to H_*(X;\mathbb{L}_{(2)})$ is an isomorphism.*

Proof. Since $\mathbb{L}_{(2)}$ is an Eilenberg-MacLane spectrum, we can take ϵ from Proposition 6.7. Then Proposition 6.6 and the fact that $H_i(M) = H_i(X) = 0$ for $i > n$ imply the required result. This last fact follows from the arguments above. Given k, one shows that there is a $\delta > 0$ so that if $d_{GH}(M^n, X) < \delta$, there is a k-connected map $M \to X$. Repeating this for a sequence of M's shows that the homology groups of X are trivial in dimensions $> n$. □

Petersen [62] correctly concludes from similar arguments that X can have no finite-dimensional subsets of dimension $> n$ and incorrectly concludes from this that X must have covering dimension $\leq n$. The limit spaces X constructed in this paper are infinite-dimensional spaces containing no finite-dimensional subspaces of dimension $> n$. See [73], [18] for further explanation.

Proof of Theorem 6.4. We take ϵ from Proposition 6.9. Let $c : N \to X$ and $q : M \to X$ be corresponding maps. We may assume that there is a homotopy lift $f : N \to M$ of c which is a homotopy equivalence. Then f induces isomorphisms $f_* : H_*(N; \mathbb{L}_{(2)}) \to H_*(M; \mathbb{L}_{(2)})$.

As was shown in [29] (P4, page 98), there are finite polyhedra P_1, P_2 and maps $p_1 : X \to P_1$, $p_2 : X \to P_2$ and $g : P_2 \to P_1$ such that $p_1 = g \circ p_2$, p_2 is $n+3$-connected and g is $(\dim P_1 + 3)$-connected. Let $q_i = p_i \circ q$, $i = 1, 2$. We note that these conditions imply that q_2 induces isomorphisms of homology in dimension $\leq n+3$ and g induces isomorphisms of homology in dimension $\leq \dim P_1 + 3$. The latter implies that $\operatorname{im} g_* = \operatorname{im}(q_1)_*$ for homology of dimension $\leq \dim P_1 + 3$. In view of Proposition 6.8 we obtain the following:

1. $(q_2)_* : H_n(M; \mathbb{L}_{(2)}) \to H_n(P_2; \mathbb{L}_{(2)})$ is a monomorphism;
2. $\operatorname{im} g_* = \operatorname{im}(q_1)_*$ for the $(n+1)$-dimensional $\mathbb{L}_{(2)}$-homology.

We claim that $g_* : H_{n+1}(P_2, M; \mathbb{L}_{(2)}) \to H_{n+1}(P_1, M; \mathbb{L}_{(2)})$ is the zero homomorphism.

Consider the commutative diagram generated by exact sequences of pairs

$$\begin{array}{ccccccc}
H_{n+1}(M; \mathbb{L}_{(2)}) & \xrightarrow{(q_2)_*} & H_{n+1}(P_2; \mathbb{L}_{(2)}) & \xrightarrow{j^2_*} & H_{n+1}(P_2, M; \mathbb{L}_{(2)}) & \xrightarrow{\partial_2} & \\
{\scriptstyle =}\downarrow & & {\scriptstyle g_*}\downarrow & & {\scriptstyle g_*}\downarrow & & \\
H_{n+1}(M; \mathbb{L}_{(2)}) & \xrightarrow{(q_1)_*} & H_{n+1}(P_1; \mathbb{L}_{(2)}) & \xrightarrow{j^1_*} & H_{n+1}(P_1, M; \mathbb{L}_{(2)}) & \xrightarrow{\partial_1} & .
\end{array}$$

Let $\alpha \in H_{n+1}(P_2, M; \mathbb{L}_{(2)})$. By the property (1) $\partial_2(\alpha) = 0$. Hence $\alpha = j^2_*(\beta)$ for some β. By the property (2) there is $\gamma \in H_{n+1}(M; \mathbb{L}_{(2)})$ such that $(q_1)_*(\gamma) = g_*(\beta)$. Hence $0 = j^1_* \circ g_*(\beta) = g_*(\alpha)$ and the claim is proven.

Since $H_*(Y; \mathbb{L}_{(2)}) = H_*(Y; \mathbb{L}) \otimes \mathbb{Z}_{(2)}$, g_* takes $H_{n+1}(P_2, M; \mathbb{L})$ to odd torsion.

Since $p_i \circ q$ is 2-connected, the space $P_2(M)$ can be constructed out of P_i by killing higher dimensional homotopy groups. Thus the inclusion $M \subset P_2(M)$

can be factored through X and P_i, $i = 1, 2$. Hence there is a commutative diagram

(7)
$$\begin{tikzcd}
H_{n+1}(P_2, M; \mathbb{L}) \arrow[r, "\cong"] \arrow[d] & \mathcal{S}_{n+1}(P_2, M) \arrow[r, "\phi_2"] \arrow[d] & \mathcal{S}_n(M) \arrow[d, "="] \\
H_{n+1}(P_1, M; \mathbb{L}) \arrow[r, "\cong"] \arrow[d] & \mathcal{S}_{n+1}(P_1, M) \arrow[r, "\phi_1"] \arrow[d] & \mathcal{S}_n(M) \arrow[d, "="] \\
H_{n+1}(P_2(M), M; \mathbb{L}) \arrow[r, "\cong"] & \mathcal{S}_{n+1}(P_2(M), M) \arrow[r, "\partial"] & \mathcal{S}_n(M).
\end{tikzcd}$$

We show that the element $[f] \in \mathcal{S}(M)$ comes from an odd torsion element of $H_{n+1}(P_2(M), M; \mathbb{L})$. By Theorem 2.6 of [29] the structure $[f]$ defined by $f : N \to M$ belongs to the kernel of the induced map $(q_2)_* : \mathcal{S}_n(M) \to \mathcal{S}_n(P_2)$. Thus $[f] \in \mathrm{im}(\phi_i)$, $i = 1, 2$. By the above ϕ_2 factors through odd torsion. Therefore $[f]$ is the image under ∂ of an odd torsion element. Hence, $[f] \in \mathrm{im}(\delta^T_{[2]})$.

Applying Theorem 2.7 shows that N and M are CE-related. $\square$

Next, we demonstrate the existence of topological injectivity functions. Our argument is an easy modification of McMillan's Cellularity Criterion [56], which says that a compact subset X of a closed manifold M^n, $n \geq 5$ is a nested intersection of open sets homeomorphic to $\mathbb{R}^n$ if and only if X is cell-like and for every open neighborhood U of X there is an open neighborhood V of X contained in U such that the inclusion induced map $\pi_1(V - X) \to \pi_1(U - X)$ is trivial.[13]

Theorem 6.10. *Let M be a closed topological n-manifold with a contractibility function $\rho : [0, R) \to [0, \infty)$. If Q is a closed k-manifold, $k \geq 1$, $n + k \geq 5$, then $M \times Q$ has a topological injectivity function.*

Proof. McMillan shows that if X is cell-like and M_i, $i = 0, 1, 2, 3$ are nested compact PL manifolds, $M_{i+1} \subset M_i$, containing X with the inclusion of M_{i+1} into M_i nullhomotopic, $i = 0, 1, 2$, and the inclusion induced map $\pi_1(\mathring{M}_3 - X) \to \pi_1(\mathring{M}_2 - X)$ is zero, then there is an open set U with

[13]Using work of Perelman and Freedman, the cellularity criterion is now known to be true in all dimensions.

$M_3 \subset U \subset M_0$ and U homeomorphic to $\mathbb{R}^n$.[14]

By immersion theory, any open subset of a topological manifold that contracts to a point in that manifold has a PL structure, so for any $x \in M$, the contractibility function allows us to find arbitrarily long nested sequences of compact PL manifold neighborhoods M_i of x with $M_{i+1} \to M_i$ nullhomotopic for all i. A bit of manipulation to fill in holes, see [56], allows us to assume that each M_i is connected with connected boundary and that $M_i - \overset{\circ}{M}_{i+1}$ is connected for each i. If $q \in Q$ is written as a nested intersection $\{q\} = \cap B_i$ of balls in Q, then the sequence $M_i \times B_i$ satisfies McMillan's conditions and guarantees the existence of a topological injectivity function. With a bit of care, this function τ can be written explicitly in terms of ρ. $\square$

Remark 6.11. For each $x \in M \times Q$, this allows us to construct a sequence of homeomorphisms between euclidean neighborhoods of x and euclidean neighborhoods of nearby points in $M' \times Q$, where M is deformable to M'. Evidently, these homeomorphisms cannot be controlled well enough to stitch them together to provide an isomorphism of tangent microbundles, since that would contradict the characteristic class computations of Corollary 2.15. This suggests that there is no reasonable way of assigning a tangent bundle to the infinite-dimensional (but finite cohomological dimensional) homology manifold X. Nevertheless, the controlled Mischenko-Ranicki symmetric signatures $\Delta_M \in H_n(M; \mathbb{L})$ and $\Delta_{M'} \in H_n(M'; \mathbb{L})$ map to the same class in $H_n(X; \mathbb{L})$ after inverting 2. This suggests that X may possess a well-defined characteristic class theory.

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