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Mixed-Spin-P fields of Fermat polynomials

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This is the first part of the project toward an effective algorithm to evaluate all genera Gromov-Witten invariants of quintic Calabi-Yau threefolds. In this paper, we introduce the notion of Mixed-Spin-P fields, construct their moduli spaces, and construct the virtual cycles of these moduli spaces.

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1. Introduction

Explicitly solving all genera Gromov-Witten invariants (in short GW invariants) of Calabi-Yau threefolds is one of the major goals in the subject of Mirror Symmetry. For quintic Calabi-Yau threefolds, the mirror formula of genus-zero GW invariants was conjectured in [CdGP] and proved in [Gi, LLY]. The mirror formula of genus-one GW invariants was conjectured in [BCOV] and proved in [LZ, Zi]. A complete determination of all genera GW invariants based on degeneration is provided in [MP] and plays a crucial role in the proof of the GW/Pairs correspondence [PP]. However, the mirror prediction on genus g GW invariants for $2 \leq g \leq 51$ in [HKQ] is still open.

The mirror prediction in [HKQ] includes both GW invariants of quintic threefolds and FJRW invariants of the Fermat quintic. In this paper, we introduce the notion of Mixed-Spin-P fields (in short MSP fields) of the Fermat quintic polynomial, construct their moduli spaces, and establish basic properties of these moduli spaces. This class of moduli spaces will be

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employed in the sequel of this paper [CLLL] toward developing an effective theory evaluating all genera GW invariants of quintic threefolds and all genera FJRW invariants of the Fermat quintic.

The theory of MSP fields, for the CY quintic polynomial $F_{5,5}$, where

$$F_{N,r}(x) = x_1^r + \cdots + x_N^r, \tag{1.1}$$

provides a transition between FJRW invariants [FJR] and GW invariants of stable maps with p-fields [CL]. It is known that the FJRW invariants of the Fermat quintic is the LG theory taking values in $[\mathbb{C}^5/\boldsymbol{\mu}_5]$ (via spin fields), and the GW invariants of stable maps with p-fields is the LG theory taking values in $K_{\mathbb{P}^4}$ (via p-fields). Our idea is to use the master space technique to study the field theories of the two GIT quotients $[\mathbb{C}^5/\boldsymbol{\mu}_5]$ and $K_{\mathbb{P}^4}$ of $[\mathbb{C}^6/\mathbb{C}^*]$, leading to the notion of MSP fields, and providing a geometric transition between the LG theories of the two GIT quotients of $[\mathbb{C}^6/\mathbb{C}^*]$.

In this paper, we will introduce the notion of MSP fields for a Fermat polynomial. We fix a Fermat polynomial $F_{N,r}$ as in (1.1) once and for all. An MSP field (of $F_{N,r}$) is a collection

$$\xi = (\Sigma^{\mathcal{C}} \subset \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu), \tag{1.2}$$

consisting of a pointed twisted curve $\Sigma^{\mathcal{C}} \subset \mathcal{C}$, invertible sheaves of $\mathcal{O}_{\mathcal{C}}$-modules $\mathcal{L}$ and $\mathcal{N}$, fields $\varphi \in H^0(\mathcal{L}^{\oplus N})$ and $\rho \in H^0(\mathcal{C}, \mathcal{L}^{\vee\otimes r} \otimes \omega_{\mathcal{C}}^{\log})$, and a gauge field $\nu = (\nu_1, \nu_2) \in H^0(\mathcal{L} \otimes \mathcal{N}) \oplus H^0(\mathcal{N})$. The numerical invariants of ξ are the genus of $\mathcal{C}$, the monodromy γ_i of $\mathcal{L}$ at the marking $\Sigma_i^{\mathcal{C}}$ (of $\Sigma^{\mathcal{C}}$), and the bi-degrees $d_0 = \deg(\mathcal{L} \otimes \mathcal{N})$ and $d_\infty = \deg \mathcal{N}$.

For a choice of g, $\gamma = (\gamma_1, \cdots, \gamma_\ell)$ and $\mathbf{d} = (d_0, d_\infty)$, we form the moduli $\mathcal{W}_{g,\gamma,\mathbf{d}}$ of equivalence classes of stable MSP fields of numerical data $(g, \gamma, \mathbf{d})$. It is a separated DM stack, locally of finite type, though usually not proper. The stack $\mathcal{W}_{g,\gamma,\mathbf{d}}$ admits a $T = \mathbb{G}_m$ action, via

$$t \cdot (\Sigma^{\mathcal{C}} \subset \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu_1, \nu_2) = (\Sigma^{\mathcal{C}} \subset \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, t\nu_1, \nu_2).$$

It has a T-equivariant perfect (relative) obstruction theory, of virtual dimension (in case $\gamma = \emptyset$)

$$\operatorname{vir.dim} \mathcal{W}_{g,\gamma=\emptyset,\mathbf{d}} = (1 + N - r)d_0 + (1 - N + r)d_\infty + (4 - N)(g - 1).$$

Theorem 1.1. *The moduli stack $\mathcal{W}_{g,\gamma,\mathbf{d}}$ (of stable MSP fields of the Fermat polynomial $F_{N,r}$) is a separated DM stack, locally of finite type. It has a* *cosection localized T-equivariant virtual cycle*

$$[\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}} \in A^T_*(\mathcal{W}^-_{g,\gamma,\mathbf{d}})^T,$$

lying in a proper substack $\mathcal{W}^-_{g,\gamma,\mathbf{d}} \subset \mathcal{W}_{g,\gamma,\mathbf{d}}$.

In the sequel [CLLL], for the quintic Fermat polynomial $F_{5,5}$, we apply the virtual localization formula to derive a doubly indexed polynomial relations among the GW invariants of quintic threefolds and the FJRW invariants of the Fermat polynomial $F_{5,5}$. These relations provide an effective algorithm in evaluating all genera GW invariants of quintics in terms of FJRW invariants of $F_{5,5}$ (with insertions $2/5$), and provide a collection of relations among all genera FJRW invariants of $F_{5,5}$ (with insertions $2/5$).

This work is inspired by Witten's vision that the "Landau-Ginzburg looks like the analytic continuation of Calabi-Yau to negative Kahler class." (See [Wi, 3.1].) One interpretation of his proposed transition of theories is that the LG theory of $[\mathbb{C}^5/\mathbb{Z}_5]$ and that of $K_{\mathbb{P}^4}$ differ by a field version of "wall-crossing". The MSP fields introduced can be viewed as a geometric construction to realize this "wall-crossing".

Around the time of the completion of the first draft of this paper, there have been other approaches for high genus LG/CY correspondence [CK, FJR2]. After this paper, in [CLLL], the torus localization formula gives aforementioned recursion relations among GW and FJRW of the quintic Fermat polynomial. Packaging the algorithms in [CLLL] gives the following notable developments as an outcome. The explicit formula of the genus one GW invariants of quintic threefolds has been recovered in [CGLZ] (it was first obtained by A. Zinger). In [GR], Guo-Ross proved the genus-one Landau-Ginzburg/Calabi-Yau conjecture of Chiodo and Ruan. Recently, in [NMSP1], the authors replace the section ν_1 of $\mathcal{L} \otimes \mathcal{N}$ by N many sections (called N-Mixed Spin P fields), and use it to recover BCOV's Feynman rule of the GW invariants of quintic Calabi-Yau threefolds [NMSP3]. The rule determines the GW potential F_g of quintics, in a simple Feynman graphical process, from $3g-3$ many initial conditions. Indeed, in [NMSP2, NMSP3], the finite generation conjecture of Yamaguchi-Yau [YY], the Yamaguchi-Yau (functional) equations, and the convergence of the GW potential of quintic Calabi-Yau threefolds for genus $g > 1$ are all established (cf. [BCOV, YY, ASYZ, LP]).

We remark that there shall be various MSP type objects giving more applications. The properness of their degeneracy loci, as that of NMSP in [NMSP1], follows from the argument in the current paper.

This paper is organized as follows. In Section 2, we will introduce the notion of Mixed-Spin-P fields of the Fermat polynomial, construct the moduli spaces of stable Mixed-Spin-P fields, and construct the cosection localized virtual cycles of these moduli spaces. These cycles lie in the degeneracy locus of the cosection mentioned. In Section 3 and 4, we will prove that these degeneration locus are proper, separated and of finite type. In this paper, all schemes are over the field $\mathbb{C}$ of complex numbers.

2. The moduli of Mixed-Spin-P fields

In this section, we introduce the notion of MSP (Mixed-Spin-P) fields, construct their moduli stacks, and form their cosection localized virtual cycles. We introduce the T-structure on it. The proof of the localization formula of cosection localized virtual cycles appeared in [CKL].

2.1. Twisted curves and invertible sheaves

We recall the basic notions and properties of twisted curves with representable invertible sheaves on them. The materials are drawn from [ACV, AJ, AF, AGV, Cad].

A prestable twisted curve with ℓ-markings is a one-dimensional proper, separated connected DM stack $\mathcal{C}$, with at most nodal singularities and markings $\Sigma_1, \cdots, \Sigma_\ell \subset \mathcal{C}$ such that $\mathcal{C}^{\mathrm{sm}} - \cup_i \Sigma_i$ is a scheme and nodes are balanced.

Here a balanced node of index a looks like the following model

$$\mathcal{V}_a := \left[\operatorname{Spec}\big(\mathbb{C}[u,v]/(uv)\big)/\boldsymbol{\mu}_a\right], \quad \zeta\cdot(u,v) = (\zeta u, \zeta^{-1} v).$$

Similarly, an index a marking of a twisted curve looks like the model

$$[\operatorname{Spec}\mathbb{C}/\boldsymbol{\mu}_a] \subset \mathcal{U}_a := \left[\operatorname{Spec}\mathbb{C}[u]/\boldsymbol{\mu}_a\right], \quad \zeta\cdot u = \zeta u.$$

Via $x \mapsto u^a$ and $y \mapsto v^a$,

$$\pi_a : \mathcal{V}_a \to V_a := \operatorname{Spec}\big(\mathbb{C}[x,y]/(xy)\big) \quad \text{and} \quad \pi_a : \mathcal{U}_a \to U_a := \mathbb{A}^1 \tag{2.1}$$

define maps to their respective coarse moduli spaces. Note that $\mathcal{V}_a$ contains two subtwisted curves $\mathcal{V}_{a,1}$ and $\mathcal{V}_{a,2}$, each isomorphic to $\mathcal{U}_a$ in (2.1). This process $\mathcal{V}_a \mapsto \mathcal{U}_{a,u} \coprod \mathcal{U}_{a,v}$ is called the decomposition of $\mathcal{V}_a$ along its nodes. The reverse process is called the gluing defined via a push out.

We comment on our convention on invertible sheaves on a twisted curve $\mathcal{C}$ near a stacky point. In the model case $\mathcal{C} = \mathcal{V}_a$, an invertible sheaf on $\mathcal{C}$ is a $\boldsymbol{\mu}_a$-module $\mathcal{M}_m$, for $0 \le m < a$, so that, when $m \neq 0$,

$$\mathcal{M}_m := u^{-(a-m)}\mathbb{C}[u] \oplus_{[0]} v^{-m}\mathbb{C}[v] := \ker\{u^{-(a-m)}\mathbb{C}[u] \oplus v^{-m}\mathbb{C}[v] \to \mathbb{C}_m\},$$

where the arrow is a homomorphism of $\boldsymbol{\mu}_a$-modules, $\boldsymbol{\mu}_a$ leaves $1 \in \mathbb{C}[u]$ and $1 \in \mathbb{C}[v]$ fixed and acts on $1 \in \mathbb{C}_m \cong \mathbb{C}$ via $\zeta \cdot 1 = \zeta^m 1$, and both $u^{-(a-m)}\mathbb{C}[u] \to \mathbb{C}_m$ and $v^{-m}\mathbb{C}[v] \to \mathbb{C}_m$ are surjective. When $m = 0$,

$$\mathcal{M}_0 := \mathbb{C}[u] \oplus_{[0]} \mathbb{C}[v] := \ker\{\mathbb{C}[u] \oplus \mathbb{C}[v] \to \mathbb{C}\},$$

where maps are defined similarly as the case of $m \neq 0$. Note that the isomorphism classes are indexed by $m \in \{0, \cdots, a-1\}$. Furthermore, in the convention (2.1), we have

$$\pi_{a*}\mathcal{M}_m = \mathbb{C}[x] \oplus \mathbb{C}[y], \text{ when } m \neq 0; \quad \pi_{a*}\mathcal{M}_m = \mathcal{O}_{V_a}, \text{ otherwise.} \tag{2.2}$$

Similarly, invertible sheaves on $\mathcal{C}$ near an index a marking in the model case $\mathcal{C} = \mathcal{U}_a$ looks like the $\boldsymbol{\mu}_a$-module

$$\mathcal{M}_m := u^{-m}\mathbb{C}[u]. \tag{2.3}$$

Let $\zeta_a = \exp(2\pi i/a) \in \boldsymbol{\mu}_a$. Under $u \mapsto \zeta_a u$, the generator $u^{-m}1_u \mapsto \zeta_a^{-m}u^{-m}1_u$.

Convention 2.1. *We call ζ_a^m the monodromy of $\mathcal{M}_m$ at the marking, and call m the monodromy index at the marking.*

Note that for $\mathcal{M}_m$ over $\mathcal{V}_a$, $\mathcal{M}_m$ restricted to $\mathcal{V}_{a,1}$ and to $\mathcal{V}_{a,2}$ have monodromies ζ_a^{-m} and ζ_a^m, at their respective stacky points.

Definition 2.2. *We call $\mathcal{M}_m$ representable if $(m, a) = 1$.*

Example 2.3. *Let $\mathcal{C} = [\mathbb{A}^1/\boldsymbol{\mu}_a]$ be the obvious global quotient twisted curve and $p \in \mathbb{A}^1$ be its origin. Then we use $\mathcal{O}_{\mathcal{C}}(\frac{m}{a}p)$ to denote the sheaf (2.3) having monodromy index m.*

2.2. Definition of MSP fields

We denote by $\boldsymbol{\mu}_a \le \mathbb{C}^*$ the subgroup of the a-th roots of unity. We let

$$\tilde{\boldsymbol{\mu}}_a^+ = \boldsymbol{\mu}_a \cup \{(1,\rho), (1,\varphi)\}, \quad \text{and} \quad \tilde{\boldsymbol{\mu}}_a = \tilde{\boldsymbol{\mu}}_a^+ - \{1\}.$$

For $\alpha \in \boldsymbol{\mu}_a$, we let $\langle \alpha \rangle \le \mathbb{C}^*$ be the subgroup generated by α; for the two exceptional element $(1,\rho)$ and $(1,\varphi) \in \tilde{\boldsymbol{\mu}}_a^+$, we agree $\langle (1,\rho)\rangle = \langle (1,\varphi)\rangle = \langle 1 \rangle$.

We fix the polynomial $F_{N,r}$ (cf. (1.1)) throughout this paper. We pick

$$g \ge 0, \quad \gamma = (\gamma_1, \cdots, \gamma_\ell) \in (\tilde{\boldsymbol{\mu}}_a)^{\times \ell}, \quad \text{and} \quad \mathbf{d} = (d_0, d_\infty) \in \mathbb{Q}^{\times 2},$$

and call the triple $(g, \gamma, \mathbf{d})$ a numerical data (for MSP fields). In case $\gamma \in (\tilde{\boldsymbol{\mu}}_a^+)^{\times \ell}$, we call $(g, \gamma, \mathbf{d})$ a broad numerical data.

For an ℓ-pointed twisted nodal curve $\Sigma^{\mathcal{C}} \subset \mathcal{C}$, denote $\omega^{\log}_{\mathcal{C}/S} \colon = \omega_{\mathcal{C}/S}(\Sigma^{\mathcal{C}})$, and for $\alpha \in \tilde{\boldsymbol{\mu}}_a^+$, let $\Sigma^{\mathcal{C}}_\alpha = \coprod_{\gamma_i = \alpha} \Sigma^{\mathcal{C}}_i$. We abbreviate $\mathcal{L}^{-r} = \mathcal{L}^{\vee \otimes r}$.

Definition 2.4. *Let S be a scheme and let $(g, \gamma, \mathbf{d})$ be a numerical data. An S-family of MSP-fields (of Fermat type $F_{N,r}$) of type $(g, \gamma, \mathbf{d})$ is a datum*

$$(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$$

such that

(1) $\cup_{i=1}^{\ell} \Sigma^{\mathcal{C}}_i = \Sigma^{\mathcal{C}} \subset \mathcal{C}$ is an ℓ-pointed genus g twisted curve over S such that the i-th marking $\Sigma^{\mathcal{C}}_i$ is banded by the group $\langle \gamma_i \rangle \le \mathbb{C}^*$;

(2) $\mathcal{L}$ and $\mathcal{N}$ are representable invertible sheaves on $\mathcal{C}$, and $\mathcal{L} \otimes \mathcal{N}$ and $\mathcal{N}$ have fiberwise degrees d_0 and d_∞ respectively. The monodromy of $\mathcal{L}$ along $\Sigma^{\mathcal{C}}_i$ is γ_i when $\langle \gamma_i \rangle \ne \langle 1 \rangle$;

(3) $\nu = (\nu_1, \nu_2) \in H^0(\mathcal{L} \otimes \mathcal{N}) \oplus H^0(\mathcal{N})$ such that (ν_1, ν_2) is nowhere vanishing;

(4) $\varphi = (\varphi_1, .., \varphi_N) \in H^0(\mathcal{L}^{\oplus N})$, (φ, ν_1) nowhere zero, and $\varphi|_{\Sigma^{\mathcal{C}}_{(1,\varphi)}} = 0$;

(5) $\rho \in H^0(\mathcal{L}^{-r} \otimes \omega^{\log}_{\mathcal{C}/S})$, (ρ, ν_2) is nowhere vanishing, and $\rho|_{\Sigma^{\mathcal{C}}_{(1,\rho)}} = 0$.

In the future, we call φ (resp. ρ) the φ-field (resp. ρ-field).

Definition 2.5. *If $(g, \gamma, \mathbf{d})$ is a broad numerical data, a similarly defined $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$ as in Def. 2.4 is called an S-family of broad MSP-fields.*

We remark that in this paper we will only be concerned with MSP fields.

Definition 2.6. *An arrow*

$$(\mathcal{C}', \Sigma^{\mathcal{C}'}, \mathcal{L}', \mathcal{N}', \varphi', \rho', \nu') \longrightarrow (\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$$

from an S'-MSP-field to an S-MSP-field consists of a morphism $S' \to S$ and a 3-tuple (a, b, c) such that

1. *$a : (\Sigma^{\mathcal{C}'} \subset \mathcal{C}') \to (\Sigma^{\mathcal{C}} \subset \mathcal{C}) \times_S S'$ is an S'-isomorphism of twisted curves;*

2. $b : a^*\mathcal{L} \to \mathcal{L}'$ *and* $c : a^*\mathcal{N} \to \mathcal{N}'$ *are isomorphisms of invertible sheaves such that the pullbacks of* φ_k, ρ *and* ν_i *are* φ'_k, ρ' *and* ν'_i *respectively, where the pullbacks and the isomorphisms are induced by* a, b *and* c.

We define $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}$ to be the category fibered in groupoids over the category of schemes, such that objects in $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$ are S-families of MSP-fields, and morphisms are given by Definition 2.6.

Definition 2.7. *We call* $\xi \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ stable *if* $\mathrm{Aut}(\xi)$ *is finite. We call* $\xi \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$ *stable if* $\xi|_s$ *is stable for every closed* $s \in S$.

Let $\mathcal{W}_{g,\gamma,\mathbf{d}} \subset \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}$ be the open substack of families of stable objects in $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}$. We introduce a $T = \mathbb{C}^*$ action on $\mathcal{W}_{g,\gamma,\mathbf{d}}$ by

$$t \cdot (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, (\nu_1, \nu_2)) = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, (t\nu_1, \nu_2)), \quad t \in T. \tag{2.4}$$

Theorem 2.8. *The stack* $\mathcal{W}_{g,\gamma,\mathbf{d}}$ *is a DM* T*-stack, locally of finite type.*

Proof. The theorem follows immediately from that the stack $\mathcal{M}^{\mathrm{tw}}_{g,\ell}$ of stable twisted ℓ-pointed curves is a DM stack, and each of its connected components is proper and of finite type (see [AJ, Ol]). □

In this paper, we will reserve the symbol $T = \mathbb{C}^*$ for this action on $\mathcal{W}_{g,\gamma,\mathbf{d}}$.

Example 2.9 (Stable maps with p-fields)**.** *A stable MSP-field* $\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}$ *having* $\nu_1 = 0$ *will have* $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$, $\nu_2 = 1$. *Then* $\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \cdots)$ *reduces to a stable map* $f = [\varphi] : \Sigma^{\mathcal{C}} \subset \mathcal{C} \to \mathbb{P}^{N-1}$ *together with a p-field* $\rho \in H^0(f^*\mathcal{O}_{\mathbb{P}^{N-1}}(-r) \otimes \omega^{\log}_{\mathcal{C}})$. *Moduli of genus* g ℓ*-pointed stable maps (to* $\mathbb{P}^{N-1}$*) with p-fields will be denoted by* $\overline{M}_{g,\ell}(\mathbb{P}^{N-1}, d)^p$.

Example 2.10 (r-spin curves with p-fields)**.** *A stable MSP-field* $\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}$ *with* $\nu_2 = 0$ *will have* $\mathcal{N} \cong \mathcal{L}^\vee$, $\nu_1 = 1$. *Then* ξ *reduces to a pair of an* r*-spin curve* $(\Sigma^{\mathcal{C}}, \mathcal{C}, \rho : \mathcal{L}^r \cong \omega^{\log}_{\mathcal{C}})$ *and* N *p-fields* $\varphi_i \in H^0(\mathcal{L})$. *Moduli of* r*-spin curves with fixed monodromy* γ *and* N *p-fields will be denoted by* $\overline{M}^{1/r,Np}_{g,\gamma}$.

2.3. Cosection localized virtual cycle

The DM stack $\mathcal{W}_{g,\gamma,\mathbf{d}}$ admits a tautological T-equivariant perfect obstruction theory.

Let $\mathcal{D}_{g,\gamma}$ be the stack of $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N})$, where $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ are ℓ-pointed genus g connected twisted curves (objects in $\mathcal{M}^{\mathrm{tw}}_{g,\ell}$), $\mathcal{L}$ and $\mathcal{N}$ are invertible sheaves on $\mathcal{C}$ such that the monodromies of $\mathcal{L}$ along marked points are given by γ.

Because $\mathcal{M}^{\mathrm{tw}}_{g,\ell}$ is a smooth DM stack, $\mathcal{D}_{g,\gamma}$ is a smooth Artin stack, locally of finite type and of dimension $(3g-3)+\ell+2(g-1) = 5g-5+\ell$. Automorphisms of $(\Sigma^{\mathcal{C}} \subset \mathcal{C}, \mathcal{L}, \mathcal{N}) \in \mathcal{D}_{g,\gamma}(\mathbb{C})$ are triples (a,b,c) as in Definition 2.6.

Let

$$\pi : \Sigma^{\mathcal{C}} \subset \mathcal{C} \to \mathcal{W}_{g,\gamma,\mathbf{d}} \quad \text{with} \quad (\mathcal{L}, \mathcal{N}, \varphi, \rho, \nu) \tag{2.5}$$

be the universal family over $\mathcal{W}_{g,\gamma,\mathbf{d}}$. Define

$$q : \mathcal{W}_{g,\gamma,\mathbf{d}} \longrightarrow \mathcal{D}_{g,\gamma} \tag{2.6}$$

to be the forgetful morphism, forgetting the fields (φ,ρ,ν). The morphism q is T-equivariant with T acting on $\mathcal{D}_{g,\gamma}$ trivially.

Let $0 \le m_i \le r-1$ be so that $\gamma_i = \zeta_r^{m_i}$. For convenience, we let $\ell_\varphi = \#\{i \,|\, \gamma_i = (1,\varphi)\}$, and let $\ell_o = \#\{i \,|\, \gamma_i \in \boldsymbol{\mu}_r\}$. We abbreviate

$$\mathcal{P} = \mathcal{L}^{-r} \otimes \omega^{\log}_{\mathcal{C}/\mathcal{W}_{g,\gamma,\mathbf{d}}}.$$

Proposition 2.11. *The pair $q : \mathcal{W}_{g,\gamma,\mathbf{d}} \to \mathcal{D}_{g,\gamma}$ admits a tautological T-equivariant relative perfect obstruction theory taking the form*

$$\Bigl(R\pi_*\bigl(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus N} \oplus \mathcal{P}(-\Sigma^{\mathcal{C}}_{(1,\rho)}) \oplus (\mathcal{L}\otimes\mathcal{N}) \oplus \mathcal{N}\bigr)\Bigr)^{\vee} \longrightarrow L^{\bullet}_{\mathcal{W}_{g,\gamma,\mathbf{d}}/\mathcal{D}_{g,\gamma}}.$$

The virtual dimension $\delta(g,\gamma,\mathbf{d}) := \mathrm{vir.dim}\,\mathcal{W}_{g,\gamma,\mathbf{d}}$ is

$$(1+N-r)d_0 + (1-N+r)d_\infty + (4-N)(g-1) + \ell + (1-N)\bigl(\ell_\varphi + \sum_{i=1}^{\ell_o} \frac{m_i}{r}\bigr).$$

Proof. The construction of the obstruction theory is parallel to that in [CL, Prop 2.5], and will be omitted. We compute its virtual dimension. Let $\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \cdots)$ be a closed point in $\mathcal{W}_{g,\gamma,\mathbf{d}}$. Observe that when $\langle\gamma_i\rangle \neq \{1\}$, $\varphi|_{\Sigma^{\mathcal{C}}_i} = 0$. Thus by that (φ,ν_1) is nowhere vanishing, we see that $\nu_1|_{\Sigma^{\mathcal{C}}_i} \neq 0$, and the monodromy of $\mathcal{L}\otimes\mathcal{N}$ along $\Sigma^{\mathcal{C}}_i$ is trivial. Therefore, let $\pi : \mathcal{C} \to C$ be the coarse moduli morphism, the degrees of $\pi_*(\mathcal{L}\otimes\mathcal{N})$, $\pi_*\mathcal{L}$ and $\pi_*\mathcal{N}$ are d_0, $d_0 - d_\infty - \sum_{i=1}^{\ell_o}\frac{m_i}{r}$ and $d_\infty - \ell_o + \sum_{i=1}^{\ell_o}\frac{m_i}{r}$, respectively.

Further, the relative virtual dimension of $\mathcal{W}_{g,\gamma,\mathbf{d}} \to \mathcal{D}_{g,\gamma}$ is

$$\chi(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus N} \oplus \mathcal{L}^{\vee\otimes r}(-\Sigma^{\mathcal{C}}_{(1,\rho)}) \otimes \omega^{\log}_{\mathcal{C}} \oplus \mathcal{L}\otimes\mathcal{N} \oplus \mathcal{N}).$$

Here we insert $\Sigma^{\mathcal{C}}_{(1,\varphi)}$ and $\Sigma^{\mathcal{C}}_{(1,\rho)}$ because of (4) and (5) in Definition 2.4. Using $\dim \mathcal{D}_{g,\gamma} = 5g-5+\ell$ and applying Riemann-Roch theorem to $\chi(\mathcal{L}) = \chi(\pi_*\mathcal{L})$, we obtain the formula of $\delta(g,\gamma,\mathbf{d})$, as stated in the proposition. $\square$

The relative obstruction sheaf of $\mathcal{W}_{g,\gamma,\mathbf{d}} \to \mathcal{D}_{g,\gamma}$ is

$$\mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}/\mathcal{D}_{g,\gamma}} := R^1\pi_*\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus N} \oplus \mathcal{P}(-\Sigma^{\mathcal{C}}_{(1,\rho)}) \oplus (\mathcal{L}\otimes\mathcal{N}) \oplus \mathcal{N}\big),$$

and the absolute obstruction sheaf $\mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}}$ is the cokernel of the tautological map $q^*T_{\mathcal{D}_{g,\gamma}} \to \mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}/\mathcal{D}_{g,\gamma}}$, fitting into the exact sequence

$$q^*T_{\mathcal{D}_{g,\gamma}} \longrightarrow \mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}/\mathcal{D}_{g,\gamma}} \longrightarrow \mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}} \longrightarrow 0. \tag{2.7}$$

We define a cosection

$$\sigma : \mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}/\mathcal{D}_{g,\gamma}} \longrightarrow \mathcal{O}_{\mathcal{W}_{g,\gamma,\mathbf{d}}} \tag{2.8}$$

by the rule that at an S-point $\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}(S)$, $\xi = (\mathcal{C}, \Sigma^{\mathcal{C}}, \cdots)$, as in (1.2),

$$\sigma(\xi)(\dot\varphi,\dot\rho,\dot\nu_1,\dot\nu_2) = r\rho\sum \varphi_i^{r-1}\dot\varphi_i + \dot\rho\sum\varphi_i^r \in H^1(\omega_{\mathcal{C}/S}) \equiv H^0(\mathcal{O}_{\mathcal{C}})^\vee, \tag{2.9}$$

where

$$(\dot\varphi,\dot\rho,\dot\nu_1,\dot\nu_2) \in H^1\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus N} \oplus \mathcal{P}(-\Sigma^{\mathcal{C}}_{(1,\rho)}) \oplus \mathcal{L}\otimes\mathcal{N} \oplus \mathcal{N}\big).$$

Note that the term $r\rho\sum\varphi_i^{r-1}\dot\varphi_i + \dot\rho\sum\varphi_i^r$ a priori lies in $H^1\big(\mathcal{C}, \omega^{\log}_{\mathcal{C}/S}(-\Sigma^{\mathcal{C}}_{(1,\rho)})\big)$. However, when $\langle\gamma_i\rangle \neq (1,\rho)$, $\varphi_j|_{\Sigma^{\mathcal{C}}_i} = 0$. Thus it lies in $H^1(\mathcal{C}, \omega_{\mathcal{C}/S})$.

Lemma 2.12. *The rule* (2.9) *defines a T-equivariant homomorphism σ as in* (2.8)*. Via* (2.7) *σ lifts to a T-equivariant cosection of $\mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}}$.*

Proof. The proof that the cosection σ lifts is exactly the same as in [CLL], and will be omitted. That the homomorphism σ is T-equivariant is because T acts on $\mathcal{W}_{g,\gamma,\mathbf{d}}$ via scaling ν_1, and that σ is independent of ν_1. $\square$

As in [KL], we define the degeneracy locus of σ to be

$$\mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C}) = \{\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}(\mathbb{C}) \mid \sigma|_\xi = 0\}, \tag{2.10}$$

endowed with the reduced structure. It is a closed substack of $\mathcal{W}_{g,\gamma,\mathbf{d}}$.

Lemma 2.13. *The closed points of $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})$ are $\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ such that*

$$(\varphi = 0) \cup (\varphi_1^r + \cdots + \varphi_N^r = \rho = 0) = \mathcal{C}. \tag{2.11}$$

Proof. Consider individual terms in (2.9). Taking the term $\rho\varphi_i^{r-1}\dot\varphi_i$, by the vanishing along $\Sigma_i^{\mathcal{C}}$ (see the statement before Lemma 2.12), we conclude that

$$\rho\varphi_i^{r-1} \in H^0\big(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}}^{\log}(-\Sigma^{\mathcal{C}})\big) = H^0\big(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}}\big).$$

By Serre duality, when $\rho\varphi_i^{r-1} \neq 0$, there is a $\dot\varphi_i \in H^1(\mathcal{L})$ so that $\rho\varphi_i^{r-1} \cdot \dot\varphi_i \neq 0 \in H^1(\omega_{\mathcal{C}})$.

Repeating this argument, we conclude that $\sigma|_\xi = 0$ if and only if

$$\rho\varphi_1^{r-1} = \cdots = \rho\varphi_N^{r-1} = \varphi_1^r + \cdots + \varphi_N^r = 0.$$

This is equivalent to that $(\varphi = 0) \cup (\varphi_1^r + \cdots + \varphi_N^r = \rho = 0) = \mathcal{C}$. □

Note that (2.10) makes sense for $\xi \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ as well. Thus we denote

$$\mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(\mathbb{C}) = \big\{\xi \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(\mathbb{C}) \mid (2.10) \text{ holds for } \xi\big\}.$$

Applying [KL, CKL], we obtain the cosection localized virtual cycle

$$[\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}} \in A^T_\delta\big(\mathcal{W}^-_{g,\gamma,\mathbf{d}}\big), \quad \delta = \delta(g,\gamma,\mathbf{d}).$$

2.4. MSP invariants

Using the universal family (2.5), we define the evaluation maps (associated to the marked sections $\Sigma_i^{\mathcal{C}}$):

$$\mathrm{ev}_i : \mathcal{W}_{g,\gamma,\mathbf{d}} \to X := \mathbb{P}^N \cup \boldsymbol{\mu}_r.$$

In case $\langle\gamma_i\rangle \neq 1$, define ev_i to be the constant map to $\gamma_i \in \boldsymbol{\mu}_r$; in case $\gamma_i = (1,\varphi)$, define $\mathrm{ev}_i(\gamma_i) = 1 \in \boldsymbol{\mu}_r$. In case $\gamma_i = (1,\rho)$, let $s_i : \mathcal{W}_{g,\gamma,\mathbf{d}} \to \mathcal{C}$ be the i-th marked section of the universal curve[1], by (2) of Definition 2.4 we have $s_i^*\rho = 0$, thus $s_i^*\nu_2$ is nowhere zero and $s_i^*\mathcal{N} \cong \mathcal{O}_{\mathcal{W}_{g,\gamma,\mathbf{d}}}$. Thus, $s_i^*(\varphi,\nu_1)$ is a nonwhere zero section of $s_i^*\mathcal{L}^{\oplus(N+1)}$, defining the evaluation morphism

$$\mathrm{ev}_i = [s_i^*\varphi_1, \cdots, s_i^*\varphi_N, s_i^*\nu_1] : \mathcal{W}_{g,\gamma,\mathbf{d}} \to \mathbb{P}^N \tag{2.12}$$

such that $\mathrm{ev}_i^*\mathcal{O}_{\mathbb{P}^N}(1) = s_i^*\mathcal{L}$.

Let T act on $\mathbb{P}^N$ by

$$t \cdot [\varphi_1, \ldots, \varphi_N, \nu_1] = [\varphi_1, \ldots, \varphi_N, t\nu_1],$$

[1] As $\gamma_i = (1,\rho)$, the i-th marking is a scheme marking.

and let T act trivially on $\boldsymbol{\mu}_r$. It makes ev_i T-equivariant.

We introduce the MSP state space. As a $\mathbb{C}$-vector space, the MSP state space and the T-equivariant MSP state space are the cohomology group and the T-equivariant cohomology group of the evaluation space $X = \mathbb{P}^N \cup \boldsymbol{\mu}_r$:

$$\mathcal{H}^{\mathrm{MSP}} = H^*(X;\mathbb{Q}), \quad \text{and} \quad \mathcal{H}^{\mathrm{MSP},T} = H^*_T(X;\mathbb{Q}).$$

In terms of generators, we have

$$H^*_T(\mathbb{P}^N;\mathbb{Q}) = \mathbb{Q}[H,\mathfrak{t}]/\langle H^N(H+\mathfrak{t})\rangle, \quad \text{and} \quad H^*_T(\boldsymbol{\mu}_r;\mathbb{Q}) = \bigoplus_{j=1}^{r} \mathbb{Q}[\mathfrak{t}]\mathbf{1}_{\frac{j}{r}},$$

and the (non-equivariant) MSP state space is by setting $\mathfrak{t} = 0$, while the grading is given by

$$\deg H = 2, \quad \deg \mathfrak{t} = 2 \quad \text{and} \quad \deg \mathbf{1}_{\frac{j}{r}} = \frac{2(N-1)}{r} j. \tag{2.13}$$

We formulate the gravitational descendants. Given

$$a_1, \dots, a_\ell \in \mathbb{Z}_{\geq 0}, \quad \phi_1, \dots, \phi_\ell \in \mathcal{H}^{\mathrm{MSP}} = H^*(X;\mathbb{Q}),$$

we define the MSP-invariants

$$\langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle^{\mathrm{MSP}}_{g,\ell,\mathbf{d}} := \int_{[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}} \prod_{k=1}^{\ell} \psi_k^{a_k} \mathrm{ev}_k^* \phi_k \in \mathbb{Q}, \tag{2.14}$$

where

$$[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}} = \sum_{\gamma \in (\tilde{\boldsymbol{\mu}}_r)^\ell} [\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}.$$

Similarly for $\phi_i \in \mathcal{H}^{\mathrm{MSP},T}$, define T-equivariant genus g MSP-invariants:

(2.15)

$$\langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle^{\mathrm{MSP},T}_{g,\ell,\mathbf{d}} := \int_{[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}} \prod_{k=1}^{\ell} \psi_k^{a_k} \mathrm{ev}_k^* \phi_k \in H^*(BT;\mathbb{Q}) = \mathbb{Q}[\mathfrak{t}],$$

$$[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}} = \sum_{\gamma \in (\tilde{\boldsymbol{\mu}}_r)^\ell} [\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}.$$

Here we use the same $[\cdot]^{\mathrm{vir}}_{\mathrm{loc}}$ to mean the T-equivariant class. Suppose $\phi_1, \dots, \phi_\ell$ are homogeneous, and let

$$\begin{aligned}\mathbf{e}(a., \phi.) := \sum_{k=1}^{\ell}\Big(a_k + \frac{\deg \phi_k}{2}\Big) - \\ - \Big((1+N-r)d_0 + (1-N+r)d_\ell + (4-N)(g-1) + \ell\Big).\end{aligned}$$

By the formula of the virtual dimension of $\mathcal{W}_{g,\ell,\mathbf{d}}$, we see that

$$\langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell\rangle^{\mathrm{MSP},T}_{g,\ell,\mathbf{d}} \in \mathbb{Q}\mathbf{t}^{\mathbf{e}(a.,\phi.)}.$$

In case $\mathbf{e}(a., \phi.) < 0$, we have vanishing

$$\Big[\mathbf{t}^{-\mathbf{e}(a.,\phi.)} \cdot \langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell\rangle^{\mathrm{MSP},T}_{g,\ell,\mathbf{d}}\Big]_0 = 0, \tag{2.16}$$

where $[\cdot]_0$ is the dimension 0 part of the pushforward to $H_0(pt)$.

By virtual localization, we will express all genera full descendant MSP invariants in terms of (1): GW invariants of the quintic threefold $Q \subset \mathbb{P}^4$; (2): FJRW invariants of the Fermat quintic; and (3): the descendant integrals on $\overline{\mathcal{M}}_{g,\ell}$. The invariants in item (1) have been solved in genus zero [Gi, LLY] and genus one for all degrees [LZ, Zi], and in all genera for degree zero. Those in item (2) have been solved in genus zero [CR], and those in item (3) have been solved in all genera. One of our goals to introduce MSP invariants is to use vanishing (2.16) to obtain recursive relations to determine (1) and (2) in all genera. This is addressed in details in [CLLL].

3. Valuative criterion for properness

In this section, we will prove that $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ satisfies the valuative criterion for properness with residue field $\mathbb{C}$.

3.1. The conventions

In this section, we denote by $\eta_0 \in S$ a closed point in an affine smooth curve, and denote $S_* = S - \eta_0$ its complement.

In using valuative criterion to prove properness, we need to take a finite base change $S' \to S$ ramified over η_0. By shrinking S if necessary, we assume there is an étale $S \to \mathbb{A}^1$ so that η_0 is the only point lying over $0 \in \mathbb{A}^1$. In this way, for any positive integer a, we can take S' to be $S_a = S \times_{\mathbb{A}^1} \mathbb{A}^1$, where $\mathbb{A}^1 \to \mathbb{A}^1$ is via $t \mapsto t^a$. Note that $\eta_0' \in S_a$ lying over $\eta_0 \in S$ is the only point lying over $0 \in \mathbb{A}^1$, of ramification index a.

For notational convenience, for a property P that holds after a finite base change $S' \to S$ of a family ξ over S, we say "after a finite base change, the family ξ has the property P", meaning that we have already done the finite base change $S' \to S$ and then replace S' by S for abbreviation of notation.

In this and the next section, for $\xi \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$, we understand

$$\xi = \big(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu\big) \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S). \tag{3.1}$$

Similarly, we will use subscript "$*$" to denote families over S_*. Hence $\xi_* \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S_*)$ takes the form

$$\xi_* = \big(\Sigma^{\mathcal{C}_*}, \mathcal{C}_*, \mathcal{L}_*, \mathcal{N}_*, \varphi_*, \rho_*, \nu_*\big). \tag{3.2}$$

We first prove a simple version of the extension result.

Proposition 3.1. *Let $\xi_* \in \mathcal{W}^{-}_{g,\gamma,\mathbf{d}}(S_*)$ be such that $\rho_* = 0$. Then after a finite base change, ξ_* extends to a $\xi \in \mathcal{W}^{-}_{g,\gamma,\mathbf{d}}(S)$.*

Proof. Since $\rho_* = 0$, ν_{2*} is nowhere vanishing and $\mathcal{N}_* \cong \mathcal{O}_*$. Thus (φ_*, ν_{1*}) is a nowhere vanishing section of $H^0(\mathcal{L}_*^{\oplus N} \oplus \mathcal{L}_*)$, and induces a morphism f_* from $(\Sigma^{\mathcal{C}_*}, \mathcal{C}_*)$ to $\mathbb{P}^N$, such that $(f_*)^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{L}_*$. By the stability assumption of ξ_*, this morphism is an S_*-family of stable maps. By the properness of moduli stack of stable maps, after a finite base change we can extend $(\Sigma^{\mathcal{C}_*}, \mathcal{C}_*)$ to $(\Sigma^{\mathcal{C}}, \mathcal{C})$ and extend f_* to an S-family of stable maps f from $(\Sigma^{\mathcal{C}}, \mathcal{C})$ to $\mathbb{P}^N$. Let $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^N}(1)$, which is an extension of $\mathcal{L}_*$. Then f is provided by a section $(\varphi, \nu_1) \in H^0(\mathcal{L}^{\oplus N} \oplus \mathcal{L})$, extending (φ_*, ν_{1*}). Since $[f, \Sigma^{\mathcal{C}}, \mathcal{C}]$ is stable, the central fiber $\mathcal{C}_0$ is a connected curve with at worst nodal singularities. Define $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$ and ν_2 to be the isomorphism $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$ extending ν_{2*}. Define $\rho = 0$. Then $\xi = \big(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \varphi, \rho, \nu\big)$ is a desired extension. $\square$

The case involving $\varphi_* = 0$ over some irreducible components is technically more involved. We will treat this case by first studying the case $\mathcal{C}_*$ is smooth. For this, we characterize stable objects in $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(\mathbb{C})$. We say that an irreducible component $\mathcal{E} \subset \mathcal{C}$ is a rational curve if it is smooth and its coarse moduli is isomorphic to $\mathbb{P}^1$. We quote a well-known result.

Lemma 3.2. *Let $p_1 \neq p_2 \in \mathbb{P}^1$ be two distinct closed points, $G \leq \mathrm{Aut}(\mathbb{P}^1)$ be the subgroup fixing p_1 and p_2, and L be a G-linearized line bundle on $\mathbb{P}^1$ such that G acts trivially on $L|_{p_1}$. Then the following holds:*

1. *any invariant* $s \in H^0(L)^G$ *with* $s(p_1) = 0$ *must be the zero section;*
2. *suppose* G *acts trivially on* $L|_{p_2}$*, then* $L \cong \mathcal{O}_{\mathbb{P}^1}$.

Lemma 3.3. *Let* $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(\mathbb{C})$*; it is unstable if and only if one of the following holds:*

1. $\mathcal{C}$ *contains a rational curve* $\mathcal{E}$ *such that* $\mathcal{E} \cap (\Sigma^{\mathcal{C}} \cup \mathcal{C}_{\mathrm{sing}})$ *contains two points, and* $\mathcal{L}^{\otimes r}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$*;*
2. $\mathcal{C}$ *contains a rational curve* $\mathcal{E}$ *such that* $\mathcal{E} \cap (\Sigma^{\mathcal{C}} \cup \mathcal{C}_{\mathrm{sing}})$ *contains one point, and either* $\mathcal{L}|_{\mathcal{E}} \cong \mathcal{N}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$ *or* $\rho|_{\mathcal{E}}$ *is nowhere vanishing;*
3. $\mathcal{C}$ *is a smooth rational curve with* $\Sigma^{\mathcal{C}} = \emptyset$, $d_0 = d_\infty = 0$*;*
4. $\mathcal{C}$ *is irreducible,* $g = 1$, $\Sigma^{\mathcal{C}} = \emptyset$*, and* $\mathcal{L}^{\otimes r} \cong \mathcal{O}_{\mathcal{C}}$ *and* $\mathcal{L}^\vee \cong \mathcal{N}$.

Proof. We first prove the necessary part. Let $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ be unstable. For each irreducible $\mathcal{E} \subset \mathcal{C}$, let $\mathrm{Aut}_{\mathcal{E}}(\xi)$ be the subgroup of $\mathrm{Aut}(\xi)$ keeping $\mathcal{E}$ invariant. Since $\mathrm{Aut}(\xi)$ is infinite, there exists an $\mathcal{E} \subset \mathcal{C}$ such that $\mathrm{Aut}_{\mathcal{E}}(\xi)$ is infinite. If the image of $\mathrm{Aut}_{\mathcal{E}}(\xi) \to \mathrm{Aut}(\mathcal{E})$ is finite, then for a finite index subgroup $G' \le \mathrm{Aut}_{\mathcal{E}}(\xi)$, G' leaves $\mathcal{C}$ fixed, thus G' acts on ξ by acting on the line bundles $\mathcal{L}$ and $\mathcal{N}$ via scaling. However, by Definition 2.4, that G' leaves (φ, ρ, ν) invariant implies that the image of $G' \to \mathrm{Aut}(\mathcal{L}) \times \mathrm{Aut}(\mathcal{N})$ is finite, a contradiction since this arrow is injective. Thus the group $G = \mathrm{im}(\mathrm{Aut}_{\mathcal{E}}(\xi) \to \mathrm{Aut}(\mathcal{E}))$ is infinite.

We now consider the case where $\mathcal{E}$ has arithmetic genus zero (thus smooth). We divide it into several cases. The first case (when $g_a(\mathcal{E}) = 0$) is when $\mathcal{E} \cap (\Sigma^{\mathcal{C}} \cup \mathcal{C}_{\mathrm{sing}})$ contains one point, say $p \in \mathcal{E}$. Suppose $\rho|_{\mathcal{E}} = 0$, then $\nu_2|_{\mathcal{E}}$ is nowhere vanishing, implying $\mathcal{N}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$. Thus $(\varphi, \nu_1)|_{\mathcal{E}}$ is a nowhere vanishing section of $H^0(\mathcal{L}^{\oplus(n+1)}|_{\mathcal{E}})$. Since G is infinite and $(\varphi, \nu_1)|_{\mathcal{E}}$ is G-equivariant, this is possible only if $\mathcal{L}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$ and $(\varphi, \nu_1)|_{\mathcal{E}}$ is a constant section. This is Case (2).

Suppose $\rho|_{\mathcal{E}} \neq 0$. We argue that $\rho|_{\mathcal{E}}$ is nowhere vanishing. Otherwise, $\nu_2|_{\mathcal{E}} \neq 0$, and then $\deg \mathcal{N}|_{\mathcal{E}} \ge 0$. Since $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(\mathbb{C})$, we have $\varphi|_{\mathcal{E}} = 0$, thus $\nu_1|_{\mathcal{E}}$ is nowhere vanishing and $\mathcal{L}^\vee|_{\mathcal{E}} \cong \mathcal{N}|_{\mathcal{E}}$. Because $\rho|_{\mathcal{E}} \neq 0$ and $\deg \omega^{\log}_{\mathcal{C}}|_{\mathcal{E}} = -1$, we must have $\deg \mathcal{L}|_{\mathcal{E}} < 0$. Thus $\nu_2 \in H^0(\mathcal{N}|_{\mathcal{E}}) = H^0(\mathcal{L}^\vee|_{\mathcal{E}})$ must vanish at some point. Let p_1 and $p_2 \in \mathcal{E}$ be such that $\rho(p_1) = 0 = \nu_2(p_2)$. Since (ρ, ν_2) is nowhere vanishing, we have $p_1 \neq p_2$. Furthermore, since G fixes p, p_1 and p_2, and is infinite, $p = p_1$ or p_2.

Suppose $p = p_1$, a similar argument shows that $\mathcal{L}^{-r} \otimes \omega^{\log}_{\mathcal{C}}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$, contradicting to $\rho \neq 0$ and vanishing somewhere. Suppose $p = p_2$, we conclude that $\deg \mathcal{L}|_{\mathcal{E}} = 0$, contradicting to $\deg \mathcal{L}|_{\mathcal{E}} < 0$. Combined, we proved that if $\rho|_{\mathcal{E}} \neq 0$, then $\rho|_{\mathcal{E}}$ is nowhere vanishing. This is Case (2).

The second case is when $\mathcal{E} \cap (\Sigma^{\mathcal{C}} \cup \mathcal{C}_{\mathrm{sing}})$ contains two points, say p_1 and $p_2 \in \mathcal{E}$. Then G fixes both p_1 and p_2. A parallel argument shows that

G acts trivially on $\mathcal{L}^r|_{p_1}$ and $\mathcal{L}^r|_{p_2}$. Applying Lemma 3.2, we conclude that $\mathcal{L}^r \cong \mathcal{O}_{\mathcal{E}}$. This is Case (1). The third case is when $\mathcal{E} \cap (\Sigma^{\mathcal{C}} \cup \mathcal{C}_{\text{sing}}) = \emptyset$. A parallel argument shows that in this case we must have $\mathcal{L} \cong \mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$. This concludes the study of the case $g_a(\mathcal{E}) = 0$.

The remaining case is when $g_a(\mathcal{E}) = 1$, then $\mathcal{E} \cap \Sigma^{\mathcal{C}} = \emptyset$, and a similar argument shows that it must belong to Case (4). Combined, this proves that if ξ is unstable, then one of (1)-(4) holds.

We now prove the other direction that whenever there is an $\mathcal{E} \subset \mathcal{C}$ that satisfies one of (1)-(4), then ξ is unstable. Most of the cases can be argued easily, except a sub-case of (2) when $\rho|_{\mathcal{E}}$ is nowhere vanishing, which we now prove. Since $\mathcal{E} \cap (\Sigma_{\mathcal{C}} \cup \mathcal{C}_{\text{sing}})$ contains one point, say $p \in \mathcal{C}$, we have $\deg \omega_{\mathcal{C}}^{\log}|_{\mathcal{E}} = -1$. Since $\rho|_{\mathcal{E}}$ is nowhere vanishing, we have $\deg \mathcal{L}^r|_{\mathcal{E}} = -1$. Thus p must be a stacky point. Hence $\mathcal{E} \cong \mathbb{P}_{1,r}$ as stacks. Let $\mathbb{P}_{1,r} = \operatorname{Proj}(k[x,y])$ where $\deg x = 1$ and $\deg y = r$. Then p corresponds to the point $[0,1]$. Let $G = \mathbb{C}$ (the additive group) act on $\mathbb{P}_{1,r}$ via $x \to x$, $y \to \lambda x^r + y$ for $\lambda \in G$. The G-action on $\mathbb{P}_{1,r}$ lifts to an action of $\omega_{\mathcal{C}}^{\log}|_{\mathcal{E}}$ as well as $\mathcal{L}^\vee|_{\mathcal{E}} \cong \mathcal{O}_{\mathbb{P}_{1,r}}(1)$. One can check via local calculations that G acts trivially on $\big(\omega_{\mathcal{C}}^{\log}|_{\mathcal{E}}\big)|_p$ as well as $\big(\mathcal{L}^{-1}|_{\mathcal{E}}\big)|_p$, thus trivially on $\big(\mathcal{L}^{-r} \otimes \omega_{\mathcal{C}}^{\log}\big)|_p$. Since $\mathcal{L}^{-r} \otimes \omega_{\mathcal{C}}^{\log}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$, and since $\mathbb{G}_a = \mathbb{C}$ has no non-trivial characters, by Prop. 1.4 in [FMK], G acts on $\big(\mathcal{L}^{-r} \otimes \omega_{\mathcal{C}}^{\log}\big)|_{\mathcal{E}}$ trivially as well. Hence G acts trivially on $\rho|_{\mathcal{E}}$. Therefore the group G is a subgroup of the automorphism group of $\xi|_{\mathcal{E}}$. $\square$

Corollary 3.4. *Let $\xi \in \mathcal{W}^{\text{pre}-}_{g,\gamma,\mathbf{d}}(\mathbb{C})$. Let $\pi : \tilde{\mathcal{C}} \to \mathcal{C}$ be the normalization of $\mathcal{C}$, let $\Sigma^{\tilde{\mathcal{C}}} = \pi^{-1}(\Sigma^{\mathcal{C}} \cup \mathcal{C}_{\text{sing}})$, and let $(\tilde{\mathcal{L}}, \tilde{\mathcal{N}}, \tilde{\varphi}, \tilde{\rho}, \tilde{\nu})$ be the pullback of $(\mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$ via π^*. Writing $\tilde{\mathcal{C}} = \coprod_a \tilde{\mathcal{C}}_a$ the connected component decomposition, and letting $\tilde{\xi}_a$ be $(\Sigma^{\tilde{\mathcal{C}}} \cap \tilde{\mathcal{C}}_a, \tilde{\mathcal{C}}_a)$ paired with $(\tilde{\mathcal{L}}, \tilde{\mathcal{N}}, \tilde{\varphi}, \tilde{\rho}, \tilde{\nu})|_{\tilde{\mathcal{C}}_a}$, then ξ is stable if and only if all $\tilde{\xi}_a$ are stable.*

Proof. If ξ is unstable, then it contains an irreducible $\mathcal{E}$ satisfying one of (1)-(4) in Lemma 3.3. This $\mathcal{E}$ (or its normalization) will appear in one of $\tilde{\xi}_a$, making it unstable. The other direction is the same. This proves the Corollary. $\square$

This lemma shows that the crucial part is to study the specialization of MSP fields with irreducible domain curves.

3.2. The baskets

We begin with a special case. Let $\xi_* \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$ be of the form (3.2). We say that it is of *φ-vanishing type* if $\mathcal{C}_*$ is smooth (and connected), $\varphi_* = 0$,

$\nu_{2*} \neq 0$ and $\rho_* \neq 0$.

Proposition 3.5. *Let ξ_* over S_* be of φ-vanishing type. Then after a finite base change,*

*(**a1**) $(\Sigma^{\mathcal{C}_*}, \mathcal{C}_*)$ extends to a pointed twisted curve $(\Sigma^{\mathcal{C}}, \mathcal{C})$ over S such that $\mathcal{C} - \Sigma^{\mathcal{C}}$ is a scheme, $\mathcal{C}$ is smooth, and the central fiber $\mathcal{C}_0$ is reduced with at worst nodal singularities and smooth irreducible components;*
*(**a2**) $\mathcal{L}_*$ and $\mathcal{N}_*$ extend to invertible sheaves $\mathcal{L}$ and $\mathcal{N}$ respectively on $\mathcal{C}$ so that ν_* extends to a surjective $\nu = (\nu_1, \nu_2) : \mathcal{L}^\vee \oplus \mathcal{O}_{\mathcal{C}} \to \mathcal{N}$;*
*(**a3**) ρ_* extends to a $\rho \in \Gamma(\mathcal{L}^{-r} \otimes \omega^{\log}_{\mathcal{C}/S}(\mathcal{D}))$ for a divisor $\mathcal{D}$ on $\mathcal{C}$ contained in the central fiber $\mathcal{C}_0$ such that ρ restricting to every irreducible component of $\mathcal{C}_0$ is non-trivial;*
*(**a4**) $\overline{(\rho_* = 0)} \cap \overline{(\nu_{2*} = 0)} = \emptyset$, $\overline{(\rho_* = 0)}$ and $\overline{(\nu_{2*} = 0)}$ intersect $\mathcal{C}_0$ transversally.*

Proof. First, possibly after a finite base change, we can assume that $\overline{(\rho_* = 0)} \cup \overline{(\nu_{2*} = 0)}$ is a union of disjoint sections of $\mathcal{C}_* \to S_*$, and that if we let Σ^{ex}_* be the union of those sections of $\overline{(\rho_* = 0)} \cup \overline{(\nu_{2*} = 0)}$ that are not contained in $\Sigma^{\mathcal{C}_*}$, then Σ^{ex}_* is disjoint from $\Sigma^{\mathcal{C}_*}$. If $(\Sigma^{\mathcal{C}_*} \cup \Sigma^{\mathrm{ex}}_*, \mathcal{C}_*)$ is a stable pointed curve, let $\Sigma^{\mathrm{au}}_* = \emptyset$. Otherwise, let Σ^{au}_* be some extra sections of $\mathcal{C}_* \to S_*$, disjoint from $\Sigma^{\mathcal{C}_*} \cup \Sigma^{\mathrm{ex}}_*$, so that after letting $\Sigma^{\mathrm{comb}}_* = \Sigma^{\mathcal{C}_*} \cup \Sigma^{\mathrm{ex}}_* \cup \Sigma^{\mathrm{au}}_*$, the pair $(\Sigma^{\mathrm{comb}}_*, \mathcal{C}_*)$ is stable.

Since $(\Sigma^{\mathrm{comb}}_*, \mathcal{C}_*)$ is stable, possibly after a finite base change, it extends to an S-family of stable twisted curves $(\Sigma^{\mathrm{comb}}, \mathcal{C}')$ such that all singular points of its central fiber $\mathcal{C}'_0$ are non-stacky. Thus after blowing up $\mathcal{C}'$ along the singular points of $\mathcal{C}'_0$ if necessary, taking a finite base change, and followed by a minimal desingularization, we can assume that the resulting family $(\Sigma^{\mathcal{C}}, \mathcal{C})$ is a family of pointed twisted curves with smooth $\mathcal{C}$ satisfying Condition (a1). Condition (a4) is satisfied due to the construction.

Since φ_* is identically zero, ν_{1*} is an isomorphism. We can extend $\mathcal{N}_*$ to an invertible sheaf $\mathcal{N}$ on $\mathcal{C}$ so that ν_{2*} extends to a section ν_2 of $\mathcal{N}$. Let $\mathcal{L} \cong \mathcal{N}^\vee$. We extend ν_{1*} to an isomorphism $\nu_1 : \mathcal{L}^\vee \to \mathcal{N}$, and extend ρ_* to a section ρ satisfying (a3), for a choice of $\mathcal{D}$. This proves the proposition. □

Definition 3.6. *An S-family of pre-stacky pointed nodal curves is a flat S-family (Σ, C) of pointed nodal curves (i.e. not twisted curves) so that each marked-section Σ_i (of Σ) is assigned an integer r_i for $r_i \in \mathbb{Z}_{>0}$. We call it a good family if in addition C is smooth, and all irreducible components of the central fiber $C_0 = C \times_S \eta_0$ are smooth.*

Note that if we let C be the coarse moduli of the $\mathcal{C}$ in Proposition 3.5, let $\Sigma_i \subset C$ be the image of $\Sigma_i^{\mathcal{C}} \subset \mathcal{C}$ under $\mathcal{C} \to C$, call Σ_i pre-stacky if $\Sigma_i^{\mathcal{C}}$ is stacky, and call it regular otherwise, then (Σ, C) with this assignment is a good S-family of pre-stacky pointed nodal curves.

Given an S-family of pointed twisted curve $(\Sigma^{\mathcal{C}}, \mathcal{C})$ so that the only non-scheme points of $\mathcal{C}$ are possibly along $\Sigma^{\mathcal{C}}$, applying the procedure described, we obtain a pre-stacky pointed nodal curve (Σ, C). We call this procedure un-stacking. Conversely, applying the root construction (cf. [AGV, Cad]) to the S-family of pre-stacky pointed curves (Σ, C), we recover the original family $(\Sigma^{\mathcal{C}}, \mathcal{C})$ after knowing the $\boldsymbol{\mu}_{r_i}$-stacky structure of $\mathcal{C}$ along $\Sigma_i^{\mathcal{C}}$. We call the latter the stacking of (Σ, C). When $r_i = 1$, $\mathcal{C}$ is a scheme along $\Sigma_i^{\mathcal{C}}$.

Definition 3.7. *Let (Σ, C) be a good S-family of pre-stacky pointed nodal curves, and let D_i, $i \in \Lambda$, be irreducible components of C_0. A pre-basket of (Σ, C) is a data*

$$\mathcal{B} = (B + \sum\nolimits_{i\in\Lambda} l_i D_i, A + \sum\nolimits_{i\in\Lambda} m_i D_i), \tag{3.3}$$

consisting of

1. *$A = \sum_{i=1}^{k_1} a_i A_i$, where $A_1, \cdots, A_{k_1}$ are disjoint sections of $C \to S$ such that for any pair (i,j), either $A_i \cap \Sigma_j = \emptyset$, or $A_i = \Sigma_j$ and $a_i \in \frac{1}{r}\mathbb{Z}_{>0}$; when $a_i = b_i/r_i$ is in reduced form, Σ_i is assigned $\boldsymbol{\mu}_{r_i}$-stacky;*
2. *$B = \sum_{i=1}^{k_2} b_i B_i$, where $b_i \in \mathbb{Z}_{>0}$, $B_1, \cdots, B_{k_2}$ are disjoint sections of $C \to S$ such that for any pair (i,j), either $B_i \cap \Sigma_j = \emptyset$ or $B_i = \Sigma_j$; when $B_i = \Sigma_j$, Σ_j is assigned regular;*
3. *$A_1, \cdots, A_{k_1}, B_1, \cdots, B_{k_2}$ are mutually disjoint and intersect C_0 transversally;*
4. *$rm_i \in \mathbb{Z}$ and $l_i \in \mathbb{Z}$;*

such that

$$\mathcal{O}_C(B + \sum l_i D_i) \cong \mathcal{O}_C(rA + \sum rm_i D_i) \otimes \omega_{C/S}^{\log}. \tag{3.4}$$

We call $\mathcal{B}$ a basket if in addition $l_i \geq 0$, $m_i \geq 0$ and $l_i m_i = 0$ for all i.

Definition 3.8. *We say a basket $\mathcal{B}$ final if it satisfies*

(i) for every $i \in \Lambda$, $B \cap D_i = \emptyset$ if $m_i \neq 0$, and $A \cap D_i = \emptyset$ if $l_i \neq 0$;

(ii) for distinct $i \neq j \in \Lambda$ such that $l_i m_j \neq 0$, $D_i \cap D_j = \emptyset$.

Let $(\mathcal{C}, \mathcal{L}, \mathcal{N}, \rho, \nu)$ and $\mathcal{D}$ be given by Proposition 3.5. Let $\{\mathcal{D}_i \,|\, i \in \Lambda\}$ be the set of irreducible components of $\mathcal{C}_0$. We form

$$\mathcal{A} = \overline{(\nu_{2*} = 0)},\ \mathcal{B} = \overline{(\rho_* = 0)},\ (\nu_2 = 0) = \mathcal{A} + \sum m_i \mathcal{D}_i,\ \mathcal{D} = -\sum l_i \mathcal{D}_i, \tag{3.5}$$

where the summations run over all $i \in \Lambda$. By the construction, ν_2 and ρ induce isomorphisms $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}(\mathcal{A} + \sum m_i \mathcal{D}_i)$ and $\mathcal{O}_{\mathcal{C}} \cong \mathcal{L}^{-r} \otimes \omega^{\log}_{\mathcal{C}/S}(\mathcal{D} - \mathcal{B})$. Using $\mathcal{L}^{\vee} \cong \mathcal{N}$, we obtain an isomorphism

$$\mathcal{O}_{\mathcal{C}}(\mathcal{B} + \sum l_i \mathcal{D}_i) \cong \mathcal{O}_{\mathcal{C}}(r\mathcal{A} + \sum r m_i \mathcal{D}_i) \otimes \omega^{\log}_{\mathcal{C}/S}. \tag{3.6}$$

Let (Σ, C) be the good S-family of pre-stacky pointed nodal curves that is the un-stacking of $(\Sigma^{\mathcal{C}}, \mathcal{C})$ as explained before Definition 3.7. Let $D_i \subset C_0$ be the image of $\mathcal{D}_i$. Since $\mathcal{C}$ away from $(\nu_2 = 0)$ is a scheme, and by the construction carried out in the proof of Proposition 3.5, we have $\mathcal{B} = \sum_{i=1}^{k_2} b_i \mathcal{B}_i$, where $\mathcal{B}_i$ are sections of $\mathcal{C} \to S$ and $b_i \in \mathbb{Z}_{>0}$, and for any (i,j) either $\mathcal{B}_i \cap \Sigma_j^{\mathcal{C}} = \emptyset$ or $\mathcal{B}_i = \Sigma_j^{\mathcal{C}}$, and in the latter case $\Sigma_j^{\mathcal{C}}$ is a scheme. We let $B_i \subset C$ be the image of $\mathcal{B}_i$. For $\mathcal{A}$, it can also be written as $\mathcal{A} = \sum_{i=1}^{k_1} a_i \mathcal{A}_i$, where $\mathcal{A}_i$ are sections of $\mathcal{C} \to S$. Let A_i be the image of $\mathcal{A}_i$. We form

$$A = \sum_{A_i \notin \{\text{pre-stacky}\, \Sigma_j\}} a_i A_i + \sum_{A_i \in \{\text{pre-stacky}\, \Sigma_j\}} \frac{a_i}{r} A_i, \quad \text{and} \quad B = \sum_{i=1}^{k_2} b_i B_i.$$

Lemma 3.9. *Let (Σ, C) be as before, let $\mathcal{B}$ in (3.3) be such that the coefficients l_i and m_i are given in (3.5), and let A and B be given in the identities above. Then $\mathcal{B}$ is a pre-basket.*

Proof. That $\mathcal{B}$ satisfies (1)-(4) in Definition 3.7 follows from the proof of Prop. 3.5. For the isomorphism (3.4), by our choice of A and B, we have

$$\mathcal{O}_{\mathcal{C}}(\mathcal{B} + \sum l_i \mathcal{D}_i) \cong \mathcal{O}_C(B + \sum l_i D_i) \otimes_{\mathcal{O}_C} \mathcal{O}_{\mathcal{C}},$$

and

$$\mathcal{O}_{\mathcal{C}}(r\mathcal{A} + \sum r m_i \mathcal{D}_i) \otimes \omega^{\log}_{\mathcal{C}/S} \cong \big(\mathcal{O}_C(rA + \sum r m_i D_i) \otimes \omega^{\log}_{C/S}\big) \otimes_{\mathcal{O}_C} \mathcal{O}_{\mathcal{C}}.$$

Therefore, (3.4) follows from (3.6). This proves the Lemma. $\square$

3.3. Restacking

In this subsection, we fix an S-family of pre-stacky ℓ-pointed nodal curves (Σ, C) and a final basket $\mathcal{B}$ on it in the notation of Definition 3.7 and 3.8. Let t be a uniformizing parameter of R, where $S = \operatorname{Spec} R$, that is the pullback of the standard coordinate variable of $\mathbb{A}^1$ via the map $S \to \mathbb{A}^1$ specified at the beginning of §3.1. Let $R_r = R[z]/(z^r - t)$, and let $S_r = \operatorname{Spec} R_r$.

Lemma 3.10. *Let R be a DVR and $S = \mathrm{Spec}R$. Let C be a flat S-family of nodal curves, let N be the singular points of the central fiber C_0, and let M be an integral effective Cartier divisor on C such that $M = rM_h + M_0$, where M_0 is an integral Weil divisor contained in C_0 and where M_h is an integral Weil divisor such that none of its irreducible components lie in C_0. Then there is an S_r-family of twisted curves $\tilde{\mathcal{C}}$ such that*

1. *let $\tilde{N} \subset \tilde{\mathcal{C}}_0$ be the singular points of $\tilde{\mathcal{C}}_0$, then $\tilde{\mathcal{C}} - \tilde{N} \cong (C - N) \times_S S_r$;*
2. *let $\tilde{\phi} : \tilde{\mathcal{C}} - \tilde{N} \to C$ be the morphism induced by (1), then $\tilde{\phi}^*(M)$ is divisible by r, and $\frac{1}{r}\tilde{\phi}^*(M)$ extends to a Cartier divisor on $\tilde{\mathcal{C}}$, denoted by $\tilde{M}_{\frac{1}{r}}$;*
3. *each $\tilde{p} \in \tilde{N}$ is either a scheme point or a $\boldsymbol{\mu}_a$-stacky point of $\tilde{\mathcal{C}}$, where $a|r$, such that the tautological map $\mathrm{Aut}(\tilde{p}) \to \mathrm{Aut}(\mathcal{O}_{\tilde{\mathcal{C}}}(\tilde{M}_{\frac{1}{r}})|_{\tilde{p}})$ is injective.*

Proof. Let $p \in N$ be a singular point of C_0. We can choose a Zariski open subset W of p in $\mathcal{C}$ such that any irreducible component of the divisor $W \cap M$ contains p, and $W \cap M$ is given by the zero locus of a regular function f. We can find an étale open neighborhood $q\colon (V, v_o) \to (C, p)$ with an étale morphism $\pi\colon (V, v_o) \to \big(Y\colon = \mathrm{Spec}\big(R[x,y]/(xy = t^k)\big), \mathbf{0}\big)$ over S. Here t is the uniformizing parameter of R. Let D_1 and D_2 be prime Weil divisors on Y corresponding to the ideals (x,t) and (y,t) respectively. We can choose V small enough such that v_o is the only pre-image of p and $q^*M_0 = \pi^*(n_1D_1 + n_2D_2)$ for some nonnegative integers n_1 and n_2.

Consider the base change $\tilde{Y} := Y \times_S S_r \to Y$. Let $\tilde{\mathbf{0}}$ be the node in the central fiber $\tilde{Y} \to \mathrm{Spec}R_r$. Then we have $\varphi\colon \tilde{Y} = \mathrm{Spec}\big(R_r[x,y]/(xy - z^{kr})\big) \to Y$, where z is the uniformizing parameter of R_r.

Let $\tilde{U}\colon = \mathrm{Spec}\big(R_r[u,v]/(uv - z)\big)$. $\tilde{U}$ is smooth. Consider the morphism $\tilde{\varphi}\colon \tilde{U} \to \tilde{Y}$ via $x \to u^{kr}$ and $y \to v^{kr}$. Consider a $\boldsymbol{\mu}_{kr}$-action on $\tilde{U}$ via $\zeta \cdot (u,v) = (\zeta u, \zeta^{-1}v)$ for $\zeta \in \boldsymbol{\mu}_{kr}$. Then the quotient $\tilde{U}/\boldsymbol{\mu}_{kr} \cong \tilde{Y}$. Let $\tilde{D}_1 = \{u = 0\}$ and $\tilde{D}_2 = \{v = 0\}$ be Cartier divisors in $\tilde{U}$. Let $\tilde{V} = V \times_Y \tilde{Y} = V \times_S S_r$ and $V' = \tilde{U} \times_{\tilde{Y}} \tilde{V}$. The $\boldsymbol{\mu}_{kr}$-action on $\tilde{U}$ lifts to a $\boldsymbol{\mu}_{kr}$-action on V', $\tilde{V}$ is étale over $\tilde{Y}$, and V' is étale over $\tilde{U}$. Hence V' is smooth. Let $\pi'\colon V' \to \tilde{U}$. Let $q'\colon (V', v_o') \to (W, p)$ be the induced morphism. We can choose a Zariski open subset of V', still denoted by V' without loss of generality, such that $q'^*(M_0) = \pi'^*(krn_1\tilde{D}_1 + rkn_2\tilde{D}_2)$ and

$$\frac{1}{r}q'^*M = \frac{1}{r}\big(rq'^*M_h + krn_1\bar{D}_1 + krn_2\bar{D}_2\big) = q'^*M_h + kn_1\bar{D}_1 + kn_2\bar{D}_2,$$

where $\bar{D}_i = \pi'^{-1}(\tilde{D}_i)$. It is a Cartier divisor and $\boldsymbol{\mu}_{kr}$-invariant. Let g be a regular function in a Zariski open neighbourhood of v_o' whose zero locus is the divisor $\frac{1}{r}q'^*M$. Clearly $q'^*(f)/g^r$ is a nowhere zero regular function. There exists a $\boldsymbol{\mu}_{kr}$-equivariant étale morphism $\bar{q}\colon X \to V'$ and a regular function h on X such that $\bar{q}^*(q'^*(f)/g^r) = h^r$. Since q'^*f is $\boldsymbol{\mu}_{kr}$-invariant, the action of an element $\zeta \in \boldsymbol{\mu}_{kr}$ on the function $(\bar{q}^*q'^*f)^\zeta$ is $\bar{q}^*q'^*f$. Thus the action of an element $\zeta \in \boldsymbol{\mu}_{kr}$ on the function $h\bar{q}^*g$ satisfies $\big((h\bar{q}^*g)^\zeta\big)^r = (h\bar{q}^*g)^r$, that is, $(h\bar{q}^*g)^\zeta = \zeta^\beta h\bar{q}^*g$ for some nonnegative integer β. Let $\boldsymbol{\mu}_b$ be the largest subgroup of $\boldsymbol{\mu}_{kr}$ such that $h\bar{q}^*g$ is invariant under the action of $\boldsymbol{\mu}_b$. Let $\boldsymbol{\mu}_a = \boldsymbol{\mu}_{kr}/\boldsymbol{\mu}_b$. Then $h\bar{q}^*g$ descends to a regular function $\tilde{g}$ on $\tilde{X}\colon = X/\boldsymbol{\mu}_b$. Let $\tilde{q}\colon (\tilde{X}, \tilde{x}_o) \to (W, p)$ be the induced morphism. $\frac{1}{r}\tilde{q}^*(M)$ is the divisor defined by the zero of $\tilde{g}$ and a $\boldsymbol{\mu}_a$-action on $\tilde{X}$ such that, for $\xi \in \boldsymbol{\mu}_a$, $\tilde{g}^\xi = \xi^\alpha \tilde{g}$ where α and a are relatively prime. $\tilde{X}/\boldsymbol{\mu}_a$ is étale over $\tilde{V}$. It is clear that $a \,|\, r$.

Let $\tilde{C}\colon = C \times_S S_r$ be the base change of C and $\tilde{p}$ be the node in $\tilde{C}$ mapped to p. The above discussions introduce possibly a stacky structure at $\tilde{p}$, that is, we use $[\tilde{X}/\boldsymbol{\mu}_a]$ as a local chart for $\tilde{p}$. By repeating this over all $p \in N$, we obtain $\tilde{\phi} : \tilde{\mathcal{C}} \to C$ such that $\mathcal{C}$ is a twisted curve defined in [AV] and $\tilde{M}_{\frac{1}{r}} = \frac{1}{r}\tilde{\phi}^*(M)$ is an integral Cartier divisor satisfying the requirements of the Lemma where $\tilde{\phi}$ is the morphism from $\tilde{\mathcal{C}}$ to C. □

Corollary 3.11. *Let C be a flat S-family of nodal curves, let N be the singular points of C_0, and $v \in H^0(\mathcal{M})$ be a section of an invertible sheaf $\mathcal{M}$ on $C - N$ so that $\mathcal{M}^{\otimes r}$ extends to an invertible sheaf on C. Then there is an S_r-family of twisted nodal curves $\tilde{\mathcal{C}}$ such that*

1. *let $\tilde{N} \subset \tilde{\mathcal{C}}_0$ be the singular points of $\tilde{\mathcal{C}}_0$, then $\tilde{\mathcal{C}} - \tilde{N} \cong (C - N) \times_S S_r$;*
2. *there is an invertible sheaf $\tilde{\mathcal{M}}$ on $\tilde{\mathcal{C}}$ and a section $\tilde{v} \in H^0(\tilde{\mathcal{M}})$ so that, letting $\tilde{\phi} : \tilde{\mathcal{C}} - \tilde{N} \to C$ be the morphism induced by (1), then $\tilde{\mathcal{M}}|_{\tilde{\mathcal{C}}-\tilde{N}} \cong \tilde{\phi}^*\mathcal{M}$, and $\tilde{v}|_{\tilde{\mathcal{C}}-\tilde{N}} = \tilde{\phi}^*v$;*
3. *each $\tilde{p} \in \tilde{N}$ is either a scheme point or a $\boldsymbol{\mu}_a$-stacky point, $a|r$, of $\tilde{\mathcal{C}}$, and the tautological map $\operatorname{Aut}(\tilde{p}) \to \operatorname{Aut}(\tilde{\mathcal{M}}|_{\tilde{p}})$ is injective.*

Proof. Since C is normal, and $\mathcal{M}^{\otimes r}$ extends to an invertible sheaf on C, v^r extends to a regular section over C, thus $M = \overline{(v^r = 0)}$ is a Cartier divisor on C. As $M|_{C-N} = r(v = 0)$, we can write $M = rM_h + M_0$, where M_0 is supported on C_0 and no irreducible components of M_h lie in C_0.

Let $\tilde{\phi} : \tilde{\mathcal{C}} \to C$ be the S_r-family of twisted curves constructed in the previous Lemma for the Cartier divisor $M = rM_h + M_0$, and $\tilde{M}_{\frac{1}{r}}$ be the Cartier divisor so that $\tilde{\phi}^{-1}(M) = r\tilde{M}_{\frac{1}{r}}$. Let $\tilde{\mathcal{M}} = \mathcal{O}_{\tilde{\mathcal{C}}}(\tilde{M}_{\frac{1}{r}})$. Then $\tilde{\mathcal{M}}$ is invertible with a tautological section $\tilde{v} \in H^0(\tilde{\mathcal{M}})$ so that $(\tilde{v} = 0) = \tilde{M}_{\frac{1}{r}}$. As

$$(\tilde{v} = 0) \cap (\tilde{\mathcal{C}} - \tilde{N}) = \big((v = 0) \times_S S_r\big) \cap (\tilde{\mathcal{C}} - \tilde{N}),$$

we conclude that we have isomorphism $\tilde{\mathcal{M}}|_{\tilde{\mathcal{C}}-\tilde{N}} \cong \tilde{\phi}^*\mathcal{M}|_{\tilde{\mathcal{C}}-\tilde{N}}$ so that $\tilde{v}|_{\tilde{\mathcal{C}}-\tilde{N}} = \tilde{\phi}^*v|_{\tilde{\mathcal{C}}-\tilde{N}}$. This proves the corollary. $\square$

Let (Σ, C) be a good S-family of pre-stacky pointed curves and $\mathcal{B}$ be a final basket in the notation of Definition 3.4 and 3.6. We shall provide a procedure to construct a family in $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$.

We first restack the pre-stacky pointed curve $(\Sigma\times_S S_r, C\times_S S_r)$ to obtain an S-family of pointed twisted curve $(\Sigma^{\mathcal{C}}, \mathcal{C})$. By abuse of notation, we still use C to represent the coarse moduli space of $\mathcal{C}$. Let $q : \mathcal{C} - (\mathcal{C}_0)_{\mathrm{sing}} \to C$ be the projection. Let $M = rA + \sum rm_i D_i$, which is a Cartier divisor on C so that D_i are supported along C_0 and no irreducible component of A lies in C_0. Thus $\frac{1}{r}q^*M$ is a Cartier divisor. We then apply Lemma 3.10 and Corollary 3.11 to $(\Sigma^{\mathcal{C}}, \mathcal{C})$ to obtain an S_r-family of pointed twisted curve $(\Sigma^{\tilde{\mathcal{C}}}, \tilde{\mathcal{C}})$ such that it is isomorphic to $(\Sigma^{\mathcal{C}}, \mathcal{C})$ away from the singular points of the central fiber, and there is an invertible sheaf $\tilde{\mathcal{M}}$ on $\tilde{\mathcal{C}}$ with a section $\tilde{v}$ so that $\tilde{\mathcal{M}}$ is the extension of $\mathcal{O}_{\tilde{\mathcal{C}}-(\tilde{\mathcal{C}}_0)_{\mathrm{sing}}}(\frac{1}{r}q^*M)$ and $\tilde{v}$ is the extension of the tautological section of the latter.

Let $\tilde{\phi} : \tilde{\mathcal{C}} \to C$ be the tautological morphism, let $\tilde{\mathcal{N}} = \tilde{\mathcal{M}}$, and let $\tilde{\nu}_2 = \tilde{v} \in H^0(\tilde{\mathcal{N}}) = H^0(\tilde{\mathcal{M}})$. Let $\tilde{\nu}_1 : \tilde{\mathcal{L}} \cong \tilde{\mathcal{N}}^\vee$. By (3.4), we conclude that

$$\mathcal{O}_{\tilde{\mathcal{C}}}(\tilde{\phi}^*(B + \sum l_i D_i)) \cong \tilde{\mathcal{L}}^{-r} \otimes \omega^{\log}_{\tilde{\mathcal{C}}/S}.$$

Let $\tilde{\rho} \in \Gamma(\tilde{\mathcal{L}}^{-r} \otimes \omega^{\log}_{\tilde{\mathcal{C}}/S})$ be induced by the above isomorphism and the tautological inclusion $\mathcal{O}_{\tilde{\mathcal{C}}} \subset \mathcal{O}_{\tilde{\mathcal{C}}}(\tilde{\phi}^*(B + \sum l_i D_i))$. Because ρ_* vanishes along $\Sigma^{\mathcal{C}_*}_{(1,\rho)}$, $\tilde{\rho}$ lifts to a section in $\Gamma(\tilde{\mathcal{L}}^{-r} \otimes \omega^{\log}_{\tilde{\mathcal{C}}/S}(-\Sigma^{\tilde{\mathcal{C}}}_{(1,\rho)}))$. This proves

Lemma 3.12. *With notation as above. Then $\tilde{\xi} = (\Sigma^{\tilde{\mathcal{C}}}, \tilde{\mathcal{C}}, \tilde{\mathcal{L}}, \tilde{\mathcal{N}}, \tilde{\varphi} = 0, \tilde{\rho}, \tilde{\nu})$ constructed from a final basket $\mathcal{B}$ belongs to $\mathcal{W}^{pre-}_{g,\gamma,\mathbf{d}}(S)$ for a choice of $(g,\gamma,\mathbf{d})$.*

3.4. Modifying brackets

In the followings, we will perform a series of blowups, base changes, and stabilizations to obtain an extension in $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(S)$. Let (Σ, C) be a good S-family of pre-stacky pointed nodal curves.

Definition 3.13. *Let $\mathcal{B}$ be a pre-basket of (Σ, C). We say $\mathcal{B}'$ is a modification of $\mathcal{B}$ if there is a finite base change $S' \to S$, a good S'-family (Σ', C') of pre-stacky pointed curves so that $\mathcal{B}'$ is a basket of (Σ', C'), $(\Sigma', C') \times_{S'} S'_* \cong (\Sigma, C) \times_S S'_*$ as pre-stacky pointed nodal curves, and under this isomorphism $\mathcal{B}' \times_{S'} S'_* = \mathcal{B} \times_S S'_*$.*

We first show that we can find a basket $\mathcal{B}'$, which is a modification of $\mathcal{B}$ on (Σ, C). Indeed, let $\bar{r}_i = \min(rm_i, l_i)$, and let

$$\mathcal{B}' = (B + \sum (l_i - \bar{r}_i) D_i, A + \sum (m_i - \bar{r}_i/r) D_i). \tag{3.7}$$

It is easy to see that $\mathcal{B}'$ is a modification of $\mathcal{B}$.

In the following, we assume $\mathcal{B}$ is a basket as in Definition 3.13. We will construct modifications of the basket $\mathcal{B}$ that will reduce the $\mathbb{Z}_{\geq 0}$-valued quantities

$$V_1(\mathcal{B}) = \sum_{B \cap D_j \neq \emptyset} rm_j, \quad V_2(\mathcal{B}) = \sum_{A \cap D_i \neq \emptyset} l_i, \quad V_3(\mathcal{B}) = \sum_{D_i \cap D_j \neq \emptyset} \frac{r l_i m_j}{\gcd(l_i, rm_j)^2}.$$

Lemma 3.14. *Let (Σ, C) and $\mathcal{B}$ be as stated. Then there is a modification $\mathcal{B}'$ of $\mathcal{B}$ such that $V_1(\mathcal{B}') = 0$.*

Proof. Let $\bar{j}$ be such that $m_{\bar{j}} > 0$ and $p \in B \cap D_{\bar{j}} \neq \emptyset$. Since $\mathcal{B}$ is a basket, $l_{\bar{j}} = 0$. By the definition of basket, C_0 is smooth at p and $p \notin A$. We let $\tau \colon \tilde{C} \to C$ be the blowup of C at p, let E be the exceptional divisor, and let $\tilde{\pi} \colon \tilde{C} \to S$ be the induced projection. In the following, for any Cartier divisor $G \subset C$, we denote by $\tilde{G}$ its strict transform in $\tilde{C}$. Because B is an integral divisor, $\tau^* B = \tilde{B} + lE$ with $1 \leq l \in \mathbb{Z}$. By the blowing up formula, we have $\omega^{\log}_{\tilde{C}/S} = \tau^* \omega^{\log}_{C/S}(\epsilon E)$, where

$$\epsilon = 0 \text{ when } p \in \Sigma; \quad \epsilon = 1 \text{ when } p \notin \Sigma. \tag{3.8}$$

We give $\tilde{\Sigma}$ the pre-stacky assignments according to that of Σ. Form

$$\tilde{\mathcal{B}} = (\tilde{B} + \sum l_i \tilde{D}_i + (l+\epsilon)E, \tilde{A} + \sum m_j \tilde{D}_j + m_{\bar{j}} E).$$

We claim that it is a pre-basket of $(\tilde{\Sigma}, \tilde{C})$. Indeed, the conditions (1)-(4) in the Definition 3.7 can be easily verified. It remains to verify the isomorphism (3.4). Obviously we have

$$\tau^* \mathcal{O}_C(B + \sum l_i D_i) \cong \mathcal{O}_{\tilde{C}}(\tilde{B} + \sum lE + \sum l_i D_i),$$
$$\tau^* \big((\mathcal{O}_C(rA + \sum rm_i D_i) \otimes \omega^{\log}_{C/S} \big) \cong \mathcal{O}_{\tilde{C}}(r\tilde{A} + \sum rm_i D_i + rm_{\bar{j}} E) \otimes \tau^* \omega^{\log}_{C/S},$$

and $\tau^* \omega^{\log}_{C/S} \cong \omega^{\log}_{\tilde{C}/S}(-\epsilon E)$. Combined, we get

$$\mathcal{O}_{\tilde{C}}(\tilde{B} + \sum (l+\epsilon)E + \sum l_i D_i) \cong \mathcal{O}_{\tilde{C}}(r\tilde{A} + \sum rm_i D_i + rm_{\bar{j}} E) \otimes \omega^{\log}_{\tilde{C}/S}.$$

Thus $\tilde{\mathcal{B}}$ is a pre-basket. Let

$$\mathcal{B}' = \big(\tilde{B} + \sum l_i \tilde{D}_i + (l + \epsilon - \bar{r})E, \tilde{A} + \sum m_j \tilde{D}_j + (m_{\bar{j}} - \frac{\bar{r}}{r})E\big),$$

where $1 \le \bar{r} = \min\{rm_{\bar{j}}, l + \epsilon\} \in \mathbb{Z}$ since $0 \ne rm_{\bar{j}} \in \mathbb{Z}$. By construction, it is a modification of $\mathcal{B}'$.

Since $\tilde{B} \cap \tilde{D}_{\bar{j}} = \emptyset$ and $0 \le m_{\bar{j}} - \bar{r}/r < m_{\bar{j}}$, we have

$$V_1(\mathcal{B}') = \sum_{\tilde{B}\cap\tilde{D}_j \ne \emptyset, j \ne \bar{j}} rm_j + r(m_{\bar{j}} - \frac{\bar{r}}{r}) = \sum_{B\cap D_j \ne \emptyset} rm_j - \bar{r} = V_1(\mathcal{B}) - \bar{r} < V_1(\mathcal{B}).$$

The lemma is proved by induction. □

Lemma 3.15. *Let (Σ, C) and $\mathcal{B}$ be as stated with $V_1(\mathcal{B}) = 0$. Then there is a modification $\mathcal{B}'$ of $\mathcal{B}$ such that $V_1(\mathcal{B}') = V_2(\mathcal{B}') = 0$.*

Proof. Suppose there is an $l_{\bar{i}} > 0$ such that $A \cap D_{\bar{i}} \ne \emptyset$. Since $\mathcal{B}$ is a basket, $m_{\bar{i}} = 0$. Pick $p \in A \cap D_{\bar{i}}$. Let $\tau \colon \tilde{C} \to C$ be the blowup of C at p. If p lies on a marking Σ_i, let $\tilde{\Sigma}_i \subset \tilde{C}$ be the strict transform of Σ_i. By transversality, $\tau^* A = \tilde{A} + mE$ where $m \in \frac{1}{r}\mathbb{Z}_{>0}$. Consider

$$\tilde{\mathcal{B}} = (\tilde{B} + \sum l_i \tilde{D}_i + (l_{\bar{i}} + \epsilon)E, \tilde{A} + \sum m_j \tilde{D}_j + mE),$$

where ϵ is as in (3.8). We have $\tilde{B} \cap E = \emptyset$, $\tilde{A} \cap \tilde{D}_{\bar{i}} = \emptyset$, and $\tilde{A}$ intersects E transversally. As in the proof of the previous Lemma, it is straight forward to verify that $\tilde{\mathcal{B}}$ is a pre-basket. As in (3.7), let

$$\mathcal{B}' = \big(\tilde{B} + \sum l_i \tilde{D}_i + (l_{\bar{i}} + \epsilon - \bar{r})E, \tilde{A} + \sum m_j \tilde{D}_j + (m - \frac{\bar{r}}{r})E\big),$$

where $\bar{r} = \min\{rm, l_{\bar{i}} + \epsilon\}$. We claim that $\bar{r} \ge 1 + \epsilon$. Indeed, when $p \notin \Sigma$, $m \ge 1$; when $p \in \Sigma$, then $rm \ge 1$ but $\epsilon = 0$. Thus $0 \le l_{\bar{i}} + \epsilon - \bar{r} < l_{\bar{i}}$, and $V_2(\mathcal{B}') < V_2(\mathcal{B})$. Repeating this construction, we prove the Lemma. □

Lemma 3.16. *Let (Σ, C) and $\mathcal{B}$ be as stated with $V_1(\mathcal{B}) = V_2(\mathcal{B}) = 0$. Then there is a final basket $\mathcal{B}'$ which is a modification of $\mathcal{B}$.*

Proof. Suppose there are pairs $D_{\bar{i}} \ne D_{\bar{j}}$ such that $p \in D_{\bar{i}} \cap D_{\bar{j}}$, and $\ell_{\bar{i}} > 0$ and $m_{\bar{j}} > 0$. Without lose of generality, we assume that C_0 has only one node, which is $p = D_{\bar{i}} \cap D_{\bar{j}}$. Take a base change $S_2 \to S$, and let $C' = S_2 \times_S C$. Then near a node of the central fiber of C', C' locally is of the form $xy = t^2$. Minimally resolve C' to get a smooth $\tilde{C}$ with a (-2)-curve corresponding to that node. Let E be the (-2)-curve, the exceptional curve of the resolution $\tilde{\tau}: \tilde{C} \to C'$. Then $\omega^{\log}_{\tilde{C}/S_2} = \tilde{\tau}^*\omega^{\log}_{C'/S_2} = \tau^*\omega^{\log}_{C/S}$, where τ is the composition of $\tilde{\tau}$ with the base change map $\tau'\colon C' \to C$.

Let $\mathcal{B} = (B + \sum \ell_i D_i, A + \sum m_j D_j)$. Since A and B do not intersect $D_{\bar{i}} \cap D_{\bar{j}}$, we have $\tau^* A = \tilde{A}$ and $\tau^* B = \tilde{B}$. From the base-change and the minimal resolution, we get a new pre-basket on $\tilde{C}$:

$$\tilde{\mathcal{B}} = (\tilde{B} + \sum 2\ell_i \tilde{D}_i + 2\ell_{\bar{i}} E, \tilde{A} + \sum 2m_j \tilde{D}_j + 2m_{\bar{j}} E).$$

$\tilde{C} \to C$ is a ramified double covering with branch locus D_i. Then let

$$\mathcal{B}' = (\tilde{B} + \sum 2\ell_i \tilde{D}_i + (2\ell_{\bar{i}} - \bar{r})E, \tilde{A} + \sum 2m_j \tilde{D}_j + (2m_{\bar{j}} - \bar{r}/r)E),$$

where $\bar{r} = \min\{2rm_{\bar{j}}, 2\ell_{\bar{i}}\}$ as in (3.7). It is a basket. E intersects $\tilde{D}_{\bar{i}}$ and $\tilde{D}_{\bar{j}}$ at the nodes, and $\tilde{D}_{\bar{i}} \cap \tilde{D}_{\bar{j}} = \emptyset$. If $\ell_{\bar{i}} > rm_{\bar{j}}$, we have

$$\gcd(2\ell_{\bar{i}} - \bar{r}, 2rm_{\bar{j}}) \geq \gcd(2\ell_{\bar{i}}, 2dm_{\bar{j}}) = 2\gcd(\ell_{\bar{i}}, rm_{\bar{j}}),$$

which imply

$$\frac{(2\ell_{\bar{i}} - \bar{r}) \cdot 2rm_{\bar{j}}}{\gcd(2\ell_{\bar{i}} - \bar{r}, 2rm_{\bar{j}})^2} < \frac{2\ell_{\bar{i}} \cdot 2rm_{\bar{j}}}{4\gcd(\ell_{\bar{i}}, rm_{\bar{j}})^2} = \frac{\ell_{\bar{i}} \cdot rm_{\bar{j}}}{\gcd(\ell_{\bar{i}}, rm_{\bar{j}})^2}.$$

Thus $V_3(\mathcal{B}') < V_3(\mathcal{B})$. If $\ell_{\bar{i}} < rm_{\bar{j}}$, same argument gives $V_3(\mathcal{B}') < V_3(\mathcal{B})$. As the central fiber of $\tilde{C}$ is reduced, the lemma follows by induction. □

Proposition 3.17. *Let $\xi_* \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$ be of φ-vanishing type. Then possibly after a finite base change, ξ_* can be extended to a family $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(S)$.*

Proof. Applying Proposition 3.5 and the discussion afterwards, possibly after a finite base change, we obtain a good S-family of pre-stacky pointed nodal curves (Σ, C), and basket $\mathcal{B}$ as in (3.3). Applying results proved in Section 3, we can modify $\mathcal{B}$ to get a final basket. By Lemma 3.12, the final basket provides us a family $\tilde{\xi} \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(S)$. □

3.5. Stabilization

We will show that the extension ξ constructed in Proposition 3.17 can be made in $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(S)$.

Lemma 3.18. *Let $\xi \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$ be such that $\xi_* \in \mathcal{W}^{-}_{g,\gamma,\mathbf{d}}(S_*)$. Suppose the central fiber $\mathcal{C}_0$ is irreducible, then $\xi_0 \in \mathcal{W}^{-}_{g,\gamma,\mathbf{d}}(\eta_0)$.*

Proof. Suppose ξ_0 is unstable. Since $\mathcal{C}_0$ is irreducible, by Lemma 3.3, either $\mathcal{C}_0$ is a smooth rational curve satisfying one of (1)-(3) in Lemma 3.3, or $\mathcal{C}_0$ satisfies (4) of the same Lemma.

In case $\mathcal{C}_0$ is a smooth rational curve and satisfies one of (1)-(3) mentioned, since the properties in (1)-(3) are deformation invariant, for general $s \in S_*$, $\mathcal{C}_s$ satisfies the same property, forcing ξ_s unstable and thus contradicting to that ξ_* is a family of stable objects.

Therefore, $\mathcal{C}_0$ must be the Case (4) in Lemma 3.3. Therefore, for a closed point $s \in S_*$, $\Sigma^{\mathcal{C}_s} = \emptyset$, and $\deg \mathcal{L}_s = \deg \mathcal{N}_s = 0$. Here $\Sigma^{\mathcal{C}_s} = \Sigma^{\mathcal{C}} \cap \mathcal{C}_s$, etc. By the non-vanishing assumption on (ρ, ν_2), (φ, ν_1) and (ν_1, ν_2), as in the proof of Lemma 3.3, when $\rho|_{\mathcal{C}_s} \neq 0$, we conclude that $\mathcal{L}^{\otimes r}|_{\mathcal{C}_s} \cong \mathcal{O}_{\mathcal{C}_s}$ and $\mathcal{L}|^{\vee}_{\mathcal{C}_s} \cong \mathcal{N}|_{\mathcal{C}_s}$; when $\varphi|_{\mathcal{C}_s} \neq 0$, we conclude that $\mathcal{L}|_{\mathcal{C}_s} \cong \mathcal{N}|_{\mathcal{C}_s} \cong \mathcal{O}_{\mathcal{C}_s}$. Therefore, ξ_s for general $s \in S_*$ belongs to Case (4) of Lemma 3.3, thus must be unstable. This proves the Lemma. $\square$

Let $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(S)$ be the extension constructed in Proposition 3.17. Suppose ξ_0 is not stable, by Lemma 3.18, $\mathcal{C}_0$ is reducible. By Lemma 3.3, we can find a rational curve $\mathcal{E} \subset \mathcal{C}$ so that either (1) or (2) of Lemma 3.3 holds.

For case (1), we further divide it into two subcases: (1a) when both $\mathcal{E} \cap \Sigma^{\mathcal{C}}$ and $\mathcal{E} \cap \mathcal{C}_{0,\mathrm{sing}}$ consist of one point; (1b) when $\mathcal{E} \cap \mathcal{C}_{0,\mathrm{sing}}$ consists of two points. For case (2), as $\mathcal{C}_0$ is reducible, we know $\mathcal{E} \cap \Sigma^{\mathcal{C}} = \emptyset$ and $\mathcal{E} \cap \mathcal{C}_{0,\mathrm{sing}}$ consists of one point.

Lemma 3.19. *The extension ξ constructed in Proposition 3.17 can be made in $\mathcal{W}^{-}_{g,\gamma,\mathbf{d}}(S)$.*

Proof. As argued, we need to treat the case when $\mathcal{C}_0$ (of the family ξ) is reducible and belongs to cases (1a), (1b), or (2). As the treatments of cases (1a) and (2) are similar, we will prove the case (1a) in detail and skip (2).

Let $\mathcal{E} \subset \mathcal{C}$ be of the case (1a), a rational curve intersecting the remaining irreducible components of $\mathcal{C}_0$ at $p \in \mathcal{E}$. Let $q = \Sigma^{\mathcal{C}} \cap \mathcal{E}$. As $\mathcal{E}$ is of case (1a), we know $\mathcal{L}^r|_{\mathcal{E}} = \mathcal{O}_{\mathcal{E}}$. Because ξ is of φ-vanishing type, $\deg \mathcal{N}|_{\mathcal{E}} = \deg \mathcal{L}|_{\mathcal{E}} = 0$. Thus, if $\rho|_{\mathcal{E}}$ (resp. $\nu_2|_{\mathcal{E}}$) is non-trivial, it is nowhere vanishing on $\mathcal{E}$.

Consider the case when $\nu_2|_{\mathcal{E}} = 0$. Let (Σ, C) with $\mathcal{B}$ being the good S-family of pre-stacky pointed nodal curves with a final basket $\mathcal{B}$ that constructs the family ξ according to the proof of Prop. 3.17. Let $E \subset C_0$ be the irreducible component corresponding to $\mathcal{E}$. Then $E \subset C$ is a (-1)-curve.

Following the notation in (3.3), $E \subset A + \sum m_i D_i$, thus $E = D_{i_0}$ for some i_0.

We blow down C along E to get $\pi\colon C \to \tilde{C}$. Let $\tilde{A}, \tilde{D}_i, \tilde{B}$ be the image of A, D_i, B under $\pi : C \to \tilde{C}$, respectively. Let D_{j_0} be the one that intersects E. Let $c = A \cdot E \geq 0$. Because $\deg \mathcal{N}|_{\mathcal{E}} = 0$, $(A + m_{i_0} D_{i_0} + m_j D_j) \cdot E = 0$, namely, $c - m_{i_0} + m_j = 0$. Thus

$$
\begin{aligned}
\pi^*(\tilde{A} + \sum_{i \neq i_0} m_i \tilde{D}_i) &= A + cE + \sum_{i \neq i_0} m_i D_i + m_j E \\
&= A + \sum_{i \neq i_0} m_i D_i + m_{i_0} D_{i_0} = A + \sum m_i D_i.
\end{aligned}
$$

Since we have

$$
\pi^*(\tilde{B} + \sum l_i \tilde{D}_i) = B + \sum l_i D_i,
$$

we obtain an isomorphism parallel to (3.4):

$$
\pi^* \mathcal{O}_{\tilde{C}}(\tilde{B} + \sum \ell_i \tilde{D}_i) \cong \pi^* \mathcal{O}_{\tilde{C}}(r\tilde{A} + r \sum_{i \neq i_0} m_i D_i) \otimes \pi^* \omega^{\log}_{\tilde{C}/S}.
$$

Note that $\pi(E) \in \tilde{C}$ is a marking on $\tilde{D}_j$ and thus $\omega^{\log}_{C/S} = \pi^* \omega^{\log}_{\tilde{C}/S}$. Pushing forward and using $\pi_*\big(\mathcal{O}_C(E)\big) = \mathcal{O}_{\tilde{C}}$, we get

$$
\mathcal{O}_{\tilde{C}}(\tilde{B} + \sum \ell_i \tilde{D}_i) \cong \mathcal{O}_{\tilde{C}}(r\tilde{A} + r \sum_{i \neq i_0} m_i D_i) \otimes \omega^{\log}_{\tilde{C}/S}.
$$

Hence $\tilde{\mathcal{B}} = (\tilde{B} + \sum l_i \tilde{D}_i, \tilde{A} + \sum_{i \neq i_0} m_i \tilde{D}_i)$ is a final basket on a good family of pointed curves $(\tilde{C}, \tilde{\Sigma})$.

To complete the study of (1a), we also need to take care of the case when $\rho|_{\mathcal{E}} = 0$, or the case both $\nu_2|_{\mathcal{E}}$ and $\rho|_{\mathcal{E}}$ are non-trivial. The study of these cases is parallel and will be omitted. Finally, by induction we conclude that we can find ξ, extending ξ_*, such that only (1b) occurs.

Therefore, to prove the Lemma, we only need to consider the case where the rational $\mathcal{E} \subset \mathcal{C}$ that makes ξ_0 unstable are all of case (1b). Let $\mathcal{D} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k \subset \mathcal{C}$ be a maximal connected chain or loop of rational curves of $\mathcal{C}_0$, of case (1b). We first rule out the loop case. If $\mathcal{D}$ is a loop, then $\mathcal{C}_0 = \mathcal{D}$, thus ξ has $g = 1$, no markings, and $\deg \mathcal{L} = \deg \mathcal{N} = 0$. Thus we have ξ_* unstable, a contradiction.

Let (C, Σ) and $\mathcal{B}$ be as in the beginning of this proof and $D \subset C$ be the union of curves $D = E_1 \cup \cdots \cup E_k$ where $E_i \subset C_0$ corresponds to $\mathcal{E}_i \subset \mathcal{C}_0$.

Then $D \subset C$ is a connected chain of (-2)-curves. Let $q : C \to C'$ be the contraction of D, $p' = q(D) \in C'$ and $\Sigma' = q(\Sigma)$. By Lemma 3.3, we have $p' \cap \Sigma' = \emptyset$.

We first assume that $\mathcal{C}$ is not a scheme along $\mathcal{D}$. As ξ is of φ-vanishing type, $\mathcal{L} = \mathcal{N}^\vee$. Further, over this chain of rational curve $\mathcal{D}$, we have $\deg \mathcal{L}|_{\mathcal{E}_i} = 0$ for any $\mathcal{E}_i \subset \mathcal{D}$. Since ν_2 vanishes at the stacky point of $\mathcal{D}$, $\nu_2|_\mathcal{D} = 0$, implying that $\rho|_\mathcal{D}$ is nowhere vanishing. Let $\mathcal{E}_0 \subset \mathcal{C}_0$ and $\mathcal{E}_{k+1}$ be the two (could be one) irreducible component of $\mathcal{C}_0$ that intersect $\mathcal{E}_1$ and $\mathcal{E}_k$, respectively. Let q_i be the node $\mathcal{E}_i \cap \mathcal{E}_{i+1}$, $i = 0, \cdots, k$. Then using that $\rho|_\mathcal{D}$ is nowhere vanishing we conclude that the monodromy of $\mathcal{L}$ at q_{i-1} and q_i along $\mathcal{E}_i$ is exactly the opposite.

To proceed, we introduce a family of twisted curves $\mathcal{C}'$ with coarse moduli C', the contraction morphism $q : C \to C'$ induces an isomorphism

$$\phi : \mathcal{C}' - p' \xrightarrow{\cong} \mathcal{C} - \mathcal{D},$$

and p' is a scheme point.

Let $(\bar{\mathcal{L}}, \bar{\mathcal{N}}, \bar\rho, \bar\varphi, \bar\nu)$ be the pullback of $(\mathcal{L}, \mathcal{N}, \rho, \varphi, \nu)$ via ϕ. Since $\mathcal{L}^r|_\mathcal{D} \cong \mathcal{O}_\mathcal{D}$, $\bar{\mathcal{L}}^{\otimes r}$ extends to an invertible sheaf on $\mathcal{C}'$. As $\bar{\mathcal{L}}$ is isomorphic to $\bar{\mathcal{N}}^\vee$ near p', $\bar{\mathcal{N}}^{\otimes r}$ extends to an invertible sheaf on $\mathcal{C}'$.

We now consider $\bar\nu_2 \in H^0(\mathcal{C}' - p', \bar{\mathcal{N}})$. Since $\bar{\mathcal{N}}^{\otimes r}$ extends to an invertible sheaf on $\mathcal{C}'$, we can apply Corollary 3.11 to $(\bar\nu_2, \bar{\mathcal{N}})$ to introduce a $\boldsymbol{\mu}_a$-stacky structure at $p' \in \mathcal{C}'$, for $a \mid r$, if necessary. After a finite base change, we continue to denote by $\mathcal{C}'$ the resulting family of twisted curves. Then $\bar{\mathcal{N}}$ extends to an invertible sheaf $\mathcal{N}'$ on $\mathcal{C}'$ so that $\bar\nu_2$ extends to a regular section ν_2' of $\mathcal{N}'$. Since $\bar{\mathcal{L}}^\vee$ is isomorphic to $\bar{\mathcal{N}}$ near p', we extend $\bar{\mathcal{L}}$ to an invertible sheaf $\mathcal{L}'$ on $\mathcal{C}'$ so that $\mathcal{L}'^\vee$ is isomorphic to $\mathcal{N}'$ near p', and extend the known isomorphism between $\bar{\mathcal{L}}^\vee$ and $\bar{\mathcal{N}}$. Because $\mathcal{C}'$ is normal near p', we can extend $\bar\varphi$, $\bar\rho$ and $\bar\nu_1$ to regular sections φ', ρ' and ν_1' over $\mathcal{C}'$.

We claim that $\xi' = (\Sigma^{\mathcal{C}'}, \mathcal{C}', \mathcal{L}', \cdots) \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(S)$. For this, we only need to check that $(\varphi', \nu_1')|_{p'}$, $(\rho', \nu_2')|_{p'}$ and $(\nu_1', \nu_2')|_{p'}$ are nonzero. First, as $\nu_1 = 1$, ν_1' is nowhere vanishing away from p', thus $\nu_1' = 1$. For the same reason, as $\rho|_\mathcal{D}$ is nowhere vanishing, $\rho'|_{\mathcal{C}'-p'}$ is nowhere vanishing near p', this ρ' is nonzero at p'. This proves that $(\varphi', \nu_1')|_{p'}$, $(\rho', \nu_2')|_{p'}$ and $(\nu_1', \nu_2')|_{p'}$ are nonzero. Therefore, $\xi' \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(S)$ and satisfies $\xi_* = \xi'_*$.

The case that $\mathcal{C}$ is a scheme along $\mathcal{D}$ can be proved similarly.

Apply induction on the number of such chains of rational curves, we prove that we can extend ξ_* to a family $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S)$. This proves the lemma. $\square$

We prove the desired existence for the general case.

Proposition 3.20. *Let $\xi_* \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$ be such that $\mathcal{C}_*$ is smooth. Then possibly after a finite base change of S, ξ_* extends to a $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S)$.*

Proof. Let $\xi_* = (\Sigma^{\mathcal{C}_*}, \mathcal{C}_*, \cdots)$. We distinguish several cases. The case $\rho_* = 0$ is proved in Proposition 3.1. The next case is when $\varphi_* = \nu_{2*} = 0$. In this case $(\rho_* = 0) = \emptyset$ and $\mathcal{L}_*^\vee \cong \mathcal{N}_*$. So we get a r-spin twisted curve $(\mathcal{C}_*, \mathcal{L}_*)$. By [AJ], we can extend $(\mathcal{C}_*, \mathcal{L}_*)$ to a r-spin twisted curve $(\mathcal{C}, \mathcal{L})$ over possibly a finite base change of S. Here the extension ξ is in $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(S)$. The last case is when $\varphi_* = 0$, $\rho_* \neq 0$ and $\nu_{2*} \neq 0$. This case is proved in Lemma 3.19. This proves the proposition. $\square$

3.6. Valuative criterion for properness

We verify the valuative criterion for properness by gluing the extensions constructed in the previous subsections, using the construction in [AGV, Appendix] (cf. [AF, Def. 1.4.1]).

Let $\mathcal{X}$ be an S-family of not necessary connected twisted nodal curves with two markings Γ_1 and Γ_2 which are $\boldsymbol{\mu}_a$-gerbes over S where $a|r$, let X be the moduli of $\mathcal{X}$ with the natural projection $\pi\colon \mathcal{X} \to X$, and $s_1, s_2\colon S \to X$ be two sections such that $s_i(S) = \pi(\Gamma_i)$. The line bundle $N^{\otimes a}_{\Gamma_i/\mathcal{X}}$ descends to the normal bundle of $\pi(\Gamma_i)$ in X.

Lemma 3.21 ([AF, Def. 1.4.1])**.** *With notation and assumptions as above. Assume $s_1^* N^a_{\Gamma_1/\mathcal{X}} \cong s_2^* N^{-a}_{\Gamma_2/\mathcal{X}}$. Then possibly after a finite base change, we can find an S-family of not necessary connected twisted nodal curves $\mathcal{X}'$ together with an S-morphism $\alpha : \mathcal{X} \to \mathcal{X}'$ that is the gluing of $\mathcal{X}$ via (an appropriate S-isomorphism) $\Gamma_1 \cong \Gamma_2$.*

Proof. Possibly after a finite base change, we can find an S-isomorphism $\gamma : \Gamma_1 \to \Gamma_2$ and $N_{\Gamma_1/\mathcal{X}} \otimes \gamma^* N_{\Gamma_2/\mathcal{X}} \cong \mathcal{O}_{\Gamma_1}$ that induces the given isomorphism $s_1^* N^a_{\Gamma_1/\mathcal{X}} \cong s_2^* N^{-a}_{\Gamma_2/\mathcal{X}}$. In case Γ_1 and Γ_2 lie in different connected components of $\mathcal{X}$, the gluing is given in [AF, Definition 1.4.1]. The case Γ_1 and Γ_2 lie in the same connected component of $\mathcal{X}$ can be deduced by adopting the construction in the Appendix of [AGV]. $\square$

We can also glue the sheaves and sections. Let the situation be as in Lemma 3.21, and let $\gamma : \Gamma_1 \to \Gamma_2$ be the isomorphism given in its proof.

Corollary 3.22. *Suppose we have an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ and an isomorphism $\gamma^*(\mathcal{L}|_{\Gamma_2}) \cong \mathcal{L}|_{\Gamma_1}$. Then the sheaf $\mathcal{L}$ glues to get an invertible sheaf $\mathcal{L}'$ on $\mathcal{X}'$ via the exact sequence*

$$0 \longrightarrow \mathcal{L}' \longrightarrow \alpha_* \mathcal{L} \xrightarrow{\epsilon} \alpha_*(\mathcal{L}|_{\Gamma_1}) \longrightarrow 0,$$

where the arrow ϵ takes the form

$$\big(\alpha_*\mathcal{L}\big)|_\Gamma \cong \alpha_*(\mathcal{L}|_{\Gamma_1}) \oplus \alpha_*(\mathcal{L}|_{\Gamma_2}) \xrightarrow{(\epsilon_1,-\epsilon_2)} \alpha_*(\mathcal{L}|_{\Gamma_1}),$$

where ϵ_1 is the identity $\alpha_*(\mathcal{L}|_{\Gamma_1}) = \alpha_*(\mathcal{L}|_{\Gamma_1})$, and ϵ_2 is the isomorphism $\alpha_*(\mathcal{L}|_{\Gamma_2}) \cong \alpha_*(\mathcal{L}|_{\Gamma_1})$ induced by the isomorphism $\gamma^*(\mathcal{L}|_{\Gamma_2}) \cong \mathcal{L}|_{\Gamma_1}$ given. Furthermore, suppose $s \in H^0(\alpha_*\mathcal{L})$ is a section so that $\epsilon(s) = 0$, then s lifts to a section $s' \in H^0(\mathcal{L}')$.

Proposition 3.23. *Let $S_* \subset S$ be as before and $\xi_* \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$. Then, possibly after a finite base change, ξ_* extends to $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S)$.*

Proof. Let $\xi_* = (\Sigma^{\mathcal{C}_*}, \mathcal{C}_*, \cdots) \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$. Possibly after a finite base change, we can assume that every connected component of the singular locus $(\mathcal{C}_*)_{\mathrm{sing}}$ is the image of a section of $\mathcal{C}_* \to S_*$. Let

$$\pi : \mathcal{C}_*^{\mathrm{nor}} = \coprod \mathcal{C}_{\alpha *} \to \mathcal{C}_*$$

be the normalization of $\mathcal{C}_*$, with every $\mathcal{C}_{\alpha*}$ connected. After a finite base change, we can assume that $\mathcal{C}_{\alpha*} \to S_*$ have connected fibers. Let $\tau_{\alpha*} : \mathcal{C}_{\alpha*} \to \mathcal{C}_*$ be the tautological morphism. For each $\mathcal{C}_{\alpha*}$, we endow it with the markings the (disjoint) union of $\tau_{\alpha*}^{-1}(\Sigma^{\mathcal{C}_*})$ and $\tau_{\alpha*}^{-1}((\mathcal{C}_*)_{\mathrm{sing}})$. Let $\mathcal{L}_{\alpha*}$, etc., be the pullbacks of $\mathcal{L}_*$, etc., via $\tau_{\alpha*}$. By Corollary 3.4,

$$\xi_{\alpha*} = (\Sigma^{\mathcal{C}_{\alpha*}}, \mathcal{C}_{\alpha*}, \mathcal{L}_{\alpha*}, \mathcal{N}_{\alpha*}, \varphi_{\alpha*}, \rho_{\alpha*}, \nu_{\alpha*}) \in \mathcal{W}^-_{g_\alpha,n_\alpha,\mathbf{d}_\alpha}(S_*),$$

for a choice of $(g_\alpha, n_\alpha, \mathbf{d}_\alpha)$.

Applying Proposition 3.20, after a finite base change $S_\alpha \to S$, we can extend $\xi_{\alpha*}$ to a $\xi'_\alpha \in \mathcal{W}^-_{g_\alpha,n_\alpha,\mathbf{d}_\alpha}(S_\alpha)$. We then pick a finite base change $\tilde{S} \to S$, factoring through all $S_\alpha \to S$, and form $\xi_\alpha = \xi'_\alpha \times_{S_\alpha} \tilde{S}$. Therefore, after denoting $\tilde{S}$ by S, we conclude that possibly after a finite base change, every $\xi_{\alpha*}$ extends to a $\xi_\alpha \in \mathcal{W}^-_{g_\alpha,n_\alpha,\mathbf{d}_\alpha}(S)$.

We now glue ξ_α's to a ξ that is a stable extension of ξ_*. Let $\tilde{\mathcal{C}} = \coprod \mathcal{C}_\alpha$, $\Upsilon_* \subset \mathcal{C}_*$ be a section of $(\mathcal{C}_*)_{\mathrm{sing}}$, and $\Upsilon_{1_*} \coprod \Upsilon_{2_*} \subset \mathcal{C}_*^{\mathrm{nor}}$ be the preimage of Υ_*. Using $(\mathcal{C}_*)^{\mathrm{nor}} \to \mathcal{C}_*$, they are markings in $\tilde{\mathcal{C}}_*$. Since markings in $\mathcal{C}_{\alpha*}$ extend to markings in $\mathcal{C}_\alpha$, after a finite base change, we can assume that all Υ_{i*} extend to sections Υ_i in $\tilde{\mathcal{C}}$ such that the S_*-isomorphisms $\Upsilon_{1_*} \cong \Upsilon_* \cong \Upsilon_{2_*}$ extend to an S-isomorphisms $\sigma : \Upsilon_1 \cong \Upsilon_2$.

Then possibly after a finite base change, we can find an isomorphism

$$\sigma^* N_{\Upsilon_2/\tilde{\mathcal{C}}} \otimes N_{\Upsilon_1/\tilde{\mathcal{C}}} \cong \mathcal{O}_{\Upsilon_1}$$

whose restriction to Υ_{1*} is consistent with the isomorphism $\mathcal{E}xt^1(\Omega_{\mathcal{C}_*},\mathcal{O}_{\mathcal{C}_*}) \cong \mathcal{O}_{\mathcal{C}_*}$. Applying Lemma 3.21, we obtain a gluing $\mathcal{C}$ of $\tilde{\mathcal{C}}$ along $\Upsilon_1 \cong \Upsilon_2$, resulting in a family of twisted pointed curves. After performing such gluing to all sections of $(\mathcal{C}_*)_{\mathrm{sing}}$, we obtain an S-family of twisted nodal curves $\mathcal{C} \to S$ that is an extension of $\mathcal{C}_* \to S_*$. Denote the gluing morphism by

$$\beta : \tilde{\mathcal{C}} \longrightarrow \mathcal{C}. \tag{3.9}$$

We next glue the sheaves and fields. Let $(\tilde{\mathcal{L}}, \tilde{\mathcal{N}}, \tilde{\varphi}, \tilde{\rho}, \tilde{\nu})$ be the sheaves and sections on $\tilde{\mathcal{C}}$ so that its restriction to $\mathcal{C}_\alpha$ is a part of the extension ξ_α. We will show that possibly after a finite base change, we can find $(\mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$ over $\mathcal{C}$ together with isomorphisms $(\tilde{\mathcal{L}}, \tilde{\mathcal{N}}) \cong \beta^*(\mathcal{L}, \mathcal{N})$ and $(\tilde{\varphi}, \tilde{\rho}, \tilde{\nu}) = \beta^*(\varphi, \rho, \nu)$.

Without loss of generality, we can assume that $(\mathcal{C}_*)_{\mathrm{sing}}$ consists of a single S_*-section. Thus $\mathcal{C}$ is the gluing of $\tilde{\mathcal{C}}$ along $\Upsilon_1 \cong \Upsilon_2$. Let

$$\iota_i : \Upsilon \longrightarrow \tilde{\mathcal{C}}$$

be the composite $\Upsilon \cong \Upsilon_i \to \tilde{\mathcal{C}}$ of the tautological maps. We first consider the case where $\iota_1^*\tilde{\varphi} \neq 0$. Then necessarily $\iota_2^*\tilde{\varphi} \neq 0$. Since $\xi_\alpha \in \mathcal{W}^-_{g_\alpha,\gamma_\alpha,\mathbf{d}_\alpha}(S)$, $\tilde{\rho} = 0$ in a neighborhood $\tilde{\mathcal{U}}$ of $\Upsilon_1 \cup \Upsilon_2$ in $\tilde{\mathcal{C}}$, thus $\tilde{\nu}_2$, which is nowhere vanishing in $\tilde{\mathcal{U}}$, induces an isomorphism $\tilde{\mathcal{N}}|_{\tilde{\mathcal{U}}} \cong \mathcal{O}_{\tilde{\mathcal{U}}}$, hence inducing $\iota_i^*\tilde{\mathcal{N}} \cong \mathcal{O}_\Upsilon$ so that $\iota_i^*\tilde{\nu}_2 = 1$. Note that in this case, $\tilde{\mathcal{C}}$ is a scheme along Υ_i.

For $i = 1$ or 2, we consider $(\iota_i^*\tilde{\varphi}, \iota_i^*\tilde{\nu}_1)$, which is a nowhere vanishing section of $H^0(\iota_i^*\tilde{\mathcal{L}}^{\oplus(N+1)})$. It induces a morphism $\beta_i : \Upsilon \to \mathbb{P}^N$. Because $\beta_1|_{\Upsilon_*} = \beta_2|_{\Upsilon_*}$, we have $\beta_1 = \beta_2$. Consequently, we have a unique isomorphism $\phi : \iota_1^*\tilde{\mathcal{L}} \cong \iota_2^*\tilde{\mathcal{L}}$ so that

$$\phi^*(\iota_2^*\tilde{\varphi}, \iota_2^*\tilde{\nu}_1) = (\iota_1^*\tilde{\varphi}, \iota_1^*\tilde{\nu}_1). \tag{3.10}$$

Using $\iota_i^*\tilde{\mathcal{N}} \cong \mathcal{O}_\Upsilon$ and $\iota_i^*\tilde{\nu}_2 = 1$, we have isomorphism $\phi' : \iota_1^*\tilde{\mathcal{N}} \cong \iota_2^*\tilde{\mathcal{N}}$ so that $\phi'^*\iota_2^*\tilde{\nu}_2 = \iota_1^*\tilde{\nu}_2$.

Applying the scheme version of Cor. 3.22, we obtain invertible sheaves $\mathcal{L}$ and $\mathcal{N}$ on $\mathcal{C}$ with isomorphisms $\beta^*\mathcal{L} \cong \tilde{\mathcal{L}}$ and $\beta^*\mathcal{N} \cong \tilde{\mathcal{N}}$ whose restrictions to Υ are ϕ and ϕ' respectively. By (3.10) and Cor. 3.22, we also obtain sections $\varphi \in H^0(\mathcal{L})^{\oplus N}$ and $\nu_2 \in H^0(\mathcal{N})$ that are liftings of $\beta_*\tilde{\varphi}$ and $\beta_*\tilde{\nu}_2$, respectively, which satisfy $\beta^*\varphi = \tilde{\varphi}$ and $\beta^*\nu_2 = \tilde{\nu}_2$, under the given isomorphisms.

It remains to check that $\tilde{\nu}_1$ and $\tilde{\rho}$ can be glued to sections over $\mathcal{C}$. In this case, since $\mathcal{C}$ is a scheme along $\beta(\Upsilon_1) = \beta(\Upsilon_2)$, using that $\tilde{\mathcal{L}}$ glues to $\mathcal{L}$ we conclude that $\tilde{\mathcal{L}}^{-r} \otimes \omega^{\log}_{\tilde{\mathcal{C}}/S}$ glues to $\mathcal{L}^{-r} \otimes \omega^{\log}_{\mathcal{C}/S}$. Since $\tilde{\rho}$ vanishes along

$\Upsilon_1 \cup \Upsilon_2$, $\beta_*\tilde{\rho}$ lifts to ρ so that $\rho|_{\beta(\Upsilon_1)} = 0$. The gluing of $\tilde{\nu}_1$ is similar. This proves the existence of gluing in this case.

The other case is when $\iota_1^*\rho \neq 0$, which implies that $\iota_2^*\tilde{\rho} \neq 0$. In this case, we must have $\tilde{\varphi}|_{\tilde{\mathcal{U}}} = 0$ over a neighborhood $\tilde{\mathcal{U}}$ of $\Upsilon_1 \cup \Upsilon_2$ in $\tilde{\mathcal{C}}$. Consequently, $\tilde{\nu}_1|_{\tilde{\mathcal{U}}}$ is nowhere vanishing, forcing $\tilde{\mathcal{L}}^\vee|_{\tilde{\mathcal{U}}} \cong \tilde{\mathcal{N}}|_{\tilde{\mathcal{U}}}$. In particular, we have the induced isomorphism $\iota_i^*(\tilde{\mathcal{L}} \otimes \tilde{\mathcal{N}}) \cong \mathcal{O}_\Upsilon$ and $\iota_i^*\tilde{\nu}_1 = 1$ under this isomorphism.

To proceed, we use the canonical isomorphisms $\iota_i^*\omega^{\log}_{\tilde{\mathcal{C}}/S} \cong \mathcal{O}_\Upsilon$ due to that $\Upsilon_i \subset \tilde{\mathcal{C}}$ is a section along smooth locus of fibers of $\tilde{\mathcal{C}} \to S$. Using this isomorphism, we can view $\iota_i^*\tilde{\rho}$ as a section in $H^0(\iota_i^*\tilde{\mathcal{L}}^{-r})$. Using $\iota_i^*\tilde{\mathcal{L}}^\vee \cong \iota_i^*\tilde{\mathcal{N}}$, $\iota_i^*\tilde{\nu}_2$ is a section in $H^0(\iota_i^*\tilde{\mathcal{L}}^\vee)$. Because $(\iota_i^*\tilde{\rho}, \iota_i^*\tilde{\nu}_2)$ is nowhere vanishing, it defines a morphism $\beta_i : \Upsilon_i \to \mathbb{P}_{(r,1)}$. Because $\beta_1 \times_S S_* = \beta_2 \times_S S_*$, we have $\beta_1 = \beta_2$. Thus there are isomorphisms

$$\phi' : \iota_1^*\tilde{\mathcal{L}} \cong \beta_1^*\mathcal{O}_{\mathbb{P}_{(r,1)}}(1) = \beta_2^*\mathcal{O}_{\mathbb{P}_{(r,1)}}(1) \cong \iota_2^*\tilde{\mathcal{L}}$$

so that $\iota_1^*\tilde{\rho} = \iota_2^*\tilde{\rho}$ and $\iota_1^*\tilde{\nu}_2 = \iota_2^*\tilde{\nu}_2$ (with the known $\iota_i^*\omega^{\log}_{\tilde{\mathcal{C}}/S} \cong \mathcal{O}_\Upsilon$).

Like before, using ϕ', and applying Corollary 3.22, we can glue $\tilde{\mathcal{L}}$ to get $\mathcal{L}$ on $\mathcal{C}$ so that, letting $\iota : \Upsilon \cong \alpha(\Upsilon_1) \subset \mathcal{C}$ be the tautological inclusion, the isomorphisms $\iota_1^*\tilde{\mathcal{L}} \cong \iota^*\mathcal{L} \cong \iota_2^*\tilde{\mathcal{L}}$ induce the isomorphism ϕ'. We next glue $\tilde{\mathcal{N}}$. Let $\mathcal{U}$ be the image of $\tilde{\mathcal{U}}$ under $\tilde{\mathcal{C}} \to \mathcal{C}$, which is a neighborhood of $\Upsilon \subset \mathcal{C}$. Then using $\tilde{\mathcal{L}}^\vee|_{\tilde{\mathcal{U}}} \cong \tilde{\mathcal{N}}|_{\tilde{\mathcal{U}}}$, we see that $\tilde{\mathcal{N}}$ glues to get $\mathcal{N}$ on $\mathcal{C}$ so that $\mathcal{N}|_{\mathcal{U}} \cong \mathcal{L}^\vee|_{\mathcal{U}}$, consistent with the isomorphism $\tilde{\mathcal{L}}^\vee|_{\tilde{\mathcal{U}}} \cong \tilde{\mathcal{N}}|_{\tilde{\mathcal{U}}}$.

As before, applying Corollary 3.22 and using that $\iota_1^*\tilde{\nu}_2 = \iota_2^*\tilde{\nu}_2$ and $\iota_1^*\tilde{\rho}_1 = \iota_2^*\tilde{\rho}_2$ under $\phi' : \iota_1^*\tilde{\mathcal{L}} \cong \iota_2^*\tilde{\mathcal{L}}$, we conclude that $\tilde{\nu}_2$ and $\tilde{\rho}$ glue to ν_2 and ρ of $\mathcal{N}$ and $\mathcal{L}^{-r} \otimes \omega^{\log}_{\mathcal{C}/S}$, respectively. Since $\tilde{\varphi}|_{\mathcal{U}} = 0$, it glues to φ such that $\varphi|_{\mathcal{U}} = 0$. For $\tilde{\nu}_1$, since it induces isomorphism $\tilde{\mathcal{N}}|_{\tilde{\mathcal{U}}} \cong \tilde{\mathcal{L}}^\vee|_{\tilde{\mathcal{U}}}$, and this isomorphism descends to $\mathcal{N}|_{\mathcal{U}} \cong \mathcal{L}'^\vee|_{\mathcal{U}}$, we see that $\tilde{\nu}_1$ glues to get ν_1. Finally, we let $\Sigma^{\mathcal{C}}$ be the image of $\Sigma^{\tilde{\mathcal{C}}} - \big(\Upsilon_1 \cup \Upsilon_2\big)$. Then

$$\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu) \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S).$$

This proves the Proposition. $\square$

Let $\mathcal{W}^\sim_{g,\gamma,\mathbf{d}} \subset \mathcal{W}_{g,\gamma,\mathbf{d}}$ be the reduced closed substack where close points are $\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ such that $(\varphi = 0) \cup (\rho = 0) = \mathcal{C}$ (cf. Lemma 2.13). As the proof doesn't use the condition $\varphi_1^r + \ldots + \varphi_N^r = 0$, we have

Corollary 3.24. *Proposition 3.23 holds with $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ replaced by $\mathcal{W}^\sim_{g,\gamma,\mathbf{d}}$.*

4. The proof of Theorem 1.1

We first verify the valuative criterion for separatedness of $\mathcal{W}_{g,\gamma,\mathbf{d}}$. We then show that $\mathcal{W}^{-}_{g,\gamma,\mathbf{d}}$ is of finite type. Combined, they prove Theorem 1.1.

4.1. Valuative criterion for separatedness

As before, $\eta_0 \in S$ is a closed point in a smooth curve over $\mathbb{C}$, and $S_* = S - \eta_0$.

Lemma 4.1. *Let ξ, $\xi' \in \mathcal{W}_{g,\gamma,\mathbf{d}}(S)$ be such that $\xi_* \cong \xi'_* \in \mathcal{W}_{g,\gamma,\mathbf{d}}(S_*)$. Suppose $\mathcal{C}_*$ is smooth. Then $\xi \cong \xi'$.*

Proof. Let $\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \cdots)$ and $\xi' = (\Sigma^{\mathcal{C}'}, \mathcal{C}', \mathcal{L}', \cdots)$, let C (resp. C') be the coarse moduli of $\mathcal{C}$ (resp. $\mathcal{C}'$), and let $\pi : X \to C$ (resp. $\pi' : X' \to C'$) be the minimal desingularization. Thus π and π' are contractions of chains of (-2)-curves.

Let $f : X \dashrightarrow X'$ be the birational map induced by $\xi_* \cong \xi'_*$, let $U_0 \subset X$ be the largest open subset over which f is well-defined. Suppose $U_0 \subsetneq X$, then $X - U_0$ is discrete. Let X_1 be the blowing up of X at $X - U_0$. Inductively, suppose $X_k \to X$ is a successive blowing up along points, let $U_k \subset X_k$ be the largest open subset over which the birational $f : X_k \dashrightarrow X'$ is well-defined, then let X_{k+1} be the blowing up of X_k along $X_k - U_k$. After finite steps, we have $X_{\bar{k}} = U_{\bar{k}}$. Denote $Y = X_{\bar{k}}$ with

$$\bar{\pi} : Y \longrightarrow X \quad \text{and} \quad \bar{f} : Y \longrightarrow X'$$

the induced projection and birational morphism.

Let $E \subset Y$ be the exceptional divisor of $\bar{\pi}$. Write $E = \sum_{k\geq 1} E_k$, where $E_k \subset E$ is the proper transform of the exceptional divisor of $X_k \to X_{k-1}$ when $k \geq 2$ and the exceptional divisor of the map $X_1 \to X$ when $k = 1$. Let $E' \subset Y$ be the exceptional divisor of $\bar{f}$. By our construction, E and E' share no common irreducible curves. Let $Y_0 = \cup_j^n D_j$ be the irreducible component decomposition of the central fiber Y_0.

Furthermore, by our construction for $V = Y - E'$, $\bar{f}|_V : V \to \bar{f}(V)$ is an isomorphism, and by the blowing up formula, we have

$$\omega^{\log}_{Y/S} \cong \bar{\pi}^* \omega^{\log}_{X/S}(\sum_i iE_i) \quad \text{and} \quad \omega^{\log}_{Y/S}|_V \cong \bar{f}^* \omega^{\log}_{X'/S}|_V. \tag{4.1}$$

Let L and M (resp. L' and M') be the line bundles on X (resp. X') which are the pullbacks of the descents of $\mathcal{L}^{\otimes r}$ and $\mathcal{N}^{\otimes r}$ (resp. $\mathcal{L}'^{\otimes r}$ and $\mathcal{N}'^{\otimes r}$) to

C (resp. C'). Using $\xi_* \cong \xi'_*$, we can find integers a_i and b_i so that

$$\bar{f}^* L' \cong \bar{\pi}^* L\big(\sum a_i D_i\big) \quad \text{and} \quad \bar{f}^* M' \cong \bar{\pi}^* M\big(\sum b_i D_i\big). \tag{4.2}$$

Let u_1, u_2 and h_j be the pullbacks of the descents of ν_1^r, ν_2^r and φ_j^r, respectively, which are sections of $L \otimes M$, M and L, respectively. Denote by u'_1, u'_2 and h'_j be the pullbacks of the descents of $\nu_1'^r$, $\nu_2'^r$ and $\varphi_j'^r$ similarly. By the same reason, we will view ρ and ρ' as sections in $H^0(X, L^{-1} \otimes \omega^{\log}_{X/S})$ and $H^0(X', L'^{-1} \otimes \omega^{\log}_{X'/S})$ respectively.

We now show that $E = \emptyset$, namely, f is a morphism. Suppose not, let $D_j \subset E$ be an irreducible component with $x = \bar{\pi}(D_j) \in X$. We first remark that x is a smooth point of the central fiber X_0. Indeed, that x is a singular point of X_0 implies that $V_0 = V \times_S \eta_0$ is not reduced. On the other hand, since $\bar{f}|_V : V \to X'$ is an S-isomorphism onto its image and X'_0 is reduced, V_0 is reduced too. This proves that x is a smooth point of X_0.

For the same $x = \bar{\pi}(D_j)$, by the construction of $\bar{\pi}$ we know $\bar{\pi}^{-1}(x)$ is a tree of rational curves. By reindexing $Y_0 = \cup_{j=1}^n D_j$, we can assume that $D_1 + \cdots + D_k$ form a maximal chain of rational curves in $\bar{\pi}^{-1}(x)$ with $D_i \subset E_i$ for $i \le k$, and $D_i \cap D_{i+1} \neq \emptyset$ for $i < k$, thus $D_k \subset Y$ is a (-1)-curve. Since x is a smooth point of X_0, by our construction of $(\bar{\pi}, \bar{f})$, $\bar{f}(D_1), \cdots, \bar{f}(D_k)$ is a chain of rational curves in X', and $\bar{f}(D_k)$ is a (-1)-curve in X'. In particular, the image of $\bar{f}(D_k)$ in $\mathcal{C}'_0$, denoted by $\mathcal{D}'$, is a rational curve. Let $z \in D_k$ be a general point and let $y = \bar{f}(z)$. Note $x = \bar{\pi}(z)$.

Sublemma 4.2. *Let the situation be as stated. Then we have $u_2(x) = 0$.*

Proof. We prove by contradiction. Suppose $u_2(x) \neq 0$. We claim that $a_k = b_k = 0$.

We divide it into two cases. The first is when $u_1(x) \neq 0$. Since $\bar{\pi}^* u_2 \in H^0(V, \bar{\pi}^* M)$ and non-trivial along $V \cap D_k$, and since $\bar{f}^* u'_2 \in H^0(V, \bar{f}^* M')$, using (4.2) and that $\bar{\pi}^* u_2|_{V \times_S S_*} = \bar{f}^* u'_2|_{V \times_S S_*}$, we conclude that $b_k \ge 0$. Similarly, using that $u_1(x) \neq 0$, we conclude that $a_k + b_k \ge 0$.

We claim that $a_k + b_k = 0$. Suppose not, then by (4.2), we have $u'_1(y) = 0$, thus $u'_2(y) \neq 0$ and $(h'_i(y))_{i=1}^N \neq 0$, which forces $b_k \le 0$ and $a_k \le 0$, contradicting to $a_k + b_k > 0$. This proves $a_k + b_k = 0$.

We now prove $a_k = b_k = 0$. Suppose not. Since $a_k + b_k = 0$ and $b_k \ge 0$, we must have $b_k > 0$. Then $a_k < 0$ and $u'_2(y) = 0$. Note that because of (4.1) and (4.2), for any $i \le k$ and a dense open $U \subset D_i$ that is disjoint from the nodes of Y_0,

$$\bar{f}^*(L'^{-1} \otimes \omega^{\log}_{X'/S})|_U \cong \bar{\pi}^*(L^{-1} \otimes \omega^{\log}_{X/S})((i - a_i) D_i)|_U. \tag{4.3}$$

Applying to $i = k$, we conclude that $\rho'(y) = 0$, contradicting to $u_2'(y) = 0$. Thus $a_k = b_k = 0$.

The other case is when $u_1(x) = 0$. Since $u_1(x) = 0$, we have $(h_i(x))_{i=1}^N \neq 0$. Similar to the argument above, using $h_i(x)) \neq 0$ (resp. $u_2(x) \neq 0$), we conclude that $a_k \geq 0$ (resp. $b_k \geq 0$).

We now show that $b_k = 0$. Suppose not, that is $b_k > 0$, then we must have $\bar{f}^* u_2'|_{D_k} = 0$, which forces $\bar{f}^*\rho'|_{D_k} \neq 0$. Applying (4.1) and (4.2), we must have $k - a_k \leq 0$, thus $a_k \geq k \geq 1$, which forces $\bar{f}^* u_1'|_{D_k} = 0$, violating that (u_1', u_2') is nowhere vanishing. This proves $b_k = 0$.

A similar argument shows that $a_k > 0$ would lead to $\bar{f}^* h_i'|_{D_k} = \bar{f}^* u_1'|_{D_k} = 0$, a contradiction. Therefore, $a_k = b_k = 0$ in this case, too.

Let $\mathcal{D}' \subset \mathcal{C}'$ be the irreducible component whose image in C' is the same as the image of D_k under $Y \to X' \to C'$. Since $\bar{f}(D_k)$ is a (-1)-curve, $\mathcal{D}'$ is a smooth rational curve in $\mathcal{C}'$ and contains exactly one node of $\mathcal{C}_0'$ and at most one marking of $\Sigma^{\mathcal{C}'}$.

Next, we use (4.1) and $a_k = b_k = 0$ to conclude that $\rho'|_{\mathcal{D}'} = 0$. Therefore, $\mathcal{N}'|_{\mathcal{D}'} \cong \mathcal{O}_{\mathcal{D}'}$, and $(\varphi_1', \cdots, \varphi_N', \nu_1')|_{\mathcal{D}'}$ defines a morphism $\beta' : \mathcal{D}' \to \mathbb{P}^N$. Let $q' \in \mathcal{D}'$ be the node of $\mathcal{C}_0'$. By (4.2) and that $a_k = b_k = 0$, we conclude that

$$\bar{f}^*(\varphi_1', \cdots, \varphi_N', \nu_1')|_{\mathcal{D}'-q'} = \bar{\pi}^*\big((\varphi_1, \cdots, \varphi_N, \nu_1)|_x\big)|_{\mathcal{D}'-q'}. \tag{4.4}$$

Thus $\beta' : \mathcal{D}' \to \mathbb{P}^N$ is a constant map. Since $\mathcal{D}'$ contains one node of $\mathcal{C}_0'$ and at most one marking in $\Sigma^{\mathcal{C}'}$, adding that $\rho'|_{\mathcal{D}'} = 0$, by Lemma 3.3 we conclude that ξ_0' is unstable, a contradiction. This proves the Sublemma. □

We continue to denote by $D_1 + \cdots + D_k$ a maximal chain of rational curves in $\bar{\pi}^{-1}(x)$ with $D_i \in E_i$.

Sublemma 4.3. *We have $a_k = k$, $a_i \leq a_{i+1} - 2$ for $i < k$, and $a_i + b_i = 0$ for all $i \leq k$.*

Proof. First, we have (4.3). We claim that for $i \leq k$,

$$i - a_i \geq 0 \quad \text{and} \quad a_i + b_i = 0. \tag{4.5}$$

In fact, by the previous Sublemma, we know $u_2(x) = 0$, thus $\rho(x) \neq 0$ and $\bar{\pi}^*\rho|_{D_i} \neq 0$. Adding that $\bar{f}^*\rho'$ is regular along V and coincides with $\bar{\pi}^*\rho$ over $V \times_S S_*$, we obtain the first inequality in (4.5). Similarly, using $u_1(x) \neq 0$, we obtain $a_i + b_i \geq 0$.

We now show $a_i + b_i = 0$. Suppose not, say $a_i + b_i > 0$, then $\bar{f}^* u_1'|_{D_i} = 0$, which forces $\bar{f}^* h_j'|_{D_i} \neq 0$ for some j, and by (4.2), we obtain $a_i \leq 0$. As

$a_i + b_i > 0$, we obtain $b_i > 0$, and hence $\bar{f}^* u_2'|_{D_i} = 0$, contradicting to $(u_1'(y), u_2'(y)) \neq 0$. This proves (4.5).

We next prove $a_k = k$. Suppose not, by (4.5) we have $k - a_k \geq 1$. Then $\bar{f}^*\rho'|_{D_k} = 0$, which forces $\bar{f}^* u_2'|_{D_k}$ nowhere vanishing. By (4.2), we have $b_k \leq 0$; adding $a_k + b_k = 0$, we have $a_k \geq 0$.

We claim $a_k > 0$. Suppose not, i.e., $a_k = 0$. Let $\mathcal{D}' \subset \mathcal{C}'$, as before, be the irreducible component whose image in C' is the same as the image of D_k under $Y \to X' \to C'$. As argued in the proof of the previous Sublemma, $\mathcal{D}'$ is a smooth rational curve in $\mathcal{C}'$ and contains one node q' of $\mathcal{C}_0'$. As $a_k = 0$, $\rho'|_{\mathcal{D}'} = 0$, $\mathcal{N}'|_{\mathcal{D}'} \cong \mathcal{O}_{\mathcal{D}'}$, and $(\varphi_1', \cdots, \varphi_N', \nu_1')|_{\mathcal{D}'}$ defines a morphism $\beta' : \mathcal{D}' \to \mathbb{P}^N$, which is a constant map by (4.4). Therefore, as $\mathcal{D}'$ contains at most one marking of $\Sigma^{\mathcal{C}'}$, ξ_0' becomes unstable, a contradiction. This proves that a_k is positive.

Adding $\bar{f}^*\rho'|_{D_k} = 0$, we have $\rho'|_{\mathcal{D}'} = \varphi'|_{\mathcal{D}'} = 0$, thus $\nu_1'|_{\mathcal{D}'}$ and $\nu_2'|_{\mathcal{D}'}$ are nowhere vanishing. Because $\mathcal{D}'$ contains one node and at most one marking of $\mathcal{C}_0'$, ξ_0' becomes unstable, a contradiction. This proves that $a_k = k$.

Finally, we prove $a_i \leq a_{i+1} - 2$. Let $\Lambda = \{1 \leq i < k \mid a_i \not\leq a_{i+1} - 2\}$. The intended inequality is equivalent to $\Lambda = \emptyset$. Suppose $\Lambda \neq \emptyset$, and let i be the largest element in Λ. Suppose $i = k - 1$. Since $a_k = k$, we have $a_{k-1} \geq a_k - 1 = k - 1$. By (4.5), we conclude that $a_{k-1} = k - 1$, which implies $\deg \bar{f}^* L'|_{D_k} = -1$.

Consequently, using (4.3), we conclude that $\bar{f}^*\rho'|_{D_k}$ is nowhere vanishing. Like before, let $\mathcal{D}' \subset \mathcal{C}_0'$ be the irreducible component associated to $D_k \subset E$, via $Y \to C'$ and $\mathcal{C}' \to C'$. Then $\mathcal{D}'$ is a rational curve, contains one node of $\mathcal{C}_0'$, and with $\varphi'|_{\mathcal{D}'} = 0$, $\rho'|_{\mathcal{D}'}$ nowhere vanishing and $\deg \mathcal{L}'|_{\mathcal{D}'} = -\frac{1}{5}$. By Lemma 3.3, this makes ξ_0' unstable, a contradiction. Therefore, $i < k - 1$.

Since $i + 1 \notin \Lambda$, $(i+1) - a_{i+1} \geq 1$, which forces $\bar{f}^*\rho'|_{D_{i+1}} = 0$, and then $\bar{f}^* u_2'|_{D_{i+1}}$ is nowhere vanishing and $\bar{f}^* M'|_{D_{i+1}} \cong \mathcal{O}_{D_{i+1}}$. We claim that $\bar{f}^* L'|_{D_{i+1}} \cong \mathcal{O}_{D_{i+1}}$. Since $\bar{\pi}(D_{i+1}) = x$, $\bar{\pi}^*(L \otimes M) \cong \mathcal{O}_{D_{i+1}}$. Because $a_{i+1} + b_{i+1} = 0$ (cf. (4.5)), by (4.2) $\bar{f}^*(L' \otimes M') \cong \mathcal{O}_{D_{i+1}}$. Using $\bar{f}^* M'|_{D_{i+1}} \cong \mathcal{O}_{D_{i+1}}$, we conclude $\bar{f}^* L'|_{D_{i+1}} \cong \mathcal{O}_{D_{i+1}}$.

Now let $D_i, D_{i+2}, D_{k_2}, \cdots, D_{k_l}$ be the irreducible components in E that intersect D_{i+1}. Since $i + 1 \notin \Lambda$, $a_{i+1} \leq a_{i+2} - 2$. Possibly by changing to a different maximal chain of rational curves in $\bar{\pi}^{-1}(x)$, we can assume without loss of generality that $a_{i+1} \leq a_{k_s} - 2$ for all $2 \leq s \leq l$. Because $\bar{f}^* L'|_{D_{i+1}} \cong \mathcal{O}_{D_{i+1}}$, we have

$$(a_i - a_{i+1}) + (a_{i+2} - a_{i+1}) + \sum_s (a_{k_s} - a_{i+1}) = 0.$$

Thus, $a_i - a_{i+1} \le -2$, a contraction. This proves $\Lambda = \emptyset$, and the Sublemma follows. $\square$

We continue our proof of Lemma 4.1. We keep the maximal chain of rational curves $D_1, \cdots, D_k \subset E$. We claim $k = 1$. Otherwise, $k \ge 2$ and $a_1 \le a_2 - 2 \le 0$. By (4.3) we obtain $\bar f^* \rho'|_{D_1} = 0$ and thus $\bar f^* u_2'|_{D_1}$ is nowhere vanishing. Since $u_2(x) = 0$ by Sublemma 4.2, we get $\bar\pi^* u_2|_{D_1} = 0$. Thus from (4.2), we get $b_1 < 0$ and hence $a_1 = -b_1 > 0$, a contradiction to $a_1 \le 0$. Thus $k = 1$.

Next we prove that every $D_j \subset Y_0$ not in E that intersects one of $D_i \subset E$ must be contracted by $\bar f$. Indeed, if $\bar f(D_j)$ is not a point, since $\bar\pi(D_j)$ is not a point as well, we must have $a_j = b_j = 0$ and $D_j \cap V$ is dense in D_j. Since D_i is a (-1)-curve, the combination of (4.1), (4.2) and $\rho(x) \ne 0$ (since $u_2(x) = 0$) implies $\bar f^* \rho'|_{D_i}$ is nowhere vanishing. From (4.2), we also have $\deg(\bar f^* L'|_{D_i}) = -1$ since $a_i = 1$ and $a_j = 0$. Thus ξ_0' satisfies the condition (2) in the Lemma 3.3 and hence is not stable, a contradiction.

In conclusion, we have proved that E is a disjoint union of (-1)-curves, likewise for E', and that every irreducible component in Y_0 not in E must be in E'. Since $Y \to X$ is by first blowing up smooth points of X_0, and since Y_0 is connected, this is possible only if both $E \cong \mathbb{P}^1$, and then Y is the blowing up of $X = S \times \mathbb{P}^1$ at a single point in X_0.

Then a direct analysis shows that this is impossible, assuming both ξ_0 and ξ_0' are stable. (As this analysis is straightforward, we omit the details here.) This proves that $E = \emptyset$ and $f : X \to X'$ is a birational morphism. By symmetry, $f^{-1} : X' \to X$ is also a birational morphism. Therefore, $f : X \cong X'$ is an isomorphism.

Knowing that f is an isomorphism, a parallel argument shows that

$$f^* L' \cong L, \quad f^* M' \cong M, \quad \text{and} \quad f^*(\rho', h_k', u_i') = (\rho, h_k, u_i). \tag{4.6}$$

We prove that this implies $C \cong C'$. Indeed, it is easy to show that a $D_i \subset X$ is contracted by $\mathrm{pr} : X \to C$ if and only if $D_i \subset X$ is a (-2)-curve and $L|_{D_i} \cong M|_{D_i} \cong \mathcal{O}_{D_i}$. Therefore, D_i is contracted by the map $X \to C$ if and only if $f(D_i)$ is contracted by the map $X' \to C'$. This proves that $f : X \cong X'$ induces an isomorphism $\bar\phi : C \cong C'$.

Let Δ (resp. Δ') be the set of singular points of $\mathcal{C}_0$ (resp. $\mathcal{C}_0'$). Let $\pi : \mathcal{C} \to C$ and $\pi' : \mathcal{C}' \to C'$ be the coarse moduli morphisms. Then the isomorphisms $\bar\phi$ and (4.6) (with $a_i = b_i = 0$) induce isomorphisms

$$\bar\phi^* p_*'(\Sigma^{\mathcal{C}'}, \mathcal{L}'^{\otimes r}, \mathcal{N}'^{\otimes r}, \varphi_k'^r, \rho', \nu_i'^r) \cong p_*(\Sigma^{\mathcal{C}}, \mathcal{L}^{\otimes r}, \mathcal{N}^{\otimes r}, \varphi_k^r, \rho, \nu_i^r); \tag{4.7}$$

and isomorphisms $\phi : \mathcal{C} - \Delta \xrightarrow{\cong} \mathcal{C}' - \Delta'$,

$$\phi^* \mathcal{L}' \cong \mathcal{L}, \quad \phi^* \mathcal{N}' \cong \mathcal{N}, \quad \text{and} \quad \phi^*(\varphi', \rho', \nu_i') = (\varphi, \rho, \nu_i), \tag{4.8}$$

extending $\xi_* \cong \xi_*'$.

Now let $p \in \Delta$ be a point and let $p' \in \Delta'$ be the corresponding point. Pick an open subset $\mathcal{U} \subset \mathcal{C}$ of $p \in \mathcal{C}$ so that $\mathcal{U} \cap \Delta = p$. Let $\mathcal{U}' \subset \mathcal{C}'$ be the open subset of $p' \in \mathcal{C}'$ so that $\mathcal{U}' \cap \Delta' = p'$ and $\phi(\mathcal{U} - p) = \mathcal{U}' - p'$.

If $\nu_2(p) \neq 0$, then $\nu_2'(p') \neq 0$. Hence both p and p' are scheme points. The uniqueness of extensions is straightforward.

Next, let's consider the case that $\nu_2(p) = \nu_2'(p') = 0$. Hence, we have $\rho(p) \neq 0$. Thus, $\nu_1(p) \neq 0$. Hence, after shrinking $\mathcal{U}$ if necessary, ν_1 is nowhere vanishing on $\mathcal{U}$, which provides an isomorphism $\mathcal{L} \cong \mathcal{N}^{-1}$. Similarly, ν_1', after shrinking $\mathcal{U}'$ if necessary, is nowhere vanishing on $\mathcal{U}'$, resulting an isomorphism $\mathcal{L}' \cong \mathcal{N}'^{-1}$. We now assume that $\nu_2 = 0$ on the whole curve $\mathcal{C}$, then $\nu_2' = 0$ on the whole curve $\mathcal{C}'$. This implies that ρ and ρ' are nowhere vanishing on the whole $\mathcal{C}$ and $\mathcal{C}'$ respectively, which reduces to the case of stable spin curves. By the result in [AJ, CLL], the extension is unique.

The other case is when $\{\nu_2 = 0\}$ and $\{\nu_2' = 0\}$ are divisors on $\mathcal{C}$ and $\mathcal{C}'$ respectively. Let R be the (strict) henselization of the local ring $\mathcal{O}_{S,\eta_0}$. For a scheme Y over $\mathbb{C}$ and a point p on Y, let Y_p^h represent the (strict) henselization of Y at p. The same notation applies to rings. Since the coarse moduli spaces C and C' are isomorphic, we can write $C_p^h \cong C_{p'}^h \cong \operatorname{Spec}\big(R[x,y]^h/(xy - t^k)\big)$. Following the description of twisted curves in §1.3 of [AJ], we have $\mathcal{C}_p^h = [\operatorname{Spec}\big(R[u,v]^h/(uv - t^b)\big)/\boldsymbol{\mu}_a]$, where $ab = k$ and the group action by $\boldsymbol{\mu}_a$ is given by $\zeta \cdot (u,v) = (\zeta u, \zeta^{-1} v)$ for $\zeta \in \boldsymbol{\mu}_a$. For the same reason, $\mathcal{C}_{p'}'^h = [\operatorname{Spec}\big(R[u',v']^h/(u'v' - t^{b'})\big)/\boldsymbol{\mu}_{a'}]$, where $k = a'b'$ and $\boldsymbol{\mu}_{a'}$-action is given by $\zeta' \cdot (u',v') = (\zeta' u', \zeta'^{-1} v')$ for $\zeta' \in \boldsymbol{\mu}_{a'}$.

By abuse of notation, we still use π for the morphism $\operatorname{Spec}\big(R[u,v]^h/(uv - t^b)\big) \to C_p^h$ via $(x,y) \to (u^a, v^a)$; π' for the morphism $\operatorname{Spec}\big(R[u',v']^h/(u'v' - t^{b'}\big) \to C_{p'}'^h$ via $(x,y) \to (u'^{a'}, v'^{a'})$. The sheaf $\mathcal{N}$ restricted to $\mathcal{C}_p^h$ can be identified with a $\boldsymbol{\mu}_a$-equivariant (structure) sheaf $\mathcal{O}$ of $\operatorname{Spec}\big(R[u,v]^h/(uv - t^b)\big)$, with the $\boldsymbol{\mu}_a$ action given by $\mathbf{1}^\zeta = \zeta^{-\alpha}\mathbf{1}$, where $\zeta \in \boldsymbol{\mu}_a$, $0 < \alpha < a$, and $\mathbf{1}$ is the constant section 1 of $\mathcal{O}$. Because $\mathcal{L} \cong \mathcal{N}$ shown before, the representability of $(\mathcal{L}, \mathcal{N})$ implies that α is relatively prime to a. Using $\mathbf{1}$, we can write the lift of ν_2 as $f \cdot \mathbf{1}$, for an $f \in R[u,v]^h/(uv - t^b)$. As the lift of ν_2 must be $\boldsymbol{\mu}_a$-invariant, we have

$$f \cdot \mathbf{1} = (f \cdot \mathbf{1})^\zeta = \zeta^{-\alpha} f^\zeta \cdot \mathbf{1}.$$

(Here for the group action, we follow the convention that for a function f on a G-scheme X, $f^g(x) = f(g^{-1}\cdot x)$ for $g \in G$ and $x \in X$.) Thus $f^\zeta = \zeta^\alpha f$.

We do the same thing for the pair $(\nu', \mathcal{N}')$. After trivializing the lift of $\mathcal{N}'$ to $\mathcal{C}'^h_{p'}$, we can write the lift of ν' as $f' \cdot \mathbf{1}'$, with the $\boldsymbol{\mu}_{a'}$ action as $(f')^{\zeta'} = \zeta'^{\alpha'} f'$, where $\zeta' \in \boldsymbol{\mu}_{a'}$ and $0 < \alpha' < a'$. As before $(a', \alpha') = 1$.

Let $d = \gcd(b, b')$, the greatest common divisor of b and b'; we write $b = \bar{b}d$ and $b' = \bar{b}'d$, thus $\gcd(\bar{b}, \bar{b}') = 1$. Let $c = \mathrm{lcm}(a, a')$. Because $ab = a'b'$, we obtain $c = a\bar{b} = a'\bar{b}'$.

Let $Z = \mathrm{Spec}\big(R[z, w]^h/(zw - t^d)\big)$. We define morphisms

$$\begin{aligned} \tau \colon Z &\to \mathrm{Spec}\big(R[u, v]^h/(uv - t^b)\big), \quad u \to z^{\bar{b}}, \quad v \to w^{\bar{b}}; \\ \tau' \colon Z &\to \mathrm{Spec}\big(R[u', v']^h/(u'v' - t^{b'})\big), \quad u' \to z^{\bar{b}'}, \quad v' \to w^{\bar{b}'}. \end{aligned}$$

We let $\zeta_m = e^{2\pi i/m}$ be the obvious generator of the group $\boldsymbol{\mu}_m$, and let the group $\boldsymbol{\mu}_c$ acts on Z via $\zeta_c \cdot (z, w) = (\zeta_c z, \zeta_c^{-1} w)$. Direct calculations give

$$(\tau^* f)^{\zeta_c} = \zeta_a^\alpha \tau^* f, \quad \text{and} \quad (\tau'^* f')^{\zeta_c} = \zeta_{a'}^{\alpha'} \tau'^* f'.$$

Since, as divisors, $\{f^r = 0\} = \pi^*\{u_2 = 0\}$ and $\{f'^r = 0\} = \pi'^*\{u_2' = 0\}$, and since $\{u_2 = 0\} = \{u_2' = 0\}$, the vanishing of $\tau^* f$ and $\tau'^* f'$ at the node are identical. This implies that $\zeta_a^\alpha = \zeta_{a'}^{\alpha'}$. Because $(a, \alpha) = (a', \alpha') = 1$, we conclude $a = a'$, and thus $\alpha = \alpha'$.

Therefore, ϕ extends to a morphism $\tilde{\phi} : (\mathcal{C} - \Delta) \cup \mathcal{U} \to (\mathcal{C}' - \Delta') \cup \mathcal{U}'$ so that (4.8) extends to

$$\tilde{\phi}^* \mathcal{L}' \cong \mathcal{L}, \quad \tilde{\phi}^* \mathcal{N}' \cong \mathcal{N}, \quad \tilde{\phi}^*(\varphi', \rho', \nu_1', \nu_2') = (\varphi, \rho, \nu_1, \nu_2), \tag{4.9}$$

and $\tilde{\phi}$ is an isomorphism. By going through this local extension throughout all points in Δ, we conclude that ϕ extends to an isomorphism $\bar{\phi} : \mathcal{C} \to \mathcal{C}'$ so that (4.8) extends to (4.9). This proves that $\xi \cong \xi'$. □

Proposition 4.4. *Lemma 4.1 holds without assuming that $\mathcal{C}_*$ is smooth.*

Proof. Let $\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \cdots)$ be as before, and let ξ_α, $\alpha \in \Xi$, be families constructed as in Corollary 3.4. Let $\xi' = (\Sigma^{\mathcal{C}'}, \mathcal{C}', \cdots)$ and likewise ξ'_α, $\alpha \in \Xi$, be the similar decomposition. Here both ξ_α and ξ'_α are indexed by the same set Ξ because $\xi_* \cong \xi'_*$. By Corollary 3.4, all ξ_α and ξ'_α are stable families of MSP-fields.

Since $\xi_* \cong \xi'_*$, we have $\xi_{\alpha*} \cong \xi'_{\alpha*}$. By Lemma 4.1, $\xi_{\alpha*} \cong \xi'_{\alpha*}$ extends to $\xi_\alpha \cong \xi'_\alpha$. Then a direct argument shows that as the isomorphisms $\xi_\alpha \cong \xi'_\alpha$ are consistent with the isomorphism $\xi_* \cong \xi'_*$, they induce an isomorphism

$\xi \cong \xi'$, extending $\xi_* \cong \xi'_*$. As the argument is straightforward, we omit the detail here. $\square$

4.2. $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ is of finite type

We know $\mathcal{W}_{g,\gamma,\mathbf{d}}$ is a T-stack, and $\mathcal{W}^-_{g,\gamma,\mathbf{d}} \subset \mathcal{W}_{g,\gamma,\mathbf{d}}$ is T-equivariant. We consider the fixed locus $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})^T$.

Let $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})^T$ be of the presentation $(\Sigma^{\mathcal{C}}, \mathcal{C}, \cdots)$. An easy argument shows that for any irreducible component $\mathcal{D} \subset \mathcal{C}$, one of the following holds: $\nu_1|_{\mathcal{C}} = 0$, or $\nu_2|_{\mathcal{D}} = 0$, or $\varphi|_{\mathcal{D}} = \rho|_{\mathcal{D}} = 0$, or $\mathcal{D}$ is a (twisted) rational curve.

We let $\mathcal{C}_0$ (resp. $\mathcal{C}_\infty$; resp. $\mathcal{C}_1$) be the dimension one part of $\mathcal{C} \cap (\nu_1 = 0)_{\mathrm{red}}$ (resp. $\mathcal{C} \cap (\nu_2 = 0)_{\mathrm{red}}$; resp. $\mathcal{C} \cap (\rho = \varphi = 0)_{\mathrm{red}}$). Let $\mathcal{C}_{01}$ (resp. $\mathcal{C}_{1\infty}$) be union of irreducible components in $\mathcal{C} \cap (\rho = 0)_{\mathrm{red}}$ (resp. $\mathcal{C} \cap (\varphi = 0)_{\mathrm{red}}$) that are not in $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_\infty$.

Lemma 4.5. *We have*

1. *The curves* $\mathcal{C}_0$, $\mathcal{C}_1$ *and* $\mathcal{C}_\infty$ *are mutually disjoint;*
2. *no two of* $\mathcal{C}_0, \mathcal{C}_{01}, \mathcal{C}_1, \mathcal{C}_{1\infty}, \mathcal{C}_\infty$ *share common irreducible components;*
3. $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_{01} \cup \mathcal{C}_1 \cup \mathcal{C}_{1\infty} \cup \mathcal{C}_\infty$.

We derive some information of the degrees of $\mathcal{L}$ and $\mathcal{N}$ along $\mathcal{C}_a$. First, since $\nu_2|_{\mathcal{C}_0 \cup \mathcal{C}_{01} \cup \mathcal{C}_1}$ is nowhere vanishing, $\mathcal{N}|_{\mathcal{C}_0 \cup \mathcal{C}_{01} \cup \mathcal{C}_1} \cong \mathcal{O}_{\mathcal{C}_0 \cup \mathcal{C}_{01} \cup \mathcal{C}_1}$. Thus

$$d_\infty = \deg \mathcal{N} = \deg \mathcal{N}|_{\mathcal{C}_{1\infty} \cup \mathcal{C}_\infty}. \tag{4.10}$$

Similarly, since ν_1 restricted to $\mathcal{C}_1 \cup \mathcal{C}_{1\infty} \cup \mathcal{C}_\infty$ is nowhere vanishing,

$$\mathcal{L} \otimes \mathcal{N}|_{\mathcal{C}_1 \cup \mathcal{C}_{1\infty} \cup \mathcal{C}_\infty} \cong \mathcal{O}|_{\mathcal{C}_1 \cup \mathcal{C}_{1\infty} \cup \mathcal{C}_\infty}.$$

Combined, we get

$$\deg \mathcal{L}|_{\mathcal{C}_0 \cup \mathcal{C}_{01}} = d_0, \quad \deg \mathcal{L}|_{\mathcal{C}_1} = 0, \quad \deg \mathcal{L}|_{\mathcal{C}_{1\infty} \cup \mathcal{C}_\infty} = -d_\infty. \tag{4.11}$$

We let Υ_ξ be the dual graph of $\Sigma^{\mathcal{C}} \subset \mathcal{C}$. Namely, vertices of Υ_ξ correspond to irreducible components of $\mathcal{C}$, edges of Υ_ξ correspond to nodes connecting two different irreducible components of $\mathcal{C}$, and legs of Υ_ξ correspond to markings $\Sigma^{\mathcal{C}}$. Let $V(\Upsilon_\xi)$ be the set of vertices, and $E(\Upsilon_\xi)$ be the set of edges. Furthermore, each vertex $v \in V(\Upsilon_\xi)$ is decorated by $g_v := g(\mathcal{C}_v)$, the arithmetic genus of the irreducible component $\mathcal{C}_v$ associated to v.

Given a decorated graph Υ as above (i.e. a connected graph with legs, vertices v decorated by $g_v \in \mathbb{Z}_{\geq 0}$, and without circular-edges), we say $v \in$

$V(\Upsilon)$ is stable (resp. semistable) if $2g_v - 2 + |E_v| \geq 1$ (resp. ≥ 0), where E_v is the set of legs and edges in Υ attached to v.

Proposition 4.6. *The set* $\Theta := \{\Upsilon_\xi \mid \xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})^T\}$ *is a finite set.*

Proof. Let $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})^T$. As the curve $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ may not be stable, knowing the total genus and the number of legs of Υ_ξ is not sufficient to bound the geometry of Υ_ξ. Our approach is to use the information of line bundles $\mathcal{L}$ and $\mathcal{N}$ on $\mathcal{C}$ given by ξ to add legs to Υ_ξ to form a semistable $\tilde{\Upsilon}_\xi$

First, let $\mathbb{I} = \{0, 01, 1, 1\infty, \infty\}$. Using (3) of Lemma 4.5, for $a \in \mathbb{I}$ we define $V(\Upsilon_\xi)_a = \{v \in V(\Upsilon_\xi) \mid \mathcal{C}_v \subset \mathcal{C}_a\}$. By the stability criterion Lemma 3.3, we know that all $v \in V(\Upsilon_\xi)_1 \cup V(\Upsilon_\xi)_\infty$ are stable.

We now add new legs to vertices in $V(\Upsilon_\xi)$, called auxiliary legs. For every $v \in V(\Upsilon_\xi)_0 \cup V(\Upsilon_\xi)_{01}$, we add $3\deg \mathcal{L}|_{\mathcal{C}_v}$ auxiliary legs to v. By (4.11), the total new legs added to all vertices in $V(\Upsilon_\xi)_0 \cup V(\Upsilon_\xi)_{01}$ is $3d_0$.

We next treat the vertices in $V(\Upsilon_\xi)_{1\infty}$. We first introduce

$$E_\infty = \{e \in E(\Upsilon_\xi) \mid e \in E_v \text{ for some } v \in V(\Upsilon_\xi)_\infty\}.$$

For $v \in V(\Upsilon_\xi)_{1\infty}$, we define

$$\delta(v) = r\deg \mathcal{N}|_{\mathcal{C}_v} - |E_v \cap E_\infty| \in \mathbb{Z}_{\geq 0}. \tag{4.12}$$

Since $|E_v \cap E_\infty| = 0$ or 1, and since $\deg \mathcal{N}|_{\mathcal{C}_v} \geq 1/r$, $\delta(v) \geq 0$. It takes value in $\mathbb{Z}_{\geq 0}$ since $\deg \mathcal{N}|_{\mathcal{C}_v} \in \frac{1}{r}\mathbb{Z}$. To each $v \in V(\Upsilon_\xi)_{1\infty}$, we add $2\delta(v)$ many auxiliary legs to v.

We now show that the number of new legs added to $V(\Upsilon_\xi)_{1\infty}$ is bounded by $2rd_\infty + 4g + 2\ell$. Let $\mathcal{C}^1_\infty, \ldots, \mathcal{C}^s_\infty$ be the connected components of $\mathcal{C}_\infty$, m_i be the number of markings on $\mathcal{C}^i_\infty$. Because $\rho|_{\mathcal{C}^i_\infty}$ is nowhere vanishing, using the discussion leading to (4.11), we have

$$0 = -r\deg \mathcal{L}|_{\mathcal{C}^i_\infty} + \deg \omega^{\log}_{\mathcal{C}}|_{\mathcal{C}^i_\infty} = r\deg \mathcal{N}|_{\mathcal{C}^i_\infty} + \big(2g(\mathcal{C}^i_\infty) - 2 + |\mathcal{C}^i_\infty \cap \mathcal{C}_{1\infty}| + m_i\big).$$

Therefore, using that $\deg \mathcal{N}|_{\mathcal{C}_v} = 0$ unless $v \in V(\Upsilon_\xi)_{1\infty} \cup V(\Upsilon_\xi)_\infty$, and using that $\sum_{v \in V(\Upsilon_\xi)_{1\infty}} |E_v \cap E_\infty| = \sum_{i=1}^r |\mathcal{C}^i_\infty \cap \mathcal{C}_{1\infty}|$, we obtain

$$\begin{aligned} d_\infty = \deg \mathcal{N} &= \sum_{i=1}^s \frac{1}{r}\big(2 - 2g(\mathcal{C}^i_\infty) - |\mathcal{C}^i_\infty \cap \mathcal{C}_{1\infty}| - m_i\big) + \sum_{v \in V(\Upsilon_\xi)_{1\infty}} \deg \mathcal{N}|_{\mathcal{C}_v} \\ &= \frac{2s}{r} - \sum_{i=1}^s \frac{1}{r}\big(m_i + 2g(\mathcal{C}^i_\infty)\big) + \frac{1}{r}\sum_{v \in V(\Upsilon_\xi)_{1\infty}} \delta(v). \end{aligned}$$

Thus the total number of auxiliary legs added to vertices in $V(\Upsilon_\xi)_{1\infty}$ is bounded by $2rd_\infty + 4g + 2\ell$; the number s of connected components of $\mathcal{C}_\infty$ is bounded by the same number, too.

Let $\tilde{\Upsilon}_\xi$ be the resulting graph after adding auxiliary legs to $v \in V(\Upsilon_\xi)$ according to the rules specified above. We now study the stability of vertices of $\tilde{\Upsilon}_\xi$. Let $v \in V(\tilde{\Upsilon}_\xi)$ be a not-stable vertex. Then $v \in V(\Upsilon_\xi)_{1\infty}$, $|E_v| = 1$ or 2, and $\delta(v) = 0$. In case $|E_v| = 1$, by (4.12) we have $\deg \mathcal{N}|_{\mathcal{C}_v} = \frac{1}{r}|E_v \cap E_{V(\Upsilon_\xi)_\infty}| \le 1$. On the other hand, $\deg \mathcal{N}|_{\mathcal{C}_v} \in \frac{1}{r}\mathbb{Z}_{>0}$. Thus $|E_v \cap E_\infty| = 1$ and $\deg \mathcal{N}|_{\mathcal{C}_v} = \frac{1}{r}$. By Lemma 3.3, ξ is not stable, impossible. Thus $|E_v| = 2$ and $\deg \mathcal{N}|_{\mathcal{C}_v} = \frac{1}{r}$, and v is a strictly semistable vertex of $\tilde{\Upsilon}_\xi$.

We now show that $\tilde{\Upsilon}_\xi$ contains no chain of strictly semistable vertices of length more than two. Indeed, suppose $v_1, v_2, v_3 \in V(\tilde{\Upsilon}_\xi)$ with edges e_1 and e_2 forming a chain of unstable vertices in $\tilde{\Upsilon}_\xi$, where e_1 connects v_1 and v_2, and e_2 connects v_2 and v_3. By our construction, all $v_i \in V(\Upsilon_\xi)_{1\infty}$. Since ξ is stable, by Lemma 3.3, $\deg \mathcal{N}|_{\mathcal{C}_{v_2}} \ge \frac{1}{r}$. Since both v_1 and v_3 are not in $V(\Upsilon_\xi)_\infty$, $E_{v_2} \cap E_\infty = \emptyset$, contradicting to $\delta(v_2) = 0$.

For $\tilde{\Upsilon}_\xi$, we let $(\tilde{\Upsilon}_\xi)^{\mathrm{st}}$ be the stabilization of $\tilde{\Upsilon}_\xi$. Since the total genus of $(\tilde{\Upsilon}_\xi)^{\mathrm{st}}$ is g, while the number of marking of $(\tilde{\Upsilon}_\xi)^{\mathrm{st}}$ is bounded by some constant N, the set

$$\Theta^{\mathrm{st}} = \{(\tilde{\Upsilon}_\xi)^{\mathrm{st}} \mid \tilde{\Upsilon}_\xi \in \Theta\}$$

is finite. Because the contraction $\tilde{\Upsilon}_\xi \rightsquigarrow (\tilde{\Upsilon}_\xi)^{\mathrm{st}}$ is by contracting chains of at most length two strictly semistable vertices, Θ^{st} is finite implies that Θ is finite. $\square$

Proposition 4.7. *The collection $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})^T$ is bounded.*

Proof. Since Θ is finite, we only need to show that to any $\Upsilon \in \Theta$, the set $\mathcal{W}_\Upsilon = \{\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C}) \mid \Upsilon_\xi \cong \Upsilon\}$ is bounded. But this is a routine check, and will be omitted. $\square$

4.3. Proof of the theorem

Proposition 4.8. *The stack $\mathcal{W}_{g,\gamma,\mathbf{d}}$ is separated.*

Proof. Because $\mathcal{W}_{g,\gamma,\mathbf{d}}$ if locally of finite type, to prove that it is separated we only need to show that any finite type open substack $\mathcal{W} \subset \mathcal{W}_{g,\gamma,\mathbf{d}}$ is separated. Then because each $\mathcal{W}$ is defined over $\mathbb{C}$ and is of finite type, to show that it is separated we only need to verify the statement in Proposition 4.4. (see [Zh].) By Proposition 4.4, $\mathcal{W}$ is separated. This proves $\mathcal{W}_{g,\gamma,\mathbf{d}}$ is separated. $\square$

Proposition 4.9. *The stack $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ is of finite type.*

Proof. Since $\mathcal{W}_{g,\gamma,\mathbf{d}}$ is of locally finite type, we can cover it by a collection of étale morphisms $f_i\colon U_i \to \mathcal{W}^-_{g,\gamma,\mathbf{d}}$, where $i \in \Lambda$, such that the index set Λ is possibly infinite and each U_i is of finite type. Since $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})^T$ is of finite type, we can find a finite subset $\Lambda_0 \subset \Lambda$ such that

$$\mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})^T = \bigcup_{i\in\Lambda_0} \big(f_i(U_i) \cap \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})^T\big).$$

In this way, the map $f\colon U = \coprod_{i\in\Lambda_0} U_i \to \mathcal{W}^-_{g,\gamma,\mathbf{d}}$ is an étale neighbourhood of $(\mathcal{W}^-_{g,\gamma,\mathbf{d}})^T$.

We claim that the morphism

$$F : U \times \mathbb{C}^* \to \mathcal{W}^-_{g,\gamma,\mathbf{d}}, \quad (u,t) \to t\cdot f(u)$$

is surjective. Indeed, let $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}} - F(U\times\mathbb{C}^*)$; let $F_\xi : \mathbb{C}^* \to \mathcal{W}^-_{g,\gamma,\mathbf{d}}$, $t \mapsto t\cdot\xi$. Then $F_\xi(\mathbb{C}^*) \cap F(U\times\mathbb{C}^*) = \emptyset$. On the other hand, by Proposition 3.23, $F_\xi(t)$ specializes to a $\xi_0 \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})$ as t specializes to 0. By the construction of F_ξ, $\xi_0 \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})^T$. But since $f(U)$ is an open neighborhood of ξ_0, $F_\xi(\mathbb{C}^*) \cap f(U) \neq \emptyset$. Finally, since $f(U) \subset F(U\times\mathbb{C}^*)$, we get $F_\xi(\mathbb{C}^*) \cap F(U\times\mathbb{C}^*) \neq \emptyset$, a contradiction. This proves the claim and, as a consequence, that $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ is of finite type. □

Proposition 4.10. *The stack $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ is proper.*

Proof. Because $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ is defined over $\mathbb{C}$, separated and of finite type, to prove it is proper we only need to prove the statement of Proposition 3.23. (cf. [Zh].) By Proposition 3.23, the stack $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ is proper. □

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