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On the Drinfeld moduli problem of p-divisible groups

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Drinfeld proved that the p-adic upper half space Ω_F^d for a p-adic local field F is the general fiber of a formal scheme which is the moduli space of *special formal O_D-modules*, where D is a central division algebra with invariant $1/d$ over F. We give examples of other moduli problems of p-divisible formal groups which have general fiber Ω_F^d. We show that, when $F/\mathbb{Q}_p$ is unramified, our moduli spaces actually agree with Drinfeld's space. Using our results in the ramified case, P. Scholze has proved this in general. We also consider a variant for the Lubin-Tate moduli problem.

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1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$, with ring of integers O_F and uniformizer π. In [5], Drinfeld defines a certain moduli problem of formal O_F-modules. Let us recall Drinfeld's theorem.

A *formal O_F-module* over a scheme S such that p is locally nilpotent in $\mathcal{O}_S$ is a p-divisible formal group X over S with an action of O_F,

$$\iota : O_F \longrightarrow \operatorname{End} X\,.$$

If X is defined over an O_F-scheme S, the Lie algebra $\operatorname{Lie} X$ is naturally an O_F-module. If this coincides with the O_F-module structure given by $d\iota$ we call X a *strict formal O_F-module.*

Let D be a central division algebra of invariant $1/n$ over F, with maximal order O_D. Drinfeld defines a *special formal O_D-module* over an O_F-scheme S to be a strict formal O_F-module of height $n^2[F:\mathbb{Q}_p]$ equipped with an action $\iota\colon O_D \to \operatorname{End}(X)$ of O_D that extends the action of O_F and such that, in each geometric point of S, the eigenspaces of $\operatorname{Lie} X$ under the action of an unramified extension $O_{\tilde{F}}$ of O_F of degree n contained in O_D are all one-dimensional.

It is easy to see that if $S = \operatorname{Spec} \bar{k}$ is the spectrum of the algebraic closure of the residue field of O_F, there is a unique special formal O_D-module $\mathbb{X}$ over S, up to O_D-linear isogeny.

Let $O_{\breve{F}}$ be the ring of integers in the completion of the maximal unramified extension $\breve{F}$ of F. Drinfeld defines as follows a set-valued functor $\mathcal{M}$ on the category $\mathrm{Nilp}_{O_{\breve{F}}}$ of $O_{\breve{F}}$-schemes S such that the ideal sheaf $\pi\mathcal{O}_S$ is locally nilpotent: the functor associates to $S \in \mathrm{Nilp}_{O_{\breve{F}}}$ the set of isomorphism classes of triples (X, ι, ρ), where (X, ι) is a special formal O_D-module over S and where $\rho\colon X \times_S \bar{S} \to \mathbb{X} \times_{\operatorname{Spec} \bar{k}} \bar{S}$ is a O_D-linear quasi-isogeny of height 0. Here $\bar{S} = S \times_{\operatorname{Spec} O_{\breve{F}}} \operatorname{Spec} \bar{k}$. Drinfeld's theorem is that this functor is representable by a very specific formal scheme, namely,

$$\mathcal{M} \simeq \hat{\Omega}^n_F \times_{\operatorname{Spf} O_F} \operatorname{Spf} O_{\breve{F}}. \tag{1.1}$$

Here $\hat{\Omega}^n_F$ is the formal scheme over $\operatorname{Spf} O_F$ defined by Deligne, Drinfeld and Mumford, cf. [5]. It has as generic fiber (associated rigid-analytic space) Drinfeld's p-adic halfspace associated to F,

$$(\hat{\Omega}^n_F)^{\mathrm{rig}} = \mathbb{P}^{n-1}_F \setminus \bigcup\nolimits_{H/F} H.$$

Here H ranges over the hyperplanes of $\mathbb{P}^{n-1}_F$ defined over F.

Drinfeld's theorem has many applications, in particular to the p-adic uniformization of Shimura varieties, cf. [21]. It also has applications to arithmetic, e.g. [16, 23].

The generic fiber of Drinfeld's formal moduli space admits a tower of finite étale coverings (via level structures on the Tate module of the universal p-divisible group). As such it is a prominent example of a *local Shimura variety*, cf. [20].

Let G be a reductive group over the local field F, let b be an element of $G(\breve{F})$ and let $\{\mu\}$ be a conjugacy class of minuscule cocharacters of $G_{\overline{F}}$. One requires that the σ-conjugacy class $[b]$ of b is *neutral acceptable*, i.e., $[b] \in B(G, \{\mu\})$, cf. [10]. The triple $(G, b, \{\mu\})$ is called a *local Shimura datum* over F, cf. [20]. One expects to be able to attach to these data a local Shimura variety which satisfies obvious functorial properties and more. This is a tower of rigid-analytic spaces $\mathbb{M}(G, b, \{\mu\}) = \{\mathbb{M}^K \mid K \subset G(F)\}$ indexed by the open compact subgroups of $G(F)$, defined over the *reflex field* $E = E(G, \{\mu\})$, cf. [20]. By [21], local Shimura varieties exist in many cases. In the PEL case, local Shimura varieties are related to Shimura varieties by *non-archimedean uniformization*, cf. [21, Thm. 6.36]. One also expects to have *integral models* over O_E of $\mathbb{M}^K$, for judicious choices of the open compact subgroup K, i.e., formal schemes over $\operatorname{Spf} O_{\breve{E}}$ with Weil descent datum to $\operatorname{Spf} O_E$ whose associated rigid-analytic space is $\mathbb{M}^K$.

In Drinfeld's case $G = D^\times$, (the linear algebraic group over F associated to) the multiplicative group of D. The cocharacter $\{\mu\}$ of $G_{\overline{F}} \simeq \mathrm{GL}_{n,\overline{F}}$ is $(1, 0, \dots, 0)$ and b is a representative of the unique element of $B(G, \{\mu\})$. Drinfeld's theorem says not only that the member $\mathbb{M}^K$, for the level subgroup $K = O_D^\times$, is the Drinfeld p-adic upper half space attached to F, but also that the moduli problem $\mathcal{M}$ defines an integral model of $\mathbb{M}^K$, which is even a π-adic formal scheme with semi-stable reduction.

In the theory of local Shimura varieties, the following question arises. Assume that b is a representative of the unique *basic* element of $B(G, \{\mu\})$. Let ε be a central cocharacter of G defined over F, and set $\{\mu'\} = \{\mu\varepsilon\}$. Then $E(G, \{\mu\}) = E(G, \{\mu'\})$. Let b' be a representative of the unique basic element of $B(G, \{\mu'\})$. The question is whether the local Shimura varieties $\mathbb{M}(G, b, \{\mu\})$ and $\mathbb{M}(G, b', \{\mu'\})$ are Galois twists of each other (unramified twists if the connected center of G splits over an unramified extension).

In a similar vein, start with a local Shimura datum $(G, b, \{\mu\})$ over F such that b is basic. Let $G' = \mathrm{Res}_{F/\mathbb{Q}_p}(G)$. Then $G' \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p$ is a product of $G \otimes_F \bar{\mathbb{Q}}_p$ indexed by the embeddings of F in $\bar{\mathbb{Q}}_p$. We fix an embedding by choosing an isomorphism $\bar{F} \cong \bar{\mathbb{Q}}_p$. Define the conjugacy class $\{\mu_0\}$ of G' to be $\{\mu\}$ for this fixed embedding and to be trivial for all other embeddings. Then

$E(G', \{\mu_0\}) \subset E(G, \{\mu\})$ with respect to the chosen isomorphism. Let ε be a central cocharacter of G' defined over $F \subset \bar{\mathbb{Q}}_p$. We set $\{\mu'\} = \{\mu_0\varepsilon\}$. We still have $E(G', \{\mu'\}) \subset E(G, \{\mu\})$. Let b' be a representative of the unique basic element in $B(G', \{\mu'\})$. The question is whether the local Shimura varieties $\mathbb{M}(G, b, \{\mu\})$ and $\mathbb{M}(G', b', \{\mu'\}) \otimes_{E(G',\{\mu'\})} E(G, \{\mu\})$ are Galois twists of each other (unramified twists if the connected center of G' splits over an unramified extension of F).

The goal of this paper is to prove that this last question has an affirmative answer for the Drinfeld datum $(G, b, \{\mu\})$. But, even better, we construct integral models for the members corresponding to the natural *maximal* level subgroups of both local Shimura varieties and show that they are isomorphic, at least when $F/\mathbb{Q}_p$ is unramified[1]. These integral models are constructed by posing a moduli problem of p-divisible groups. This is substantially different from Drinfeld's moduli problem (unless $F = \mathbb{Q}_p$), which is a moduli problem of strict formal O_F-modules. This contrast between *relative* and *absolute* Rapoport-Zink spaces is important also in other contexts: in the work of A. Mihatsch on the Arithmetic Fundamental Lemma [15] and in joint work of us with S. Kudla [13] on p-adic uniformization of Shimura curves. In fact, the approach in [13] is modelled on the present paper, but involves in addition a polarization.

There is another well-known local Shimura datum $(G, b, \{\mu\})$, referred to as the *Lubin-Tate datum*. Here $G = \mathrm{GL}_n$, $\{\mu\} = (1, 0, \dots, 0)$, and $[b] \in B(G, \{\mu\})$ is the unique basic element. In this case, the corresponding local Shimura variety again has an explicit integral model for its member $\mathbb{M}^K$, where $K = \mathrm{GL}_n(O_F)$. For this local Shimura variety, we also give a positive answer to the question raised above, again in the strong form pertaining to integral models. This theorem is applied in the work of B. Smithling, W. Zhang and the second author on the Arithmetic Gan-Gross-Prasad conjecture [22].

It should be pointed out that the two cases of integral models of $\mathbb{M}^K$ considered here are essentially the only ones *known explicitly*, which justifies singling out these special cases of a general problem.

The lay-out of the paper is as follows. In section 2 we formulate our moduli functor and state our main results. In section 3, we discuss the conditions on the Lie algebras in the formulation of the moduli problem. In section 4 we establish an isomorphism between our moduli functor when

[1] In his talk 14 July 2016 in the Bonn Arbeitsgemeinschaft Arithmetische Geometrie, P. Scholze explained his proof of Conjecture 2.6 below, i.e., how to remove the unramifiedness hypothesis. His proof is based on our Theorem 2.8, but uses in addition the *integral p-adic Hodge theory* of B. Bhatt, M. Morrow and P. Scholze (arXiv:1602.03148), and more. Scholze will publish his proof elsewhere.

restricted to $\bar{k}$-schemes with the original Drinfeld moduli functor. The main tool here is the theory of displays. In section 5 we determine the local structure of our moduli scheme. Here the main tool is the theory of local models of Rapoport-Zink spaces. In section 6 we prove the compatibility theorem with the Drinfeld moduli functor in the generic fiber. This proof is due to P. Scholze, and uses his theory of p-divisible groups over O_C, cf. [24]. In section 7 we prove our *integral* representability conjecture in the case where $F/\mathbb{Q}_p$ is unramified. The proof uses the theory of relative displays of T. Ahsendorf. In the final section 8 we give the Lubin-Tate variant of our main theorem.

2. Formulation of the main results

Let F be a field extension of degree d of $\mathbb{Q}_p$. We denote by O_F the ring of integers and by κ the residue field. We write $f = [\kappa : \mathbb{F}_p]$ for the inertia index and $d = ef$.

Fix $n \geq 2$. Let $\Phi = \mathrm{Hom}_{\mathbb{Q}_p}(F, \bar{\mathbb{Q}}_p)$ be the set of field embeddings. We fix an embedding $\varphi_0 : F \to \bar{\mathbb{Q}}_p$. Let $r : \Phi \to \mathbb{Z}$ be a function such that

$$r_\varphi = \begin{cases} 1, & \text{if } \varphi = \varphi_0 \\ 0 \text{ or } n, & \text{if } \varphi \neq \varphi_0. \end{cases} \tag{2.1}$$

The reflex field $E \subset \bar{\mathbb{Q}}_p$ of r is characterized by

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/E) = \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \mid r_{\sigma\varphi} = r_\varphi \text{ for all } \varphi\}.$$

We have $\varphi_0(F) \subset E$, and we consider E as a field extension of F via φ_0.

Let D be a central division algebra of invariant $1/n$ over F. We will consider p-divisible groups X of height $n^2 d$ over O_E-schemes S, with an action of the ring of integers O_D in D,

$$\iota : O_D \longrightarrow \mathrm{End}(X).$$

We will need to impose conditions on the induced action of O_D on $\mathrm{Lie}\, X$. The first condition is the *Kottwitz condition*

$$\mathrm{char}(\iota(x) | \,\mathrm{Lie}\, X) = \prod_\varphi \varphi\big(\mathrm{chard}\ (x)(T)\big)^{r_\varphi}, \quad \forall x \in O_D\,. \tag{2.2}$$

Here $\mathrm{chard}(x)$ denotes the *reduced characteristic polynomial* of x, a polynomial of degree n with coefficients in O_F. The RHS is a polynomial in $O_E[T]$. It becomes a polynomial with coefficients in $\mathcal{O}_S$ via the structure morphism.

As we will show (cf. Proposition 2.2), the Kottwitz condition is all we need to yield a good moduli problem when $F/\mathbb{Q}_p$ is unramified. When $F/\mathbb{Q}_p$ is ramified, the Kottwitz condition is too weak. To state the additional condition we impose, we need some preparation.

Let F^t be the maximal unramified subfield of F. We will write $\Psi = \mathrm{Hom}_{\mathbb{Q}_p}(F^t, \bar{\mathbb{Q}}_p)$ for the set of field embeddings. Let $\psi_0 = \varphi_{0|F^t}$. For an embedding $\psi : F^t \to \bar{\mathbb{Q}}_p$ we set

$$
\begin{aligned}
A_\psi &= \{\varphi : F \to \bar{\mathbb{Q}}_p \mid \varphi_{|F^t} = \psi, \text{ and } r_\varphi = n\} \\
B_\psi &= \{\varphi : F \to \bar{\mathbb{Q}}_p \mid \varphi_{|F^t} = \psi, \text{ and } r_\varphi = 0\}.
\end{aligned} \tag{2.3}
$$

Also, let $a_\psi = |A_\psi|$ and $b_\psi = |B_\psi|$. For any O_E-scheme S we have a decomposition of $O_{F^t} \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules

$$
O_{F^t} \otimes_{\mathbb{Z}_p} \mathcal{O}_S = \bigoplus\nolimits_{\psi \in \Psi} \mathcal{O}_S \,, \tag{2.4}
$$

where the action of $\mathcal{O}_{F^t}$ on the ψ-th factor is via ψ. Hence for (X, ι) over S, we obtain a decomposition into locally free $\mathcal{O}_S$-modules,

$$
\mathrm{Lie}\, X = \bigoplus\nolimits_{\psi \in \Psi} \mathrm{Lie}_\psi X \,. \tag{2.5}
$$

The rank of $\mathrm{Lie}_\psi X$ is given by (2.2) as

$$
\mathrm{rank}\ \mathrm{Lie}_\psi X = a_\psi n^2 + \epsilon_\psi \, n \,, \tag{2.6}
$$

where ϵ_ψ is equal to 1 if $\psi = \psi_0$, and is equal to 0 if $\psi \neq \psi_0$.

Let π be a uniformizer in O_F. Consider the Eisenstein polynomial $Q(T)$ of π in $O_{F^t}[T]$. We consider the image $Q_\psi(T)$ of $Q(T)$ in $\bar{\mathbb{Q}}_p[T]$ under ψ, for $\psi \in \Psi$. In $\bar{\mathbb{Q}}_p[T]$ this has a decomposition into linear factors,

$$
Q_\psi(T) = \prod_{\{\varphi \mid \; \varphi_{|F^t} = \psi\}} (T - \varphi(\pi)). \tag{2.7}
$$

Note that $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E)$ acts on the index set of this product, as is clear since the LHS is a polynomial in $O_E[T]$. We therefore obtain a decomposition in $O_E[T]$

$$
\begin{aligned}
Q_{\psi_0}(T) &= Q_0(T) \cdot Q_{A_{\psi_0}}(T) \cdot Q_{B_{\psi_0}}(T), && \text{resp.} \\
Q_\psi(T) &= Q_{A_\psi}(T) \cdot Q_{B_\psi}(T), && \text{for } \psi \neq \psi_0.
\end{aligned} \tag{2.8}
$$

Here

$$Q_0(T) = T - \varphi_0(\pi),\ Q_{A_\psi}(T) = \prod_{\varphi\in A_\psi}(T-\varphi(\pi)),\ Q_{B_\psi}(T) = \prod_{\varphi\in B_\psi}(T-\varphi(\pi)).$$

Indeed, the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E)$ stabilizes the corresponding subsets in the index set on the RHS of (2.7). Using the structure morphism $O_E \to \mathcal{O}_S$, we obtain an endomorphism $Q_{A_\psi}(\iota(\pi))$ of the $\mathcal{O}_S$-module $\mathrm{Lie}_\psi X$ that we denote by $Q_{A_\psi}(\iota(\pi)|\,\mathrm{Lie}_\psi X)$. We similarly define $Q_{B_\psi}(\iota(\pi)|\,\mathrm{Lie}_\psi X)$ and $Q_0(\iota(\pi)|\,\mathrm{Lie}_{\psi_0} X) = \iota(\pi)|\,\mathrm{Lie}_{\psi_0} X - \varphi_0(\pi)\mathrm{Id}_{\mathrm{Lie}_{\psi_0} X}$.

The additional conditions we impose, that we call the *Eisenstein conditions*, are now the following identities of endomorphisms

$$\begin{gathered}\big(Q_0\cdot Q_{A_{\psi_0}}\big)(\iota(\pi)|\,\mathrm{Lie}_{\psi_0} X) = 0,\\ \bigwedge^{n+1}\big(Q_{A_{\psi_0}}(\iota(\pi)|\,\mathrm{Lie}_{\psi_0} X)\big) = 0,\\ Q_{A_\psi}(\iota(\pi)|\,\mathrm{Lie}_\psi X) = 0,\ \forall\psi\neq\psi_0.\end{gathered}\tag{2.9}$$

Remark 2.1. We note that the Eisenstein conditions only depend on the restriction of the O_D-action to O_F. It will follow a posteriori from Corollary 5.9 (flatness) that the moduli problem formulated using the Eisenstein conditions is independent of the choice of the uniformizer π.

We first note the following statement.

Proposition 2.2. *If $F/\mathbb{Q}_p$ is unramified, the Eisenstein conditions are implied by the Kottwitz condition.*

Proof. When $F = F^t$ is unramified over $\mathbb{Q}_p$, the uniformizer π lies in F^t and $Q(T) = T - \pi$ is a linear polynomial. Furthermore, A_ψ has at most one element for $\psi\neq\psi_0$, and $A_{\psi_0} = \emptyset$. Let $\psi\neq\psi_0$. If $A_\psi = \emptyset$, then $\mathrm{Lie}_\psi X = (0)$ and the Eisenstein condition relative to the index ψ is empty; if A_ψ has one element, the Eisenstein condition relative to the index ψ is just equivalent to the definition of the ψ-th eigenspace in the decomposition (2.5). Something analogous applies to the index ψ_0. □

The following statement shows that the moduli problems considered in this paper are indeed generalizations of Drinfeld's moduli problem. We call the *Drinfeld function* the function r° with $r^\circ_\varphi = 0, \forall\varphi\neq\varphi_0$. In this case $E_{r^\circ} = F$.

Proposition 2.3. *Assume that $r = r^\circ$. Then a p-divisible group (X,ι) as above, i.e., satisfying the Kottwitz condition (2.2) and the Eisenstein conditions (2.9), is a special formal O_D-module in the sense of Drinfeld [5].*

Proof. In this case, $A_\psi = \emptyset, \forall\psi$. Hence $\operatorname{Lie}_\psi X = (0)$ for $\psi \neq \psi_0$, and $\operatorname{Lie}_{\psi_0} X$ is a locally free $\mathcal{O}_S$-module of rank n. Also, the endomorphism $Q_{A_{\psi_0}}(\iota(\pi)|\operatorname{Lie}_{\psi_0} X)$ is the identity automorphism. Hence the first Eisenstein condition implies that $Q_0(\iota(\pi)|\operatorname{Lie}_{\psi_0} X) = 0$. Since $Q_0(T) = T - \varphi_0(\pi)$, it follows that $\iota(\pi)$ acts on $\operatorname{Lie}_{\psi_0} X$ through the structure morphism $O_E \to \mathcal{O}_S$. The same is true for all elements of O_{F^t}. Hence X is a strict formal O_F-module. Now the Kottwitz condition (2.2) tells us that the action of O_D on X is *special*, which proves the claim. □

Definition 2.4. Fix a function $r : \Phi \to \mathbb{Z}_{\geq 0}$, with corresponding reflex field $E = E_r$. A p-divisible group X with action ι by O_D over a O_E-scheme S is called an *r-special O_D-module*, if X is of height $n^2 d$ and (X, ι) satisfies the Kottwitz condition and the Eisenstein conditions relative to r.

Hence the previous proposition shows that a r°-special formal O_D-module is just a special formal O_D-module in the sense of Drinfeld [5].

For the formulation of the moduli problem we will make use of the following lemma. The lemma follows from section 4, more precisely, Corollary 4.13. Alternatively, the lemma follows from the fact that $B(G, \{\mu\})$ has only one element, cf. [10], §6. Here $G = \operatorname{Res}_{F/\mathbb{Q}_p}(D^\times)$ is the linear algebraic group over $\mathbb{Q}_p$ associated to $D^\times$, and $\{\mu\}$ is the conjugacy class of cocharacters with component $(1, 0^{(n-1)})$ for φ_0 and component $(1^{(n)})$, resp. $(0^{(n)})$ for $\varphi \neq \varphi_0$, depending on whether $r_\varphi = n$ or $r_\varphi = 0$.

Lemma 2.5. *Fix r. Let $\bar{k}$ be an algebraic closure of the residue field κ_E of O_E. Any two r-special p-divisible groups over $\bar{k}$ are isogenous by a O_D-linear isogeny (which may be taken to be of height 0).* □

Now fix such a pair $(\mathbb{X}, \iota_{\mathbb{X}})$ over $\bar{k}$. Denote by $O_{\breve{E}}$ the ring of integers in the completion of the maximal unramified extension of E. Then $\bar{k}$ is the residue field of $O_{\breve{E}}$. We consider the following set-valued functor $\mathcal{M}_r$ on $\operatorname{Nilp}_{O_{\breve{E}}}$. It associates to $S \in \operatorname{Nilp}_{O_{\breve{E}}}$ the set of isomorphism classes of triples (X, ι, ϱ), where (X, ι) is an r-special O_D-module over S, and where

$$\varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{k}} \bar{S} \tag{2.10}$$

is a O_D-linear quasi-isogeny of height zero. Here $\bar{S} = S \otimes_{O_{\breve{E}}} \bar{k}$. Our main conjecture can now be stated as follows.

Conjecture 2.6. [2] *The functor $\mathcal{M}_r$ is represented by $\hat{\Omega}^n_F \hat{\otimes}_{O_F} O_{\breve{E}}$.*

Our main results towards this conjecture are the following.

[2]See the footnote 1.

Theorem 2.7. *The conjecture is true if $F/\mathbb{Q}_p$ is unramified.*

If $F/\mathbb{Q}_p$ is ramified, we can still prove the following properties of $\mathcal{M}_r$ which are analogous to the properties of $\hat{\Omega}^n_F \hat{\otimes}_{O_F} O_{\breve{E}}$.

Theorem 2.8. *The formal scheme $\mathcal{M}_r$ is flat over* $\mathrm{Spf}\, O_{\breve{E}}$*, and is π-adic. All its completed local rings are normal. Furthermore,*

(i) *there is an isomorphism between the special fibers*

$$\mathcal{M}_r \times_{\mathrm{Spf}\, O_{\breve{E}}} \mathrm{Spec}\; \bar{k} \simeq \hat{\Omega}^n_F \times_{\mathrm{Spf}\, O_F} \mathrm{Spec}\; \bar{k}.$$

(ii) *there is an isomorphism between the generic fibers*

$$\mathcal{M}_r^{\mathrm{rig}} \simeq (\hat{\Omega}^n_F \times_{\mathrm{Spf}\, O_F} \mathrm{Spf}\, O_{\breve{E}})^{\mathrm{rig}}.$$

Here the proof of point (ii) is due to P. Scholze.

We also prove the following variant of this theorem in the Lubin-Tate context. Let F and φ_0 be as before, fix an integer $n \geq 2$ and let r and $E = E_r$ have the same meaning as before. We now consider p-divisible formal groups X of height nd over O_E-schemes S with an action $\iota : O_F \longrightarrow \mathrm{End}(X)$. We impose the following Kottwitz condition,

$$\mathrm{char}\,(\iota(x) \mid \mathrm{Lie}\, X) = \prod\nolimits_{\varphi} (T - \varphi(x))^{r_\varphi}\,,\ x \in O_F\,. \tag{2.11}$$

In addition, we impose Eisenstein conditions (where the second condition in (2.9) is changed into

$$\bigwedge^2 \big(Q_{A_{\psi_0}}(\iota(\pi)|\, \mathrm{Lie}_{\psi_0} X)\big) = 0,$$

see (8.2)).

We fix a pair $(\mathbb{X}, \iota_{\mathbb{X}})$ over $\bar{k}$ as above. It is easy to see that $\mathbb{X}$ is unique up to O_F-linear isogeny. We may therefore define a functor $\mathcal{M}^F_r$ on $\mathrm{Nilp}_{O_{\breve{E}}}$ analogous to the functor $\mathcal{M}_r$ above. The formal scheme representing this functor will be denoted by the same symbol. If $r = r^\circ$ is the Drinfeld function, then $\mathcal{M}^F_{r^\circ}$ can be identified with the Lubin-Tate deformation space (this follows from Proposition 2.3). Our main result in this context is that this continues to hold for arbitrary r.

Theorem 2.9. *The functor $\mathcal{M}^F_r$ is representable by* $\mathrm{Spf}\, O_{\breve{E}}[[t_1, \cdots, t_{n-1}]]$.

3. The Kottwitz and Eisenstein conditions

In this section, we analyze the conditions that can be put on a locally free module with O_D-action. We continue with the same notation as before. In addition, let $\tilde{E} \subset \bar{\mathbb{Q}}_p$ be a normal extension of $\mathbb{Q}_p$ which contains the images of all $\mathbb{Q}_p$-algebra homomorphisms $\tilde{F} \to \bar{\mathbb{Q}}_p$. We have $E \subset \tilde{E}$.

We denote by $\mathrm{Nrd}_\varphi : D \otimes_{F,\varphi} \tilde{E} \to \tilde{E}$ the reduced norm. Using it, we define the polynomial function

$$\mathrm{Nrd}_r : D \otimes_{\mathbb{Q}_p} \tilde{E} \cong \prod_\varphi D \otimes_{F,\varphi} \tilde{E} \overset{\prod_\varphi \mathrm{Nrd}_\varphi^{r_\varphi}}{\longrightarrow} \tilde{E}. \tag{3.1}$$

If $\mathcal{M}$ is a quasicoherent sheaf on a scheme S we denote by $\mathbb{V}_S(\mathcal{M})(T) = \Gamma(T, \mathcal{M}_T)$ the corresponding flat sheaf on the category of S-schemes T. This sheaf is representable by a scheme over S if $\mathcal{M}$ is a finite locally free $\mathcal{O}_S$-module.

We write simply $\mathbb{V}(D)$ for the affine space over $\mathbb{Q}_p$ associated to D. Then we may regard Nrd_r as a polynomial function (= morphism of schemes). It is defined over E,

$$\mathrm{Nrd}_r : \mathbb{V}(D)_E \to \mathbb{A}^1_E. \tag{3.2}$$

Clearly this function is homogeneous of degree $\sum_\varphi n r_\varphi$.

Let O_D be the ring of integers of D. Let $\tilde{F} \subset D$ be an unramified extension of F of degree n. We denote by $\tau \in \mathrm{Gal}(\tilde{F}/F)$ the Frobenius automorphism. Then we may write

$$O_D = O_{\tilde{F}}[\Pi], \quad \Pi^n = \pi, \quad \Pi a = \tau(a)\Pi, \text{ for } a \in O_{\tilde{F}}. \tag{3.3}$$

Here Π is a prime element of O_D and π is a prime element of O_F.

We denote by $\mathbb{V}(O_D)$ the corresponding affine space over $\mathbb{Z}_p$. We will now define an integral version of (3.2).

We begin with a general remark. Let S be an O_E-scheme and let $\mathcal{L}$ be a finite locally free $\mathcal{O}_S$-module with an action of O_D, i.e. a morphism of $\mathbb{Z}_p$-algebras

$$\iota : O_D \to \mathrm{End}_{\mathcal{O}_S}(\mathcal{L}). \tag{3.4}$$

We will call $(\mathcal{L}, \iota)$ an *O_D-module over S*.

If T is an S-scheme and $\alpha \in \Gamma(T, O_D \otimes_{\mathbb{Z}_p} \mathcal{O}_T)$, we can take the determinant $\det(\alpha|\mathcal{L}_T)$. This defines a morphism

$$\det{}_{\mathcal{L}} : \mathbb{V}(O_D)_S \to \mathbb{A}^1_S.$$

Let $\varphi \in \Phi$. We define an embedding

$$O_D \otimes_{O_F,\varphi} O_{\tilde{E}} \to M(n \times n, O_{\tilde{E}}).$$

For this we choose an embedding $\tilde{\varphi} : \tilde{F} \to \bar{\mathbb{Q}}_p$ which extends φ and define for $x \in \tilde{F}$,

(3.5)

$$x \longmapsto \begin{pmatrix} \tilde{\varphi}(x) & 0 & \dots & 0 \\ 0 & \tilde{\varphi}(\tau(x)) & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & \tilde{\varphi}(\tau^{n-1}(x)) \end{pmatrix}, \ \Pi \longmapsto \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ \varphi(\pi) & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Consider the standard $M(n \times n, O_{\tilde{E}})$-module $O_{\tilde{E}}^n$. Via restriction of scalars

$$O_D \otimes_{\mathbb{Z}_p} O_{\tilde{E}} \to O_D \otimes_{O_F,\varphi} O_{\tilde{E}} \to M(n \times n, O_{\tilde{E}}),$$

we obtain a $O_D \otimes_{\mathbb{Z}_p} O_{\tilde{E}}$-module P_φ. We define the $O_D \otimes_{\mathbb{Z}_p} O_{\tilde{E}}$-module

$$P_r = \bigoplus P_\varphi^{r_\varphi}.$$

This module defines a polynomial function $\mathrm{Nrd}_r : \mathbb{V}(O_D)_{O_{\tilde{E}}} \to \mathbb{A}^1_{O_{\tilde{E}}}$,

$$\mathrm{Nrd}_r(\xi) = \det(\xi | P_r \otimes_{O_{\tilde{E}}} R), \quad \xi \in O_D \otimes_{\mathbb{Z}_p} R,$$

where R is an arbitrary $O_{\tilde{E}}$-algebra. Similarly, the module P_φ defines a polynomial function $\mathrm{Nrd}_\varphi : \mathbb{V}(O_D)_{O_{\tilde{E}}} \to \mathbb{A}^1_{O_{\tilde{E}}}$. The polynomial function Nrd_r is invariant under $\mathrm{Gal}(\tilde{E}/E)$ and therefore defines a polynomial function

$$\mathrm{Nrd}_r : \mathbb{V}(O_D)_{O_E} \to \mathbb{A}^1_{O_E}. \tag{3.6}$$

It follows from (3.5) that Π acts on $P_\varphi \otimes_{O_{\tilde{E}}} \kappa_{\tilde{E}}$ as zero. Therefore the latter is a module over $(O_D/\Pi O_D) \otimes_{O_{\tilde{F}}} \kappa_{\tilde{E}}$. We see that the base change Nrd_φ factors through a polynomial function

$$(\mathrm{Nrd}_\varphi)_{\kappa_{\tilde{E}}} : \mathbb{V}(O_D/\Pi O_D)_{\kappa_{\tilde{E}}} \to \mathbb{A}^1_{\kappa_{\tilde{E}}}.$$

We obtain that $(\mathrm{Nrd}_r)_{\kappa_{\tilde{E}}}$ factors too,

$$(\mathrm{Nrd}_r)\kappa_{\tilde{E}} : \mathbb{V}(O_D/\Pi O_D) \times_{\mathrm{Spec}\,\mathbb{F}_p} \mathrm{Spec}\,\kappa_{\tilde{E}} \to \mathbb{A}^1_{\kappa_{\tilde{E}}}.$$

The affine algebra on the left hand side is an integral domain. Therefore $(\mathrm{Nrd}_\varphi)_{\kappa_{\tilde{E}}}$ is a non zero-divisor as an element of this algebra. This remains true after base change to any $\kappa_{\tilde{E}}$-algebra R.

Definition 3.1. Let S be an O_E-scheme and let $(\mathcal{L}, \iota)$ be an O_D-module over S. We say that $(\mathcal{L}, \iota)$ satisfies the *Kottwitz condition* $(\mathbf{K}_r)$ *with respect to* r, if

$$\det{}_{\mathcal{L}} = (\mathrm{Nrd}_r)_S, \tag{3.7}$$

where the right hand side is the base change with respect to $S \to \operatorname{Spec} O_E$.

Remark 3.2. By [9], Prop. 2.1.3, the condition is equivalent to the identity of polynomials in $\mathcal{O}_S[T]$ (comp. (2.2))

$$\mathrm{char}(\iota(x)|\mathcal{L}) = \prod\nolimits_\varphi \varphi\big(\mathrm{chard}\ (x)(T)\big)^{r_\varphi}, \quad \forall x \in O_D\,; \tag{3.8}$$

(this uses Amitsur's formula, comp. [4], Lemma 1.12).

It is clear that the Kottwitz condition is a closed condition. If the Kottwitz condition is fulfilled, we have

$$\mathrm{rank}_S\, \mathcal{L} = n \sum_\varphi r_\varphi,$$

because Nrd_r is homogeneous of this degree.

Recall the maximal unramified subfield $F^t \subset F$ and its residue field κ. We denote by $\tilde{\kappa}$ the residue field of $\tilde{F}$. The maximal unramified subfield of $\tilde{F}$ is denoted by $\tilde{F}^t$. In addition to $\Psi = \mathrm{Hom}_{\mathbb{Q}_p}(F^t, \bar{\mathbb{Q}}_p)$, we introduce $\tilde{\Psi} = \mathrm{Hom}_{\mathbb{Q}_p}(\tilde{F}^t, \bar{\mathbb{Q}}_p)$.

We now introduce another condition which will turn out to be weaker than the Kottwitz condition. Let R be an $O_{\tilde{E}}$-algebra. Let (L, ι) be an O_D-module over R. Then we have the decompositions

$$L = \bigoplus_{\psi \in \Psi} L_\psi, \quad L = \bigoplus_{\tilde{\psi} \in \tilde{\Psi}} L_{\tilde{\psi}}. \tag{3.9}$$

For example, the second of these decompositions is induced by

$$O_{\tilde{F}^t} \otimes_{\mathbb{Z}_p} O_{\tilde{E}} = \prod\nolimits_{\tilde{\psi} \in \tilde{\Psi}} O_{\tilde{E}}.$$

For $\psi \in \Psi$ let Φ_ψ the set of all embeddings $\varphi : F \to \bar{\mathbb{Q}}_p$ whose restriction to F^t is ψ. We define

$$r_\psi = \sum_{\varphi \in \Phi_\psi} r_\varphi \tag{3.10}$$

Definition 3.3. Let R be an $O_{\tilde{E}}$-algebra. We say that an O_D-module (L, ι) over R satisfies the *rank condition* $(\mathbf{R}_r)$ *with respect to* r if, for all $\tilde{\psi} \in \tilde{\Psi}$,

$$\operatorname{rank}_R L_{\tilde{\psi}} = r_\psi, \tag{3.11}$$

where ψ denotes the restriction of $\tilde{\psi}$ to F^t. We will write $r_{\tilde{\psi}} := r_\psi$.

The rank condition is obviously an open condition (the rank goes up under specialization). It is also a closed condition since $\sum_{\tilde{\psi}} \operatorname{rank}_R L_{\tilde{\psi}} = \operatorname{rank} L$ is constant on the base. To check the rank condition it is enough to check it for the geometric points of $\operatorname{Spec} R$. If R is an arbitrary O_E-algebra, we say that the rank condition is fulfilled, if it is fulfilled with respect to any base change $R \to \tilde{R}$ and an extension of the O_E-algebra structure to an $O_{\tilde{E}}$-algebra structure on $\tilde{R}$. The rank condition is then independent of the last choice.

Remark 3.4. The rank condition is independent of the choice of the chosen isomorphism (3.3). Indeed let $U \subset D$ be an unramified extension of degree n of F. Then we could reformulate the rank condition using the action of $O_{U^t} \otimes_{\mathbb{Z}_p} O_{\tilde{E}}$. However, this yields the same condition. Indeed, since U and $\tilde{F}$ are isomorphic field extensions of F, there is by Skolem-Noether an element $u \in D$, such that $u\tilde{F}u^{-1} = U$. Replacing u by $u\Pi^m$ for a suitable integer m, we may assume that $u \in O_D^\times$. We denote by $L_{[u]}$ the R-module L with the new O_D-action

$$a \cdot_{new} \ell = uau^{-1}\ell.$$

But the decomposition (3.9) for $L_{[u]}$,

$$L_{[u]} = \bigoplus_{\tilde{\psi} \in \tilde{\Psi}} (L_{[u]})_{\tilde{\psi}}\,,$$

is exactly the decomposition coming from the O_{U^t}-action on L. Since the multiplication by u $L \to L_{[u]}$ is an isomorphism of O_D-modules we obtain the independence.

We denote by $\kappa_{\tilde{E}}$ the residue class field of $O_{\tilde{E}}$. We consider a polynomial function over a *reduced* $\kappa_{\tilde{E}}$-algebra R,

$$\chi : \mathbb{V}(O_D)_R \to \mathbb{A}^1_R, \tag{3.12}$$

which is multiplicative with respect to the ring structures of these schemes. We are given for each R-algebra A a multiplicative map

$$\chi_A : O_D/pO_D \otimes_{\mathbb{F}_p} A \to A. \tag{3.13}$$

Let $a, b \in O_D/pO_D \otimes_{\mathbb{F}_p} A$. We claim that

$$\chi((a + (\Pi \otimes 1)b)) = \chi(a). \tag{3.14}$$

We regard this as an identity of polynomial functions on $\mathbb{V}(O_D)_R \times_R \mathbb{V}(O_D)_R$. We consider the units of O_D as a subscheme $G \subset \mathbb{V}(O_D)_R$. This is dense in each fiber over R. Therefore it suffices to show (3.14) in the case where a is a unit. By multiplicativity it suffices to show that

$$\chi((1 + (\Pi \otimes 1)b)) = \chi(1). \tag{3.15}$$

We may restrict our attention to the universal case where $A = R[Y_1, \ldots, Y_t]$, which is also reduced.

We have

$$(1 + (\Pi \otimes 1)b)^{p^s} = 1^{p^s}$$

for some p-power p^s. Since A is reduced and χ is multiplicative, we deduce (3.15).

Therefore (3.13) is equivalent to a functorial map

$$O_D/\Pi O_D \otimes_{\mathbb{F}_p} A \to A.$$

We have $\tilde{\kappa} = O_D/\Pi O_D$. Therefore for each $\tilde{\psi} \in \tilde{\Psi} = \mathrm{Hom}_{\mathbb{F}_p}(\tilde{\kappa}, \kappa_{\tilde{E}}) = \mathrm{Hom}_{\mathbb{Q}_p}(\tilde{F}^t, \bar{\mathbb{Q}}_p)$, we obtain a polynomial function

$$\chi_{\tilde{\psi}} : O_D \to \tilde{\kappa} \xrightarrow{\tilde{\psi}} \kappa_{\tilde{E}} \to A. \tag{3.16}$$

If $\operatorname{Spec} R$ is connected and reduced, we deduce that the only multiplicative polynomial functions (3.12) have the form

$$\prod_{\tilde{\psi}} \chi_{\tilde{\psi}}^{e_{\tilde{\psi}}}, \tag{3.17}$$

for suitable exponents $e_{\tilde{\psi}}$. These functions are also defined if R is an arbitrary $\kappa_{\tilde{E}}$-algebra.

Lemma 3.5. *Let R be a reduced κ_E-algebra. Let (L, ι) be an O_D-module over R. In particular L is a finitely generated locally free R-module.*

Then the Kottwitz condition $(\mathbf{K}_r)$ *for* $(\mathcal{L}, \iota)$ *is equivalent to the rank condition* $(\mathbf{R}_r)$*. For an arbitrary* O_E*-algebra* R *the condition* $(\mathbf{K}_r)$ *implies the condition* $(\mathbf{R}_r)$*.*

Proof. To prove this we make a base change $R \mapsto R \otimes_{O_E} O_{\tilde{E}}$ which is reduced if R is a reduced κ_E-algebra. Therefore we may assume that R is a $O_{\tilde{E}}$-algebra.

The first assertion follows because a polynomial function is uniquely determined by the numbers $e_{\tilde{\psi}}$ in (3.17).

The last assertion depends only on the geometric fibers. In this case we have either a $\kappa_{\tilde{E}}$-algebra or an O_E-algebra of characteristic 0. We have already seen the first case. The characteristic 0 case is clear. □

Proposition 3.6. *Let us assume that* $r = r^\circ$*, i.e.,* $r_\varphi = 0$ *for* $\varphi \neq \varphi_0$*. Let* R *be a* $\kappa_{\tilde{E}}$*-algebra. Let* (L, ι) *be a* O_D*-module over* R*. Then* (L, ι) *satisfies the condition* $(\mathbf{K}_r)$ *iff* π *annihilates* L *and the rank condition* $(\mathbf{R}_r)$ *is fulfilled.*

The rank condition says in this case that for all $\tilde{\psi} : \tilde{\kappa} \to \kappa_{\tilde{E}}$

$$\operatorname{rank} L_{\tilde{\psi}} = 1, \ \text{if } \tilde{\psi}_{|F^t} = \psi_0 = \varphi_{0|F^t}, \quad \operatorname{rank} L_{\tilde{\psi}} = 0, \ \text{else.}$$

Proof. By Lemma 3.5 we already know that the Kottwitz condition implies the rank conditions. If $a \in (O_D/pO_D) \otimes R$ we have $\operatorname{Nrd}_r(a\pi) = 0$. Therefore the Kottwitz condition implies $\det(\pi | L_{\tilde{\psi}}) = 0$. Since $L_{\tilde{\psi}}$ has rank 1, this shows that π annihilates L.

Assume conversely that the ranks are as indicated and that π annihilates L. Then we deduce the result from the following Lemma. □

Lemma 3.7. *Let* R *be a ring, such that* $pR = 0$*. Let* n *be a natural number. Let* $L_1, \ldots, L_n$ *be locally free* R*-modules of rank* 1*. Assume we are given a chain of homomorphisms* $\Pi_i : L_i \to L_{i+1}$ *and* $\Pi_n : L_n \to L_1$ *such that*

$$\Pi_n \circ \Pi_{n-1} \circ \ldots \circ \Pi_1 = 0.$$

We set $L = L_1 \bigoplus \ldots \bigoplus L_n$*, and* $\Pi = \Pi_1 \bigoplus \ldots \bigoplus \Pi_n$*. This is an endomorphism of* L *such that* $\Pi^n = 0$*.*

Let $v \in M(n \times n, R)$ *be a diagonal matrix. It induces an endomorphism* $v : L \to L$*. We consider an endomorphism of* L *of the form*

$$v_0 + v_1\Pi + v_2\Pi^2 + \ldots + v_{n-1}\Pi^{n-1},$$

where the v_i *are diagonal matrices.*

Then

$$\det{}_F(v_0 + v_1\Pi + v_2\Pi^2 + \ldots + v_{n-1}\Pi^{n-1}) = \det{}_F v_0. \tag{3.18}$$

Proof. We may assume $L_i = R$. Then Π_i is the multiplication by some element $y_i \in R$. Our assumption says

$$y_n y_{n-1} \cdot \ldots \cdot y_1 = 0.$$

One deduces that $\Pi^n = 0$. We note that $\Pi v = v'\Pi$, where v' is another diagonal matrix, which is obtained from v by cyclically permuting the diagonal entries.

We may reduce to the universal case where R is the quotient of a polynomial ring $R = \mathbb{F}_p[x_{ij}, y_k]/y_1 \cdot \ldots \cdot y_n$, where $i \in [0, n-1]$, $j \in [1, n]$, $k \in [1, n]$. For fixed i, the x_{ij} are the diagonal entries of v_i. In this case the ring R is reduced.

Consider first the case where $v_0 = E$ is the unit matrix. We take the universal case where the ring R is reduced (as above but no indeterminates x_{0j}). Then we have two commuting operators $\rho = v_1\Pi + v_2\Pi^2 + \ldots + v_{n-1}\Pi^{n-1}$ and v_0. Therefore

$$(E + \rho)^{p^s} = E^{p^s} + \rho^{p^s}$$

But $\rho^{p^s} = 0$ for $p^s > n$. This implies $\det(E + \rho)^{p^s} = 1$. Because R is reduced in the universal case, this implies (3.18) in the case where $v_0 = E$. Clearly this shows also the case where v_0 is an invertible diagonal matrix.

In the general case we consider the universal R as above and its localization,

$$R \subset R[x_{01}^{-1}, x_{02}^{-1}, \ldots, x_{0n}^{-1}].$$

Over the bigger ring v_0 becomes invertible and therefore the relation (3.18) holds. Hence it holds also over the subring R. $\square$

Let us recall the Eisenstein conditions. We recall the following notation: For an embedding $\psi : F^t \to \bar{\mathbb{Q}}_p$ we set

$$A_\psi = \{\varphi : F \to \bar{\mathbb{Q}}_p \mid \varphi_{|F^t} = \psi, \text{ and } r_\varphi = n\}$$
$$B_\psi = \{\varphi : F \to \bar{\mathbb{Q}}_p \mid \varphi_{|F^t} = \psi, \text{ and } r_\varphi = 0\}.$$

We set $\psi_0 = \varphi_{0|F^t}$. This gives a partition of the set Φ_ψ,

$$\Phi_{\psi_0} = A_{\psi_0} \cup B_{\psi_0} \cup \{\varphi_0\}, \quad \Phi_\psi = A_\psi \cup B_\psi \text{ for } \psi \neq \psi_0.$$

Let $a_\psi = |A_\psi|$, $b_\psi = |B_\psi|$. Then we have

$$a_\psi + b_\psi + \epsilon_\psi = e = [F : F^t],$$

where $\epsilon_\psi = 0$ if $\psi \neq \psi_0$, and $\epsilon_{\psi_0} = 1$.

We find (compare (3.10)) that

$$r_\psi = \begin{cases} na_\psi + 1 & \text{if } \psi = \psi_0 \\ na_\psi & \text{if } \psi \neq \psi_0. \end{cases} \tag{3.19}$$

Let $Q(T) \in O_{F^t}[T]$ be the Eisenstein polynomial of a fixed prime element $\pi \in F$. We set $Q_\psi(T) = \psi(Q(T)) \in O_E[T]$. We set

$$Q_{A_\psi}(T) = \prod_{\varphi \in A_\psi} (T - \varphi(\pi)), \quad Q_{B_\psi}(T) = \prod_{\varphi \in B_\psi} (T - \varphi(\pi)).$$

These are polynomials in $O_E[T]$. Moreover we set $Q_0(T) = T - \varphi_0(\pi)$. Then we have the decompositions

$$\begin{aligned} Q_\psi(T) &= Q_{A_\psi}(T) \cdot Q_{B_\psi}(T), \quad \text{for } \psi \neq \psi_0 \\ Q_{\psi_0}(T) &= Q_0(T) \cdot Q_{A_{\psi_0}}(T) \cdot Q_{B_{\psi_0}}(T). \end{aligned} \tag{3.20}$$

Let R be an O_E-algebra. We will introduce the Eisenstein condition on a O_D-module (L, ι) over R. We have decompositions

$$O_{F^t} \otimes_{\mathbb{Z}_p} O_E \cong \prod_{\psi \in \Psi} O_E, \qquad O_{F^t} \otimes_{\mathbb{Z}_p} R \cong \prod_{\psi \in \Psi} R.$$

This gives a decomposition

$$L = \bigoplus_{\psi \in \Psi} L_\psi.$$

Definition 3.8. We say that (L, ι) satisfies the *Eisenstein condition* $(\mathbf{E}_r)$ *with respect to* r if

$$\begin{aligned} ((Q_0 \cdot Q_{A_{\psi_0}})(\iota(\pi)) | L_{\psi_0}) &= 0, \\ \textstyle\bigwedge^{n+1} (Q_{A_{\psi_0}}(\iota(\pi) | L_{\psi_0})) &= 0, \\ (Q_{A_\psi}(\iota(\pi)) | L_\psi) &= 0, \quad \text{for } \psi \neq \psi_0. \end{aligned} \tag{3.21}$$

This definition applies to any $\mathcal{O}_E$-scheme S and any O_D-module $(\mathcal{L}, \iota)$ over S.

Remarks 3.9. (i) The condition $(\mathbf{E}_r)$ only depends on the restriction to O_F of the action ι of O_D on $\mathcal{L}$. The condition depends on the choice of the uniformizer π.

(ii) Let $r = r^\circ$, i.e., $r_\varphi = 0$ for $\varphi \neq \varphi_0$. In this case $E = F$. We have two actions of O_F on $\mathcal{L}$. The first is given by $O_F \xrightarrow{\varphi_0} O_E \to \mathcal{O}_S$ and the second by ι. By the Eisenstein condition these actions coincide. Indeed, $\mathcal{L} = \mathcal{L}_{\psi_0}$, and $a_{\psi_0} = 0$ for all $\psi \in \Psi$. Therefore the Eisenstein condition (3.21) implies that $\iota(\pi)$ acts on $\mathcal{L}$ as $\varphi_0(\pi)$. The conditions (3.11) imply moreover that $\iota(a)$ for $a \in O_{F^t}$ acts via $\varphi_0(a)$ on $\mathcal{L}$.

(iii) Let $F/\mathbb{Q}_p$ be unramified, i.e., $F = F^t$. Then the first and the last Eisenstein conditions are empty. The second Eisenstein condition just says that $\operatorname{rank} \mathcal{L}_{\varphi_0} \leq n$.

(iv) Let us assume that S is a κ_E-scheme. The images of the three polynomials $Q_{A_\psi}(T)$, $Q_{B_\psi}(T)$ and $Q_0(T)$ in $\kappa_E[T]$ are respectively

$$T^{a_\psi}, \quad T^{b_\psi}, \quad T.$$

Therefore the Eisenstein conditions are in this case:

$$\begin{aligned} \iota(\pi)^{a_{\psi_0}+1} &= 0 \text{ on } \mathcal{L}_{\psi_0}, \\ \wedge^{n+1}(\iota(\pi)^{a_{\psi_0}}) &= 0 \text{ on } \wedge^{n+1} \mathcal{L}_{\psi_0}, \\ \iota(\pi)^{a_\psi} &= 0 \text{ on } \mathcal{L}_\psi \text{ for } \psi \neq \psi_0. \end{aligned} \tag{3.22}$$

Definition 3.10. Let $(\mathcal{L}, \iota)$ be a O_D-module over an O_E-scheme S. We say that $(\mathcal{L}, \iota)$ satisfies the *Drinfeld condition* $(\mathbf{D}_r)$ *with respect to* r if it satisfies the Eisenstein condition $(\mathbf{E}_r)$ and the rank condition $(\mathbf{R}_r)$.

The rank condition makes sense by the remark after Definition 3.3.

Lemma 3.11. *Assume that $r = r^\circ$, i.e., $r_\varphi = 0$ for $\varphi \neq \varphi_0$. Let S be a κ_E-algebra.*

Then the condition $(\mathbf{D}_r)$ implies $(\mathbf{K}_r)$.

This is a reformulation of Proposition 3.6.

4. Formal O_D-modules

Definition 4.1. Let S be an O_E-scheme such that $p\mathcal{O}_S$ is a nilpotent ideal sheaf. A *p-divisible O_D-module over S* is a p-divisible group X of height $[D : \mathbb{Q}_p] = n^2 d$ with an action

$$\iota : O_D \to \operatorname{End} X. \tag{4.1}$$

In this case $\operatorname{Lie} X$ is an O_D-module in the sense of (3.4). We say that X *satisfies* $(\mathbf{K}_r)$ *(3.7), resp.* $(\mathbf{R}_r)$ *(3.11), resp.* $(\mathbf{E}_r)$ *(3.21), resp.* $(\mathbf{D}_r)$ (Definition 3.10) if the $\mathcal{O}_S$-module $\operatorname{Lie} X$ with the O_D-action $d\iota$ does.

A p-divisible O_D-module X which satisfies $(\mathbf{D}_r)$ is called an *r-special formal O_D-module*. The name is justified because we will prove that X has no étale part. A formal O_D-module which satisfies $(\mathbf{D}_{r^\circ})$ is also called a *special formal O_D-module.*

Remarks 4.2. (i) The definition of a special formal O_D-module above coincides with Drinfeld's definition in [5]. Indeed, in this case $r = r^\circ$, and it follows from Remarks 3.9, (ii) that X is a strict formal O_F-module (the induced action of O_F on $\operatorname{Lie} X$ is via the structure morphism $O_F = O_E \to \mathcal{O}_S$).

(ii) Let A be a p-adic O_E-algebra. Let X be a p-divisible group over A with an action (4.1). Then we define $\operatorname{Lie} X = \varprojlim \operatorname{Lie} X \otimes A/p^n$. It is still a O_D-module and the definition above makes sense.

Assume that p is locally nilpotent on S. We denote by $\mathbb{D}(X)$ the covariant crystal associated to X. The evaluation $\mathbb{D}(X)_S$ at S coincides with the Lie algebra of the universal extension of X.

Let $a \in O_D$. We will often write simply a when we mean the action $\iota(a)$ on $\mathbb{D}(X)_S$, or the derived action $d\iota(a)$ on $\operatorname{Lie} X$.

Proposition 4.3. *Assume that $p\mathcal{O}_S$ is a locally nilpotent ideal sheaf. Let X be a r-special formal O_D-module of height n^2d.*

Then $\mathbb{D}(X)_S$ is locally on S a free $O_D \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-module of rank 1. Moreover X is a formal Lie group.

Proof. We begin with the case where $S = \operatorname{Spec} k$ where k is a perfect field with a $\kappa_{\tilde{E}}$-structure. Let $W(k)$ be the ring of Witt vectors and $M = \mathbb{D}(X)_{W(k)}$ be the covariant Dieudonné module.

We have a bijection of field embeddings

$$\tilde{\Psi} = \operatorname{Hom}(\tilde{\kappa}, k) = \operatorname{Hom}(\tilde{F}^t, W(k) \otimes \mathbb{Q}).$$

We regard $\tilde{\psi} \in \tilde{\Psi}$ as a homomorphism $\tilde{\psi} : O_{\tilde{F}^t} \to W(k)$. We set

$$M_{\tilde{\psi}} = \{m \in M \mid \iota(a)m = \tilde{\psi}(a)m, \text{ for } a \in O_{\tilde{F}^t}\}.$$

We have a direct decomposition

$$M = \bigoplus M_{\tilde{\psi}}, \quad \tilde{\psi} \in \tilde{\Psi}. \tag{4.2}$$

We denote by σ the Frobenius acting on $W(k)$. The Verschiebung induces a map

$$V : M_{\sigma\tilde{\psi}} \to M_{\tilde{\psi}}.$$

Therefore the rank of the $W(k)$-module $M_{\tilde{\psi}}$ is independent of $\tilde{\psi}$ and therefore equal to ne. For each $\tilde{\psi} : \tilde{F}^t \to W(k) \otimes \mathbb{Q}$ we have $\tilde{\psi}\tau = \sigma^f \tilde{\psi}$, where τ is from (3.3). Since $r_{\tilde{\psi}}$ depends only on the restriction of $\tilde{\psi}$ to F^t, we find $r_{\tilde{\psi}} = r_{\tilde{\psi}\tau}$. Obviously Π induces a map

$$\Pi : M_{\tilde{\psi}\tau} \to M_{\tilde{\psi}}.$$

We consider the commutative diagram

$$\begin{array}{ccc} M_{\sigma\tilde{\psi}\tau} & \xrightarrow{\Pi} & M_{\sigma\tilde{\psi}} \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle V} \\ M_{\tilde{\psi}\tau} & \xrightarrow{\Pi} & M_{\tilde{\psi}}\,. \end{array} \tag{4.3}$$

By the rank condition, the cokernels of the vertical maps are k-vector spaces of dimension $r_{\tilde{\psi}\tau} = r_{\tilde{\psi}}$. Therefore the lengths of the cokernels of both horizontal $\iota(\Pi)$ are also the same. On the other hand we have

$$\sum\nolimits_{\tilde{\psi}} \operatorname{length} M_{\tilde{\psi}}/\Pi M_{\tilde{\psi}\tau} = \operatorname{length} M/\Pi M = nf.$$

The last equality holds since by assumption $\operatorname{length} M/pM = n^2 d$. We conclude that

$$\operatorname{length} M_{\tilde{\psi}}/\Pi M_{\tilde{\psi}\tau} = 1, \quad \text{for } \tilde{\psi} \in \tilde{\Psi}.$$

The last equation tells us that $M/\Pi M$ is a free $O_D/\Pi O_D \otimes_{\mathbb{F}_p} k$-module of rank 1. By Nakayama's lemma it follows that M is a free $O_D \otimes_{\mathbb{Z}_p} W(k)$-module. This completes the proof of the first assertion in the case $S = \operatorname{Spec} k$.

To show that X is a formal group we have to show that V is nilpotent on $M/\iota(\Pi)M$. We consider the map induced by V on the cokernels of the horizontal maps of the diagram (4.3). For each $\tilde{\psi}$ this map is either a bijection or it is zero. It suffices to see that this map is zero for some $\tilde{\psi}$. If not, we would have a bijection for each $\tilde{\psi}$. We denote the cokernels of the vertical maps by $\operatorname{Lie}_{\tilde{\psi}} X$. Then

$$\operatorname{Lie} X = \bigoplus_{\tilde{\psi}} \operatorname{Lie}_{\tilde{\psi}} X.$$

We conclude that $\iota(\Pi) : \operatorname{Lie}_{\tilde{\psi}\tau} X \to \operatorname{Lie}_{\tilde{\psi}} X$ is bijective for each $\tilde{\psi}$. This is a contradiction since $\iota(\Pi)$ is nilpotent on $\operatorname{Lie} X$. Therefore X is a formal group.

A base change argument shows that our result is true if S is the spectrum of a field. In the general case we consider a point $s \in S$. We fix a basis m of the $O_D \otimes \kappa(s)$-module $\mathbb{D}(X)_S \otimes \kappa(s)$. We may assume that $S = \operatorname{Spec} R$ and that m is the image of an element $n \in \mathbb{D}(X)_R$. It follows from Nakayama's Lemma that the homomorphism induced by n

$$O_D \otimes R \to \mathbb{D}(X)_R$$

is an isomorphism in an open neighbourhood of s. □

We state the following consequence separately:

Corollary 4.4. *For each $\tilde{\psi} \in \tilde{\Psi}$ the cokernel of*

$$\Pi : \mathbb{D}(X)_{S,\tilde{\psi}\tau} \to \mathbb{D}(X)_{S,\tilde{\psi}}$$

is a locally free $\mathcal{O}_S$-module of rank 1. □

Proposition 4.5. *Let A be an O_E-algebra. Assume moreover that A is a p-adic integral domain, with fraction field of characteristic* 0*. Let X be an r-special formal O_D-module over A.*

Then $\operatorname{Lie} X$ *satisfies* $(\mathbf{K}_r)$.

Proof. Let K denote the fraction field of A. By Proposition 4.3, $H = \mathbb{D}(X)_K$ is a free $O_D \otimes K$-module of rank 1. We consider the decomposition

$$H = \bigoplus_{\psi \in \Psi} H_\psi.$$

We can assume that A is a $O_{\tilde{E}}$-algebra. Let $\varphi \in \Phi$ be an extension of ψ. We write $\varphi|\psi$. Then π acts semisimply on H_ψ and has eigenvalues $\varphi(\pi)$ for $\varphi|\psi$, each with multiplicity n^2. Now $L = (\operatorname{Lie} X)_K$ is a quotient of H. It has the decomposition $L = \bigoplus L_\psi$. Assume first that $\psi \neq \psi_0$. By the Eisenstein condition, the eigenvalues of π acting on L_ψ are among $\varphi(\pi)$ with $\varphi \in A_\psi$. The multiplicity of the eigenvalues is at most n^2. But since by (3.11) $\operatorname{rank} L_\psi = n^2 a_\psi$, each of these eigenvalues must have exactly multiplicity n^2. We assume now that $\psi = \psi_0$. Then again the eigenspaces of $\varphi(\pi)$ have dimension $\leq n^2$. But the second of the Eisenstein conditions says that for $\varphi = \varphi_0$ this multiplicity is $\leq n$. Since $\operatorname{rank} L_{\psi_0} = a_\psi n^2 + n$, this implies that the multiplicity of $\varphi(\pi)$ is n^2 for $\varphi \in A_{\psi_0}$ and is n for $\varphi = \varphi_0$. Altogether the multiplicity of the eigenvalue $\varphi(\pi)$ of π acting on L is $r_\varphi n$. This implies the Kottwitz condition on L_K. Since $(\mathbf{K}_r)$ is a closed condition, the assertion follows. □

We will now assume that S is a $\kappa_{\tilde{E}}$-scheme. The O_D-module $O_D \otimes \mathcal{O}_S$ defines a polynomial function

$$\rho : \mathbb{V}(O_D)_S \to \mathbb{A}^1_S.$$

The module $O_D \otimes \mathcal{O}_S$ has a composition series with factors

$$O_D/\Pi O_D \otimes_{\tilde{\kappa},\tilde{\psi}} \mathcal{O}_S,$$

where $\tilde{\psi} \in \tilde{\Psi}$. This last module defines the polynomial function $\chi_{\tilde{\psi}}$ from (3.16). One deduces easily that the determinant of the module $O_D \otimes \mathcal{O}_S$ is

$$\rho = \prod_{\tilde{\psi} \in \tilde{\Psi}} \chi_{\tilde{\psi}}^{ne} = \prod_{\varphi \in \Phi} \mathrm{Nrd}_{\varphi}^{n}.$$

Analogously to (4.2) we have a decomposition

$$\mathbb{D}(X)_S = \bigoplus_{\tilde{\psi} \in \tilde{\Psi}} \mathbb{D}(X)_{S,\tilde{\psi}},$$

where all summands are locally free $\mathcal{O}_S$-modules of rank ne. We set for $\psi \in \Psi$

$$\mathbb{D}(X)_{S,\psi} = \bigoplus_{\tilde{\psi} \in \tilde{\Psi}} \mathbb{D}(X)_{S,\tilde{\psi}},$$

where the sum runs over all $\tilde{\psi}$ such that $\tilde{\psi}_{|F^t} = \psi$. It follows from Proposition 4.3 that we have locally free $\mathcal{O}_S$-modules with ranks

$$\mathrm{rank}\, \mathbb{D}(X)_{S,\tilde{\psi}}/\pi^i \mathbb{D}(X)_{S,\tilde{\psi}} = ni, \quad \text{for } 0 \leq i \leq e.$$

We note that the $\mathcal{O}_S$-modules $\mathbb{D}(X)_{S,\tilde{\psi}}$ are defined for each $O_{\tilde{E}}$-scheme with p locally nilpotent.

Proposition 4.6. *Let (X, ι) be a p-divisible O_D-module over a κ_E-scheme S. There are natural surjective maps*

$$\mathbb{D}(X)_{S,\psi} \to \mathrm{Lie}_{\psi} X, \quad \textit{for } \psi \in \Psi. \tag{4.4}$$

(i) Assume that (X, ι) is r-special. Then the maps (4.4) induce isomorphisms

$$\begin{aligned} \mathbb{D}(X)_{S,\psi}/\pi^{a_\psi}\mathbb{D}(X)_{S,\psi} &\to \mathrm{Lie}_{\psi} X, \ \textit{for } \psi \neq \psi_0 \\ \mathbb{D}(X)_{S,\psi_0}/\pi^{a_{\psi_0}}\mathbb{D}(X)_{S,\psi_0} &\to \mathrm{Lie}_{\psi_0} X/\pi^{a_{\psi_0}} \mathrm{Lie}_{\psi_0} X. \end{aligned} \tag{4.5}$$

In particular, the cokernel of any power of π on $\mathrm{Lie}\, X$ is a locally free $\mathcal{O}_S$-module.

(ii) Conversely, assume that the following conditions on (X, ι) are satisfied.

1. $\mathrm{Lie}\, X$ *satisfies the rank condition* $(\mathbf{R}_r)$.

2. The natural map $\mathbb{D}(X)_S \to \mathrm{Lie}\, X$ *induces isomorphisms (4.5)*

3. Lie_{ψ_0} *is annihilated by* $\pi^{a_{\psi_0}+1}$.

Then X is r-special.

We will prove this together with the following Corollary:

Corollary 4.7. *An r-special formal O_D-module X over a κ_E-scheme S satisfies the Kottwitz condition* $(\mathbf{K}_r)$.

Remark 4.8. In the case where $r = r^\circ$, this corollary follows from Lemma 3.11.

Proof. Clearly we can restrict to the case where S is a scheme over $\kappa_{\tilde{E}}$.

We first prove (i). The last condition of (3.22) says that for $\psi \neq \psi_0$ the first arrow of (4.5) exists. By (3.11) we have on both sides locally free $\mathcal{O}_S$-modules of the same rank. Therefore this arrow is an isomorphism.

For the second line in (4.5), we begin with the case where $S = \mathrm{Spec}\, k$. The second condition of (3.22) says that the rank of the following homomorphism of vector spaces

$$\pi^{a_{\psi_0}} : \mathrm{Lie}_{\psi_0} X \to \mathrm{Lie}_{\psi_0} X$$

is at most n. This shows

$$\dim_k(\mathrm{Lie}_{\psi_0} X/\pi^{a_{\psi_0}} \mathrm{Lie}_{\psi_0} X) \geq \dim_k \mathrm{Lie}_{\psi_0} X \; - n = a_{\psi_0} n^2.$$

Therefore the second arrow of (4.5) is an isomorphism because on the left hand side we have a vector space of dimension $n^2 a_{\psi_0}$.

It follows that the $\mathcal{O}_S$-module $\mathrm{Lie}_{\psi_0} X/\pi^{a_{\psi_0}} \mathrm{Lie}_{\psi_0}$ has in each point of S the same rank $n^2 a_{\psi_0}$. This already proves assertion (i) in the case where S is a reduced scheme.

The general case is a consequence of Lemma 4.9 below, applied to $L = \mathrm{Lie}_{\psi_0} X$, $f = \pi^{a_{\psi_0}}$, $r = a_{\psi_0} n^2$, $m = a_{\psi_0} n^2 + n$, and $s = n$.

Now we prove the corollary. We remark that the Kottwitz condition $(\mathbf{K}_r)$ is satisfied for $\mathrm{Lie}\, X$. This is clear by the first isomorphism of (4.5) for the part $\mathrm{Lie}_\psi X$, for $\psi \neq \psi_0$. By the second isomorphism it suffices to show that the determinant morphism for the O_D-module $\pi^{a_{\psi_0}} \mathrm{Lie}_{\psi_0} X$ is $\mathrm{Nrd}_{\varphi_0} = \prod_{\tilde{\psi}} \chi_{\tilde{\psi}}$, where $\tilde{\psi}$ extends φ_0. But it follows from the isomorphism (4.5) and (3.11) that $\mathrm{rank}\, \pi^{a_{\psi_0}} \mathrm{Lie}_{\tilde{\psi}} X = 1$ for each $\tilde{\psi} \in \tilde{\Psi}$ which extends φ_0. Therefore we conclude by Lemma 3.7.

Now we prove (ii). We have to prove the Eisenstein conditions. For $\psi \neq \psi_0$ they are clear. We denote by K_{ψ_0} the kernel of the natural map $\mathbb{D}(X)_{S,\psi_0} \to \mathrm{Lie}_{\psi_0} X$. We obtain $\pi^{a_{\psi_0}}\mathbb{D}(X)_{S,\psi_0} \supset K_{\psi_0} \supset \pi^{a_{\psi_0}+1}\mathbb{D}(X)_{S,\psi_0}$. Then the rank condition shows that

$$\pi^{a_{\psi_0}}\mathbb{D}(X)_{S,\psi_0}/K_{\psi_0} = \pi^{a_{\psi_0}} \mathrm{Lie}_{\psi_0} X$$

has rank n. This proves the second condition of (3.22). The other Eisenstein conditions are trivially satisfied. □

In the preceding proof, we used the following lemma.

Lemma 4.9. *Let R be a local ring with residue field k. Let L be a finitely generated free R-module of rank m. Let $f : L \to L$ be an endomorphism.*

Let $r = \dim_k(L/f(L)) \otimes k$ and let $s = m - r$. We assume that

$$\bigwedge^{s+1} f = 0.$$

Then $L/f(L)$ is a free R-module of rank r.

Proof. The exact sequence

$$L \otimes k \to L \otimes k \to (L/f(L)) \otimes k \to 0$$

shows that there is a basis of $L \otimes k$ of the form

$$f(\bar{y}_1), \ldots, f(\bar{y}_s), \bar{e}_1, \ldots, \bar{e}_r\,, \tag{4.6}$$

where $\bar{y}_1, \ldots, \bar{y}_s, \bar{e}_1, \ldots, \bar{e}_r \in L \otimes k$, and where the images of $\bar{e}_1, \ldots, \bar{e}_r$ in $(L/f(L)) \otimes k$ form a basis.

Lifting the elements $\bar{y}_1, \ldots, \bar{y}_s, \bar{e}_1, \ldots, \bar{e}_r$ to L we obtain a basis of this R-module,

$$f(y_1), \ldots, f(y_s), e_1, \ldots e_r.$$

The elements $\bar{y}_1, \ldots, \bar{y}_s \in L \otimes k$ are linearly independent. Therefore we find a second basis of L,

$$y_1, \ldots, y_s, x_1, \ldots, x_r. \tag{4.7}$$

We write the matrix of f with respect to the basis (4.6) and (4.7) as a $(s + r) \times (s + r)$ block matrix:

$$\begin{pmatrix} E_s & A \\ \mathbf{0} & B \end{pmatrix}$$

The assumption $\bigwedge^{s+1} f = 0$ implies that the matrix B is zero. We find

$$f(x_i) = \sum_j a_{ji} f(y_j), \quad a_{ji} \in R.$$

We set $z_i = x_i - \sum_j a_{ji} y_j \in \operatorname{Ker} f$. Clearly

$$y_1, \dots, y_s, z_1, \dots, z_r$$

is also a basis of L, i.e., we may assume WLOG that $x_i = z_i$. But then the matrix of f becomes

$$\begin{pmatrix} E_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

From this our assertion is obvious. □

Before continuing, we add a lemma needed later which is proved in the same manner.

Lemma 4.10. *Let n, m, and r be natural numbers. Let R be a commutative ring. Let W be a locally free R-module of rank n. Let $f : W \to W$ be an endomorphism such that* $\operatorname{Coker} f$ *is a locally free R-module of rank r.*
Let $V \subset W$ be a direct summand of rank m. We assume that $s = m - r \geq 0$ and that

$$\bigwedge^{s+1} (f_{|V}) = 0.$$

Then $\operatorname{Ker} f \subset V$.

Proof. We note that the assumptions of the lemma are compatible with base change $R \to S$. The situation of the lemma is always defined over a noetherian subring. Therefore we may assume that R is noetherian. The desired inclusion may be checked over the localizations of R. Therefore we may assume that R is a local noetherian ring with maximal ideal $\mathfrak{m}$. Finally the matrix of $\operatorname{Ker} f \to W/V$ is zero if it is zero modulo $\mathfrak{m}^t$ for all $t \in \mathbb{N}$. Therefore we may assume that R is an artinian local ring. We also note that under the assumptions $\operatorname{Ker} f$ is a direct summand of W of rank r.

If R is a field, the assumption of the lemma implies $\operatorname{rank} f_{|V} \leq s$. Hence $\dim \operatorname{Ker} f_{|V} \geq r = \dim \operatorname{Ker} f$. This implies $\operatorname{Ker} f_{|V} = \operatorname{Ker} f$ and the lemma.

Now let $(R, \mathfrak{m})$ be any artinian local ring. Let $e_1, \dots, e_r$ be a basis of $\operatorname{Ker} f$. It follows from the case of a field that V has a basis of the form

$$v_1, \dots, v_s, e_1 + \rho_1, \dots, e_r + \rho_r, \tag{4.8}$$

where $\rho_i \in \mathfrak{m}W$. If the lemma is false we can choose $t \in \mathbb{N}$ maximal such that there is a basis of the form (4.8) with $\rho_i \in \mathfrak{m}^t W$.

By assumption $f(v_1), \ldots, f(v_s)$ are linearly independent modulo $\mathfrak{m}$. Therefore we find a basis of V of the form

$$f(v_1), \ldots, f(v_s), u_1, \ldots, u_r. \tag{4.9}$$

We write the matrix of $f_{|V}$ with respect to the matrices (4.8) and (4.9) as a block matrix,

$$\begin{pmatrix} E_s & * \\ \mathbf{0} & X \end{pmatrix}.$$

By the assumption $\wedge^{s+1}(f_{|V}) = 0$ all determinants of $(s+1) \times (s+1)$ minors of this matrix are zero. Therefore the matrix X is zero. We obtain equations

$$f(\rho_i) = f(e_i + \rho_i) = \sum_j a_{ji} f(v_j).$$

Since the left hand side is in $\mathfrak{m}^t W$ we conclude that $a_{ji} \in \mathfrak{m}^t$. We find $\rho_i - \sum_j a_{ji} v_j \in \operatorname{Ker} f$. We may write

$$\rho_i - \sum_j a_{ji} v_j = \sum_k c_{ki} e_k,$$

where necessarily $c_{ki} \in \mathfrak{m}^t$. The last equation gives:

$$(e_i + \rho_i) - \sum_j a_{ji} v_j - \sum_k c_{ki}(e_k + \rho_k) = e_i - \sum_k c_{ki} \rho_k.$$

The RHS is an element of V. But this shows that we may replace in (4.8) the element $e_i + \rho_i$ by $e_i + \rho'_i$ with $\rho'_i = \sum_k c_{ki} \rho_k \in \mathfrak{m}^{2t} W$. This is a contradiction. □

Remark 4.11. If R is a local ring such that each element of the maximal ideal $\mathfrak{m}$ is nilpotent, we can replace the condition that "$\operatorname{Coker} f$ is a free R-module of rank r" by the weaker assumption that "$\operatorname{Ker} f \subset W$ is a direct summand of rank r". Indeed, over R any free submodule of a free module is a direct summand. This shows that $\operatorname{Coker} f$ is free. The weaker assumption also suffices if R is a ring such that $R \to \prod R_{\mathfrak{p}}$ is injective, where $\mathfrak{p}$ runs over all minimal prime ideals of R, since then we may reduce to $R = R_{\mathfrak{p}}$.

We will now study the display of a formal O_D-module over a κ_E-scheme S which satisfies $(\mathbf{D}_r)$. To ease the notation we will assume that $S = \operatorname{Spec} R$.

We denote by $\mathcal{P} = (P, Q, F, \dot{F})$ the display of X. We use the notation $I := I_R := \ker(W(R) \to R)$. Recall that $\mathbb{D}(X)_R = P/IP$.

Again write $\Psi = \mathrm{Hom}(\kappa, \kappa_E)$ for the set of field embeddings. We obtain the decompositions

$$P = \bigoplus_{\psi \in \Psi} P_\psi, \quad Q = \bigoplus_{\psi \in \Psi} Q_\psi.$$

By Proposition 4.6 we obtain:

$$\begin{aligned} \pi^{a_{\psi_0}+1} P_{\psi_0} + IP_{\psi_0} \subset Q_{\psi_0} \subset \pi^{a_{\psi_0}} P_{\psi_0} + IP_{\psi_0}, \\ Q_\psi = \pi^{a_\psi} P_\psi + IP_\psi, \qquad \text{for } \psi \neq \psi_0. \end{aligned} \tag{4.10}$$

The maps F and $\dot{F}$ induce maps

$$F_\psi : P_\psi \to P_{\psi\sigma}, \quad \dot{F}_\psi : P_\psi \to P_{\psi\sigma}.$$

Here σ denotes the Frobenius automorphism of F^t over $\mathbb{Q}_p$. We set $Q'_\psi = P_\psi$ for $\psi \neq \psi_0$ and we define Q'_{ψ_0} to be the unique submodule of P_{ψ_0} such that $P_{\psi_0} \supset Q'_{\psi_0} \supset Q_{\psi_0}$ and such that Q'_{ψ_0}/Q_{ψ_0} is the kernel of the homomorphism

$$\pi^{a_{\psi_0}} : P_{\psi_0}/Q_{\psi_0} \to P_{\psi_0}/Q_{\psi_0}.$$

By Lemma 4.9 we know that this kernel is a direct summand of P_{ψ_0}/Q_{ψ_0}.

We set

$$Q' = \bigoplus_\psi Q'_\psi, \quad F'_\psi = F_\psi \pi^{a_\psi}, \quad \dot{F}'_\psi = \dot{F}_\psi \pi^{a_\psi}.$$

We obtain Frobenius-linear homomorphisms

$$F' = \bigoplus_\psi F_\psi \pi^{a_\psi} : P \to P, \quad \dot{F} = \bigoplus_\psi \dot{F}_\psi \pi^{a_\psi} : Q' \to P. \tag{4.11}$$

We claim that the quadruple $\mathcal{P}' = (P, Q', F', \dot{F}')$ is the display of a special formal O_D-module.

Theorem 4.12. *Let R be a κ_E-algebra. We assume that the nilradical of R is a nilpotent ideal. Let $\mathcal{C}_{r,R}$ be the category of r-special formal O_D-modules, and let $\mathcal{C}_{\mathbf{0},R}$ the category of special formal O_D-modules (Definition 4.1).*

The construction $\mathcal{P} \mapsto \mathcal{P}'$ is an equivalence of categories

$$\mathcal{C}_{r,R} \to \mathcal{C}_{\mathbf{0},R}. \tag{4.12}$$

Proof. We begin with the case $S = \operatorname{Spec} k$, where k is a perfect field. Let the covariant Dieudonné M_X be identified with P. In this case (4.10) is equivalent with

$$\begin{aligned} \pi^{a_{\psi_0}+1} M_{\psi_0,X} \subset & V M_{\psi_0\sigma,X} \subset \pi^{a_{\psi_0}} M_{\psi_0}, \\ V M_{\psi\sigma,X} &= \pi^{a_\psi} M_{\psi,X} \quad \text{for } \psi \neq \psi_0. \end{aligned} \tag{4.13}$$

We define

$$\begin{aligned} V' &= \pi^{-a_\psi} V \colon M_{\psi\sigma,X} \to M_{\psi,X} \\ F' &= \pi^{a_\psi} F \colon M_{\psi,X} \to M_{\psi\sigma,X} \,. \end{aligned} \tag{4.14}$$

Then the Dieudonné module (M_X, F', V') corresponds to the display above. From this we see that $\mathcal{P}'$ is the Dieudonné module of a special formal O_D-module. Indeed, by the remark after (3.22) we need only to verify $(\mathbf{R}_r)$ (3.11) for $\mathcal{P}'$. But this follows easily from (3.19) and (4.13).

If conversely (M, F', V') is the Dieudonné module of a special formal O_D-module, then we find

$$F' M_{\psi_0,X} \subset \pi^{e-1} M_{\psi_0\sigma,X}, \quad F' M_{\psi,X} \subset \pi^{e} M_{\psi\sigma,X}, \text{ for } \psi \neq \psi_0.$$

This follows because $V' M_{\psi\sigma} = M_\psi$ for $\psi \neq \psi_0$ and $M_{\psi_0}/V' M_{\psi_0\sigma}$ is annihilated by π. Therefore the formulas $V = \pi^{a_\psi} V'$ and $F = \pi^{-a_\psi} F'$ define a Dieudonné module structure on M such that (4.13) is satisfied. This shows that (M, F, V) is the Dieudonné module of an r-special formal O_D-module. This proves the theorem in the case of a perfect field.

In the general case we need first to verify that $\mathcal{P}'$ is a display. The only non-trivial property is that $\dot{F}'$ is a Frobenius-linear epimorphism. To show this, we take locally on $\operatorname{Spec} R$ a normal decomposition of $\mathcal{P}'$ and consider the matrix of $F' \bigoplus \dot{F}'$. We have to show that the image of the determinant of this matrix in R is a unit. But this property follows since we know it for a perfect field. The same argument shows that $\mathcal{P}'$ is nilpotent. Therefore we have defined a functor (4.12).

We construct first a quasi-inverse functor in the case that the ring R is reduced. Let $\mathcal{P}'$ be the display of a special formal O_D-module. We note that P'/IP' is a locally free $O_D \otimes R$-module of rank 1. In particular, it has a filtration by direct summands as R-modules,

$$0 = \pi^e (P'_\psi / IP'_\psi) \subset \pi^{e-1} (P'_\psi / IP'_\psi) \subset \ldots \subset \pi (P'_\psi / IP'_\psi) \subset P'_\psi / IP'_\psi$$

The multiplication by π gives an isomorphism between the subquotients of this filtration.

If we have a direct R-module summand $L \subset P'_\psi/IP'_\psi$ with the property that $\pi(P'_\psi/IP'_\psi) \subset L \subset P'_\psi/IP'_\psi$, we obtain therefore a direct R-module summand $\pi^{a_\psi+1}(P'_\psi/IP'_\psi) \subset \pi^{a_\psi}L \subset \pi^{a_\psi}P'_\psi/IP'_\psi$.

This gives the possibility to invert our construction $\mathcal{P} \to \mathcal{P}'$. We set $P = P'$. We note that $Q'_\psi = P'_\psi$ if $\psi \neq \psi_0$. We set in general

$$Q_\psi = \pi^{a_\psi} Q'_\psi + IP_\psi. \tag{4.15}$$

We want to define F and $\dot{F}$ by the formulas

$$F_\psi = \pi^{-a_\psi} F'_\psi, \quad \dot{F}_\psi = \pi^{-a_\psi} \dot{F}'_\psi.$$

We note that $F'_\psi Q'_\psi = p\dot{F}'_\psi Q'_\psi$. This implies for $\psi \neq \psi_0$ that $F'_\psi P_\psi \subset \pi^e P_{\psi\sigma}$. From $\pi P'_{\psi_0} \subset Q'_{\psi_0}$ we conclude that $F'_{\psi_0} P_{\psi_0} \subset \pi^{e-1} P_{\psi_0\sigma}$. Since R is reduced, π operates injectively on $W(R)$ and therefore the definition of F_ψ makes sense. From (4.15) we see that also the definition of $\dot{F}$ makes sense. We have to show that $\mathcal{P} = (P, Q, F, \dot{F})$ is indeed a display. But this follows from the case of a perfect field treated above.

Now we treat the case of a nonreduced ring R. We assume that we have a divided power thickening $R \to S$, and that the theorem is already known for S. We denote by X an r-special formal O_D-module over S, and by X' the corresponding special formal O_D-module over S. We show that our functor gives a bijection between the liftings of X to an r-special formal O_D-module over R and the liftings of X' to a special formal O_D-module over R. This will prove the theorem by induction. By Grothendieck-Messing, the liftings of X correspond to liftings of the Hodge-filtration,

$$\begin{array}{ccc} \mathbb{D}(X)_{R,\psi} & \longrightarrow & L_\psi \\ \downarrow & & \downarrow \\ \mathbb{D}(X)_{S,\psi} & \longrightarrow & \mathrm{Lie}_\psi X\,. \end{array} \tag{4.16}$$

If $\psi \neq \psi_0$ we have no choice for L_ψ because, by Proposition 4.6, $L_\psi = \mathbb{D}(X)_{R,\psi}/\pi^{a_\psi}\mathbb{D}(X)_{R,\psi}$. As a special case this holds also for X'. Now let $\psi = \psi_0$. Let $\bar{Q}_R$ and $\bar{Q}_S$ the kernels of the two horizontal maps in (4.16). Then we have $\pi^{a_\psi}\mathbb{D}(X)_{R,\psi} \supset \bar{Q}_R \supset \pi^{a_\psi+1}\mathbb{D}(X)_{R,\psi}$. We replace $\bar{Q}_R$ by $\bar{Q}'_R := \pi^{-a_\psi}\bar{Q}_R$. This makes sense because we have a bijection

$$\mathbb{D}(X)_{R,\psi}/\pi\mathbb{D}(X)_{R,\psi} \xrightarrow{\pi^{a_\psi}} \pi^{a_\psi}\mathbb{D}(X)_{R,\psi}/\pi^{a_\psi+1}\mathbb{D}(X)_{R,\psi}.$$

Then $L' = \mathbb{D}(X)_{R,\psi}/\bar{Q}'_R$ is a lifting of $\mathrm{Lie}_\psi X'$ which defines a lifting of the special formal O_D-module X'. This sets up the desired bijection of liftings. $\square$

Corollary 4.13. *Let k be an algebraically closed field which is at the same time a κ_E-algebra. Any two r-special formal O_D-modules over k are isogenous by a O_D-linear isogeny.*

Proof. By Theorem 4.12, we are reduced to the case of special formal O_D-modules, i.e., the case $r = r^\circ$. In this case, the assertion follows from [5], §2, comp. [3], Prop. 5.2. $\square$

Let $\breve{E}$ the completion of the maximal unramified extension of E. Its residue class field $\bar{k}$ is an algebraic closure of κ_E. We fix an r-special formal O_D-module $(\mathbb{X}, \iota_{\mathbb{X}})$ over $\bar{k}$ (a *framing object*).

Definition 4.14. We define the set-valued functor $\mathcal{M}_r$ on the category of $O_{\breve{E}}$-schemes as follows[3]. Then $\mathcal{M}_r$ associates to scheme $S \in (\mathrm{Sch}/O_{\breve{E}})$ the set of isomorphism classes of triples (X, ι, ρ). Here (X, ι) is an r-special formal O_D-module over S, and ρ denotes a O_D-linear isogeny $X \times_{\mathrm{Spec}\, O_{\breve{E}}} \mathrm{Spec}\, \bar{k} \to \mathbb{X} \times_{\mathrm{Spec}\, \bar{k}} S$ of height zero.

We write $\overline{\mathcal{M}}_r$ for the restriction of this functor to $\bar{k}$-schemes S. Theorem 4.12 now implies the following corollary.

Corollary 4.15. *The functor $\overline{\mathcal{M}}_r$ is representable by a scheme over $\bar{k}$ which is isomorphic to $\overline{\mathcal{M}}_{r^\circ}$. Hence there is an isomorphism $\overline{\mathcal{M}}_r \simeq \hat{\Omega}^n \otimes_{O_{\breve{E}}} \bar{k}$.*

Proof. The isomorphism $\overline{\mathcal{M}}_r \simeq \overline{\mathcal{M}}_{r^\circ}$ follows from Theorem 4.12. The last assertion follows from [5], which also implies that $\overline{\mathcal{M}}_{r^\circ}$ is a scheme. $\square$

5. The local model

In this section we consider the local structure of the formal scheme $\mathcal{M}_r$ (Definition 4.14). By the general theory [18], this comes down to considering the *local model* of $\mathcal{M}_r$. Let us define it.

Recall that D is the central division algebra with invariant $1/n$ over F. Let V be a D-vector space of dimension 1 and let Λ be an O_D-lattice in V.

The local model in question represents the following functor on (Sch/O_E):

---

[3] We will prove in Proposition 5.8 that this definition coincides with the one in section 2.

$$\mathbb{M}_r(S) = \{\mathcal{F} \subset \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \mid O_D\text{-stable } \mathcal{O}_S\text{-submodule, locally on } S \text{ a direct summand, } (\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S)/\mathcal{F} \text{ satisfies conditions } (\mathbf{R}_r) \text{ and } (\mathbf{E}_r)\}.$$

Lemma 5.1. *The functor $\mathbb{M}_r$ is representable by a projective scheme over* $\operatorname{Spec} O_E$. *The geometric generic fiber is isomorphic to $\mathbb{P}^{n-1}$.*

Let S be an E-scheme. Consider a $\mathcal{O}_S$-submodule $\mathcal{F} \subset \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ which is locally a direct summand and which is O_D-stable. Then $(\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S)/\mathcal{F}$ satisfies conditions $(\mathbf{R}_r)$ *and* $(\mathbf{E}_r)$ *if and only if it satisfies the condition* $(\mathbf{K}_r)$.

Proof. The first assertion is obvious since the rank condition is a closed condition, cf. the remark after Definition 3.3.

The implication " $\Rightarrow$ " in the last assertion follows as in Proposition 4.5. To show the converse, let R be a $\bar{\mathbb{Q}}_p$-algebra and $S = \operatorname{Spec} R$. Let $\mathcal{F}$ be a direct summand of $\Lambda \otimes_{\mathbb{Z}_p} R$ that is O_D-stable and such that $(\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S)/\mathcal{F}$ satisfies $(\mathbf{K}_r)$. There are decompositions

$$\Lambda \otimes_{\mathbb{Z}_p} R = \bigoplus_{\varphi} \Lambda_\varphi\,, \quad \mathcal{F} = \bigoplus_{\varphi} \mathcal{F}_\varphi\,,$$

where φ runs through the embeddings of F into $\bar{\mathbb{Q}}_p$. Here O_F acts on the summand corresponding to φ via $\varphi : F \to \bar{\mathbb{Q}}_p \to R$. Each summand is stable under the action of D. The condition $(\mathbf{K}_r)$ just says that rank $\Lambda_\varphi/\mathcal{F}_\varphi = nr_\varphi$, in which case $\Lambda_\varphi/\mathcal{F}_\varphi$ is locally on S isomorphic to the direct sum of r_φ copies of the simple representation $F^n \otimes_\varphi R$ of $D \otimes_{F,\varphi} R \simeq \mathrm{M}_n(R)$. On the summand $\Lambda_\varphi/\mathcal{F}_\varphi$, $\iota(\pi)$ acts as $\varphi(\pi)\,\mathrm{Id}_{\mathcal{F}_\varphi}$. Let $\psi = \psi_0$. It then follows that $Q_{A_{\psi_0}}(\iota(\pi))$ annihilates all summands $\Lambda_\varphi/\mathcal{F}_\varphi$, for those φ with $\varphi_{|_{F^t}} = \psi_0$ and $\varphi \neq \varphi_0$, and $Q_{A_{\psi_0}}(\iota(\pi))$ induces an isomorphism on $\Lambda_{\varphi_0}/\mathcal{F}_{\varphi_0}$, which implies the second Eisenstein condition. The first and third Eisenstein conditions are proved in an analogous way.

For φ with $\varphi_{|_{F^t}} \neq \psi_0$, the subspace $\mathcal{F}_\varphi$ is trivial, i.e., either equal to (0) or to Λ_φ. On the other hand, using Morita equivalence, the $\mathrm{M}_n(R)$-stable summand $\mathcal{F}_{\varphi_0}$ of Λ_{φ_0} corresponds to a hyperplane of $F^n \otimes_{F,\varphi_0} R$. It now follows that the geometric generic fiber of $\mathbb{M}_r$ is isomorphic to the projective space of hyperplanes in $\bar{\mathbb{Q}}_p^n$, i.e., to $\mathbb{P}^{n-1}$ (Grothendieck's convention). □

The geometric special fiber $\overline{\mathbb{M}}_r = \mathbb{M}_r \otimes_{O_E} \bar{k}$ can be described as follows. Let $W_\psi = \Lambda \otimes_{O_{F^t},\psi} \bar{k}$, an en^2-dimensional vector space with its endomorphism $\Pi = \iota(\Pi)$. Let $S = \operatorname{Spec} R$, for a $\bar{k}$-algebra R, and let $(\mathcal{F}_\psi)_{\psi\in\Psi}$ be a point in $\overline{\mathbb{M}}_r(S)$. Let first $\psi \neq \psi_0$. By the third Eisenstein condition,

$\mathcal{F}_\psi$ is a direct summand of rank $(e-a_\psi)n^2$ containing the image of $\Pi^{a_\psi n}$. Since these two submodules are direct summands of the same rank, they are equal.

Now let $\psi=\psi_0$, and set $W_0=W_{\psi_0}$ and $a_0=a_{\psi_0}$. Then, due to the action of $O_{\tilde{F}}$, we obtain a $\mathbb{Z}/n$-grading

$$W_0=\bigoplus\nolimits_{k\in\mathbb{Z}/n} W_{0,k}, \tag{5.1}$$

and Π is an endomorphism of degree one. Forgetting the subspaces $\mathcal{F}_\psi$ with $\psi\neq\psi_0$, we have an identification

$$\begin{aligned}\overline{\mathbb{M}}_r(S)=\{\mathcal{F}_0\subset W_0\otimes_{\bar{k}}\mathcal{O}_S \mid{}& \Pi\text{-stable } \textit{graded} \text{ direct summand,}\\ & \operatorname{rank}(W_{0,k,S}/\mathcal{F}_{0,k})=a_0n+1\,,\forall k\in\mathbb{Z}/n,\\ & \text{and } 1')\text{ and } 2')\}\,.\end{aligned}$$

Here we have set $W_{0,S}=W_0\otimes_{\bar{k}}\mathcal{O}_S$ and $W_{0,k,S}=W_{0,k}\otimes_{\bar{k}}\mathcal{O}_S$, and 1′) and 2′) are as follows:

$$\begin{aligned} 1')\quad & \Pi^{(a_0+1)n}|(W_{0,S}/\mathcal{F}_0) &&=0\\ 2')\quad & \bigwedge^{n+1}(\Pi^{a_0n}|(W_{0,S}/\mathcal{F}_0)) &&=0.\end{aligned} \tag{5.2}$$

Of course, we have used here (3.22).

Let us now apply Lemma 4.10 to the $\mathcal{O}_S$-dual $W_{0,S}^*$, its endomorphism f induced by $(\Pi^*)^{a_0n}$ and its submodule $V=(W_{0,S}/\mathcal{F}_0)^*$, in which case $\operatorname{rank}W_{0,S}^*=en^2$, and $\operatorname{rank}V=(a_0n+1)n$, and $r=a_0n^2$, and $s=n$. We conclude that $\operatorname{Ker}(\Pi^*)^{a_0n}\otimes_{\bar{k}}\mathcal{O}_S\subset V$. Translated back into $\mathcal{F}_0$, we obtain a chain of inclusions of direct summands of $W_{0,S}$,

$$\operatorname{Im}(\Pi^{(a_0+1)n})\otimes_{\bar{k}}\mathcal{O}_S\subset\mathcal{F}_0\subset\operatorname{Im}(\Pi^{a_0n})\otimes_{\bar{k}}\mathcal{O}_S\,. \tag{5.3}$$

In the Drinfeld case $r=r^\circ$ we have $a_0=0$. Let us write $\mathbb{M}^\circ$ for $\mathbb{M}_{r^\circ}$.

Let us identify $\operatorname{Im}\Pi^{(a_0+1)n}/\operatorname{Im}\Pi^{a_0n}$ with $W_0/\Pi W_0$. Associating now to an S-valued point $\mathcal{F}_0$ of $\overline{\mathbb{M}}_r$ the locally direct summand

$$\mathcal{F}_0/\operatorname{Im}(\Pi^{(a_0+1)n})\otimes_{\bar{k}}\mathcal{O}_S\subset(\operatorname{Im}\Pi^{a_0n}/\operatorname{Im}\Pi^{(a_0+1)n})\otimes_{\bar{k}}\mathcal{O}_S=(\Lambda\otimes_{O_F,\varphi_0}\bar{k})\otimes_{\bar{k}}\mathcal{O}_S\,,$$

we have obtained an S-valued point of the local model $\mathbb{M}^\circ$, more precisely an S-valued point of $\mathbb{M}^\circ\otimes_{O_F,\varphi_0}O_E\otimes_{O_E}\bar{k}$. Letting S vary, this induces obviously an isomorphism of schemes over $\bar{k}$,

$$\mathbb{M}_r \otimes_{O_E} \bar{k} \simeq \mathbb{M}^\circ \otimes_{O_F} \bar{k}. \tag{5.4}$$

Therefore we obtain from the Drinfeld case:

Corollary 5.2. *The geometric special fiber $\mathbb{M}_r \otimes_{O_E} \bar{k}$ is a reduced scheme, which has n irreducible components, all of which have dimension $n-1$. Furthermore, the local rings of closed points of $\mathbb{M}_r \otimes_{O_E} \bar{k}$ are isomorphic to localizations in closed points of the $\bar{k}$-algebra $\bar{k}[X_1, \dots, X_n]/(X_1 \cdot \ldots \cdot X_n)$.* □

Remark 5.3. We point out that $\mathbb{M}^\circ$ coincides with the *standard local model* for the triple $(G, \{\mu\}, K)$ consisting of GL_n and $\mu_{(1,0^{(n-1)})}$ and the Iwahori subgroup, after extension of scalars to $\operatorname{Spec} O_{\breve{E}}$, cf. [7], comp. also [17, 18].

Corollary 5.4. *$\mathbb{M}_r$ is flat over O_E.*

Proof. Since the special fiber is reduced, it suffices by [8], Prop. 14.16 to show that a generic point of an irreducible component of the special fiber is in the closure of the general fibre. Since the general fiber and the special fibre of $\mathbb{M}_r$ have the same dimension $n-1$ and since $\mathbb{M}_r$ is proper it follows that at least one irreducible component of the special fiber is contained in the closure of the generic fiber. The claim therefore follows from the following lemma. □

Lemma 5.5. *There is an action of $\mathbb{Z}/n$ on $\mathbb{M}_r \otimes_{O_E} O_{\breve{E}}$, which induces a transitive action on the set of irreducible components of $\mathbb{M}_r \otimes_{O_E} \bar{k}$.*

Proof. The action is given by sending $\mathcal{F} = \oplus_{\tilde{\psi} \in \tilde{\Psi}} \mathcal{F}_{\tilde{\psi}}$ to $\mathcal{F}'$ with

$$(\mathcal{F}')_{\tilde{\psi}} = \mathcal{F}_{\tilde{\psi}\tau}\,, \quad \tilde{\psi} \in \tilde{\Psi}\,.$$

Here τ is taken from the presentation of O_D in (3.3). Indeed, since the O_F-structure of $(\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S)/\mathcal{F}'$ coincides with that of $(\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S)/\mathcal{F}$, the Eisenstein conditions are satisfied for $(\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S)/\mathcal{F}'$, since they are satisfied for $(\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S)/\mathcal{F}$. On the other hand, for any $\psi \in \Psi$,

$$\operatorname{rank}(\mathcal{F}')_\psi = \sum_{\tilde{\psi} \in \tilde{\Psi}_\psi} \operatorname{rank}(\mathcal{F}')_{\tilde{\psi}} = \sum_{\tilde{\psi} \in \tilde{\Psi}_\psi} \operatorname{rank} \mathcal{F}_{\tilde{\psi}\tau} = \sum_{\tilde{\psi} \in \tilde{\Psi}_\psi} \operatorname{rank} \mathcal{F}_{\tilde{\psi}} = r_\psi.$$

Therefore, $\mathcal{F}'$ also satisfies the condition $(\mathbf{R}_r)$.

That the action of $\mathbb{Z}/n$ on the set of irreducible components of $\mathbb{M}_r \otimes_{O_E} \bar{k}$ is transitive, follows from the corresponding fact for $\mathbb{M}^\circ$ (the Drinfeld case) (the isomorphism (5.4) is obviously equivariant for the action of $\mathbb{Z}/n$). □

Corollary 5.6. *The scheme $\mathbb{M}_r$ is normal.*

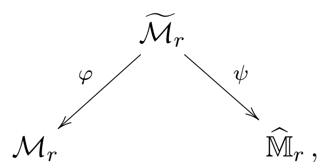
Proof. Indeed, $\mathbb{M}_r$ is flat over O_E, with normal generic fiber (even regular, cf. Lemma 5.1), and reduced special fiber. These properties imply that $\mathbb{M}_r$ is normal, cf. [19], Prop. 9.2. □

Remark 5.7. If the Conjecture 2.6 were true, it would follow that $\mathbb{M}_r$ has semi-stable reduction, in particular $\mathbb{M}_r$ would be regular. However, we are unable to prove these stronger assertions.

Let $\mathcal{M}'_r$ be the closed formal subscheme of $\mathcal{M}_r$ which is given by the Kottwitz condition $(\mathbf{K}_r)$. By Corollary 4.7 the special fibers of $\mathcal{M}_r$ and $\mathcal{M}'_r$ are identical.

Proposition 5.8. *The two formal schemes $\mathcal{M}_r$ and $\mathcal{M}'_r$ are identical. Both are p-adic and flat over* $\operatorname{Spf} O_{\breve{E}}$*, with special fiber $\mathcal{M}_r \times_{\operatorname{Spf} O_{\breve{E}}} \operatorname{Spec} \bar{k}$ a reduced scheme. All their completed local rings are normal.*

Proof. We use the *local model diagram*



where $\widehat{\mathbb{M}}_r$ denotes the formal completion of $\mathbb{M}_r \times_{\operatorname{Spec} O_E} \operatorname{Spec} O_{\breve{E}}$ along its special fiber. Here $\widetilde{\mathcal{M}}_r$ and the morphism φ is obtained from $\mathcal{M}_r$ by adding to (X, ι, ρ) an $O_D \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-linear isomorphism with the value at S of the covariant crystal associated to X,

$$\alpha \colon \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \longrightarrow \mathbb{D}(X)_S .$$

The morphism ψ maps the S-valued point (X, ι, ρ, α) of $\widetilde{\mathcal{M}}_r$ to the submodule $\alpha^{-1}(\operatorname{Ker}(\mathbb{D}(X) \to \operatorname{Lie} X))$ of $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$. The theory of local models [21] tells us that the completed local ring of a point $x \in \mathcal{M}_r$ is isomorphic to the completed local ring of $\psi(\tilde{x})$, where $\tilde{x}$ is any point of $\widetilde{\mathcal{M}}_r$ mapping under φ to x. Hence all completed local rings of points of $\mathcal{M}_r$ are isomorphic to completed rings of points of $\mathbb{M}_r \times_{\operatorname{Spec} O_E} \operatorname{Spec} O_{\breve{E}}$. Hence by Corollaries 5.4 and 5.2, the formal scheme $\mathcal{M}_r$ is flat over $\operatorname{Spf} O_{\breve{E}}$ with all completed local rings normal. That $\mathcal{M}_r \times_{\operatorname{Spf} O_{\breve{E}}} \operatorname{Spec} \bar{k}$ is a reduced scheme follows from Corollary 4.15. Now the equality of $\mathcal{M}'_r$ and $\mathcal{M}_r$ follows from Proposition 4.5. □

Corollary 5.9. *The definition of $\mathcal{M}_r$ is independent of the choice of the uniformizer π of O_F.*

Proof. Consider the formal scheme $\mathcal{N}_r$ that represents the moduli problem where the Kottwitz condition $(\mathbf{K}_r)$ is imposed but the Eisenstein conditions $(\mathbf{E}_r)$ are dropped. Let $\tilde{\pi}$ be another uniformizer, and let $\tilde{\mathcal{M}}_r$ be the corresponding formal scheme defined using the Eisenstein condition for $\tilde{\pi}$ instead of π. What has to be shown is that the formal subschemes $\mathcal{M}_r$ and $\tilde{\mathcal{M}}_r$ of $\mathcal{N}$ are identical. Let $\mathbb{N}_r$ be the local model corresponding to $\mathcal{N}_r$; then the local models $\mathbb{M}_r$ and $\tilde{\mathbb{M}}_r$ of $\mathcal{M}_r$ and $\tilde{\mathcal{M}}_r$ are closed subschemes of $\mathbb{N}_r$. It suffices to prove that $\mathbb{M}_r = \tilde{\mathbb{M}}_r$. But by Lemma 5.1 the generic fibers of $\mathbb{N}_r, \mathbb{M}_r$ and $\tilde{\mathbb{M}}_r$ all coincide and, by Corollary 5.4, $\mathbb{M}_r$ and $\tilde{\mathbb{M}}_r$ are equal to the flat closure of the generic fiber inside $\mathbb{N}_r$. □

6. The generic fiber (after Scholze)

In this section, we prove the last point in Theorem 2.8, in the following form. For convenience, we introduce for a function $r\colon \varphi \mapsto r_\varphi$ the formal scheme $\tilde{\mathcal{M}}_r$ over $\operatorname{Spf} O_{\breve{E}}$ that represents the same moduli problem as $\mathcal{M}_r$, except that we drop the condition that the height of ρ be zero. Then our original formal scheme $\mathcal{M}_r$ is an open and closed formal subscheme of $\tilde{\mathcal{M}}_r$.

Let r° be the Drinfeld function, i. e., $r^\circ_\varphi = 0, \forall \varphi \neq \varphi_0$. We write $\tilde{\mathcal{M}}^\circ = \tilde{\mathcal{M}}_{r^\circ}$.

We will prove the following theorem. We use the embedding $\breve{F} \hookrightarrow \breve{E}$ defined by the natural map $\breve{F} = F \otimes_{F^t} \breve{\mathbb{Q}}_p \xrightarrow{\varphi_0 \otimes \mathrm{id}} E \otimes_{E^t} \breve{\mathbb{Q}}_p = \breve{E}$.

Theorem 6.1. *There is an isomorphism of adic spaces over* $\operatorname{Spa}(\breve{E}, O_{\breve{E}})$,

$$(\tilde{\mathcal{M}}_r)^{ad} \simeq (\tilde{\mathcal{M}}^\circ \hat{\otimes}_{O_{\breve{F}}} O_{\breve{E}})^{ad}$$

This theorem implies the last point in Theorem 2.8. Indeed, passing to the open and closed sublocus where the universal quasi-isogeny ϱ has height zero, we obtain a similar isomorphism when $\tilde{\mathcal{M}}_r$ is replaced by $\mathcal{M}_r$ and $\tilde{\mathcal{M}}^\circ$ by $\mathcal{M}^\circ$ (the proof of Theorem 6.1 will show that the isomorphism in question is compatible with the decompositions according to the height). Since by Drinfeld's theorem $(\mathcal{M}^\circ)^{ad} \simeq \Omega^n_F$, we deduce the desired isomorphism

$$(\mathcal{M}_r)^{ad} \simeq \hat{\Omega}_F \otimes_F \breve{E} = (\hat{\Omega}_F \otimes_{O_F} O_{\breve{E}})^{ad}\,.$$

In the proof of Theorem 6.1, we will use the following notation. We denote by $(\mathbb{X}, \iota_{\mathbb{X}})$ the framing object for the moduli problem $\tilde{\mathcal{M}}_r$. Let $M(\mathbb{X})_{\mathbb{Q}_p}$ be its rational Diendonné module.

Let V be a free D-module of rank one, and let Λ be an O_D-lattice in V. Let $G = \mathrm{GL}_D(V)$, considered as a linear algebraic group over $\mathbb{Q}_p$. The function r defines a G-homogeneous projective variety $\boldsymbol{\mathcal{F}}$ over E. If R is a $\bar{\mathbb{Q}}_p$-algebra, then $\boldsymbol{\mathcal{F}}(R)$ parametrizes D-linear surjective homomorphisms into locally free R-modules

$$V \otimes_{\mathbb{Q}_p} R \longrightarrow \mathcal{F}$$

such that under the decomposition $\mathcal{F} = \bigoplus_\varphi \mathcal{F}_\varphi$, we have $\operatorname{rank}(\mathcal{F}_\varphi) = r_\varphi n$. In other words, $\boldsymbol{\mathcal{F}}$ coincides with the generic fiber of the local model, $\mathbb{M}_r \times_{\operatorname{Spec} O_E} \operatorname{Spec} E$, cf. Lemma 5.1.

Let $K_0 \subset G(\mathbb{Q}_p)$ be the stabilizer of the lattice Λ. For any open subgroup $K \subset K_0$, we obtain the corresponding member $\mathbb{M}_K$ of the RZ-tower over $(\tilde{\mathcal{M}}_r)^{ad}$. These coverings of $(\tilde{\mathcal{M}}_r)^{ad}$ parametrize level-K-structures on the universal object $X/\tilde{\mathcal{M}}_r$,

$$\alpha : \Lambda \longrightarrow T(X) \bmod K \, .$$

We denote by

$$\pi_K : \mathbb{M}_K \longrightarrow (\mathcal{F} \otimes_E \breve{E})^{ad}$$

the crystalline period maps, which are compatible with changes in K.

In [24] Scholze and Weinstein define a preperfectoid space $\mathbb{M}_\infty$ by imposing a full level structure on $T(X)$. In particular, there is a morphism

$$\mathbb{M}_\infty \longrightarrow \varprojlim \mathbb{M}_K \, ,$$

which induces a bijection for any algebraically closed extension C of $\bar{\mathbb{Q}}_p$ which is complete for a p-adic valuation,

$$\mathbb{M}_\infty(C) \simeq \varprojlim \mathbb{M}_K(C) \, .$$

We denote by $\pi_\infty : \mathbb{M}_\infty \longrightarrow (\boldsymbol{\mathcal{F}} \otimes_E \breve{E})^{ad}$ the induced period mapping. In [24], the following description of $\mathbb{M}_\infty(C)$ is given. Let $\mathcal{O}_C$ be the ring of integers in C.

Let $B^+_{\text{cris}} = A_{\text{cris}}(\mathcal{O}_C/p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ be Fontaine's ring attached to C. The Fargues-Fontaine curve[4] is defined as $Y = \operatorname{Proj} P$, where P is the graded ring

$$P = \bigoplus_{d \geq 0} (B^+_{\text{cris}})^{\phi = p^d} \, .$$

[4]The Fargues-Fontaine curve is usually denoted by X; since this notation is already in use for the universal p-divisible group, we use the notation Y instead.

Then Y is a connected separated regular noetherian scheme of dimension 1, equipped with the point $\infty \in Y$ corresponding to Fontaine's homomorphism

$$\theta : B^+_{\text{cris}} \longrightarrow C\,.$$

In [24] appears a description of p-divisible groups X over $\mathcal{O}_C$, in terms of two vector bundles, $\mathcal{E}$ and $\mathcal{F}$. Here

- $\mathcal{F} = T \otimes_{\mathbb{Z}_p} \mathcal{O}_Y$,
- $\mathcal{E}$ corresponds to the graded P-module $\bigoplus_{d \geq 0} (M_{\mathbb{Q}_p})^{\phi = p^{d+1}}$.

Here $T = T(X)$ denotes the Tate module of the generic fibre of X, and $M_{\mathbb{Q}_p} = M(X)_{\mathbb{Q}_p}$ the rational Dieudonné module of the reduction modulo p of X.

We now apply this description to the fibers of the universal p-divisible group X at points of $\mathbb{M}_\infty(C)$, noting that the universal full level structure induces an isomorphism $T(X) = \Lambda$, and the universal quasi-isogeny an isomorphism of rational Dieudonné modules $M(X)_{\mathbb{Q}_p} = M(\mathbb{X})_{\mathbb{Q}_p} \otimes_{\breve{\mathbb{Q}}_p} B^+_{\text{cris}}$. Accordingly we set $\mathcal{F} = \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_Y$, and let $\mathcal{E}$ correspond to the graded module

$$\Big(\bigoplus_{d \geq 0} M(\mathbb{X})_{\mathbb{Q}_p} \otimes_{\breve{\mathbb{Q}}_p} B^+_{\text{cris}}\Big)^{\phi = p^{d+1}}\,.$$

We also fix a D-linear isomorphism $M(\mathbb{X})_{\mathbb{Q}_p} = V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$. The Scholze-Weinstein description therefore implies the following fact, cf. [24], Cor. 6.3.10 (and its extension to the EL-case in Thm. 6.5.4).

Theorem 6.2. *There is an identification of* $\mathbb{M}_\infty(C)$ *with the set of injective* D*-linear homomorphisms of vector bundles on* Y*,*

$$f : \mathcal{F} \longrightarrow \mathcal{E}$$

such that $\operatorname{supp}(\operatorname{Coker} f) = \{\infty\}$ *and such that* $\mathfrak{m}_{Y,\infty}$ *kills* $\operatorname{Coker} f$ *and such that the induced surjective map*

$$\mathcal{E} \otimes_{\mathcal{O}_{X,\infty}} C = M(\mathbb{X})_{\mathbb{Q}_p} \otimes_{\breve{\mathbb{Q}}_p} C = V \otimes_{\mathbb{Q}_p} C \xrightarrow{\varphi_f} \operatorname{Coker} f$$

defines a point in $\boldsymbol{\mathcal{F}}(C)$*. Furthermore, the period morphism* π_∞ *sends the point corresponding to* f *to* $[V \otimes_{\mathbb{Q}_p} C \xrightarrow{\varphi_f} \operatorname{Coker} f]$. □

It will be convenient to reformulate this last description. Let $Y_F = Y \times_{\operatorname{Spec} \mathbb{Q}_p} \operatorname{Spec} F$. This is a finite étale cover $\psi : Y_F \longrightarrow Y$ of degree d, and the fiber $\psi^{-1}(\{\infty\})$ can be identified with $\{\infty_\varphi \mid \varphi : F \longrightarrow \bar{\mathbb{Q}}_p\}$. Since the vector bundles $\mathcal{F}$ and $\mathcal{E}$ are equipped with F-actions, they are of the form $\mathcal{F} = \psi_*(\mathcal{F}_F)$ and $\mathcal{E} = \psi_*(\mathcal{E}_F)$, with vector bundles $\mathcal{F}_F$ and $\mathcal{E}_F$ over Y_F.

Corollary 6.3. *There is a natural identification of* $\mathbb{M}_\infty(C)$ *with the set of* D*-linear injective morphisms of vector bundles on* Y_F*,*

$$f_F : \mathcal{F}_F \longrightarrow \mathcal{E}_F\,,$$

such that $\operatorname{supp}\operatorname{Coker} f_F \subset \psi^{-1}(\{\infty\})$*, with* $\operatorname{Coker} f_F$ *killed by* $\mathfrak{m}_{Y_F,\infty_\varphi}\ \forall\varphi$*, and such that*

$$\dim_C(\operatorname{Coker} f_F)_{\infty_\varphi} = r_\varphi \cdot n\,,\ \forall\varphi\,.$$

Furthermore, the period morphism π_∞ *sends the point corresponding to* f_F *to the family of surjections* $[\mathcal{E}_F \otimes_{Y_F,\infty_\varphi} C = V \otimes_{F,\varphi} C \longrightarrow (\operatorname{Coker} f_F)_{\infty_\varphi}]$*, considered as a point in* $\boldsymbol{\mathcal{F}}(C)$*.* □

Now we compare the previous descriptions for the given function r, and for the Drinfeld function r°. Let $\mathbb{X}^\circ$ denote the framing object for $\tilde{\mathcal{M}}_{r^\circ}$. Then we may choose D-linear isomorphisms

$$V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p = M(\mathbb{X}^\circ)_{\mathbb{Q}_p} = V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p = M(\mathbb{X})_{\mathbb{Q}_p}$$

such that for the respective Frobenius endomorphisms

$$F = \pi^m \cdot F^\circ\,,$$

where

$$m = \#\{\varphi \mid r_\varphi = n\}\,.$$

For the corresponding vector bundles on Y_F, we get

$$\mathcal{E}_F = \mathcal{E}_F^\circ \otimes_{\mathcal{O}_{X_F}} \mathcal{L}^m\,,$$

where $\mathcal{L}$ is the line bundle on Y_F corresponding to the graded module

$$\bigoplus_{d\geq 0} (F \otimes_{\mathbb{Q}_p} B^+_{\mathrm{cris}})^{\phi=\pi p^d}\,.$$

We have a natural identification

$$\mathcal{F}_F = \Lambda \otimes_{O_F} \mathcal{O}_{Y_F} = \mathcal{F}_F^\circ\,.$$

On the other hand, we have a natural identification

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}^\circ \otimes_F E\,.$$

Here, for a $\bar{\mathbb{Q}}_p$-algebra R, a point $[V \otimes_{\mathbb{Q}_p} R \longrightarrow \mathcal{F}] = [\{V \otimes_{F,\varphi} R \longrightarrow \mathcal{F}_\varphi \mid \varphi\}]$ of $\boldsymbol{\mathcal{F}}$ is sent to $[\{V \otimes_{F,\varphi} R \longrightarrow \mathcal{F}^\circ_\varphi \mid \varphi\}]$, with

$$\mathcal{F}^\circ_\varphi = \begin{cases} \mathcal{F}_{\varphi_0} & \text{if } \varphi = \varphi_0 \\ V \otimes_{F,\varphi} R & \text{if } \varphi \neq \varphi_0 \,. \end{cases}$$

The general case of an E-algebra R follows by descent. We point out that, by Morita equivalence, $\boldsymbol{\mathcal{F}}^\circ \cong \mathbb{P}^{n-1}_F$.

Lemma 6.4. *For a given φ, there exists a global section*

$$LT_\varphi \in \Gamma(Y_F, \mathcal{L}) = (F \otimes_{\mathbb{Q}_p} B^+_{\text{cris}})^{\phi=\pi}$$

such that LT_φ vanishes to first order at ∞_φ and is non-vanishing at all other points. Furthermore, LT_φ is unique up to $F^\times$.

Proof. The homomorphism $LT_\varphi : \mathcal{O}_{Y_F} \longrightarrow \mathcal{L}$ corresponds in the Scholze-Weinstein description to the Lubin-Tate formal group corresponding to the triple (F, φ, π). □

Lemma 6.5. *Let $i : \mathcal{G}_F \longrightarrow \mathcal{E}_F$ be the injection of vector bundles on Y_F,*

$$\mathcal{G}_F = \{x \in \mathcal{E}_F \mid x \equiv 0 \mod \infty_\varphi \,, \forall \varphi \text{ with } r_\varphi = n\} \,.$$

Then the map $x \longmapsto x \cdot \Pi_{\{\varphi \mid r_\varphi = n\}} LT_\varphi$ defines an isomorphism $\mathcal{E}^\circ_F \simeq \mathcal{G}_F$.

Proof. The map

$$x \longmapsto x \prod\nolimits_{\{\varphi \mid r_\varphi = n\}} LT_\varphi$$

obviously identifies $\mathcal{E}^\circ_F$ with a subbundle of $\mathcal{G}_F$. Comparing the degrees of $\mathcal{E}^\circ_F$ and $\mathcal{G}$, the result follows. (There is a formalism of degrees of vector bundles on Y resembling the usual theory over smooth projective curves.) □

Proposition 6.6. *Under the identification $\mathcal{E}^\circ_F = \mathcal{G}_F$ and $\mathcal{F}_F = \mathcal{F}^\circ_F$, the map sending $f^\circ_F \in \mathrm{Hom}(\mathcal{F}^\circ_F, \mathcal{E}^\circ_F)$ to $f_F = i \circ f^\circ_F \in \mathrm{Hom}(\mathcal{F}_F, \mathcal{E}_F)$ defines a bijection $\mathbb{M}^\circ_\infty(C) = \mathbb{M}_\infty(C)$ which commutes with the period map π°_∞, resp. π_∞, to $\boldsymbol{\mathcal{F}}^\circ(C) = \boldsymbol{\mathcal{F}}(C)$.*

Proof. It is clear that $\mathrm{Coker}\, f^\circ_F$ has support in $\{\infty_{\varphi_0}\} \cup \{\infty_\varphi \mid r_\varphi = n\}$, and that for φ with $r_\varphi = n$, one has that $(\mathrm{Coker}\, f^\circ_F)_{\infty_F} = \mathcal{E}_F \otimes_{\mathcal{O}_{Y_F}, \infty_\varphi} C$. Conversely, any $f_F \in \mathbb{M}_\infty(C)$ has to factor through $\mathcal{G}_F$ and has the correct cokernel at ∞_{φ_0}. The assertion regarding the period map is obvious from the way that $\boldsymbol{\mathcal{F}}(C)$ is identified with $\boldsymbol{\mathcal{F}}^\circ(C)$. □

Corollary 6.7. *Under the identification $\mathcal{F}^{ad} = (\mathcal{F}^\circ \otimes_F E)^{ad}$, the images of the period morphisms π_∞ and $\pi^\circ_\infty \otimes_{\breve F} \breve E$, coincide.*

Proof. All maps are partially proper [6], hence it suffices to prove for all algebraically closed complete extensions C of $\bar{\mathbb{Q}}_p$ that

$$\mathrm{Im}\ \pi_\infty(C) = \mathrm{Im}\ \pi^\circ_\infty(C)\,.$$

This follows from the previous proposition. □

Proof of Theorem 6.1. We need to construct an isomorphism

$$\mathbb{M}_{K_0} \simeq \mathbb{M}^\circ_{K_0} \otimes_{\breve F} \breve E\,.$$

It suffices to construct the isomorphism on the open and closed subloci $\mathbb{M}^{(n)}_{K_0}$, resp. $\mathbb{M}^{\circ(n)}_{K_0}$, where the height of ϱ is a fixed integer n. But the fibers of the period morphisms π_{K_0}, resp. $\pi^\circ_{K_0}$, through $\mathbb{M}^{(n)}_{K_0}$, resp. $\mathbb{M}^{\circ(n)}_{K_0}$, can both be identified with $G(\mathbb{Q}_p)^\circ/K_0$, where under the identification $G(\mathbb{Q}_p) = D^\times$, we have

$$G(\mathbb{Q}_p)^\circ = \{x \in D^\times \mid \mathrm{ord}\det x = 0\}\,.$$

Since $K_0 = G(\mathbb{Q}_p)^\circ$, the period maps identify $\mathbb{M}^{(n)}_{K_0}$ and $\mathbb{M}^{\circ(n)}_{K_0}$ with open adic subsets of $(\mathcal{F} \otimes_E \breve E)^{ad} = (\mathcal{F}^\circ \otimes_F \breve E)^{ad}$. The assertion therefore follows from Corollary 6.7. □

7. The unramified case

In this section we prove Theorem 2.7. Hence in this section $F/\mathbb{Q}_p$ is unramified. Since we fixed an embedding $\varphi_0 : F \longrightarrow \bar{\mathbb{Q}}_p$, we may identify

$$\mathrm{Hom}_{\mathbb{Q}_p}(F, \bar{\mathbb{Q}}_p) = \{\sigma^i \mid i \in \mathbb{Z}/d\}\,.$$

In particular, $E = F$, via φ_0. We abbreviate

$$r_i = r_{\varphi_0 \circ \sigma^i}\,, \quad i \in \mathbb{Z}/d\,.$$

Let S be a O_F-scheme, such that p is nilpotent on S. We have a decomposition

$$O_F \otimes_{\mathbb{Z}_p} \mathcal{O}_S = \bigoplus_{i \in \mathbb{Z}/d} \mathcal{O}_S\,,$$

where the action of O_F on the i-th summand is via σ^i. If (X, ι) is a formal O_D-module over S, we correspondingly obtain a decomposition,

$$\operatorname{Lie} X = \bigoplus_{i \in \mathbb{Z}/d} \operatorname{Lie}_i X \,,$$

where $t \in \operatorname{Lie}_i X$ iff $\iota(x)(t) = \sigma^i(x)\, t$, $\forall x \in O_F$.

Lemma 7.1. *The condition* ($\mathbf{R}_r$) *implies the conditions* ($\mathbf{K}_r$) *and* ($\mathbf{E}_r$).

Proof. For the implication ($\mathbf{R}_r$) $\Rightarrow$ ($\mathbf{E}_r$), we refer to Remarks 3.9, (iii). The condition ($\mathbf{D}_r$) implies ($\mathbf{K}_r$) by Proposition 5.8. □

We will now consider the display of X. To simplify the notation, we assume that $S = \operatorname{Spec} R$, where R is an O_F-algebra. Let $W(O_F)$ be the ring of Witt vectors of O_F, and define a ring homomorphism by

$$\lambda : O_F \longrightarrow W(O_F)\,, \quad w_m(\lambda(x)) = \sigma^m(x)\,.$$

Then λ is Frobenius equivariant, i.e., $\lambda(\sigma(x)) =^F(\lambda(x))$. For a O_F-algebra R, we obtain

$$\bar{\lambda} : O_F \longrightarrow W(O_F) \longrightarrow W(R)\,.$$

Let

$$\bar{\lambda}^{(i)} = \bar{\lambda} \circ \sigma^i =^{F^i} \circ \bar{\lambda} : O_F \longrightarrow W(R).$$

Let $\mathcal{P} = (P, Q, F, \dot{F})$ be the display of X over R. Hence P is a finitely generated projective $W(R)$-module, and Q is submodule of P, and

$$F : P \longrightarrow P\,,\ \dot{F} : Q \longrightarrow P.$$

The action of O_F on $\mathcal{P}$ defines decompositions,

$$P = \bigoplus P_i\,, \quad Q = \bigoplus Q_i\,,$$

where O_F acts on the i-th summand via $\bar{\lambda}^{(i)}$. The operators F and $\dot{F}$ are of degree 1 with respect to this $\mathbb{Z}/d$-grading,

$$F : P_i \longrightarrow P_{i+1}\,, \quad \dot{F} : Q_i \longrightarrow P_{i+1}\,.$$

Then rank $P_i = n^2$, $\forall i$, by Proposition 4.3. The rank condition says:

$$\begin{aligned} &P_0/Q_0 \textit{ is a locally free } R\textit{-module of rank } n \\ &P_i = Q_i \textit{ if } r_i = 0\,,\ I_R \cdot P_i = Q_i \textit{ if } r_i = n\,. \end{aligned} \tag{7.1}$$

We choose a normal decomposition,

$$P_i = T_i \oplus L_i\,, \text{for all } i\,.$$

Then

$$F \oplus \dot{F} : T_i \oplus L_i \longrightarrow P_{i+1}$$

is a ^{F}-linear isomorphism, for all i. Let $i \neq 0$. When $r_i = 0$, then $P_i = Q_i$ and we obtain an isomorphism

$$\dot{F}^{\#} : W(R) \otimes_{F,W(R)} P_i \longrightarrow P_{i+1}\,. \tag{7.2}$$

When $r_i = n$, then $L_i = (0)$, and we obtain an isomorphism

$$F^{\#} : W(R) \otimes_{F,W(R)} P_i \longrightarrow P_{i+1}\,. \tag{7.3}$$

We now define ${}^{F^d}$-linear homomorphisms,

$$F^{\#}_{\mathrm{rel}} : P_0^{(p^d)} \longrightarrow P_0\,,\ \dot{F}^{\#}_{\mathrm{rel}} : Q_0^{(p^d)} \longrightarrow P_0\,,$$

as compositions,

$$F^{\#}_{\mathrm{rel}} : P_0^{(p^d)} \xrightarrow{F^{\#}} P_1^{(p^{d-1})} \xrightarrow{\sim} \ldots \xrightarrow{\sim} P_{d-1}^{(p)} \xrightarrow{\sim} P_0\,,$$

resp.

$$\dot{F}^{\#}_{\mathrm{rel}} : Q_0^{(p^d)} \longrightarrow P_1^{(p^{d-1})} \xrightarrow{\sim} \ldots \xrightarrow{\sim} P_{d-1}^{(p)} \xrightarrow{\sim} P_0\,.$$

Here the isomorphisms in the last two lines are either (7.2) or (7.3). We note that P_0 is a finitely generated projective $W(R)$-module, that Q_0 is a submodule and F_{rel} and $\dot{F}_{\mathrm{rel}}$ satisfy the following relations,

$$\begin{aligned} &\dot{F}_{\mathrm{rel}}({}^V\xi x) = {}^{F^{d-1}}\xi \cdot F_{\mathrm{rel}}(x), \quad && x \in P_0\,. \\ &p \cdot \dot{F}_{\mathrm{rel}}(y) = F_{\mathrm{rel}}(y), && y \in Q_0\,. \end{aligned} \tag{7.4}$$

Indeed, the first identity reflects the F^d-linearity of $\dot{F}_{\mathrm{rel}}$; the second identity comes from the fact that a similar identity holds for $\dot{F}$ and F. The quadruple $(P_0, Q_0, F_{\mathrm{rel}}, \dot{F}_{\mathrm{rel}})$ is a d-display in the following sense:

Definition 7.2. Let $d \geq 1$ be a natural number. Let R be a ring such that p is a nilpotent in R. An *d-display over* R is a quadruple $(P, Q, F, \dot{F})$, where P is a finitely generated projective $W(R)$-module, Q a submodule of P and $F : P \to Q$ and $\dot{F} : Q \to P$ are ${}^{F^d}$-linear maps such that the following properties are satisfied:

1. $I_R P \subset Q$, and P/Q is a direct summand of the R-module $P/I_R P$.

2. The linearization of $\dot{F}$,

$$\dot{F}^{\sharp} : W(R) \otimes_{F^d, W(R)} Q \longrightarrow P$$

is surjective.

3. For $x \in P$ and $w \in W(R)$,

$$\dot{F}({}^V w x) = {}^{F^{d-1}} w F(x).$$

We will now use the theory of relative Witt vectors and relative displays. We denote by $q = p^d$ the number of elements in the residue class field κ of O_F. Also, when we use O_F as a subscript, we simply write O. For an O_F-algebra R, we denote by $W_O(R)$ the ring of relative Witt vectors defined by the Witt polynomials

$$w'_n(x_0, \ldots, x_n) = x_0^{q^n} + p x_1^{q^{n-1}} + \cdots + p^n x_n.$$

We have a canonical morphism $u : W(R) \to W_O(R)$ such that

$$\begin{aligned} u({}^{F^d}\xi) &= {}^{F'} u(\xi) \\ u({}^{V}\xi) &= {}^{V'}(u({}^{F^{d-1}}\xi)), \end{aligned}$$

cf. [5], Prop. 1.2. Here we denoted by a prime the operators on $W_O(R)$.

We now show how to associate to $(P_0, Q_0, F_{\mathrm{rel}}, \dot{F}_{\mathrm{rel}})$ a relative display $(P', Q', F', \dot{F}')$ with respect to $W_O(R)$ (replace the Witt vectors by the relative Witt vectors in the definition of a display).

We set

$$\begin{aligned} P' &= W_O(R) \otimes_{u, W(R)} P_0 \\ Q' &= \mathrm{Ker}\left(W_O(R) \otimes_{u, W(R)} P_0 \longrightarrow P_0/Q_0 \right). \end{aligned}$$

Here the last homomorphism is given by the composition

$$\begin{aligned} W_O(R) \otimes_{u, W(R)} P_0 \longrightarrow & W_O(R)/I_O(R) \otimes_{u, W(R)} P_0 = R \otimes_{u, W(R)} P_0 \\ &= P_0/I(R)P_0 \longrightarrow P_0/Q_0 . \end{aligned}$$

$$F' = {}^{F'} \otimes F_{\mathrm{rel}} : P' \longrightarrow P'.$$

Note that this makes sense because of (7.4), and defines a ${}^{F^d}$-linear endomorphism of P'. It remains to define $\dot{F}' : Q' \longrightarrow P'$ with the following properties,

$$\begin{aligned}\dot{F}'({}^{V'}\xi x) &= \xi \cdot F'(x), \qquad x \in P' \\ \dot{F}'(\xi \otimes y) &= {}^{F'}\xi \otimes \dot{F}_{\mathrm{red}}(y), \quad y \in Q\,.\end{aligned} \tag{7.5}$$

More precisely, consider the normal decomposition $P_0 = T_0 \oplus L_0$. Then

$$Q' = \big(I_O(R) \otimes_{W(R)} T_0\big) \oplus \big(W_O(R) \otimes_{W(R)} L_0\big)\,.$$

We define $\dot{F}'$ on the first, resp. second summand by

$$\begin{aligned}\dot{F}'({}^{V'}\xi \otimes t_0) &= \xi \otimes F_{\mathrm{rel}}(t_0)\,, \qquad t_0 \in T_0 \\ \dot{F}'(\xi \otimes l_0) &= {}^{F'}\xi \otimes F_{\mathrm{rel}}(l_0)\,, \quad l_0 \in L_0\,.\end{aligned}$$

Claim: *The identities* (7.5) *are satisfied.*

We start with the second identity. Let

$$y = {}^{V'}\eta \cdot t_0 + l_0 \quad , t_0 \in T_0,\, l_0 \in L_0\,.$$

For the second summand, the identity to be checked is the definition of $\dot{F}'$. So we may take $l_0 = 0$. Now

$$\begin{aligned}\xi \otimes {}^{V'}\eta \cdot t_0 = \xi \cdot u({}^{V'}\eta) \otimes t_0 &= \xi \cdot {}^{V'}u({}^{F^{d-1}}\eta) \otimes t_0 \\ &= {}^{V'}({}^{F'}\xi \cdot u({}^{F^{d-1}}\eta)) \otimes t_0\,.\end{aligned}$$

Hence the LHS of the identity to be checked is

$$\begin{aligned}\dot{F}'(\xi \otimes {}^{V'}\eta\, t_0) &= {}^{F'}\xi u({}^{F^{d-1}}\eta) \otimes F_{\mathrm{rel}}(t_0) \\ &= {}^{F'}\xi \otimes {}^{F^{d-1}}\eta \cdot F_{\mathrm{rel}}(t_0) \\ &= {}^{F'}\xi \otimes \dot{F}_{\mathrm{rel}}({}^{V'}\eta\, t_0)\,,\end{aligned}$$

where in the last equation we used (7.4). The second identity of (7.5) is proved.

Now we check the first identity. It suffices to check that

$$\dot{F}'({}^{V'}\xi \otimes x) = \xi \otimes F_{\mathrm{rel}}(x)\,, \quad x \in P_0\,.$$

If $x = t_0 \in T_0$, this holds by definition. Let $x = l_0 \in L_0$. Then

$$\begin{aligned}\dot{F}'({}^{V'}\xi \otimes l_0) = p\xi \otimes \dot{F}_{\mathrm{rel}}(l_0) &= \xi \otimes p\dot{F}_{\mathrm{rel}}(l_0) \\ &= \xi \otimes F_{\mathrm{rel}}(l_0)\,,\end{aligned}$$

where in the last equation we used (7.4).

We now have checked that $(P', Q', F', \dot{F}')$ is a relative display relative to O_F. By functoriality this relative display has an O_D-action. Its Lie algebra P'/Q' coincides with $\mathrm{Lie}_0\, X = P_0/Q_0$. Therefore the action of O_D on P'/Q' is special in the sense of Drinfeld (satisfies condition $(\mathbf{R}_{r^\circ})$), i.e., we are in the case of Proposition 3.6. This implies automatically that the relative display is nilpotent.

Let R be an O_F-algebra where πR is a nilpotent ideal. By a theorem of Ahsendorf [1], Thm. 5.3.8, there is an equivalence of categories between the category of p-divisible formal O_F-modules over R and the category of nilpotent relative displays (this holds even without the hypothesis that $F/\mathbb{Q}_p$ is unramified). We therefore obtain a formal O_F-module X' over $S = \mathrm{Spec}\, R$, which is a special formal O_D-module because $\mathrm{Lie}\, X' = P'/Q'$.

Applying the above construction to the framing object $(\mathbb{X}, \iota_{\mathbb{X}})$, we obtain a special formal O_D-module $(\mathbb{X}', \iota_{\mathbb{X}'})$ that we use as a framing object for the Drinfeld functor $\mathcal{M}_{r^\circ}$. Since the above construction is functorial in S, we obtain a morphism of formal schemes over $\mathrm{Spf}\, O_{\breve{F}}$,

$$\mathcal{M}_r \longrightarrow \mathcal{M}_{r^\circ}\,. \tag{7.6}$$

Theorem 7.3. *The morphism (7.6) is an isomorphism. In particular, there is an isomorphism of formal schemes over* $\mathrm{Spf}\, O_{\breve{F}}$*,*

$$\mathcal{M}_r \simeq \hat{\Omega}^n_F \hat{\otimes}_{O_F} O_{\breve{F}}\,.$$

Proof. Let R be a κ_F-algebra. We have (compare (4.11)) described a functor $\mathcal{P} \mapsto \mathcal{P}'$ which associates to the display of a formal O_D-module with condition $(\mathbf{D}_r)$ the display of a special formal O_D-module with condition $(\mathbf{D}_{r^\circ})$. In the unramified case we take $\pi = p$. Since $a_\psi = 0$ or 1, we see that

$$\dot{F}'_\psi = \dot{F}_\psi, \text{ or } \dot{F}'_\psi = p\dot{F}_\psi = F_\psi.$$

Therefore starting from $\mathcal{P}$ or $\mathcal{P}'$ the construction (7.4) yields the same d-display. Therefore over κ_F the functor morphism (7.6) factors

$$\mathcal{M}_r \otimes_{O_F} \kappa_F \longrightarrow \mathcal{M}_{r^\circ} \otimes_{O_F} \kappa_F \longrightarrow \mathcal{M}_{r^\circ} \otimes_{O_F} \kappa_F.$$

The first morphism is given by Theorem 4.12 and the second morphism is defined by associating to a display of a formal O_D-module satisfying $(\mathbf{D}_{r^\circ})$ the display relative to O_F. Therefore the last morphism is the identity. Since the first arrow is an isomorphism by Theorem 4.12, we deduce that (7.6) is an isomorphism over κ_F. But since both schemes are flat by Proposition 5.8, it is an isomorphism. ∎

The only thing we use of Ahsendorf's work is: to each relative display $\mathcal{P}'$ there is a formal group $BT(\mathcal{P}')$. Let X be a formal O_D-module satisfying $(\mathbf{D}_{r^\circ})$. Let $\mathcal{P}$ be the display of X. Let $\mathcal{P}'$ be the relative display given by construction (7.5). Then $BT(\mathcal{P}')$ is canonically isomorphic to X.

8. The Lubin-Tate moduli problem

In this section we sketch that a modification of our method is applicable to a variant of the Lubin-Tate moduli problem. Let again F be an extension of degree d of $\mathbb{Q}_p$. We again fix an embedding $\varphi_0 : F \longrightarrow \bar{\mathbb{Q}}_p$. We also fix an integer $n \geq 2$, and function $r : \Phi \longrightarrow \mathbb{Z}_{\geq 0}$ with the same properties as in (2.1). Let E be the corresponding reflex field.

Let S be an O_E-scheme. Let $(\mathcal{L}, \iota)$ be a locally free $\mathcal{O}_S$-module of finite rank with an action by O_F. We say that $(\mathcal{L}, \iota)$ is an *O_F-module over S*.

Let $\tilde{E}$ be the normal closure of E. Then each $\varphi \in \Phi$ factors through $\varphi : O_F \to O_{\tilde{E}}$. The induced homomorphism $O_F \otimes_{\mathbb{Z}_p} O_{\tilde{E}} \to O_{\tilde{E}}$ defines a map

$$\mathrm{Nm}_\varphi : \mathbb{V}(O_F)_{O_{\tilde{E}}} \longrightarrow \mathbb{A}^1_{O_{\tilde{E}}}.$$

We set $\mathrm{Nm}_r = \prod_\varphi \mathrm{Nm}_\varphi^{r_\varphi}$, comp. (3.6). This is a polynomial function defined over O_E,

$$\mathrm{Nm}_r : \mathbb{V}(O_F)_{O_E} \longrightarrow \mathbb{A}^1_{O_E}.$$

We use the notations Ψ, a_ψ from (2.3). For an O_E-module $\mathcal{L}$ we have a natural decomposition

$$\mathcal{L} = \bigoplus_{\psi \in \Psi} \mathcal{L}_\psi.$$

We say that $(\mathcal{L}, \iota)$ satisfies the rank condition $(\mathbf{R}_r^F)$ if

$$\text{(8.1)} \qquad \mathrm{rank}_{\mathcal{O}_S} \mathcal{L}_{\psi_0} = a_{\psi_0} n + 1, \quad \mathrm{rank}_{\mathcal{O}_S} \mathcal{L}_\psi = a_\psi n, \ \text{for}\, \psi \neq \psi_0.$$

This implies $\mathrm{rank}_{\mathcal{O}_S} \mathcal{L} = \sum_{\varphi \in \Phi} r_\varphi$. With the notations (2.8) we introduce the Eisenstein conditions $(\mathbf{E}_r^F)$ (compare Definition 3.8):

$$\text{(8.2)} \qquad \begin{aligned} ((Q_0 \cdot Q_{A_{\psi_0}})(\iota(\pi))|\mathcal{L}_{\psi_0}) &= 0, \\ \bigwedge\nolimits^2 (Q_{A_{\psi_0}}(\iota(\pi)|\mathcal{L}_{\psi_0})) &= 0, \\ (Q_{A_\psi}(\iota(\pi))|\mathcal{L}_\psi) &= 0, \quad \text{for}\ \psi \neq \psi_0. \end{aligned}$$

We say that $\mathcal{L}$ satisfies $(\mathbf{LT}_r^F)$ if $(\mathbf{R}_r^F)$ and $(\mathbf{E}_r^F)$ are satisfied.

We say that an O_F-module $(\mathcal{L}, \iota)$ over an O_E-scheme S satisfies the *Kottwitz condition* $(\mathbf{K}_r^F)$ if

$$\mathrm{Nm}_{\mathcal{L}} = \mathrm{Nm}_r\,, \tag{8.3}$$

an equality of two morphisms from $\mathbb{V}(O_F) \times_{\mathrm{Spec}\,\mathbb{Z}_p} S$ to $\mathbb{A}^1_S$. The condition is equivalent to the identity of polynomials with coefficients in $\mathcal{O}_S$,

$$\mathrm{char}\,(\iota(x) \mid \mathcal{L}) = \prod\nolimits_{\varphi} (T - \varphi(x))^{r_\varphi}\,,\ \forall x \in O_F\,, \tag{8.4}$$

cf. Remark 3.2.

We will consider p-divisible groups X of height nd over O_E-schemes S with an action $\iota : O_F \longrightarrow \mathrm{End}(X)$. We assume that the action on $\mathrm{Lie}\,X$ satisfies the condition $(\mathbf{LT}_r^F)$. We will also say that X is of *type* r.

Let us assume that $S = \mathrm{Spec}\,k$ is the spectrum of a perfect field of characteristic p. Let M be the Dieudonné module of X. We obtain a decomposition

$$M = \bigoplus_{\psi \in \Psi} M_\psi,$$

where M_ψ is a free $W(k)$-module of rank $n[F : F^t]$. By the condition $(\mathbf{LT}_r^F)$ we find (compare Proposition 4.6):

$$\begin{aligned} \pi^{a_{\psi_0}+1} M_{\psi_0} \subset V M_{\sigma\psi_0} &\subset \pi^{a_{\psi_0}} M_{\psi_0}, \\ V M_{\sigma\psi} &= \pi^{a_\psi} M_\psi, \qquad\qquad \text{for } \psi \neq \psi_0. \end{aligned}$$

If we replace (M, F, V) by $(M, \oplus\pi^{a_\psi} F, \oplus\pi^{-a_\psi} V)$, we obtain a Dieudonné module of a p-divisible group Y of type r° for the Drinfeld function r°. We see that X is a formal p-divisible group if and only if Y is. Then (Y, ι) is of Lubin-Tate type, cf [5], i.e., Y is a strict formal O_F-module of dimension one and O_F-height n. We conclude that over an algebraically closed field $\bar{k}$ there is up to isomorphism a unique formal p-divisible group (X, ι) of type r.

In general we say that (X, ι) over a scheme S is a *Lubin-Tate group of type* r if it is a formal p-divisible group of height nd such that the action of O_F on $\mathrm{Lie}\,X$ satisfies $(\mathbf{LT}_r^F)$.

Remark 8.1. In the case $r = r^\circ$, the Eisenstein conditions $(\mathbf{E}_r^F)$ in $(\mathbf{LT}_r^F)$ are redundant. This follows from the flatness of the *naive local model* in [17], in the case where $d = n, r = 1$ in the notation of loc. cit. But it is also easy to see this directly.

From the case $r = r^\circ$ we deduce:

Proposition 8.2. *Any Lubin-Tate group (X, ι) of type r over $\bar{k}$ is isoclinic. Any two Lubin-Tate groups of type r over $\bar{k}$ are isomorphic. Any O_F-linear quasi-isogeny of height zero between Lubin-Tate groups of type r is an isomorphism.* □

More generally the proof of Theorem 4.12 shows the following fact.

Proposition 8.3. *Let S be a $\bar{k}$-scheme. Then there is an equivalence between the category of Lubin-Tate groups of type r and the category of Lubin-Tate groups of type r°.* □

We fix a Lubin-Tate group $(\mathbb{X}, \iota_{\mathbb{X}})$ of type r over the residue class field $\bar{k}$ of $\bar{\mathbb{Q}}_p$. It is unique up to isomorphism. We may therefore define a functor $\mathcal{M}_r^F$ on $\mathrm{Nilp}_{O_{\breve{E}}}$. Namely, fixing a framing object $(\mathbb{X}, \iota_{\mathbb{X}})$ over $\bar{k}$, $\mathcal{M}_r^F$ associates to $S \in \mathrm{Nilp}_{O_{\breve{E}}}$ the set of isomorphism classes of triples (X, ι, ρ) where (X, ι) is a Lubin-Tate group of type r over S and $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, \bar{k}} \bar{S}$ is an O_F-linear quasi-isogeny of height zero. The formal scheme representing this functor will be denoted by the same symbol.

The main theorem in this section is the following.

Theorem 8.4. *The formal scheme $\mathcal{M}_r^F$ is isomorphic to the formal spectrum of $O_{\breve{E}}[[t_1, \ldots, t_{n-1}]]$. A Lubin-Tate group of type r satisfies the Kottwitz condition $(\mathbf{K}_r^F)$.*

We will show now how this follows from the results of previous sections. The Kottwitz condition follows as in the proof of Proposition 4.5. We first note the following consequence of Corollary 8.2.

Corollary 8.5. *$\mathcal{M}_r^F(\bar{k}) = \{\mathrm{pt}\}$, hence $\mathcal{M}_r^F$ is the formal spectrum of a local $O_{\breve{E}}$-algebra R w.r.t. its maximal ideal.* □

In order to complete the proof of Theorem 8.4, it remains to show that $\mathcal{M}_r^F$ is formally smooth of relative dimension $n-1$ over $O_{\breve{E}}$. This will follow from the theory of local models.

Let V be a F-vector space of dimension n and Λ an O_F-lattice in V. Let $\mathbb{M}_r^F$ be the functor on $(\mathrm{Sch}\,/O_E)$ such that

$$\mathbb{M}_r^F(S) = \{\mathcal{F} \subset \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \mid O_F \text{ -stable direct summand such that } (\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S)/\mathcal{F} \text{ satisfies } (\mathbf{R}_r^F) \text{ and } (\mathbf{E}_r^F)\}.$$

Then R is isomorphic to the completion of $\mathbb{M}_r^F$ at a point of $\mathbb{M}_r^F(\bar{k})$. Hence Theorem 8.4 follows from the next lemma.

Lemma 8.6. *$\mathbb{M}_r^F$ is smooth of relative dimension $n-1$ over O_E. In fact, $\mathbb{M}_r^F \otimes_{O_E} \bar{k} \simeq \mathbb{P}_{\bar{k}}^{n-1}$.*

Proof. The last statement implies the first. Indeed, then the general and the special fibre are smooth and irreducible of the same dimension $n-1$. The flatness of $\mathbb{M}_r^F$ follows as in Corollary 5.4. This proves smoothness.

To study the geometric special fiber $\overline{\mathbb{M}^F} = \mathbb{M}^F \otimes_{O_E} \bar{k}$, we follow the method of section 5. Let $W_0 = \Lambda \otimes_{O_{F^t},\psi_0} \bar{k}$. Let $a_0 = |A_{\psi_0}|$. Then, as in (5.2), we have an identification

$$\overline{\mathbb{M}^F}(S) = \{\mathcal{F}_0 \subset W_0 \otimes_{\bar{k}} \mathcal{O}_S \mid \pi\text{-stable direct summand}, \\ \text{rank } W_{0,S}/\mathcal{F}_0 = a_0 n + 1, \text{ and } 1'), 2')\}$$

Here we have set $W_{0,S} = W_0 \otimes_{\bar{k}} \mathcal{O}_S$ and 1′) and 2′) are as in (5.2), i.e.,

$$\begin{aligned} 1')\quad & \pi^{(a_0+1)}|(W_{0,S}/\mathcal{F}_0) &&= 0 \\ 2')\ & \wedge^2(\pi^{a_0}|(W_{0,S}/\mathcal{F}_0)) &&= 0. \end{aligned} \tag{8.5}$$

Applying Lemma 4.10, we obtain that for $\mathcal{F}_0 \in \overline{\mathbb{M}^F}(S)$, there is a chain of inclusions,

$$W_0^{a_0+1} \otimes_{\bar{k}} \mathcal{O}_S \subset \mathcal{F}_0 \subset W_0^{a_0} \otimes_{\bar{k}} \mathcal{O}_S\,,$$

where $W_0^{a_0} = \operatorname{Im}\pi^{a_0}$, resp. $W_0^{a_0+1} = \operatorname{Im}\pi^{a_0+1}$, is a $\bar{k}$-subspace of dimension $(e-a_0)n$, resp. $(e-a_0-1)n$ of W_0. Associating now to $\mathcal{F}_0$ the submodule $\mathcal{F}_0/(W_0^{a_0+1} \otimes_{\bar{k}} \mathcal{O}_S)$ of $(W_0^{a_0}/W_0^{a_0+1}) \otimes_{\bar{k}} \mathcal{O}_S$, we obtain a direct summand of codimension one, i.e., an S-valued point of $\mathbb{P}(W_0^{a_0}/W_0^{a_0+1})$. Since this association is functorial and bijective, the last assertion of Lemma 8.6 follows. □

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