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Patching and the p-adic local Langlands correspondence*

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We use the patching method of Taylor–Wiles and Kisin to construct a candidate for the p-adic local Langlands correspondence for $\mathrm{GL}_n(F)$, F a finite extension of $\mathbb{Q}_p$. We use our construction to prove many new cases of the Breuil–Schneider conjecture.

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1. Introduction

Our goal in this paper is to use global methods (specifically, the Taylor–Wiles–Kisin patching method) to construct a candidate for the p-adic local

Langlands correspondence for $\mathrm{GL}_n(F)$, where F is an arbitrary finite extension of $\mathbb{Q}_p$, and $p \nmid 2n$. At present, the existence of such a correspondence is only known for $\mathrm{GL}_1(F)$ (where it is given by local class field theory), and for $\mathrm{GL}_2(\mathbb{Q}_p)$ (*cf.* [Col10; Paš13]). We do not prove that our construction gives a purely local correspondence (and it would perhaps be premature to conjecture that it should), but we are able to say enough about our construction to prove many new cases of the Breuil–Schneider conjecture, and to reduce the general case of the Breuil–Schneider conjecture (under some mild technical hypotheses) to standard conjectures related to automorphy lifting theorems.

The idea that global methods could be used to construct the correspondence is a natural one; the only proofs at present of the classical local Langlands correspondence [HT01; Hen00; Sch13] are by global means, and indeed the first proofs of local class field theory were global. The basic idea is to embed a local situation into a global one, apply a global correspondence (for example, the association of Galois representations to certain automorphic forms), and then to prove that the construction is independent of the choice of global situation. In this paper, we carry out the first half of this idea (although, in contrast to the constructions of [HT01], the direction of the correspondence we construct is from representations of the local Galois group G_F to representations of $\mathrm{GL}_n(F)$). We intend to return to the second half (investigating the question of independence of the global situation) in subsequent work. (See Section 6 for a discussion of the relationship of our construction to conjectural extensions of the existing p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.)

1.1. A candidate for p-adic local Langlands

Let $p \nmid 2n$ be prime, and let $\mathbb{F}$ be the residue field of the ring of integers $\mathcal{O}$ in some fixed field of coefficients E, a finite extension of $\mathbb{Q}_p$. The main construction of the paper associates to any representation $\bar{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$ a compact, torsion free $\mathcal{O}$-module M_∞ equipped with commuting actions of $\mathrm{GL}_n(F)$ and of a local $\mathcal{O}$-algebra R_∞, which is a formal power series ring over the universal lifting $\mathcal{O}$-algebra $R_{\bar{r}}^{\square}$.

If $r : G_F \to \mathrm{GL}_n(E)$ is a continuous lifting of $\bar{r}$ (with respect to some suitable integral structure on r), and $y : R_\infty \to \mathcal{O}$ is a homomorphism compatible with the homomorphism $x : R_{\bar{r}}^{\square} \to E$ arising from (an appropriate choice of integral structure on) r, then we define $V(r) := (M_\infty \otimes_{R_\infty, y} \mathcal{O})^d[1/p]$ (where d denotes the Schikhof dual); this is a continuous, unitary E-Banach space representation of $\mathrm{GL}_n(F)$, which we show is furthermore admissible (Proposition 2.13 below).[1]

This construction is the key point of this paper, and the most optimistic of us expect that $r \mapsto V(r)$ will realize the p-adic Langlands correspondence for $\mathrm{GL}_n(F)$. At the moment we do not have enough control over the representations $V(r)$ for an arbitrary Galois representation r to say anything definitive in this direction in general. (Indeed, *a priori*, $V(r)$ depends not just on r, but on the choices of x and of y; furthermore, it is not evidently nonzero.) However, if r satisfies the assumptions of Theorem B below, namely is potentially crystalline and generic with regular Hodge–Tate weights, then we can show (Theorem 4.35) that the subspace of locally algebraic vectors in $V(r)$ is isomorphic to the locally algebraic representation $\mathrm{BS}(r)$ associated to r by Breuil and Schneider [BS07]. This result is very much in the spirit of a Langlands correspondence, as $\mathrm{BS}(r)$ is a tensor product of an algebraic representation, which encodes the information about the Hodge–Tate weights of r, and a smooth representation, which corresponds to the Weil–Deligne representation of r by the classical Langlands correspondence.

1.2. The Breuil–Schneider conjecture

Before giving more details of our construction, we discuss an application of it to the Breuil–Schneider conjecture [BS07]; this conjecture predicts that locally algebraic representations of $\mathrm{GL}_n(F)$ admit invariant norms (and thus nonzero completions to unitary Banach representations) if and only if they arise from regular de Rham Galois representations by applying (a generic version of) the classical local Langlands correspondence to the corresponding Weil–Deligne representations. (This conjecture is motivated by the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, where it is an immediate consequence of known properties of the p-adic local Langlands correspondence.) In one direction, Hu [Hu09] showed that if such a norm exists, the locally algebraic representation necessarily comes from a regular de Rham representation.

The converse direction is largely open. We recall the conjecture in more detail in Section 5 below, to which the reader should refer for any unfamiliar notation or terminology. As we remarked above, given a de Rham representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ of regular weight, in [BS07] there is associated to r a locally algebraic $\overline{\mathbb{Q}}_p$-representation $\mathrm{BS}(r)$ of $\mathrm{GL}_n(F)$. The following is [BS07, Conjecture 4.3] (in the open direction).

[1] In the definition of $V(r)$ given there, certain additional restrictions are placed on the choice of extension of x to y; we suppress this technical point in the present discussion.

A Conjecture. *If* $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ *is de Rham and has regular weight, then* $\mathrm{BS}(r)$ *admits a nonzero unitary Banach completion.*

In fact, the known properties of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ suggest that there should even be a nonzero admissible completion. Conjecture A was proved in the case that $\pi_{\mathrm{sm}}(r)$ is supercuspidal in [BS07, Theorem 5.2], and in the more general case that $\mathrm{WD}(r)$ is indecomposable in [Sor13]. (The notation is defined in §1.8). The argument of [Sor13] is global. It makes use of a strategy of one of us (M.E.) who observed that if r arises as the local Galois representation coming from an automorphic representation, then one can obtain an admissible completion from the completed cohomology of [Eme06b], *cf.* Proposition 4.6 of [Eme05] (and also [Sor15]). However, as there are only countably many automorphic representations, it is not possible to say anything about most principal series representations in this way; indeed, as was already remarked in [BS07] (see the discussion before Remark 5.7), the principal series case seems to be the deepest case of the conjecture.

Other than for $\mathrm{GL}_2(\mathbb{Q}_p)$, the only previous results in the general principal series case that we are aware of are those of [AKS13; Ies12; KS12; Vig08], which prove the conjecture for certain principal series cases for $\mathrm{GL}_2(F)$, under additional restrictions on the Hodge filtration of r. The methods of these papers do not seem to shed any light on the stronger question of the existence of admissible completions. Under the assumption that $p \nmid 2n$, which we make from now on, we have associated an admissible unitary Banach representation $V(r)$ to any continuous representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$. In order to prove Conjecture A, it would be enough to establish that $V(r)$ contains a copy of $\mathrm{BS}(r)$ when r is de Rham of regular weight. We expect this to be true in general, and we are able to show that it is equivalent to proving a certain automorphy lifting theorem. The following is our main result in this direction. (See sections 2 and 5 for any unfamiliar terminology; note in particular that the hypothesis that r lies on an automorphic component does not imply that r arises from the Galois representation associated to an automorphic representation, but is rather the much weaker condition that it lies on the same component of a local deformation ring as some such representation. It is a folklore conjecture (closely related to the problem of deducing the Fontaine–Mazur conjecture from generalisations of Serre's conjecture via automorphy lifting theorems) that every de Rham representation of regular weight satisfies this condition. Note also that we call a potentially crystalline representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ *generic* if the smooth representation of $\mathrm{GL}_n(F)$ corresponding via the classical local Langlands correspondence to the Weil–Deligne representation underlying r

is generic, i.e. admits a Whittaker model; see Section 2.3 of [Kud94] for more details on this notion.)

B Theorem (Theorem 5.3 and Remark 4.20)**.** *Suppose that $p \nmid 2n$, that $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is potentially crystalline of regular weight, and that r is generic. Suppose further that r lies on an automorphic component of the corresponding potentially crystalline deformation ring. Then* $\mathrm{BS}(r)$ *admits a nonzero unitary admissible Banach completion.*

By taking known automorphy lifting theorems, in particular those proved in [BLGGT14], we are able to deduce new cases of Conjecture A. In particular, we deduce the following result (Corollary 5.5).

C Theorem. *Suppose that $p > 2$, that $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is de Rham of regular weight, and that r is generic. Suppose further that either*

1. *$n = 2$, and r is potentially Barsotti–Tate, or*
2. *$F/\mathbb{Q}_p$ is unramified and r is crystalline with Hodge–Tate weights in the extended Fontaine–Laffaille range, and $n \neq p$.*

Then $\mathrm{BS}(r)$ *admits a nonzero unitary admissible Banach completion.*

Actually, we prove a more general result (Corollary 5.4) which establishes the conjecture for potentially diagonalisable representations; conjecturally, every potentially crystalline representation is potentially diagonalisable. We remark that while we expect these results to extend to potentially semistable (rather than just potentially crystalline) representations, and to non-generic representations, we have restricted to the potentially crystalline case for two reasons: we can use the main theorems of [BLGGT14] without modification, and we do not have to consider issues related to the possible reducibility of $\mathrm{BS}(r)$.

1.3. The patching construction

In the proof of the classical local Langlands correspondence [HT01; Hen00], the globalisation argument uses a reduction to the supercuspidal case (via the classification of irreducible smooth representations of $\mathrm{GL}_n(F)$ given in [BZ77; Zel80]), and then uses trace formula methods to realise supercuspidal representations as the local components of cuspidal automorphic representations. No such argument is possible in our setting; there are only countably many automorphic representations, but already for $\mathrm{GL}_2(\mathbb{Q}_p)$ there are uncountably many irreducible p-adic Galois representations (even up to twist).

It is natural to hope that in the p-adic setting, one could carry out an analogous globalisation using "p-adic automorphic representations", such as those arising from the completed cohomology of [Eme06b]. However, since the locally algebraic vectors in completed cohomology are computed by classical automorphic representations, one cannot expect to see any regular de Rham Galois representations in completed cohomology other than those arising from classical automorphic representations.

The globalisation argument in the proof of classical local Langlands is effectively a result showing the Zariski-density of automorphic points in the Bernstein spectrum; the analogous result for p-adic local Langlands (or rather, for the part of it pertaining to regular de Rham representations) would be a Zariski-density result for automorphic points in the corresponding local Galois deformation rings. This is not known in general, but strong results in this direction follow from the Taylor–Wiles–Kisin patching method, which provides a Zariski-density result for a non-empty collection of components of a local deformation ring (and in general shows that each component either contains no automorphic points, or a Zariski-dense set of points; as mentioned above, the problem of showing that each component contains an automorphic point is closely related to the problem of deducing the Fontaine–Mazur conjecture from generalisations of Serre's conjecture, *cf.* Remark 5.5.3 of [EG14]).

The Taylor–Wiles–Kisin method patches together spaces of automorphic forms with varying tame level. Traditionally, the weight and the p-part of the level of these forms is fixed, and one obtains a patched module for a certain universal local deformation ring corresponding to de Rham representations of fixed Hodge–Tate weights, and a fixed inertial type. In the present paper, we instead vary over all weights and levels at p, obtaining a module M_∞ over the unrestricted local deformation ring (with some power series variables adjoined). By construction, M_∞ naturally has an action of $\mathrm{GL}_n(\mathcal{O}_F)$; by keeping track of the action of the Hecke operators at p, we are able to promote this to an action of $\mathrm{GL}_n(F)$. Dualising the fibre of this patched module at the point corresponding to a particular Galois representation r, and inverting p, gives the unitary admissible Banach representation $V(r)$ that we seek. The condition that $p \nmid 2n$ is needed to employ the Taylor–Wiles–Kisin method (for example, this condition is necessary in order to be able to appeal to various results from [BLGGT14]), but we suspect that it is not ultimately needed to carry out variants of these constructions.

We do not know whether it is reasonable to expect that our construction is purely local, and thus defines a p-adic local Langlands correspondence; this amounts to the problem of showing that the patched modules that we construct are purely local objects. For some weak evidence in this direction, see [EGS15], which proves a related result for lattices corresponding to certain 2-dimensional tamely potentially Barsotti–Tate representations. It can also be shown that our construction recovers the known correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, without needing to use the full strength of the p-adic local Langlands correspondence. We are currently writing a paper which will explain this and for which the main steps are: showing that our module M_∞ is projective as a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation, computing its cosocle via the weight part of Serre's conjecture, and exploiting the density of crystabelline points in local deformation rings. Finally, we refer to Section 6 below for a slightly more detailed discussion of how the construction of this paper might relate to a hypothetical p-adic local Langlands correspondence in the general case of $\mathrm{GL}_n(F)$.

1.4. Inertial local Langlands, the Bernstein centre, and local-global compatibility

In Sections 3 and 4 we relate the theory of the Bernstein centre and the so-called inertial local Langlands correspondence to the theory of potentially crystalline deformation rings, and the Taylor–Wiles–Kisin patched modules which lie over them.

More precisely, in Section 3 we synthesise and expand on results of Bernstein and Bernstein–Zelevinsky, Bushnell–Kutzko, Schneider–Zink, and Dat, to draw the following conclusions: for any *inertial type* τ (i.e. a representation $\tau : I_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ with open kernel which extends to the Weil group of F), there is an associated *smooth type* $\sigma(\tau)$, which is a smooth $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_n(\mathcal{O}_F)$, with the following properties:

- (i) The Hecke algebra $\mathcal{H}\big(\sigma(\tau)\big) := \mathrm{End}\big(\text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}\sigma(\tau)\big)$ is commutative.
- (ii) $\sigma(\tau)$ appears as a $\mathrm{GL}_n(\mathcal{O}_F)$-subrepresentation of an irreducible smooth $\mathrm{GL}_n(F)$-representation π if and only if the Weil–Deligne representation attached to π via the local Langlands correspondence is isomorphic to τ when restricted to I_F, and in addition satisfies $N = 0$. Furthermore, for such π, the representation $\sigma(\tau)$ appears in π with multiplicity one.

Suppose that π is an irreducible smooth $\mathrm{GL}_n(F)$-representation whose associated Weil–Deligne representation is isomorphic to τ when restricted to I_F, and in addition satisfies $N = 0$, so that $\sigma(\tau)$ appears with multiplicity one in π, by (ii) above. By Frobenius reciprocity, the Hecke algebra

$\mathcal{H}(\sigma(\tau))$ then acts on the associated one-dimensional multiplicity space via a character $\chi_\pi : \mathcal{H}(\sigma(\tau)) \to \overline{\mathbb{Q}}_p$, and hence there is an induced surjection $\text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)} \sigma(\tau) \otimes_{\mathcal{H}(\sigma(\tau)),\chi_\pi} \overline{\mathbb{Q}}_p \to \pi$. In this context, we establish one further result.

(iii) If π is generic then the preceding surjection is an isomorphism.

From these results, we deduce that the connected components of the Spec of the Bernstein centre for $\mathrm{GL}_n(F)$ are identified with the various $\operatorname{Spec} \mathcal{H}(\sigma(\tau))$, as τ ranges over all (isomorphism classes of) inertial types. Over any such component we have the universal $\mathrm{GL}_n(F)$-representation $\text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)} \sigma(\tau)$, whose fibre over the point χ_π arising from a generic π whose associated Weil–Deligne representation satisfies $N = 0$ is isomorphic to the representation π.

For both the comparison with Galois deformation rings that we make in Section 4, and for the connections that we draw with the theory of Banach space representations of $\mathrm{GL}_n(F)$, it is technically important to work over a finite extension of $\mathbb{Q}_p$ rather than over $\overline{\mathbb{Q}}_p$, so we also explain how to descend the preceding results from $\overline{\mathbb{Q}}_p$ to such finite extensions.

In Section 4, we consider so-called *locally algebraic types*, which are representations of $\mathrm{GL}_n(\mathcal{O}_F)$, defined over some finite extension E of $\overline{\mathbb{Q}}_p$, of the form $\sigma_{\mathrm{sm}} \otimes \sigma_{\mathrm{alg}}$, where σ_{sm} is the smooth type attached to some inertial type τ, and σ_{alg} is an irreducible algebraic representation of $\operatorname{Res}_{F/\mathbb{Q}_p} \mathrm{GL}_n$. Attached to σ we have a Hecke algebra $\mathcal{H}(\sigma) := \operatorname{End}(\text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)} \sigma)$; it is isomorphic to $\mathcal{H}(\sigma_{\mathrm{sm}})$. Attached to σ and any continuous representation $\bar{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$ as above, there is a universal lifting ring $R_{\bar{r}}^{\square}(\sigma)$, parameterising potentially crystalline lifts of $\bar{r}$ whose associated inertial type coincides with τ, and whose Hodge–Tate weights match with the highest weight of σ_{alg} after applying the usual ρ-shift.

One of the main results of Section 4, which may be of independent interest, is the existence of a homomorphism $\eta : \mathcal{H}(\sigma) \to R_{\bar{r}}(\sigma)^{\square}[1/p]$ which interpolates the local Langlands correspondence. (This gives an algebraic extension of an analogous rigid-analytic result proved in [Che09]). Namely, if $x : R_{\bar{r}}(\sigma)^{\square}[1/p] \to \overline{\mathbb{Q}}_p$ corresponds to a crystalline lift r_x of $\bar{r}$, if π is the irreducible smooth representation of $\mathrm{GL}_n(F)$ associated to the Weil–Deligne representation underlying r_x via the local Langlands correspondence, and if $\chi_\pi : \mathcal{H}(\sigma) \cong \mathcal{H}(\sigma_{\mathrm{sm}}) \to \overline{\mathbb{Q}}_p$ is the character of $\mathcal{H}(\sigma)$ associated to π, then we have the equality $\chi_\pi = x \circ \eta$.

The second main result of this section is a key reciprocity law related to the Hecke action on locally algebraic vectors in M_∞, which we refer to as *local-global compatibility* (in analogy with the classical local-global compatibility results for cohomology of Shimura varieties [Car14]). Given a locally algebraic type σ, we may form the R_∞-module

$$M_\infty(\sigma) := \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{GL}_n(\mathcal{O}_F)}(M_\infty, \sigma^*)^*,$$

where * denotes continuous dual. We show that the $R^\square_{\bar{r}}$-action on this module factors through $R^\square_{\bar{r}}(\sigma)$. It is thus equipped with two natural $\mathcal{H}(\sigma)$-actions: one via its very definition together with Frobenius reciprocity (and so related to the structure of M_∞ as a $\mathrm{GL}_n(F)$-representation), and one via the homomorphism η (and so related to the structure of M_∞ as an $R^\square_{\bar{r}}$-module); local-global compatibility is the statement that these two actions coincide.

This reciprocity law is crucial to our analysis of the locally algebraic vectors in the representations $V(r)$, and to our study of the Breuil–Schneider conjecture.

1.5. The relationship with [Sch15]

We briefly discuss how our work relates to some other recent progress in the field. In [Sch15], Scholze provides more evidence for the existence of a purely local p-adic local Langlands correspondence for $\mathrm{GL}_n(F)$, by studying the cohomology of the Lubin–Tate tower. In the classical case [HT01], when $l \neq p$, it is known that the Lubin–Tate tower simultaneously realizes the local Langlands correspondence and the Jacquet-Langlands correspondence, between representations of $\mathrm{GL}_n(F)$ and representations of $D^\times$, where D/F is the central division algebra of invariant $1/n$. Scholze uses the Lubin–Tate tower to construct a purely local functor

$$\pi \mapsto F(\pi)$$

from admissible, smooth $\mathbb{F}_p$-representations of $\mathrm{GL}_n(F)$ to admissible representations of $D^\times$ equipped with an action of G_F. This functor goes in the opposite direction from our construction.

However, when $n = 2$, Scholze proves that it is compatible with our patching construction (Corollary 9.3 of [Sch15]), in the following sense. As in the case of unitary groups and $\mathrm{GL}_n(F)$, one can patch the cohomology of locally symmetric spaces coming from a quaternion algebra (which is split at p and ramified at all infinite places) to obtain a representation π_∞ of $\mathrm{GL}_2(F)$. One can also patch the cohomology of certain Shimura curves (corresponding to a quaternion algebra which is ramified at p, but split at one infinite place) to get a representation ρ_∞ of $D^\times \times G_F$. Then Scholze shows that

$$F(\pi_\infty) = \rho_\infty.$$

(In fact, Scholze employs a variant of the patching construction used in this paper, making use of ultrafilters to reduce the amount of bookkeeping needed to obtain the action of Hecke operators at p.) We remark that it should be possible to adapt his strategy to p-adic representations and to general n (using Shimura varieties of Harris–Taylor type for the latter step). It seems reasonable to expect that this will lead to a proof that one can recover the G_F-representation r from the Banach space $V(r)$ that we associate to it (at least in cases where it can be shown that $V(r)$ is nonzero; for example, this will be the case for the representations considered in Theorems B and C, where we even prove that the locally algebraic vectors in $V(r)$ are nonzero.)

1.6. Outline of the paper

In Section 2, we carry out our patching construction. Section 3 contains an introduction to the results of Bernstein–Zelevinsky, Bushnell–Kutzko, Schneider–Zink and Dat on types and the local Langlands correspondence for GL_n. We then refine some of these results, as described above, and explain how to descend them from algebraically closed coefficient fields to finite extensions of $\mathbb{Q}_p$. In Section 4 we begin by establishing our interpolation of the classical local Langlands correspondence over a (potentially crystalline) local deformation ring, and then apply this to establish local-global compatibility for our patched modules. Finally, in Section 5 we combine our local-global compatibility result with automorphy lifting theorems to prove our results on the Breuil–Schneider conjecture.

1.7. Acknowledgements

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1.8. Notation

We fix a prime p, and an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Throughout the paper we work with a finite extension $E/\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$, which will be our coefficient field. We write $\mathcal{O} = \mathcal{O}_E$ for the ring of integers in E, $\varpi = \varpi_E$ for a uniformiser, and $\mathbb{F} := \mathcal{O}/\varpi$ for the residue field. At any particular moment E is fixed, but we allow ourselves to modify E (typically via an extension of scalars) during the course of our arguments. Furthermore, we will often assume without further comment that E and $\mathbb{F}$ are sufficiently large, and in particular that if we are working with representations of the absolute Galois group of a p-adic field F, then the images of all embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$ are contained in E.

If F is a field, we let G_F denote its absolute Galois group. Let ε denote the p-adic cyclotomic character, and $\overline{\varepsilon}$ the mod p cyclotomic character. If F is a finite extension of $\mathbb{Q}_p$ for some p, we write I_F for the inertia subgroup of G_F, and ϖ_F for a uniformiser of the ring of integers $\mathcal{O}_F$ of F. If $\widetilde{F}$ is a number field and v is a finite place of $\widetilde{F}$ then we let Frob_v denote a geometric Frobenius element of $G_{\widetilde{F}_v}$.

If F is a p-adic field, W is a de Rham representation of G_F over E, and $\kappa : F \hookrightarrow E$, then we will write $\mathrm{HT}_\kappa(W)$ for the multiset of Hodge–Tate weights of W with respect to κ. By definition, the multiset $\mathrm{HT}_\kappa(W)$ contains i with multiplicity $\dim_E(W \otimes_{\kappa,F} \widehat{\overline{F}}(i))^{G_F}$. Thus for example $\mathrm{HT}_\kappa(\varepsilon) = \{-1\}$.

We say that W has *regular* Hodge–Tate weights if for each κ, the elements of $\mathrm{HT}_\kappa(W)$ are pairwise distinct. Let $\mathbb{Z}^n_+$ denote the set of tuples $(\xi_1, \dots, \xi_n)$ of integers with $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$. Then if W has regular Hodge–Tate weights, there is a $\xi = (\xi_{\kappa,i}) \in (\mathbb{Z}^n_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(F,E)}$ such that for each $\kappa : F \hookrightarrow E$,

$$\mathrm{HT}_\kappa(W) = \{\xi_{\kappa,1} + n - 1, \xi_{\kappa,2} + n - 2, \dots, \xi_{\kappa,n}\},$$

and we say that W *is regular of weight* ξ. For any $\xi \in \mathbb{Z}^n_+$, view ξ as a dominant weight (with respect to the upper triangular Borel subgroup) of the algebraic group GL_n in the usual way, and let M'_ξ be the algebraic $\mathcal{O}_F$-representation of GL_n given by

$$M'_\xi := \mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\xi)_{/\mathcal{O}_F}$$

where B_n is the Borel subgroup of upper-triangular matrices of GL_n, and w_0 is the longest element of the Weyl group (see [Jan03] for more details of these notions, and note that M'_ξ has highest weight ξ). Write M_ξ for the $\mathcal{O}_F$-representation of $\mathrm{GL}_n(\mathcal{O}_F)$ obtained by evaluating M'_ξ on $\mathcal{O}_F$. For any $\xi \in (\mathbb{Z}^n_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(F,E)}$ we write L_ξ for the $\mathcal{O}$-representation of $\mathrm{GL}_n(\mathcal{O}_F)$ defined by

$$L_\xi := \otimes_{\kappa:F\hookrightarrow E} M_{\xi_\kappa} \otimes_{\mathcal{O}_F,\kappa} \mathcal{O}.$$

If F is a p-adic field, then an *inertial type* is a representation $\tau : I_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ with open kernel which extends to the Weil group W_F.

Then we say that a de Rham representation $\rho : G_F \to \mathrm{GL}_n(E)$ has inertial type τ if the restriction to I_F of the Weil–Deligne representation $\mathrm{WD}(\rho)$ associated to ρ is equivalent to τ. Given an inertial type τ, there is a finite-dimensional smooth irreducible $\overline{\mathbb{Q}}_p$-representation $\sigma(\tau)$ of $\mathrm{GL}_n(\mathcal{O}_F)$ associated to τ by the "inertial local Langlands correspondence"; see Theorem 3.7 below. (Note that by the results of Section 3.13 below, we will be able to replace $\sigma(\tau)$ by a model defined over a finite extension of E in our main arguments.)

Let F be a finite extension of $\mathbb{Q}_p$, and let rec denote the local Langlands correspondence from isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(F)$ over $\mathbb{C}$ to isomorphism classes of n-dimensional Frobenius semisimple Weil–Deligne representations of W_F as in the introduction to [HT01]. Fix once and for all an isomorphism $\imath : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. We define the local Langlands correspondence rec_p over $\overline{\mathbb{Q}}_p$ by $\imath \circ \mathrm{rec}_p = \mathrm{rec} \circ \imath$. This depends only on $\imath^{-1}(\sqrt{p})$, and if we define $r_p(\pi) := \mathrm{rec}_p(\pi \otimes |\det|^{(1-n)/2})$, then r_p is independent of the choice of $\imath$. Furthermore, if V is a Frobenius semisimple Weil–Deligne representation of W_F over E, then $r_p^{-1}(V)$ is also defined over E by [Clo90, Prop 3.2] and the fact that r_p commutes with automorphisms of $\mathbb{C}$. (The claims about the dependence of rec_p and r_p on the choice of $\imath$ follow from the main theorem of [Hen93], together with a study of the behaviour of ε-factors under automorphisms of $\mathbb{C}$. In the case $n = 2$, this is explained in [BH06, §35], and the same argument goes through in general, with the required input on ε-factors being provided by [BH00, Thm. 3.2].[2])

Recall that a linear-topological $\mathcal{O}$-module is a topological $\mathcal{O}$-module (that is, it has a topology for which both addition and the action of $\mathcal{O}$ are continuous) which also has a fundamental system of open neighborhoods of

[2]Alternatively, perhaps more conceptually, the claims are also implied by the geometric realisation of r_p (up to dualising π) for supercuspidal representations in the cohomology of the Lubin-Tate tower. See Lemma VII.1.6 and the definition of rec_l on page 237 of [HT01].

the identity which are $\mathcal{O}$-submodules. If A is a linear-topological $\mathcal{O}$-module, we write $A^\vee$ for its Pontrjagin dual $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(A, E/\mathcal{O})$, where $E/\mathcal{O}$ has the discrete topology, and we give $A^\vee$ the compact open topology.

By the proof of Theorem 1.2 of [ST02], the functor given by $A \mapsto A^d := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(A, \mathcal{O})$ induces an anti-equivalence of categories between the category of compact, $\mathcal{O}$-torsion-free linear-topological $\mathcal{O}$-modules A and the category of ϖ-adically complete and separated $\mathcal{O}$-torsion-free $\mathcal{O}$-modules. A quasi-inverse is given by $B \mapsto B^d := \mathrm{Hom}_{\mathcal{O}}(B, \mathcal{O})$, where the target is given the weak topology of pointwise convergence. We refer to this duality as Schikhof duality. Note that if A is an $\mathcal{O}$-torsion free profinite linear-topological $\mathcal{O}$-module, then A^d is the unit ball in the E-Banach space $\mathrm{Hom}_{\mathcal{O}}(A, E)$.

If $r : G_F \to \mathrm{GL}_n(E)$ is de Rham of regular weight a, then we write $\pi_{\mathrm{alg}}(r) := L_a^d \otimes_{\mathcal{O}} E$, and $\pi_{\mathrm{sm}}(r) := r_p^{-1}(\mathrm{WD}(r)^{F-\mathrm{ss}})$, both of which are E-representations of $\mathrm{GL}_n(F)$. (The $\mathrm{GL}_n(\mathcal{O}_F)$-action on L_a^d extends linearly to a $\mathrm{GL}_n(F)$-action on $\pi_{\mathrm{alg}}(r)$.) As the names suggest, $\pi_{\mathrm{alg}}(r)$ is an algebraic representation, and $\pi_{\mathrm{sm}}(r)$ is a smooth representation. Note that $\pi_{\mathrm{alg}}(r) = L_\xi \otimes_{\mathcal{O}} E$ for $\xi_{\kappa,i} := -a_{\kappa,n+1-i}$.

We let $\mathrm{Art}_F : F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ be the isomorphism provided by local class field theory, which we normalise so that uniformisers correspond to geometric Frobenius elements.

We write all matrix transposes on the left; so ${}^t g$ is the transpose of g. We let $\mathcal{G}_n$ denote the group scheme over $\mathbb{Z}$ defined to be the semidirect product of $\mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, j\}$, which acts on $\mathrm{GL}_n \times \mathrm{GL}_1$ by

$$j(g, \mu)j^{-1} = (\mu \cdot {}^t g^{-1}, \mu).$$

We have a homomorphism $\nu : \mathcal{G}_n \to \mathrm{GL}_1$, sending (g, μ) to μ and j to -1.

Further notation is introduced in the course of our arguments; we mention just some of it here, for the reader's convenience.

From Subsection 2.8 on, we will have fixed a particular finite extension F of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$ and uniformiser ϖ_F. To ease notation we will typically write $G := \mathrm{GL}_n(F)$, $K := \mathrm{GL}_n(\mathcal{O}_F)$, and $Z := Z(G)$. For each $m \geq 0$, we write $\Gamma_m = \mathrm{GL}_n(\mathcal{O}_F/\varpi_F^m)$ and $K_m := \ker\big(\mathrm{GL}_n(\mathcal{O}_F) \to \mathrm{GL}_n(\mathcal{O}_F/\varpi_F^m)\big)$, so that $K/K_m \xrightarrow{\sim} \Gamma_m$.

Furthermore, throughout Section 2, a large amount of notation is introduced related to automorphic forms on a definite unitary group, and Taylor–Wiles–Kisin patching. Here we merely signal that the main construction of this section, and our major object of study in the paper, is a patched G-representation that we denote by M_∞. Beginning in Section 4, we will also write

$$M_\infty(\sigma^\circ) := \left(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ)^d)\right)^d,$$

when σ° is a K-invariant $\mathcal{O}_E$-lattice in a K-representation σ of finite dimension over E.

We use Ind to denote induction, and c-Ind to denote induction with compact supports. In Sections 3 and 4, we use i_P^G to denote normalised parabolic induction.

If σ is a representation of K, then we will write $\mathcal{H}(\sigma) := \mathrm{End}_G(\text{c-Ind}_K^G \sigma)$ to denote the Hecke algebra of G with respect to σ. Sometimes, when it is helpful to emphasise the role of G, we will write $\mathcal{H}(G, \sigma)$ instead. We also use obvious variants with G and K replaced by another p-adic group and compact open subgroup.

2. The patching argument

In this section we will carry out our patching argument on definite unitary groups. The key difference between the construction presented here and previous patching constructions is that the object we end up with is not simply a module over a certain Galois deformation ring, but rather a $\mathrm{GL}_n(F)$-representation over that ring; we refer to the discussion at the beginning of Subsection 2.8 below for a more detailed account of this difference.

Our construction uses the same general framework as that used in section 5 of [EG14] (which in turn is based on the approach of [CHT08], [BLGG11] and [Tho12]); we recall the key elements of this framework in the first several subsections that follow. We follow the notation of [EG14] as closely as possible, and we indicate explicitly where we deviate from it.

The construction itself is the subject of Subsection 2.8, and the key fact that it actually produces a $\mathrm{GL}_n(F)$-representation is verified in Proposition 2.10. In Subsection 2.12 we explain how our patched representation of $\mathrm{GL}_n(F)$ gives rise to admissible unitary Banach representations attached to local Galois representations.

2.1. Globalisation

Let $F/\mathbb{Q}_p$ be a finite extension, and fix a continuous representation $\bar{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$. Our goal in this subsection is to give a criterion for $\bar{r}$ to be obtained as the restriction of a global Galois representation that is automorphic, in a suitable sense (and satisfies some additional convenient properties).

We will assume that the following hypotheses are satisfied:

- $p \nmid 2n$, and
- $\bar{r}$ admits a potentially crystalline lift of regular weight, which is potentially diagonalisable.[3]

Conjecturally, the second hypothesis is always satisfied; this is Conjecture A.3 of [EG14]. In this direction, we note the following result.

2.2 Lemma. *After possibly making a finite extension of scalars, the second hypothesis is satisfied if either $n = 2$ or $\bar{r}$ is semisimple.*

Proof. If $n = 2$, this is Remark A.4 of [EG14]. If $\bar{r}$ is semisimple, then after extending scalars, we may write it as a sum of inductions of characters, and it is easy to see that by lifting these characters to crystalline characters, we can find a potentially crystalline lift which has regular weight, and is a sum of inductions of characters. Such a lift is obviously potentially diagonalisable (indeed, after restriction to some finite extension, it is a sum of crystalline characters). □

Having assumed these hypotheses, Corollary A.7 of [EG14] (with K our F, and F our $\widetilde{F}$) provides us with an imaginary CM field $\widetilde{F}$ with maximal totally real subfield $\widetilde{F}^+$, and a continuous irreducible representation $\overline{\rho} : G_{\widetilde{F}^+} \to \mathcal{G}_n(\mathbb{F})$ such that $\overline{\rho}$ is a *suitable globalisation* of $\bar{r}$ in the sense of Section 5.1 of [EG14]. Here we say that $\overline{\rho}$ is irreducible if $\overline{\rho}|_{G_{\widetilde{F}}}$, which is regarded as a representation valued in $\mathrm{GL}_n(\mathbb{F})$, is irreducible. We recall the properties that $(\widetilde{F}, \widetilde{F}^+, \overline{\rho})$ need to satisfy for this definition:

- each place $v \mid p$ of $\widetilde{F}^+$ splits in $\widetilde{F}$, and has $\widetilde{F}^+_v \cong F$; we fix a choice of such isomorphisms.

and

- $\overline{\rho}$ is automorphic (see, for example, Definition 5.3.1 of [EG14]) and unramified at primes $v \nmid p$.
- the inverse image of $\mathrm{GL}_n(\mathbb{F}) \times \mathrm{GL}_1(\mathbb{F})$ under $\overline{\rho}$ is $G_{\widetilde{F}}$.
- $\overline{\rho}(G_{\widetilde{F}(\zeta_p)})$ is adequate in the sense of Definition 2.3 of [Tho12].[4]

[3] Recall that, as in [BLGGT14], a potentially crystalline representation r of G_F is *potentially diagonalisable* if there exists a finite extension F'/F such that $r|_{G_{F'}}$ is crystalline and lies on the same irreducible component of the universal crystalline lifting ring of $\bar{r}|_{G_{F'}}$ (with fixed Hodge–Tate weights) as a sum of characters lifting $\bar{r}|_{G_{F'}}$.

[4] We will not need the precise definition of an *adequate subgroup* of $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$; we will only need to know that this property is satisfied in order to apply the machinery developed in [Tho12]. See Section 2.6 for more details.

- For each place $v \mid p$ of $\widetilde{F}^+$, there is a place $\widetilde{v}$ of $\widetilde{F}$ lying over v with $\overline{\rho}|_{G_{\widetilde{F}_{\widetilde{v}}}}$ isomorphic to $\bar{r}$.
- $\overline{\widetilde{F}}^{\ker \mathrm{ad}\, \overline{\rho}|_{G_{\widetilde{F}}}}$ does not contain $\widetilde{F}(\zeta_p)$.

Corollary A.7 of [EG14] guarantees that all these properties can be satisfied simultaneously. (The only difference is that the last property is replaced by the fact that $\overline{\widetilde{F}}^{\ker \overline{\rho}}$ does not contain $\widetilde{F}(\zeta_p)$, which is stronger, since $\ker \overline{\rho} \subset G_{\widetilde{F}}$ by the second property above.) In order to arrange that our patched modules have a certain multiplicity one property, we will also demand that:

- $\overline{\rho}(G_{\widetilde{F}}) = \mathrm{GL}_n(\mathbb{F}')$ for some subfield $\mathbb{F}' \subseteq \mathbb{F}$ with $\#\mathbb{F}' > 3n$.

To see that we can arrange this, note that the proof of Proposition A.2 of [EG14] (which is the main input to Corollary A.7 of *op. cit.*, together with the potential automorphy results of [BLGGT14]) allows us to arrange that $\overline{\rho}(G_{\widetilde{F}}) = \mathrm{GL}_n(\mathbb{F}_{p^m})$ for any sufficiently large m.

Finally, after making a solvable base change, we can and do assume that $\widetilde{F}/\widetilde{F}^+$ is unramified at all finite places.

2.3. Unitary groups

We now use the globalisation $\overline{\rho}$ of our local Galois representation $\bar{r}$ to carry out the Taylor–Wiles–Kisin patching argument as in Section 5 of [EG14]. The definitions of Hecke algebras, the choices of auxiliary primes and so on are essentially identical to the arguments made in [EG14], and rather than repeating them verbatim, we often refer the reader to [EG14] for the details of these definitions, indicating only the differences in our construction.

As in Sections 5.2 and 5.3 of [EG14], we fix a certain definite unitary group $\widetilde{G}/\widetilde{F}^+$ together with a model (which we will also denote by $\widetilde{G}$) over $\mathcal{O}_{\widetilde{F}^+}$. (The group $\widetilde{G}$ is denoted G in [EG14], but we will later use G to denote $\mathrm{GL}_n(F)$.) This model has the property that for each place v of $\widetilde{F}^+$ which splits as ww^c in $\widetilde{F}$, there is an isomorphism $\iota_w : \widetilde{G}(\mathcal{O}_{\widetilde{F}^+_v}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{\widetilde{F}_w})$; we fix a choice of such isomorphisms. We also choose a finite place v_1 of $\widetilde{F}^+$ which is prime to p, with the properties that

- v_1 splits in $\widetilde{F}$, say as $v_1 = \widetilde{v}_1\widetilde{v}_1^c$,
- v_1 does not split completely in $\widetilde{F}(\zeta_p)$, and
- $\overline{\rho}(\mathrm{Frob}_{\widetilde{F}_{\widetilde{v}_1}})$ has distinct $\mathbb{F}$-rational eigenvalues, no two of which have ratio $(\mathbf{N}v_1)^{\pm 1}$.

(It is possible to find such a place v_1 by the Cebotarev density theorem, and our assumptions that $\overline{\rho}(G_{\widetilde{F}}) = \mathrm{GL}_n(\mathbb{F}')$ with $\#\mathbb{F}' > 3n$, and that $\overline{\widetilde{F}}^{\ker \operatorname{ad} \overline{\rho}|_{G_{\widetilde{F}}}}$ does not contain $\widetilde{F}(\zeta_p)$. Indeed, choosing a conjugacy class in $\mathrm{Gal}(\widetilde{F}(\zeta_p)/\widetilde{F})$, we have a positive density set of places v_1 satisfying the first two conditions, and with $\mathbf{N}v_1$ taking a fixed value λ modulo p; if we then choose a diagonal matrix in $\mathrm{GL}_n(\mathbb{F}')$ with distinct diagonal entries, none of whose ratios are $\lambda^{\pm 1}$, then another application of the Cebotarev density theorem produces the required place v_1.

Note that this differs slightly from the choice of place v_1 in the first paragraph of Section 5.3 of [EG14], where the third condition is replaced by the requirement that $\operatorname{ad} \overline{\rho}|(\mathrm{Frob}_{\widetilde{F}_{\widetilde{v}_1}}) = 1$. However, it is still the case that any deformation of $\overline{\rho}|_{G_{\widetilde{F}_{\widetilde{v}_1}}}$ is unramified (see Lemma 2.5 below). We have made this choice in order to be able to arrange that our patched modules satisfy multiplicity one.

Let S_p denote the set of primes of $\widetilde{F}^+$ dividing p. We now fix a place $\mathfrak{p} \mid p$ of $\widetilde{F}^+$, and for each integer $m \geq 0$ we consider the compact open subgroup $U_m = \prod_v U_{m,v}$ of $\widetilde{G}(\mathbb{A}^\infty_{\widetilde{F}^+})$, where

- $U_{m,v} = \widetilde{G}(\mathcal{O}_{\widetilde{F}^+_v})$ for all v which split in $\widetilde{F}$ other than v_1 and $\mathfrak{p}$;
- U_{m,v_1} is the preimage of the upper triangular matrices under

$$\widetilde{G}(\mathcal{O}_{\widetilde{F}^+_{v_1}}) \to \widetilde{G}(k_{v_1}) \xrightarrow[\iota_{\widetilde{v}_1}]{\sim} \mathrm{GL}_n(k_{v_1})$$

- $U_{m,\mathfrak{p}}$ is the kernel of the map $\widetilde{G}(\mathcal{O}_{\widetilde{F}^+_{\mathfrak{p}}}) \to \widetilde{G}(\mathcal{O}_{\widetilde{F}^+_{\mathfrak{p}}}/\varpi^m_{\widetilde{F}^+_{\mathfrak{p}}})$;
- $U_{m,v}$ is a hyperspecial maximal compact subgroup of $\widetilde{G}(\widetilde{F}^+_v)$ if v is inert in $\widetilde{F}$.

By the choice of v_1 and U_{m,v_1} we see that U_m is sufficiently small (in the sense of Section 5.2 of [EG14]). Write $U := U_0$. In order to make the patching argument, we will need to consider certain compact open subgroups of the U_m corresponding to choices of sets of auxiliary primes Q_N that will be introduced in Section 2.6. Specifically, for each integer $N \geq 1$, we will have a finite set of primes Q_N of $\widetilde{F}^+$ disjoint from $S_p \cup \{v_1\}$ as well as open compact subgroups $U_i(Q_N)_v$ of $\widetilde{G}(\mathcal{O}_{\widetilde{F}^+_v})$ for each $v \in Q_N$ and $i = 0, 1$. We then define subgroups $U_i(Q_N)_m = \prod_v U_i(Q_N)_{m,v} \subset U_m$, for $i = 0, 1$ by setting $U_0(Q_N)_{m,v} = U_1(Q_N)_{m,v} = U_{m,v}$ for $v \notin Q_N$, and $U_i(Q_N)_{m,v} = U_i(Q_N)_v$ for $v \in Q_N$.

By assumption, $\bar{r}$ has a potentially diagonalisable lift of regular weight, say $r_{\mathrm{pot.diag}} : G_F \to \mathrm{GL}_n(\mathcal{O})$. Suppose that $r_{\mathrm{pot.diag}}$ has weight ξ and inertial type τ (in the sense of Section 1.8). Extending E if necessary, we may assume that the $\mathrm{GL}_n(\mathcal{O}_F)$-representation $\sigma(\tau)$ is defined over E. Then we have two representations L_ξ and $L_{\tau^\vee}$ of $\mathrm{GL}_n(\mathcal{O}_F)$ on finite free $\mathcal{O}$-modules in the following way: the representation L_ξ is the one defined in the notation section, and $L_{\tau^\vee}$ is a choice of $\mathrm{GL}_n(\mathcal{O}_F)$-stable lattice in $\sigma(\tau)^\vee$. Set $L_{\xi,\tau} := L_{\tau^\vee} \otimes_{\mathcal{O}} L_\xi$, a finite free $\mathcal{O}$-module with an action of $\mathrm{GL}_n(\mathcal{O}_F)$.

Returning to our global situation, let $W_{\xi,\tau}$ denote the finite free $\mathcal{O}$-module with an action of $\prod_{v\in S_p\setminus\{\mathfrak{p}\}} U_{m,v}$ given by $W_{\xi,\tau} = \otimes_{v\in S_p\setminus\{\mathfrak{p}\},\mathcal{O}} L_{\xi,\tau}$ where $U_{m,v}$ acts on the factor corresponding to v via $U_{m,v} = \widetilde{G}(\mathcal{O}_{\widetilde{F}_v^+}) \xrightarrow[\iota_{\tilde{v}}]{\sim} \mathrm{GL}_n(\mathcal{O}_{\widetilde{F}_{\tilde{v}}}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_F)$. In order to avoid duplication of definition, we allow $Q_N = \varnothing$ in the definitions we now make. For any finite $\mathcal{O}$-module V with a continuous action of $U_{m,\mathfrak{p}}$, we have spaces of algebraic modular forms $S_{\xi,\tau}(U_i(Q_N)_m, V)$; these are just the functions

$$f : \widetilde{G}(\widetilde{F}^+)\backslash\widetilde{G}(\mathbb{A}^\infty_{\widetilde{F}^+}) \to W_{\xi,\tau} \otimes_{\mathcal{O}} V$$

with the property that if $g \in \widetilde{G}(\mathbb{A}^\infty_{\widetilde{F}^+})$ and $u \in U_i(Q_N)_m$ then $f(gu) = u^{-1}f(g)$, where $U_i(Q_N)_m$ acts on $W_{\xi,\tau} \otimes_{\mathcal{O}} V$ via projection to $\prod_{v\in S_p} U_{m,v}$. (For example: when $V = \mathcal{O}$ is the trivial representation, then after extending scalars from $\mathcal{O}$ to $\mathbb{C}$ via $\mathcal{O} \subset \overline{\mathbb{Q}}_p \xrightarrow[\imath]{\sim} \mathbb{C}$, this space corresponds to classical automorphic forms of fixed type $\sigma(\tau)$ at the places in $S_p \setminus \{\mathfrak{p}\}$, full level $\mathfrak{p}^m$ at $\mathfrak{p}$, and whose weight (via our fixed isomorphism $\imath : \overline{\mathbb{Q}}_p \to \mathbb{C}$) is 0 at places above $\mathfrak{p}$, and given by ξ at each of the places in $S_p \setminus \{\mathfrak{p}\}$.)

We let $\mathbb{T}^{S_p\cup Q_N,\mathrm{univ}}$ be the commutative $\mathcal{O}$-polynomial algebra generated by formal variables $T_w^{(j)}$ for all $1 \le j \le n$, w a place of $\widetilde{F}$ lying over a place v of $\widetilde{F}^+$ which splits in $\widetilde{F}$ and is not contained in $S_p \cup Q_N \cup \{v_1\}$, together with formal variables $T_{\tilde{v}_1}^{(j)}$ for $1 \le j \le n$. The algebra $\mathbb{T}^{S_p\cup Q_N,\mathrm{univ}}$ acts on $S_{\xi,\tau}(U_i(Q_N)_m, V)$ via the Hecke operators

$$T_w^{(j)} := \left[U_{m,w}\ \iota_w^{-1} \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} U_{m,w} \right]$$

where ϖ_w is a fixed uniformiser in $\mathcal{O}_{\widetilde{F}_w}$.

Choose an ordering $\delta_1, \ldots, \delta_n$ of the (distinct) eigenvalues of $\overline{\rho}(\mathrm{Frob}_{\tilde{v}_1})$. Since $\overline{\rho}$ is a suitable globalisation of $\bar{r}$, it is in particular automorphic in the sense of Definition 5.3.1 of [EG14], and we let $\mathfrak{m}_{Q_N}$ be the maximal ideal of $\mathbb{T}^{S_p\cup Q_N,\mathrm{univ}}$ corresponding to $\overline{\rho}$, and containing each of the elements

$$\big(T_{\tilde{v}_1}^{(j)} - (\mathbf{N}v_1)^{j(1-j)/2}(\delta_1 \cdots \delta_j)\big),$$

for $1 \le j \le n$. We will write $\mathfrak{m}$ for $\mathfrak{m}_{\varnothing}$.

2.4. Galois deformations

Let S be a set of places of $\widetilde{F}^+$ which split in $\widetilde{F}$, with $S_p \subseteq S$. As in [CHT08], we will write $\widetilde{F}(S)$ for the maximal extension of $\widetilde{F}$ unramified outside S, and from now on we will write $G_{\widetilde{F}^+,S}$ for $\mathrm{Gal}(\widetilde{F}(S)/\widetilde{F}^+)$. We will freely make use of the terminology (of liftings, framed liftings etc.) of Section 2 of [CHT08].

Let $T = S_p \cup \{v_1\}$. For each $v \in S_p$, we let $\widetilde{v}$ be a choice of a place of $\widetilde{F}$ lying over v, with the property that $\overline{\rho}|_{G_{\widetilde{F}_{\widetilde{v}}}} \cong \bar{r}$. (Such a choice is possible by our assumption that $\overline{\rho}$ is a suitable globalisation of $\bar{r}$.) We let $\widetilde{T}$ denote the set of places $\widetilde{v}$, $v \in T$. For each $v \in T$, we let $R^{\square}_{\widetilde{v}}$ denote the maximal reduced and p-torsion free quotient of the universal $\mathcal{O}$-lifting ring of $\overline{\rho}|_{G_{\widetilde{F}_{\widetilde{v}}}}$. For each $v \in S_p \setminus \{\mathfrak{p}\}$, we write $R^{\square,\xi,\tau}_{\widetilde{v}}$ for the reduced and p-torsion free quotient of $R^{\square}_{\widetilde{v}}$ corresponding to potentially crystalline lifts of weight ξ and inertial type τ. (Such a quotient of $R^{\square}_{\widetilde{v}}$ exists by Corollary 2.7.7 of [Kis08]. It has the property that for any E-algebra A, an E-algebra map $R^{\square}_{\widetilde{v}}[1/p] \to A$ factors through $R^{\square,\xi,\tau}_{\widetilde{v}}$ if and only if the pullback of the universal lifting along this map is potentially crystalline of weight ξ and inertial type τ.)

Consider (in the terminology of [CHT08]) the deformation problem

$$\mathcal{S} := \big(\widetilde{F}/\widetilde{F}^+, T, \widetilde{T}, \mathcal{O}, \overline{\rho}, \varepsilon^{1-n}\delta^n_{\widetilde{F}/\widetilde{F}^+}, \{R^{\square}_{\widetilde{v}_1}\} \cup \{R^{\square}_{\widetilde{\mathfrak{p}}}\} \cup \{R^{\square,\xi,\tau}_{\widetilde{v}}\}_{v \in S_p \setminus \{\mathfrak{p}\}}\big).$$

There is a corresponding universal deformation $\rho^{\mathrm{univ}}_{\mathcal{S}} : G_{\widetilde{F}^+,T} \to \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{S}})$ of $\overline{\rho}$. In addition, there is a universal T-framed deformation ring $R^{\square_T}_{\mathcal{S}}$ in the sense of Proposition 2.2.9 of [CHT08], which parameterises deformations of $\overline{\rho}$ of type $\mathcal{S}$ together with particular local liftings for each $\widetilde{v} \in \widetilde{T}$.

2.5 Lemma. *$R^{\square}_{\widetilde{v}_1}$ is formally smooth over $\mathcal{O}$, and all of the corresponding Galois representations are unramified.*

Proof. By our assumptions on v_1, this is immediate from Lemma 2.4.9 and Corollary 2.4.21 of [CHT08]. □

2.6. Auxiliary primes

Recall that the globalisation $\overline{\rho}$ constructed in Section 2.1 satisfies the property that $\overline{\rho}(G_{\widetilde{F}(\zeta_p)})$ is adequate. This property is needed in order to apply the version of the Taylor–Wiles patching argument given in [Tho12] (see also

Section 5.5 of [EG14]). More precisely, Proposition 4.4 of [Tho12] allows us to choose an integer $q \geq [\widetilde{F}^+ : \mathbb{Q}]n(n-1)/2$ and for each $N \geq 1$ sets of primes $Q_N, \widetilde{Q}_N$ with the following properties (as well as a crucial property about the generation of global Galois deformation rings over local ones that we will recall below):

- Q_N is a finite set of finite places of $\widetilde{F}^+$ of cardinality q which is disjoint from T and consists of places which split in $\widetilde{F}$;
- $\widetilde{Q}_N$ consists of a single place $\widetilde{v}$ of $\widetilde{F}$ above each place v of Q_N;
- $\mathbf{N}v \equiv 1 \mod p^N$ for $v \in Q_N$;
- for each $v \in Q_N$, $\overline{\rho}|_{G_{\widetilde{F}_{\widetilde{v}}}} \cong \bar{s}_{\widetilde{v}} \oplus \overline{\psi}_{\widetilde{v}}$ where $\overline{\psi}_{\widetilde{v}}$ is an eigenspace of Frobenius on which Frobenius acts semisimply.

We remark that, for each v, any (generalized) eigenspace $\overline{\psi}_{\widetilde{v}}$ of Frobenius on which the action is in fact semisimple can be chosen. Part of the content of Proposition 4.4 of [Tho12] is that the adequacy of $\overline{\rho}(G_{\widetilde{F}(\zeta_p)})$ implies the existence of such an eigenspace for appropriately chosen primes v.

For each N, $v \in Q_N$ and $i = 0, 1$, we let $U_i(Q_N)_v \subset \widetilde{G}(\mathcal{O}_{\widetilde{F}^+_v})$ denote the parahoric open compact subgroups defined in Section 5.5 of [EG14], following [Tho12]. (We briefly recall their definition here: they are the inverse images under $\iota_{\widetilde{v}}$ of certain parahoric subgroups $\mathfrak{p}^{\widetilde{v}}_N, \mathfrak{p}^{\widetilde{v}}_{N,1}$ of $\mathrm{GL}_n(\mathcal{O}_{\widetilde{F}_{\widetilde{v}}})$. These parahoric subgroups correspond to the partition $n = (n - d^{\widetilde{v}}_N) + d^{\widetilde{v}}_N$, where $d^{\widetilde{v}}_N$ is the dimension of the eigenspace $\overline{\psi}_{\widetilde{v}}$: $\mathfrak{p}^{\widetilde{v}}_N$ is the standard parahoric and $\mathfrak{p}^{\widetilde{v}}_{N,1}$ is the kernel of the map

$$\mathfrak{p}^{\widetilde{v}}_N \to \mathrm{GL}_{d^{\widetilde{v}}_N}(k_{\widetilde{v}}) \to k^\times_{\widetilde{v}} \to k^\times_{\widetilde{v}}(p).$$

The first map in the sequence is given by projection to the $d^{\widetilde{v}}_N$-block and reduction to $k_{\widetilde{v}}$, the second map is taking the determinant and the last one is projection onto the maximal p-power order quotient of $k^\times_{\widetilde{v}}$. These parahoric subgroups are roughly supposed to be analogous to levels $\Gamma_0(\widetilde{v}), \Gamma_1(\widetilde{v})$ in the case of modular curves and modular forms.)

For each $v \in Q_N$, a quotient $R^{\overline{\psi}_{\widetilde{v}}}_{\widetilde{v}}$ of $R^{\square}_{\widetilde{v}}$ is defined in Section 5.5 of [EG14] (following [Tho12]). We let $\mathcal{S}_{Q_N}$ denote the deformation problem

$$\mathcal{S}_{Q_N} := \big(\widetilde{F}/\widetilde{F}^+, T \cup Q_N, \widetilde{T} \cup \widetilde{Q}_N, \mathcal{O}, \overline{\rho}, \varepsilon^{1-n}\delta^n_{\widetilde{F}/\widetilde{F}^+}, \\ \{R^{\square}_{\widetilde{v}_1}\} \cup \{R^{\square}_{\widetilde{\mathfrak{p}}}\} \cup \{R^{\square,\xi,\tau}_{\widetilde{v}}\}_{v \in S_p \setminus \{\mathfrak{p}\}} \cup \{R^{\overline{\psi}_{\widetilde{v}}}_{\widetilde{v}}\}_{v \in Q_N}\big).$$

We let $R^{\mathrm{univ}}_{\mathcal{S}_{Q_N}}$ denote the corresponding universal deformation ring, and we let $R^{\square_T}_{\mathcal{S}_{Q_N}}$ denote the corresponding universal T-framed deformation ring. We define

$$R^{\mathrm{loc}} := R^{\square}_{\widetilde{\mathfrak{p}}} \widehat{\otimes} \left(\widehat{\otimes}_{v \in S_p \setminus \{\mathfrak{p}\}} R^{\square, \xi, \tau}_{\widetilde{v}} \right) \widehat{\otimes} R^{\square}_{\widetilde{v}_1}$$

where all completed tensor products are taken over $\mathcal{O}$. By the choice of the sets of primes Q_N, we also know that

- the ring $R^{\square_T}_{\mathcal{S}_{Q_N}}$ can be topologically generated over R^{loc} by $q - [\widetilde{F}^+ : \mathbb{Q}]n(n-1)/2$ elements.

For each $v \in Q_N$ we choose a uniformiser $\varpi_{\widetilde{v}} \in \mathcal{O}_{F_{\widetilde{v}}}$, so that we have the projection operator $\mathrm{pr}_{\varpi_{\widetilde{v}}} \in \mathrm{End}_{\mathcal{O}}(S_{\xi,\tau}(U_i(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}})$ defined as in Proposition 5.9 of [Tho12]. We briefly recall that, at level $U_0(Q_N)_v$, $\mathrm{pr}_{\varpi_{\widetilde{v}}}$ is defined as (the pullback along $\iota_{\widetilde{v}}$ of) a polynomial in the Hecke operators corresponding to the block $\mathrm{GL}_{d^{\widetilde{v}}_N}$ inside the parahoric subgroup $\mathfrak{p}^{\widetilde{v}}_N \subset \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$. The same formula also gives rise to an element of the Hecke algebra for $\mathfrak{p}^{\widetilde{v}}_{N,1} \subset \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$. This gives a compatible projection operator at level $U_1(Q_N)_v$, which we will also call $\mathrm{pr}_{\varpi_{\widetilde{v}}}$ by abuse of notation. In the case of level $U_0(Q_N)_v$, $\mathrm{pr}_{\varpi_{\widetilde{v}}}$ is the projection onto a one-dimensional subspace, which is identified with the spherical vector, according to Proposition 5.9 of *op. cit.* At level $U_1(Q_N)_v$, $\mathrm{pr}_{\varpi_{\widetilde{v}}}$ is best understood in terms of the associated Galois representation: it only allows tamely ramified deformations of the subrepresentation of $\overline{\rho}$ corresponding to $\overline{\psi}_{\widetilde{v}}$. See Proposition 5.12 of *op. cit.* for more details.

We define pr to be the composite of the projections $\mathrm{pr}_{\varpi_{\widetilde{v}}}$. (These projections commute among themselves, and so it doesn't matter in which order we compose them. Whenever we use pr it will be clear from the context what the underlying set Q_N is.) Then, as in Section 5.5 of [EG14]:

1. The map

$$\mathrm{pr} : S_{\xi,\tau}(U_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}} \to \mathrm{pr}\left(S_{\xi,\tau}(U_0(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)$$

is an isomorphism. Moreover, since pr is defined using Hecke operators at places in Q_N, it commutes with the action of $\widetilde{G}(\widetilde{F}^+_{\mathfrak{p}})$ on the spaces of algebraic automorphic forms. More precisely, if $g \in \widetilde{G}(\widetilde{F}^+_{\mathfrak{p}})$ satisfies $g^{-1}U_{m',\mathfrak{p}}g \subseteq U_{m,\mathfrak{p}}$ for some positive integers $m \leq m'$, then we have a commutative diagram

$$\begin{array}{ccc} S_{\xi,\tau}(U_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}} & \xrightarrow{\mathrm{pr}} & \mathrm{pr}\left(S_{\xi,\tau}(U_0(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right) \\ \big\downarrow g & & \big\downarrow g \\ S_{\xi,\tau}(U_{m'}, \mathcal{O}/\varpi^r)_{\mathfrak{m}} & \xrightarrow{\mathrm{pr}} & \mathrm{pr}\left(S_{\xi,\tau}(U_0(Q_N)_{m'}, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right) \end{array}$$

where both vertical arrows are induced by the action of $g \in \widetilde{G}(\widetilde{F}^+_{\mathfrak{p}})$ (namely: $(gf)(g') = f(g'g)$).

2. Let

$$\Gamma_m = \mathrm{GL}_n(\mathcal{O}_F/\varpi_F^m) \cong U/U_m$$

and

$$\Delta_{Q_N} = \prod_{v \in Q_N} U_0(Q_N)_v/U_1(Q_N)_v.$$

Then $U_0(Q_N)_0$ acts on $S_{\xi,\tau}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)$ via $(gf)(g') = g_p f(g'g)$, and this action factors through $\Delta_{Q_N} \times \Gamma_m$. With respect to this action, $\mathrm{pr}\left(S_{\xi,\tau}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)$ is a projective $(\mathcal{O}/\varpi^r)[\Delta_{Q_N}][\Gamma_m]$-module, and there is a natural Γ_m-equivariant isomorphism

$$\mathrm{pr}\left(S_{\xi,\tau}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)^{\Delta_{Q_N}} \xrightarrow{\sim} S_{\xi,\tau}(U_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}}.$$

(The projectivity follows from the proof of Lemma 3.3.1 of [CHT08]. The fact that there is a Γ_m-equivariant isomorphism follows immediately from point (1) and the definitions. We shall not need the analogous statement about coinvariants which is recalled in [EG14] and proved in [Tho12]; see Remark 2.9 below for an indication as to why not. We remark that, by the explicit construction of $U_i(Q_N)_v$ above, Δ_{Q_N} is a finite abelian group of p-power order.)

3. Let $\mathbb{T}^{S_p \cup Q_N}_{\xi,\tau}(U_i(Q_N)_m, \mathcal{O}/\varpi^r)$ be the image of $\mathbb{T}^{S_p \cup Q_N, \mathrm{univ}}$ in the ring $\mathrm{End}_{\mathcal{O}}\left(\mathrm{pr}\left(S_{\xi,\tau}(U_i(Q_N)_m, \mathcal{O}/\varpi^r)\right)\right)$. As in Proposition 5.3.2 of [EG14], there exists a deformation

$$G_{\widetilde{F}^+, T \cup Q_N} \to \mathcal{G}_n\left(\mathbb{T}^{S_p \cup Q_N}_{\xi,\tau}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)$$

of $\overline{\rho}$ which is of type $\mathcal{S}_{Q_N}$. In particular, $\mathrm{pr}\left(S_{\xi,\tau}(U_i(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)$ is a finite $R^{\mathrm{univ}}_{\mathcal{S}_{Q_N}}$-module.

2.7 Remark. The construction of the above deformation of $\overline{\rho}$ follows the outline of the proof of Proposition 3.4.4 of [CHT08], but one can appeal to Corollaire 5.3 of [Lab11] for the necessary base change results and to the main results of [Car12; Car14] for local-global compatibility in the conjugate-self-dual case, which will show that the deformation is of type $\mathcal{S}_{Q_N}$. (For the places in $S_p \setminus \{\mathfrak{p}\}$, the argument is similar to the one in the proof of Lemma 4.17 (1), which works at the place $\mathfrak{p}$. In particular, the argument relies on Theorem 3.7.)

As in Section 5.5 of [EG14], there is a homomorphism $\Delta_{Q_N} \to (R^{\mathrm{univ}}_{\mathcal{S}_{Q_N}})^\times$ obtained by identifying Δ_{Q_N} with the product of the inertia subgroups in the maximal abelian p-power order quotient of $\prod_{v\in Q_N} G_{\widetilde{F}_{\tilde{v}}}$, and thus a homomorphism $\mathcal{O}[\Delta_{Q_N}] \to R^{\mathrm{univ}}_{\mathcal{S}_{Q_N}}$. The $R^{\mathrm{univ}}_{\mathcal{S}_{Q_N}}$-module structure coming from the existence of Galois representations thus induces an action of $\mathcal{O}[\Delta_{Q_N}]$ on $\mathrm{pr}\left(S_{\xi,\tau}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)$, which agrees with the one in (2) above.

2.8. Patching

We now make our patching construction, by applying the Taylor–Wiles–Kisin method. Before doing so, we provide a brief comparison and contrast with the patching constructions in some previous papers, such as [EG14] and [EGS15]. In the latter paper, we employ Taylor–Wiles–Kisin patching to construct what we call *patching functors*, which are (essentially) certain exact functors from the category of continuous $\mathrm{GL}_n(\mathcal{O}_F)$-representations on finitely generated $\mathcal{O}$-modules to the category of coherent sheaves on an appropriate deformation space of local Galois representations (perhaps with some auxiliary patching variables added). Although this is not discussed in [EGS15], such a functor can be (pro-)represented by an object M_∞, which is a continuous $\mathrm{GL}_n(\mathcal{O}_F)$-representation over the local deformation ring (again, perhaps with patching variables added). More precisely, in terms of such a $\mathrm{GL}_n(\mathcal{O}_F)$-representation M_∞, the patching functor can be defined as $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[\mathrm{GL}_n(\mathcal{O}_F)]]}(M_\infty, V^\vee)^\vee$, if V is a continuous representation of $\mathrm{GL}_n(\mathcal{O}_F)$ on a finitely generated $\mathcal{O}$-module. The exactness of the patching functor can be encoded in the requirement that M_∞ be a projective $\mathcal{O}[[\mathrm{GL}_n(\mathcal{O}_F)]]$-module.

In this paper our approach is to construct the representing object M_∞ directly, and (most importantly) to promote it from being merely a $\mathrm{GL}_n(\mathcal{O}_F)$-representation to being a representation of the full p-adic group $\mathrm{GL}_n(F)$. (In terms of patching functors, one can somewhat loosely think of this as extending the patching functor from the category of $\mathrm{GL}_n(\mathcal{O}_F)$-representations to a category that we might call the *Hecke category*, whose objects are the same, but in which the morphisms between any two $\mathrm{GL}_n(\mathcal{O}_F)$-representations U and V are defined to be $\mathrm{Hom}_{\mathrm{GL}_n(F)}\left(\text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)} U, \text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)} V\right)$.)

Obtaining this additional structure on M_∞ requires us to keep track of additional data ("partial actions" of the non-compact directions in G) in the course of the patching process. Before presenting the details of our construction, we remark that Scholze has simplified this aspect of our construction by a reinterpretation of patching in terms of ultraproducts [Sch15, §§8, 9], which obviates the need for keeping track of this extra data. We have chosen to keep the original form of our argument here, however.

From now on, to ease notation we write $K = \mathrm{GL}_n(\mathcal{O}_{\widetilde{F}_{\widetilde{\mathfrak{p}}}}) = \mathrm{GL}_n(\mathcal{O}_F)$, $G = \mathrm{GL}_n(\widetilde{F}_{\widetilde{\mathfrak{p}}}) = \mathrm{GL}_n(F)$, and $Z = Z(G)$. For each integer $N \geq 0$, we set $K_N := \ker\bigl(\mathrm{GL}_n(\mathcal{O}_F) \to \mathrm{GL}_n(\mathcal{O}_F/\varpi_F^N)\bigr)$, so that $K/K_N \xrightarrow{\sim} \Gamma_N$. We have the Cartan decomposition $G = KAK$, where A is the set of diagonal matrices whose diagonal entries are powers of the uniformiser ϖ_F, and we let A_N be the subset of A consisting of matrices with the property that the ratio of any two diagonal entries is of the form ϖ_F^r with $|r| \leq N$, and set $G_N = KA_NK$. Note that G_N is not a subgroup of G unless $N = 0$, but that each $K\backslash G_N/KZ$ is finite, and $G = \cup_{N\geq 0} G_N$.

If (σ, W) is a representation of KZ, then we write $\mathrm{Ind}_{KZ}^{G_N} \sigma$ for the space of functions $f : G_N \to W$ with $f(kg) = \sigma(k)f(g)$ for all $g \in G_N$, $k \in KZ$; this is naturally a KZ-representation via $(kf)(g) := f(gk)$. We define $\mathrm{Ind}_{KZ}^{G} \sigma$ in the same way; then $\mathrm{Ind}_{KZ}^{G} \sigma$ is a representation of G via $(gf)(g') := f(g'g)$.

For each N, we set

$$M_{i,Q_N} := \mathrm{pr}\bigl(S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathcal{O}/\varpi^N)_{\mathfrak{m}_{Q_N}}\bigr)^\vee.$$

Note that M_{i,Q_N} depends on the integer N as well as on the set of primes Q_N (it could happen that $Q_M = Q_N$ for $M \neq N$), but we will only include Q_N in the notation for the sake of simplicity. Note also that we could have equivalently defined

$$M_{i,Q_N} := \mathrm{pr}^\vee\bigl(S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathcal{O}/\varpi^N)^\vee_{\mathfrak{m}_{Q_N}}\bigr),$$

since pr is an endomorphism of $S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathcal{O}/\varpi^N)_{\mathfrak{m}_{Q_N}}$ and Pontrjagin duality is an exact contravariant functor.

Let Δ_{Q_N} be as above; it is of p-power order by the definitions of the $U_i(Q_N)_{2N}$. It follows from point (2) in the previous section that M_{1,Q_N} is a finite projective $(\mathcal{O}/\varpi^N)[\Delta_{Q_N}][\Gamma_{2N}]$-module. Since Z centralises $U_1(Q_N)_{2N}$, there is also a natural action of Z on M_{1,Q_N}.

2.9 Remark. The reason for including a Pontrjagin dual in the definition of M_{i,Q_N} is that $S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathcal{O}/\varpi^N)$ is a space of automorphic forms, and so is most naturally thought of as being contravariant in the level, while patching is a process that involves passing to a projective limit over the level (rather than a direct limit). Now since $S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathcal{O}/\varpi^N)$ is a space of automorphic forms on the definite unitary group $\widetilde{G}$, it is a space of functions on a finite set, and so has a natural self-duality. Thus, by exploiting this self-duality to convert its contravariant functoriality into a covariant functoriality, we could omit the Pontrjagin dual in the preceding definition, and indeed it *is* traditionally omitted (see e.g. [CHT08], [BLGG11], [Tho12], and [EG14]). However, we have found it conceptually clearer to include this duality in our definitions and constructions.

We now define a KZ-equivariant map

$$\alpha_N : M_{1,Q_N} \to \mathrm{Ind}_{KZ}^{G_N}((M_{1,Q_N})_{K_N})$$

(where $(M_{1,Q_N})_{K_N}$ denotes the K_N-coinvariants in M_{1,Q_N}) in the following way. Note firstly that there is a natural identification

$$(M_{1,Q_N})_{K_N} = \mathrm{pr}^\vee\big(S_{\xi,\tau}(U_1(Q_N)_N, \mathcal{O}/\varpi^N)^\vee_{\mathfrak{m}_{Q_N}}\big),$$

so it suffices to define a KZ-equivariant map

$$\alpha_N : S_{\xi,\tau}(U_1(Q_N)_{2N}, \mathcal{O}/\varpi^N)^\vee \to \mathrm{Ind}_{KZ}^{G_N} S_{\xi,\tau}(U_1(Q_N)_N, \mathcal{O}/\varpi^N)^\vee.$$

Now, given $g \in G_N$, we have $g^{-1}K_{2N}g \subseteq K_N$, so that there is a natural map

$$g^* : S_{\xi,\tau}(U_1(Q_N)_N, \mathcal{O}/\varpi^N) \to S_{\xi,\tau}(U_1(Q_N)_{2N}, \mathcal{O}/\varpi^N)$$

given by $(g^* \cdot f)(x) := f(xg)$, and a map

$$g_* := \big((g^{-1})^*\big)^\vee : S_{\xi,\tau}(U_1(Q_N)_{2N}, \mathcal{O}/\varpi^N)^\vee \to S_{\xi,\tau}(U_1(Q_N)_N, \mathcal{O}/\varpi^N)^\vee.$$

(The latter is well defined since G_N is stable under taking inverses. We note that g^* (resp. g_*) may be interpreted as the natural pullback (resp. pushforward) map on cohomology (resp. homology) under the natural right (resp. left) action of $\widetilde{G}(\mathbb{A}^\infty)$ on the tower of arithmetic quotients of $\widetilde{G}$.) We have $(gh)^* = g^* \circ h^*$ and hence $(gh)_* = g_* \circ h_*$, whenever all are defined. We also remark that we have the relation

$$\alpha_N(\mathrm{pr}^\vee(f))(g) = \mathrm{pr}^\vee(\alpha_N(f)(g)),$$

which follows from the fact that pr and $\mathrm{pr}^\vee$ are defined using Hecke operators away from p and hence commute with g_* and, respectively, g^*. Then we define α_N by

$$\big(\alpha_N(x)\big)(g) := g_*(x).$$

In order to check that this is KZ-equivariant, we must check that for all $k \in KZ$ we have $\big(\alpha_N(kx)\big)(g) = (k\alpha_N(x))(g)$; this is equivalent to checking that $g_*(kx) = (gk)_*(x)$, which is immediate from the definition.

Set $M^{\square}_{1,Q_N} := M_{1,Q_N} \otimes_{R^{\mathrm{univ}}_{\mathcal{S}_{Q_N}}} R^{\square_T}_{\mathcal{S}_{Q_N}}$. We have an induced KZ-equivariant map $\alpha_N : M^{\square}_{1,Q_N} \to \mathrm{Ind}^{G_N}_{KZ}(M^{\square}_{1,Q_N})_{K_N}$. Define

$$R_\infty := R^{\mathrm{loc}}[[x_1, \dots, x_{q-[\tilde{F}^+:\mathbb{Q}]n(n-1)/2}]],$$

$$S_\infty := \mathcal{O}[[z_1, \dots, z_{n^2\#T}, y_1, \dots, y_q]],$$

for formal variables $x_1, \dots, x_{q-[\tilde{F}^+:\mathbb{Q}]n(n-1)/2}$, $y_1, \dots, y_q$ and $z_1, \dots, z_{n^2\#T}$. For each N, we fix a surjection $R_\infty \twoheadrightarrow R^{\square_T}_{\mathcal{S}_{Q_N}}$ of R^{loc}-algebras (which, as recalled in Section 2.6, is possible by the choice of the sets Q_N). These choices allow us to regard each $M^{\square}_{1,Q_N}$ as an R_∞-module. Also, we fix choices of lifts representing the universal deformations over $R^{\mathrm{univ}}_{\mathcal{S}}$ and each $R^{\mathrm{univ}}_{\mathcal{S}_{Q_N}}$ such that our chosen lift over each $R^{\mathrm{univ}}_{\mathcal{S}_{Q_N}}$ reduces to our chosen lift over $R^{\mathrm{univ}}_{\mathcal{S}}$. These choices give rise to isomorphisms $R^{\square_T}_{\mathcal{S}_{Q_N}} \xrightarrow{\sim} R^{\mathrm{univ}}_{\mathcal{S}_{Q_N}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[z_1, \dots, z_{n^2\#T}]]$ compatible with a fixed isomorphism $R^{\square_T}_{\mathcal{S}} \xrightarrow{\sim} R^{\mathrm{univ}}_{\mathcal{S}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[z_1, \dots, z_{n^2\#T}]]$; they also allow us to regard each $M^{\square}_{1,Q_N}$ as an $\mathcal{O}[[z_1, \dots, z_{n^2\#T}]]$-module. Finally, for each N, choose a surjection $\mathcal{O}[[y_1, \dots, y_q]] \twoheadrightarrow \mathcal{O}[\Delta_{Q_N}]$ with kernel contained in the ideal generated by $(1+y_i)^{p^N} - 1$ for $i = 1, \dots, q$. This gives each $M^{\square}_{1,Q_N}$ the structure of an S_∞-module and hence the structure of an $S_\infty[[K]]$-module (where the action of K factors through Γ_{2N}). We note that the action of S_∞ on $M^{\square}_{1,Q_N}$ factors through that of $R^{\square_T}_{\mathcal{S}_{Q_N}}$.

We now apply the Taylor–Wiles method in the usual way to pass to a subsequence, and patch the modules $M^{\square}_{1,Q_N}$ together with the maps $\alpha_N : M^{\square}_{1,Q_N} \to \mathrm{Ind}^{G_N}_{KZ}(M^{\square}_{1,Q_N})_{K_N}$. More precisely, for each $N \geq 1$, let $\mathfrak{b}_N$ denote the ideal of S_∞ generated by ϖ^N, z_i^N and $(1+y_i)^{p^N} - 1$. Let $\mathfrak{a}$ denote the ideal of S_∞ generated by the z_i and the y_i. Fix a sequence $(\mathfrak{d}_N)_{N \geq 1}$ of open ideals of $R^{\mathrm{univ}}_{\mathcal{S}}$ such that

- $\varpi^N R^{\mathrm{univ}}_{\mathcal{S}} \subset \mathfrak{d}_N \subset \mathrm{Ann}_{R^{\mathrm{univ}}_{\mathcal{S}}}(S_{\xi,\tau}(U_{2N}, \mathcal{O}/\varpi^N)^\vee_{\mathfrak{m}})$;
- $\mathfrak{d}_N \supset \mathfrak{d}_{N+1}$ for all N;
- $\cap_{N \geq 1} \mathfrak{d}_N = (0)$.

(For example, we may take $\mathfrak{d}_N = \mathfrak{m}^N_{R^{\mathrm{univ}}_{\mathcal{S}}} \cap \mathrm{Ann}_{R^{\mathrm{univ}}_{\mathcal{S}}}(S_{\xi,\tau}(U_{2N}, \mathcal{O}/\varpi^N)^\vee_{\mathfrak{m}})$.)

At level N, we consider tuples consisting of

- a surjective homomorphism of R^{loc}-algebras $\phi : R_\infty \twoheadrightarrow R^{\mathrm{univ}}_{\mathcal{S}}/\mathfrak{d}_N$;

- a finite projective $(S_\infty/\mathfrak{b}_N)[\Gamma_{2N}]$-module $M^\square$ which carries a commuting action of $R_\infty \otimes_{\mathcal{O}} \mathcal{O}[Z]$ such that the action of S_∞ can be factored through that of R_∞;
- the action of $Z \cap K$ should factor through that of $Z_{2N} := (Z\cap K)/Z\cap K_{2N}$ and be compatible with the action of Γ_{2N};
- an isomorphism $\psi : (M^\square/\mathfrak{a}M^\square) \xrightarrow{\sim} S_{\xi,\tau}(U_{2N}, \mathcal{O}/\varpi^N)^\vee_{\mathfrak{m}}$ that is both Γ_{2N}- and Z-equivariant, and is compatible with ϕ;
- a KZ-equivariant and $R_\infty \otimes_{\mathcal{O}} S_\infty$-linear map

$$\alpha : M^\square \to \mathrm{Ind}_{KZ}^{G_N}[(M^\square)_{K_N}],$$

whose reduction modulo $\mathfrak{a}$ is compatible with the map

$$S_{\xi,\tau}(U_{2N}, \mathcal{O}/\varpi^N)^\vee_{\mathfrak{m}} \to \mathrm{Ind}_{KZ}^{G_N}[S_{\xi,\tau}(U_N, \mathcal{O}/\varpi^N)^\vee_{\mathfrak{m}}]$$

induced by the action of G_N on the spaces of algebraic modular forms.

We consider two such tuples $(\phi, M^\square, \psi, \alpha)$ and $(\phi', M^{\square'}, \psi', \alpha')$ to be equivalent if $\phi = \phi'$ and if there is an isomorphism of $(S_\infty/\mathfrak{b}_N)[\Gamma_{2N}] \otimes_{\mathcal{O}} R_\infty \otimes_{\mathcal{O}} \mathcal{O}[Z]$-modules $M^\square \xrightarrow{\sim} M^{\square'}$ which identifies ψ with ψ' and α with α'. Note that there are at most finitely many equivalence classes of such tuples. (Even though the algebra $\mathcal{O}[Z]$ is not of finite type over $\mathcal{O}$, the compatibility between the Z-action and the Γ_{2N}-action gives only finitely many possibilities for the $\mathcal{O}[Z]$-action. Also, for a given $M^\square$ there are only finitely many KZ-equivariant homomorphisms $\alpha : M^\square \to \mathrm{Ind}_{KZ}^{G_N}[(M^\square)_{K_N}]$, because $M^\square$ and $K\backslash G_N/KZ$ are finite.) Note also that a tuple $(\phi, M^\square, \psi, \alpha)$ of level N gives rise to a tuple $(\phi', M^{\square'}, \psi', \alpha')$ of level $(N-1)$ by setting $\phi' := \phi \mod \mathfrak{d}_{N-1}$, $M^{\square'} := (M^\square/\mathfrak{b}_{N-1})_{K_{2(N-1)}}$ and $\psi' := \psi \mod \varpi^{N-1}$. The map α' is defined by the formula $\alpha'(\overline{m})(g) = \overline{\alpha(m)(g)}$. Here $\overline{m}$ denotes the image of $m \in M^\square$ in $M^{\square'}$ and $\overline{\alpha(m)(g)}$ denotes the image of $\alpha(m)(g)$ in $(M^{\square'})_{K_{N-1}}$. Note that for any $m \in M^\square$, $\gamma \in K_{2(N-1)}$ and $g \in G_{N-1}$, we have

$$\begin{aligned}\alpha\big((\gamma-1)m\big)(g) &= \alpha(m)\big(g(\gamma-1)\big) \\ &= \alpha(m)\big((g\gamma g^{-1}-1)g\big) = (g\gamma g^{-1}-1)\alpha(m)(g),\end{aligned}$$

by the K-equivariance of α. Using the fact that $g\gamma g^{-1} \in K_{N-1}$ it is straightforward to see that α' is well-defined.

For each pair of integers $N' \geq N \geq 1$, we define a tuple $(\phi, M^\square, \psi, \alpha)$ of level N as follows: we set ϕ equal to $R_\infty \twoheadrightarrow R^{\square_T}_{\mathcal{S}_{Q_{N'}}} \twoheadrightarrow R^{\mathrm{univ}}_{\mathcal{S}}/\mathfrak{d}_N$ and we set $M^\square = (M^\square_{1,Q_{N'}} \otimes_{S_\infty} S_\infty/\mathfrak{b}_N)_{K_{2N}}$. The map ψ comes from points (1) and (2) in the previous section. The map α comes from $\alpha_{N'}$ defined above (in the same way that α' is defined in terms of α in the previous paragraph) and the compatibility it is required to satisfy comes from the commutative diagram in point (1) of the previous section. Since there are only finitely many isomorphism classes of tuples at each level N, but N' is allowed to be arbitrarily large, we can apply a diagonal argument to find a subsequence of pairs $(N'(N), N)_{N\geq 1}$ indexed by N such that for each $N \geq 2$, the tuple indexed by N reduced to level $(N-1)$ is isomorphic to the tuple indexed by $(N-1)$. For each $N \geq 2$, we fix a choice of such an isomorphism.

We now define

$$M_\infty := \varprojlim_N (M^{\square}_{1,Q_{N'(N)}} \otimes_{S_\infty} S_\infty/\mathfrak{b}_N)_{K_{2N}},$$

where the transition maps are induced by the isomorphisms fixed in the previous paragraph. (We drop the square from the notation here in order to avoid notational overload in later sections.) Each of the terms in the projective limit is a (literally) finite $\mathcal{O}$-module, endowed with commuting actions of $S_\infty[[K]]$ and $R_\infty \otimes_{\mathcal{O}} \mathcal{O}[Z]$, and by construction the transition maps in the projective limit respect these actions. Thus M_∞ is naturally a profinite topological $S_\infty[[K]]$-module which carries a commuting action of $R_\infty \otimes_{\mathcal{O}} \mathcal{O}[Z]$, the topology on M_∞ being the projective limit topology (where each of the terms in the projective limit is endowed with the discrete topology). Moreover, the action of S_∞ on M_∞ can be factored through some map $S_\infty \to R_\infty$. This follows from the fact that the image of S_∞ in $\mathrm{End}_{S_\infty}(M_\infty)$ is closed and the analogous statement holds at each finite level N. (Recall that the action of S_∞ on $M^{\square}_{1,Q_N}$ factors through that of $R^{\square_T}_{\mathcal{S}_{Q_N}}$.) We remark that M_∞ and its extra structures depend on the many choices we have made, in particular on the subsequence of pairs $(N'(N), N)$ and on the choice of isomorphisms between different levels for $N \geq 2$.

The module M_∞ is the key construction of the paper; the remainder of this section is devoted to recording some additional properties that it enjoys. Firstly, since the transition maps in the projective limit are given simply by reducing from level N to level $N-1$, it is easily verified that the natural map induces an isomorphism

$$(M_\infty/\mathfrak{b}_N)_{K_{2N}} \xrightarrow{\sim} (M^{\square}_{1,Q_{N'(N)}}/\mathfrak{b}_N)_{K_{2N}}.$$

Next, from this, it follows from the topological form of Nakayama's lemma that M_∞ is in fact a finite $S_\infty[[K]]$-module. It follows that the topology on M_∞ coincides with the quotient topology obtained by writing it as a quotient of $S_\infty[[K]]^r$, where $S_\infty[[K]]$ is endowed with its natural profinite topology. Crucially, there is a KZ-equivariant and R_∞-linear map

$$\alpha_\infty : M_\infty \to \operatorname{Ind}_{KZ}^G M_\infty$$

by taking the projective limit of the maps

$$(M^\square_{1,Q_{N'(N)}} \otimes_{S_\infty} S_\infty/\mathfrak{b}_N)_{K_{2N}} \to \operatorname{Ind}_{KZ}^{G_N}\Big((M^\square_{1,Q_{N'(N)}} \otimes_{S_\infty} S_\infty/\mathfrak{b}_N)_{K_N}\Big).$$

induced by $\alpha_{N'(N)} : M_{1,Q_{N'(N)}} \to \operatorname{Ind}_{KZ}^{G_{N'(N)}}\big((M_{1,Q_{N'(N)}})_{K_{N'(N)}}\big)$. We denote this induced map by $\overline{\alpha}_{N'(N)}$.

The following proposition establishes the additional key properties of the patched module M_∞ that we will need.

2.10 Proposition. *M_∞ is finitely generated and projective over $S_\infty[[K]]$, and consequently is finitely generated over $R_\infty[[K]]$. Furthermore, α_∞ is injective, and its image is G-stable, so that α_∞ induces an action of G on M_∞.*

Proof. As we already noted above, M_∞ is finitely generated over $S_\infty[[K]]$; in particular we may choose a surjection $S_\infty[[K]]^r \twoheadrightarrow M_\infty$ for some $r \geq 1$. In order to check that M_∞ is a projective $S_\infty[[K]]$-module, it is enough to check that this surjection splits.

Since each M_{1,Q_N} is a projective $(\mathcal{O}/\varpi^N)[\Delta_{Q_N}][\Gamma_{2N}]$-module, we see that

$$(M_\infty/\mathfrak{b}_N)_{K_{2N}} \cong (M^\square_{1,Q_{N'(N)}}/\mathfrak{b}_N)_{K_{2N}}$$

is a projective $S_\infty/\mathfrak{b}_N[\Gamma_{2N}]$-module, so we have a cofinal system of ideals (namely, the ideals generated by $\mathfrak{b}_N + \ker(\mathcal{O}[[K]] \to \mathcal{O}[\Gamma_{2N}])$) defining the topology of $S_\infty[[K]]$ modulo which the surjection splits. The sets of possible splittings at these finite levels then give us a projective system of non-empty finite sets, and an element of the projective limit of this projective system gives the required splitting.

Since, as observed above, the S_∞-action on M_∞ factors through the R_∞-action, we see that M_∞ is also finitely generated over $R_\infty[[K]]$.

We now check that α_∞ is injective. Note that by definition, for each $\alpha_N : M_{1,Q_N} \to \operatorname{Ind}_{KZ}^{G_N}\big((M_{1,Q_N})_{K_N}\big)$ we have $\big(\alpha_N(m)\big)(1) = 1_*(m) = \overline{m}$, where $\overline{m}$ denotes the image of m in $(M_{1,Q_N})_{K_N}$. From this (with N replaced by $N'(N)$) we deduce that $\big(\overline{\alpha}_{N'(N)})(m)\big)(1) = \overline{m}$ where $\overline{m}$ is the image of $m \in (M^\square_{1,Q_{N'(N)}}/\mathfrak{b}_N)_{K_{2N}}$ in $(M^\square_{1,Q_{N'(N)}}/\mathfrak{b}_N)_{K_N}$. This then implies that $\big(\alpha_\infty(m)\big)(1) = m$ for each $m \in M_\infty$, and α_∞ is certainly injective.

In order to show that the image of α_∞ is G-stable, we will show that for all $g \in G$, $m \in M_\infty$ we have

$$g\big(\alpha_\infty(m)\big) = \alpha_\infty\big(\big(\alpha_\infty(m)\big)(g)\big).$$

In other words, we will show that for all $g, h \in G$ and $m \in M_\infty$, we have

$$\big(\alpha_\infty(m)\big)(hg) = \big(\alpha_\infty\big((\alpha_\infty(m))(g)\big)\big)(h).$$

Let m be an element of M_∞ and let N be any integer large enough so that $g, h, gh \in G_N$. This certainly means that $g, h, gh \in G_{2N}$ as well. Since N can be arbitrarily large, it is enough to show that both sides of the equation above become equal in $(M_\infty/\mathfrak{b}_N)_{K_N}$ and we do this by explicit computation.

We let $\pi_N : M_\infty \to (M_\infty/\mathfrak{b}_N)_{K_{2N}}$ and $\sigma_N : M_\infty \to (M_\infty/\mathfrak{b}_N)_{K_N}$ denote the projection maps. Then by definition, we have

$$\sigma_N\big(\alpha_\infty(m)(hg)\big) = \overline{\alpha}_{N'(N)}\big(\pi_N(m)\big)(hg)$$

and

$$\begin{aligned}\sigma_N\big(\alpha_\infty(\alpha_\infty(m)(g))(h)\big) &= \overline{\alpha}_{N'(N)}\big(\pi_N(\alpha_\infty(m)(g))\big)(h)\\ &= \overline{\alpha}_{N'(N)}\big(\sigma_{2N}(\alpha_\infty(m)(g)) \bmod \mathfrak{b}_N\big)(h)\\ &= \overline{\alpha}_{N'(N)}\big(\overline{\alpha}_{N'(2N)}(\pi_{2N}(m))(g) \bmod \mathfrak{b}_N\big)(h).\end{aligned}$$

Now, for integers $N, \widetilde{N}, N'' \geq 1$ with $\widetilde{N} \leq N'(N)$, we let $U_1(Q_{N'(N)}, \widetilde{N})_{N''}$ be the open compact subgroup lying between $U_1(Q_{N'(N)})_{N''}$ and $U_0(Q_{N'(N)})_{N''}$ for which $U_0(Q_{N'(N)})_{N''}/U_1(Q_{N'(N)}, \widetilde{N})_{N''} \cong (\mathbb{Z}/p^{\widetilde{N}}\mathbb{Z})^q$. Then we have a commutative diagram

$$\begin{array}{ccc} S_{\xi,\tau}\big(U_1(Q_{N'(2N)}, 2N)_{4N}, \mathcal{O}/\varpi^{2N}\big)^\vee & \xrightarrow{g_*} & S_{\xi,\tau}\big(U_1(Q_{N'(2N)}, 2N)_{2N}, \mathcal{O}/\varpi^{2N}\big)^\vee \\ \| & & \downarrow{\scriptstyle \mathrm{nat}} \\ S_{\xi,\tau}\big(U_1(Q_{N'(2N)}, 2N)_{4N}, \mathcal{O}/\varpi^{2N}\big)^\vee & & S_{\xi,\tau}\big(U_1(Q_{N'(2N)}, N)_{2N}, \mathcal{O}/\varpi^{N}\big)^\vee \\ \downarrow{\scriptstyle \mathrm{nat}} & & \downarrow{\scriptstyle h_*} \\ S_{\xi,\tau}\big(U_1(Q_{N'(2N)}, N)_{2N}, \mathcal{O}/\varpi^{N}\big)^\vee & \xrightarrow{(hg)_*} & S_{\xi,\tau}\big(U_1(Q_{N'(2N)}, N)_{N}, \mathcal{O}/\varpi^{N}\big)^\vee. \end{array}$$

First localising at $\mathfrak{m}_{Q_{N'(2N)}}$, then applying the projector $\mathrm{pr}^\vee$, then tensoring over $R^{\mathrm{univ}}_{\mathcal{S}_{Q_{N'(N)}}}$ with $R^{\square_T}_{\mathcal{S}_{Q_{N'(N)}}}$, and finally reducing modulo $\mathfrak{b}_{2N}$ or $\mathfrak{b}_N$ as appropriate, we obtain a commutative diagram:

$$\begin{array}{ccc} (M_\infty/\mathfrak{b}_{2N})_{K_{4N}} & \xrightarrow{g_*} & (M_\infty/\mathfrak{b}_{2N})_{K_{2N}} \\ \| & & \downarrow \text{nat} \\ (M_\infty/\mathfrak{b}_{2N})_{K_{4N}} & & (M_\infty/\mathfrak{b}_{N})_{K_{2N}} \\ \downarrow \text{nat} & & \downarrow h_* \\ (M_\infty/\mathfrak{b}_{N})_{K_{2N}} & \xrightarrow{(hg)_*} & (M_\infty/\mathfrak{b}_{N})_{K_{N}}. \end{array}$$

Now, on the one hand, the element $\pi_{2N}(m)$ lies in the space in the upper left corner of this diagram. By definition, we have

$$\overline{\alpha}_{N'(N)}\big(\overline{\alpha}_{N'(2N)}(\pi_{2N}(m))(g) \bmod \mathfrak{b}_N\big)(h) = h_* \circ \mathrm{nat} \circ g_*\big(\pi_{2N}(m)\big)$$

On the other hand, $\pi_N(m)$ lies in the space in the lower left corner of the diagram, and we have

$$\overline{\alpha}_{N'(N)}\big(\pi_N(m)\big)(hg) = (hg)_*\big(\pi_N(m)\big) = (hg)_* \circ \mathrm{nat}\big(\pi_{2N}(m)\big).$$

The desired equality now follows from the commutativity of the above diagrams. □

Note that for each positive integer m, the compact open subgroups U_m have the same level away from $\mathfrak{p}$. Let $U^{\mathfrak{p}} \subset \widetilde{G}(\mathbb{A}^{\infty,\mathfrak{p}}_{\widetilde{F}^+})$ denote that common level. Define the ϖ-adically completed cohomology space

$$\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} := \varprojlim_s\big(\varinjlim_m S_{\xi,\tau}(U_m,\mathcal{O}/\varpi^s)_{\mathfrak{m}}\big).$$

The space $\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}$ is equipped with a natural G-action, induced from the action of G on algebraic automorphic forms. Moreover, $\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}$ is equipped with a natural action of the Hecke algebra

$$\mathbb{T}^{S_p}_{\xi,\tau}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} := \varprojlim_m \mathbb{T}^{S_p}_{\xi,\tau}(U_m,\mathcal{O})_{\mathfrak{m}}.$$

By taking the inverse limit of the $\mathbb{T}^{S_p}_{\xi,\tau}(U_m,\mathcal{O})_{\mathfrak{m}}$-valued deformations of $\bar{\rho}$ of type $\mathcal{S}$ we obtain a map $R^{\mathrm{univ}}_{\mathcal{S}} \to \mathbb{T}^{S_p}_{\xi,\tau}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}$. Therefore, $\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}$

is also equipped with an action of the local deformation ring $R^{\square}_{\tilde{\mathfrak{p}}}$ via the composition $R^{\square}_{\tilde{\mathfrak{p}}} \to R^{\mathrm{univ}}_{\mathcal{S}} \to \mathbb{T}^{S_p}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$.

2.11 Corollary. *There is a G-equivariant isomorphism*

$$(M_\infty/\mathfrak{a}M_\infty) \xrightarrow{\sim} \tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}},$$

which is compatible with the $R^{\square}_{\tilde{\mathfrak{p}}}$-action on both sides.

Proof. We have a KZ-equivariant isomorphism

$$\psi_\infty : (M_\infty/\mathfrak{a}M_\infty) \xrightarrow{\sim} \tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}$$

obtained by taking the projective limit of the Γ_{2N}- and Z-equivariant isomorphisms ψ at level N.

We first show that ψ_∞ is compatible with the $R^{\square}_{\tilde{\mathfrak{p}}}$-action on both sides. The $R^{\square}_{\tilde{\mathfrak{p}}}$-action on the left hand side is via the map $R^{\square}_{\tilde{\mathfrak{p}}} \to R_\infty$ and each ψ at level N is compatible with $\phi : R_\infty \twoheadrightarrow R^{\mathrm{univ}}_{\mathcal{S}}/\mathfrak{d}_N$. On the other hand, the action of $R^{\square}_{\tilde{\mathfrak{p}}}$ on the right hand side is via the map $R^{\square}_{\tilde{\mathfrak{p}}} \to R^{\mathrm{univ}}_{\mathcal{S}}$ and $\cap_{N\geq 1}\mathfrak{d}_N = (0)$. The desired compatibility follows.

Moreover, ψ_∞ fits into a commutative diagram

$$\begin{array}{ccc} (M_\infty/\mathfrak{a}M_\infty) & \xrightarrow{\psi_\infty} & \tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}} \\ \downarrow g & & \downarrow g \\ \mathrm{Ind}^G_{KZ}\big(M_\infty/\mathfrak{a}M_\infty\big) & \xrightarrow{\psi_\infty} & \mathrm{Ind}^G_{KZ}\big(\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}\big) \end{array}$$

where both vertical maps are induced by the action of $g \in G$. (The fact that the right vertical map has G-stable image follows from the analogue of Proposition 2.10 for the completed cohomology space $\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}$. Since M_∞ can be thought of as a patched version of completed cohomology, the arguments for $\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}$ are analogous, but easier.) The G-equivariance of ψ_∞ now follows from Proposition 2.10. □

2.12. Admissible unitary Banach representations

An *E-Banach space representation* V of G is an E-Banach space V together with a G-action by continuous linear automorphisms such that the corresponding map $G \times V \to V$ is continuous. A Banach space representation V is called *unitary* if there exists a G-invariant norm defining the topology on V. The existence of such a norm is equivalent to the existence of an open, bounded G-invariant $\mathcal{O}$-lattice $\Theta \subset V$. A unitary L-Banach space representation is *admissible* if $\Theta \otimes_{\mathcal{O}} \mathbb{F}$ is an admissible (smooth) representation of G. (This means that the space of invariants $(\Theta \otimes_{\mathcal{O}} \mathbb{F})^H$ is finite-dimensional for every open subgroup $H \subset G$.) This definition of admissibility is equivalent to that of [ST06] by Proposition 6.5.7 of [Eme]. (The latter definition requires Θ^d to be a finitely generated module over $\mathcal{O}[[H]]$.)

Fix a lifting $r : G_F \to \mathrm{GL}_n(\mathcal{O})$ of $\bar{r}$. We now explain how M_∞ allows us to associate to r an admissible unitary Banach representation $V(r)$ of $\mathrm{GL}_n(F)$.

As above, we identify F with $\widetilde{F}_{\widetilde{\mathfrak{p}}}$. By definition, r comes from a homomorphism of $\mathcal{O}$-algebras $x : R^{\square}_{\widetilde{\mathfrak{p}}} \to \mathcal{O}$. We extend this to a homomorphism of $\mathcal{O}$-algebras

$$x' : R^{\square}_{\widetilde{\mathfrak{p}}} \widehat{\otimes} \left(\widehat{\otimes}_{v \in S_p \setminus \{\mathfrak{p}\}} R^{\square,\xi,\tau}_{\widetilde{v}} \right) \to \mathcal{O}$$

by using the homomorphisms $R^{\square,\xi,\tau}_{\widetilde{v}} \to \mathcal{O}$ corresponding to our given potentially diagonalisable representation $r_{\mathrm{pot.diag}}$, and then extend x' arbitrarily to a homomorphism of $\mathcal{O}$-algebras $y : R_\infty \to \mathcal{O}$. We set $V(r) := (M_\infty \otimes_{R_\infty, y} \mathcal{O})^d[1/p]$.

2.13 Proposition. *The representation $V(r)$ is an admissible unitary Banach representation of $\mathrm{GL}_n(F)$.*

2.14 Remark. Note that we do not know if $V(r)$ is independent of either the global setting or the choice of y. We also do not know that $V(r)$ is necessarily nonzero, although we will prove that $V(r) \neq 0$ for many regular de Rham representations r in Section 5 below, as a consequence of the stronger result that (for the particular choice of r under consideration) the subspace of locally algebraic vectors in $V(r)$ is nonzero.

Proof of Proposition 2.13. The image of $(M_\infty \otimes_{R_\infty, y} \mathcal{O})^d$ in $V(r)$ is a unit ball stable under $\mathrm{GL}_n(F)$, and so $V(r)$ is a unitary representation.

In order to see that $V(r)$ is admissible, we must show that for each $N \geq 0$, the $\mathbb{F}$-vector space $\left((M_\infty \otimes_{R_\infty, y} \mathcal{O})^d \otimes \mathbb{F}\right)^{K_N}$ is finite-dimensional. Writing $\overline{y}$ for the composite $R_\infty \xrightarrow{y} \mathcal{O} \to \mathbb{F}$, we must check that $(M_\infty \otimes_{R_\infty, \overline{y}} \mathbb{F})_{K_N}$ is finite-dimensional. Since M_∞ is a finite $S_\infty[[K]]$-module, and $\ker \overline{y}$ induces an open ideal of S_∞, this is immediate. ∎

2.15 Remark. While we have assumed throughout this section that $\bar{r}$ has a potentially diagonalisable lift with regular Hodge–Tate weights, this hypothesis is not needed for our main results, which concern representations

$r : G_F \to \mathrm{GL}_n(E)$. Indeed, possibly after making a ramified extension of E, it is easy to see that any such representation can be conjugated to a representation r' which factors through $\mathrm{GL}_n(\mathcal{O})$ and whose reduction $\bar{r}'$ is semisimple, so that $\bar{r}'$ has a lift of the required kind (possibly after further extending E) by Lemma 2.2.

3. Hecke algebras and types

In the following two sections we will use the local Langlands correspondence and the theory of types to establish local-global compatibility results for our patched modules. In particular, we will make use of the results of [BK99] and [SZ99] in order to study spaces of automorphic forms which correspond to fixed inertial types. As explained in the introduction, in some of our results we will for simplicity restrict ourselves to the case of Weil–Deligne representations with $N = 0$; this means that we will limit ourselves to considering potentially crystalline Galois representations. However, some of our results are more naturally expressed in the more general context of representations with arbitrary monodromy, so we will make it clear when we impose this restriction. We begin by collecting and explaining various results from the literature that we will need.

Let $F/\mathbb{Q}_p$ be a finite extension. Recall that we write $K = \mathrm{GL}_n(\mathcal{O}_F)$, $G = \mathrm{GL}_n(F)$, and that W_F is the Weil group of F. Although several of our references work over $\mathbb{C}$, we work consistently over $\overline{\mathbb{Q}}_p$ (except in Subsection 3.13, where we fix a single Bernstein component, and work over a finite extension of $\mathbb{Q}_p$). The various results over $\mathbb{C}$ are transferred to our context over $\overline{\mathbb{Q}}_p$ via our fixed choice of $\imath : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$.

As recalled in Section 1.8, the local Langlands correspondence rec_p gives a bijection between the isomorphism classes of irreducible smooth representations of G over $\overline{\mathbb{Q}}_p$, and the n-dimensional Frobenius semisimple Weil–Deligne representations of W_F, independently of the choice of $\imath$; thus none of our results below depend on the choice of $\imath$.

3.1. Bernstein–Zelevinsky theory

We now recall some details of the local Langlands correspondence and its relationship to Bernstein–Zelevinsky theory, following [Rod82].

Given a partition $n = n_1 + \cdots + n_r$, let $P = MN$ be the corresponding standard parabolic subgroup of G with Levi subgroup M and unipotent radical N (standard with respect to the Borel subgroup of upper triangular matrices), so that $M \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$. For any smooth representations π_i of $\mathrm{GL}_{n_i}(F)$, the tensor product $\pi_1 \otimes \cdots \otimes \pi_r$ is naturally a representation of M and thus of P, and we denote by $\pi_1 \times \cdots \times \pi_r$ the normalised induction $i_P^G(\pi_1 \otimes \cdots \otimes \pi_r)$. If $\pi_1, \ldots, \pi_r$ are irreducible supercuspidal representations then $\pi_1 \times \cdots \times \pi_r$ has finite length by [Rod82, Proposition 4].

Any smooth irreducible representation π of G is necessarily a subrepresentation of some $\pi_1 \times \cdots \times \pi_r$ for some P and some irreducible supercuspidal representations $\pi_1, \ldots, \pi_r$. By [Rod82, Proposition 5], π determines the multiset $\{\pi_1, \ldots, \pi_r\}$ (and thus the corresponding partition of n, up to reordering) uniquely, and we refer to it as the *cuspidal support* of π. For any representation π_i of $\mathrm{GL}_{n_i}(F)$, and any integer s, we write $\pi_i(s) := \pi_i \otimes |\det|^s$. Then by [Rod82, Théorème 1] (and the remark which follows it), $\pi_1 \times \cdots \times \pi_r$ is reducible if and only if there are some i, j with $n_i = n_j$ and $\pi_j \cong \pi_i(1)$.

We define a *segment* to be a set of isomorphism classes of irreducible cuspidal representations $\mathrm{GL}_{n_i}(F)$ of the form $\Delta = \{\pi_i, \pi_i(1), \ldots, \pi_i(r-1)\}$ for some $r \geq 1$, and we write $\Delta = [\pi_i, \pi_i(r-1)]$. We say that two segments Δ_1, Δ_2 are *linked* if neither contains the other, and $\Delta_1 \cup \Delta_2$ is also a segment. If $\Delta_1 = [\pi_i, \pi_i']$ and $\Delta_2 = [\pi_i'', \pi_i''']$ are linked, we say that Δ_1 *precedes* Δ_2 if $\pi_i'' = \pi_i(r)$ for some $r \geq 0$.

If $\Delta = [\pi_i, \pi_i(r-1)]$ is a segment, then $\pi_i \times \cdots \times \pi_i(r-1)$ has a unique irreducible subrepresentation $Z(\Delta)$ and a unique irreducible quotient $L(\Delta)$. By [Rod82, Théorème 2], if $\Delta_1, \ldots, \Delta_r$ are segments such that if $i < j$, then Δ_i does not precede Δ_j, then the representation $Z(\Delta_1) \times \cdots \times Z(\Delta_r)$ has a unique irreducible subrepresentation, which we denote $Z(\Delta_1 \times \cdots \times \Delta_r)$. Every irreducible smooth representation of G is of the form $Z(\Delta_1 \times \cdots \times \Delta_r)$ for some segments $\Delta_1, \ldots, \Delta_r$, which are uniquely determined up to reordering. By [Rod82, Théorème 3], the analogous statement also holds for the $L(\Delta_i)$. By [Rod82, Théorème 5], $Z(\Delta_1, \ldots, \Delta_r)$ occurs with multiplicity one as a subquotient of $Z(\Delta_1) \times \cdots \times Z(\Delta_r)$.

The link with the local Langlands correspondence is as follows [Rod82, §4.4]. The representation $\mathrm{rec}_p(\pi)$ is irreducible if and only if π is supercuspidal. For each integer $r \geq 1$ there is an explicitly defined r-dimensional Weil–Deligne representation $\mathrm{Sp}(r)$ whose restriction to the Weil group is unramified and whose monodromy operator has rank $r-1$ (see page 213 of [Rod82]). Then if $\Delta = [\pi_i, \pi_i(r-1)]$ is a segment, we have $\mathrm{rec}_p\big(L(\Delta)\big) = \mathrm{rec}_p(\pi_i) \otimes \mathrm{Sp}(r)$, and more generally we have $\mathrm{rec}_p\big(L(\Delta_1, \ldots, \Delta_r)\big) = \oplus_{i=1}^r \mathrm{rec}_p\big(L(\Delta_i)\big)$.

3.2. The Bernstein Centre

We now briefly recall some of the results of [Ber84] (in the special case that the reductive group under consideration there is GL_n) in a fashion adapted to our needs. The *Bernstein spectrum* is the set of G-orbits of pairs (M, ω), where M is a Levi subgroup of G, ω is an irreducible supercuspidal representation of M, and the action of G is via conjugation; note that up to conjugacy, (M, ω) is of the form $(\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F), \pi_1 \otimes \cdots \otimes \pi_r)$, as in the preceding section. As we will explain below, the Bernstein spectrum naturally has the structure of an algebraic variety over $\overline{\mathbb{Q}}_p$ (with infinitely many connected components), the *Bernstein variety.* Given an irreducible smooth G-representation π, we obtain a point of the Bernstein spectrum by passing to the cuspidal support of π. This map is surjective; any point (M, ω) of the Bernstein spectrum is equal to the cuspidal support of π for any Jordan–Hölder factor π of $i_P^G \omega$ (and indeed these are precisely the π for which (M, ω) arises as the cuspidal support; here P is any parabolic subgroup of G admitting M as a Levi quotient — the collection of Jordan–Hölder factors of $i_P^G \omega$ is independent of the choice of P).

The connected components of the Bernstein variety are as follows. Fix a pair (M, ω) as above; then the component of the Bernstein variety containing (M, ω) is the union of the G-orbits of the pairs $(M, \alpha\omega)$, where α is an unramified quasicharacter of M. We say that two pairs (M, ω) and (M', ω') are *inertially equivalent* if they are in the same Bernstein component, and write $[M, \omega]$ for the equivalence class. Fixing one such pair (M, ω), it is easy to see that there is a natural algebraic structure on the inertial equivalence class, because the set of unramified quasicharacters of M has a natural algebraic structure, and thus so does any quotient of it by a finite group; this gives the structure of the Bernstein variety. Given an irreducible smooth G-representation π, we will refer to the inertial equivalence class of its cuspidal support as the *inertial support* of π.

For any connected component Ω of the Bernstein variety, we have a corresponding full subcategory of the category of smooth G-representations, whose objects are the smooth representations all of whose irreducible subquotients have cuspidal support in Ω. Such a subcategory is called a *Bernstein component* of the category of smooth G-representations, and in fact the category of smooth G-representations is a direct product of the Bernstein components. Given a Bernstein component Ω, the *centre* $\mathfrak{z}_\Omega$ of Ω is the centre of the corresponding Bernstein component (that is, the endomorphism ring of the identity functor), so that $\mathfrak{z}_\Omega$ acts naturally on each irreducible smooth representation $\pi \in \Omega$. Since π is irreducible, each element of $\mathfrak{z}_\Omega$ will act on π through a scalar. In fact this scalar depends only on the cuspidal support of π, and in this way $\mathfrak{z}_\Omega$ is identified with the ring of regular functions on the connected component Ω of the Bernstein variety.

The above notions extend in an obvious way to products of groups $\mathrm{GL}_{n_i}(F)$, and we will make use of this extension below without further comment (in order to compare representations of G with representations of a Levi subgroup).

Finally, we note that from the link between the local Langlands correspondence and Bernstein–Zelevinsky theory explained in Section 3.1, it is immediate that two irreducible smooth representations π, π' of G lie in the same Bernstein component if and only if $\mathrm{rec}_p(\pi)|_{I_F} \cong \mathrm{rec}_p(\pi')|_{I_F}$ (where we ignore the monodromy operators).

3.3. Bushnell–Kutzko theory

Fix a Bernstein component Ω. In [BK99], there is the definition of a *semisimple Bushnell–Kutzko type* (J, λ) for Ω, where $J \subseteq K$ is a compact open subgroup, and λ is a smooth irreducible $\overline{\mathbb{Q}}_p$-representation of J. The pair (J, λ) has the property that if π is an irreducible smooth representation of G, then $\mathrm{Hom}_J(\lambda, \pi) \neq 0$ if and only if $\pi \in \Omega$, and in fact the functor $\mathrm{Hom}_J(\lambda, *)$ induces an equivalence of categories between Ω and the category of left modules of the Hecke algebra $\mathcal{H}(G, \lambda) := \mathcal{H}(G, J; \lambda) := \mathrm{End}_G(\text{c-Ind}_J^G \lambda)$, with the inverse functor being given by $\text{c-Ind}_J^G \lambda \otimes_{\mathcal{H}(G,\lambda)} (*)$.

Let π be an irreducible representation in Ω, and let (M, ω) be a representative for the inertial support of π. We can and do suppose that M is a standard Levi subgroup $\prod_{i=1}^r \prod_{j=1}^{d_i} \mathrm{GL}_{e_i}(F)$, and that $\omega = \otimes_{i=1}^r \pi_i^{\otimes d_i}$, where π_i and $\pi_{i'}$ are not inertially equivalent (that is, do not differ by a twist by an unramified quasicharacter) if $i \neq i'$. Then by the construction of (J, λ) in [BK99], there is a pair $(J \cap M, \lambda_M)$ which is a semisimple Bushnell–Kutzko type for the Bernstein component Ω_M of M determined by ω, in the sense that if π_M is an irreducible smooth representation of M, then $\mathrm{Hom}_{J\cap M}(\lambda_M, \pi_M) \neq 0$ if and only if $\pi_M \in \Omega_M$, and the functor $\mathrm{Hom}_{J\cap M}(\lambda_M, *)$ induces an equivalence of categories between Ω_M and the category of left modules of $\mathcal{H}(M, \lambda_M)$ in the same way as above.

Let P be a parabolic subgroup of G with Levi factor M. Then the normalised induction i_P^G restricts to a functor from Ω_M to Ω, and there is a unique injective algebra homomorphism $t_P : \mathcal{H}(M, \lambda_M) \to \mathcal{H}(G, \lambda)$ (which is denoted j_Q on page 55 of [BK99]) such that under the equivalences of categories explained above, i_P^G corresponds to the pushforward along t_P (given by $\mathrm{Hom}_{\mathcal{H}(M,\lambda_M)}(\mathcal{H}(G, \lambda), *)$).

It will be useful to us to have a somewhat more explicit description of the pair $(J \cap M, \lambda_M)$. Recall that we may write $M = \prod_{i=1}^r \prod_{j=1}^{d_i} \mathrm{GL}_{e_i}(F)$, and that $\omega = \otimes_{i=1}^r \pi_i^{\otimes d_i}$. Then as is explained in [BK99, §1.4-5] (see also the paragraph after Lemma 7.6.3 of [BK93]), we may write $J \cap M = \prod_{i=1}^r \prod_{j=1}^{d_i} J_i$, $\lambda_M = \bigotimes_{i=1}^r \lambda_i^{\otimes d_i}$, where (J_i, λ_i) is a maximal simple type occurring in π_i in the sense of [BK98]. There is a corresponding isomorphism $\mathcal{H}(M, \lambda_M) \cong \otimes_{i=1}^r \mathcal{H}\big(\mathrm{GL}_{e_i}(F), \lambda_i\big)^{\otimes d_i}$, and each $\mathcal{H}\big(\mathrm{GL}_{e_i}(F), \lambda_i\big)$ is commutative, so that $\mathcal{H}(M, \lambda_M)$ is commutative. The following somewhat technical lemma will be useful to us in Section 4. We remind the reader that the algebra $\mathcal{H}(M, \lambda_M)$ is naturally isomorphic to the convolution algebra of compactly supported functions $f : M \to \mathrm{End}_{\overline{\mathbb{Q}}_p}(\lambda_M)$ such that $f(jmj') = j \circ f(m) \circ j'$ for all $m \in M, j, j' \in J \cap M$ (see [BL94, §2.2]).

3.4 Lemma. *There is an integer $e \geq 1$ and a $\overline{\mathbb{Q}}_p$-basis for $\mathcal{H}(M, \lambda_M)$ with the property that if ν is an element of this basis, then the e-fold convolution of ν with itself is supported on $t_\nu(J \cap M)$ for some $t_\nu \in Z(M)$.*

Proof. By the above remarks, we need only prove the corresponding result for the Hecke algebra of a maximal simple type $\mathcal{H}\big(\mathrm{GL}_{e_i}(F), \lambda_i\big)$. In this case, the proof of Theorem 7.6.1 of [BK93] shows that we can take a basis given by Hecke operators supported on cosets of the form tJ_i where $t \in \boldsymbol{D}(\mathfrak{B})$ in the notation of [BK93]. By Lemma 7.6.3 of [BK93], it suffices to show that there is a positive integer e such that if $t \in \boldsymbol{D}(\mathfrak{B})$ then the e-fold composition of a Hecke operator ψ_t supported on tJ_i with itself is supported on sJ_i, where s is a scalar matrix. But $\boldsymbol{D}(\mathfrak{B})$ is a cyclic group, generated by a uniformiser ϖ_E of some finite extension of fields E/F inside $\mathrm{GL}_{e_i}(F)$, so we can just take e to be the ramification degree $e(E : F)$. In that case, the e-fold composition of a Hecke operator supported on $\varpi_E J_i$ with itself is supported on $\varpi_F J_i$, as remarked on the bottom of page 201 of [BK93]. □

3.5 Remark. If λ_i is a maximal simple type for $\mathrm{GL}_{n_i}(F)$, then the ramification degree $e(E : F)$ is equal to n_i/f_i, where f_i is the number of unramified characters of $F^\times$ which preserve a supercuspidal $\mathrm{GL}_{n_i}(F)$-representation containing λ_i. This follows from Lemma 6.2.5 of [BK93]. Therefore, the element $\det(\varpi_E)$ of $F^\times$ has valuation f_i.

3.6. Results of Schneider–Zink and Dat

We will need a slight refinement of the Bernstein centre and of the theory of Bushnell–Kutzko, which is constructed in the paper [SZ99]. Note that there is not quite a bijection between irreducible smooth representations of

G and characters of the Bernstein centre; as explained in Section 3.2, any two Jordan–Hölder constituents of a parabolic induction from an irreducible cuspidal representation correspond to the same character of the centre, so that for example the trivial representation and the Steinberg representation correspond to the same character. Furthermore, as recalled in Section 3.2, two irreducible representations lie in the same Bernstein component if and only if the corresponding Weil–Deligne representations agree on inertia, but have possibly differing monodromy operators, and it can be useful to have a finer decomposition. In particular, we wish to be able to consider only the representations with $N = 0$.

Let Ω be a Bernstein component, with (J, λ) being the corresponding semisimple Bushnell–Kutzko type. Let $\mathrm{Irr}(\Omega)_0$ denote the irreducible elements π of Ω with the property that $N = 0$ on $\mathrm{rec}_p(\pi)$. In [SZ99, §2], the material recalled in Section 3.1 above is recast in terms of certain partition valued functions, which depend on the inertial support Ω. In [SZ99, §3], a partial ordering is defined on these functions, and there is a unique maximal element for this ordering, which we will denote by $\mathcal{P}$ from now on. In [SZ99, §6] an irreducible direct summand $\sigma_{\mathcal{P}}(\lambda)$ of $\mathrm{Ind}_J^K \lambda$ is constructed, with the property that if π is an irreducible smooth G-representation, then $\mathrm{Hom}_K(\sigma_{\mathcal{P}}(\lambda), \pi) \neq 0$ if and only if $\pi \in \mathrm{Irr}(\Omega)_0$, in which case $\sigma_{\mathcal{P}}(\lambda)$ occurs in $\pi|_K$ with multiplicity one. (This follows immediately from Proposition 6.2 of [SZ99], using the relationship between the partial ordering on partition valued functions in [SZ99] and the monodromy operators, which is explained in the proof of Proposition 6.5.3 of [BC09].)

If τ is the inertial type corresponding to Ω, then we write $\sigma(\tau)$ for $\sigma_{\mathcal{P}}(\lambda)$. As remarked on page 201 of [SZ99], since $\mathcal{P}$ is maximal, $\sigma(\tau)$ coincides with the representation $\rho_{\mathfrak{s}} = e_K \,\mathrm{Ind}_J^K \lambda$ constructed in Theorem 4.1 of [Dat99]. Here e_K is a certain idempotent in $\mathcal{H}(G, \lambda)$ which is constructed in [Dat99, §4.2].

Let $\mathfrak{Z}_\Omega$ be the centre of Ω. By Theorem 4.1 of [Dat99], the action of $\mathfrak{Z}_\Omega$ on $\text{c-Ind}_K^G \sigma(\tau)$ induces an isomorphism

$$\mathfrak{Z}_\Omega \cong \mathrm{End}_G\big(\text{c-Ind}_K^G \sigma(\tau)\big) =: \mathcal{H}\big(G, \sigma(\tau)\big),$$

so in particular $\mathcal{H}\big(G, \sigma(\tau)\big)$ is commutative.

Let $W_{[M,\omega]} := N_G([M,\omega])/M$ be the relative normaliser of the inertial equivalence class of (M, ω), with (M, ω) as in Section 3.2. Then by Proposition 2.1 of [Dat99], the natural algebra homomorphism $t_P : \mathcal{H}(M, \lambda_M) \to \mathcal{H}(G, \lambda)$ induces an isomorphism $t_P : \mathcal{H}(M, \lambda_M)^{W_{[M,\omega]}} \xrightarrow{\sim} Z\big(\mathcal{H}(G, \lambda)\big)$. It follows from [Dat99, Theorem 4.1] that there is an algebra homomorphism

$s_P : \mathcal{H}(G,\lambda) \to \mathcal{H}\bigl(G,\sigma(\tau)\bigr)$, which makes $\mathcal{H}\bigl(G,\sigma(\tau)\bigr)$ a direct summand of $\mathcal{H}(G,\lambda)$, and the composite $s_P \circ t_P$ induces an isomorphism

$$\mathcal{H}(M,\lambda_M)^{W_{[M,\omega]}} \xrightarrow{\sim} \mathcal{H}\bigl(G,\sigma(\tau)\bigr).$$

(The map s_P is just given by $h \mapsto e_K * h * e_K$, where e_K is the idempotent mentioned above.)

We summarise much of the preceding discussion as the following theorem.

3.7 Theorem. *If τ is an inertial type, then there is a finite-dimensional smooth irreducible $\overline{\mathbb{Q}}_p$-representation $\sigma(\tau)$ of K such that if π is any irreducible smooth $\overline{\mathbb{Q}}_p$-representation of G, then the restriction of π to K contains (an isomorphic copy of) $\sigma(\tau)$ as a subrepresentation if and only if $\mathrm{rec}_p(\pi)|_{I_F} \sim \tau$ and $N = 0$ on $\mathrm{rec}_p(\pi)$. Furthermore, in this case the restriction of π to K contains a unique copy of $\sigma(\tau)$. The ring $\mathcal{H}\bigl(G,\sigma(\tau)\bigr) := \mathrm{End}_G\bigl(\text{c-Ind}_K^G\sigma(\tau)\bigr)$ is commutative.*

3.8 Remark. We may think of this result as an *inertial local Langlands correspondence* for GL_n, in the potentially crystalline case. For GL_2/F, the correspondence between τ and $\sigma(\tau)$ matches the *inertial local Langlands correspondence* of Henniart (see the appendix to [BM02]), restricted to the potentially crystalline case.

3.9 Proposition. *Let π_i be an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(F)$ for $1 \le i \le r$, and $n = n_1 + \cdots + n_r$. If $\pi_j \not\cong \pi_i(1)$ (the condition being empty if $n_i \ne n_j$) for every $i < j$ then the G-socle and the G-cosocle of $\pi_1 \times \cdots \times \pi_r$ are irreducible. Moreover, the socle occurs as a subquotient with multiplicity one and is the only generic subquotient of $\pi_1 \times \cdots \times \pi_r$.*

Proof. This is an exercise in Bernstein–Zelevinsky theory [BZ77; Zel80]. We will make use of the material recalled from [Rod82] at the beginning of this section. If we let $\Delta_1 = \{\pi_1\}, \dots, \Delta_r = \{\pi_r\}$, then by assumption the Δ_i are segments such that Δ_i does not precede Δ_j for $i < j$. Since the segments are of length one, we have $L(\Delta_i) \cong Z(\Delta_i) \cong \pi_i$ by definition, so that as recalled above $\pi_1 \times \cdots \times \pi_r$ has a unique irreducible subrepresentation $Z(\Delta_1, \dots, \Delta_r)$ and a unique irreducible quotient $L(\Delta_1, \dots, \Delta_r)$. In addition, $Z(\Delta_1, \dots, \Delta_r)$ occurs as a subquotient with multiplicity one, so we only need to show that it is the unique generic subquotient.

Let $U_n \subset \mathrm{GL}_n(F)$ be the subgroup of unipotent upper-triangular matrices, and let $\theta_n : U_n \to \overline{\mathbb{Q}}_p^{\times}$ be the character $(u_{ij}) \mapsto \psi(\sum_{i=1}^{n-1} u_{i,i+1})$, where $\psi : F \to \overline{\mathbb{Q}}_p^{\times}$ is a fixed smooth non-trivial character. If π is a representation of G, we let π_{θ_n} be the largest quotient of π on which U_n acts by θ_n. If π is an irreducible representation of G then the dimension of π_{θ_n} is at most one, and is equal to one if and only if π is generic. Since U_n is equal to the union of its compact open subgroups, the functor $\pi \mapsto \pi_{\theta_n}$ is exact. Thus it is enough to show that $(\pi_1 \times \cdots \times \pi_r)_{\theta_n}$ is one dimensional and $Z(\Delta_1, \ldots, \Delta_r)_{\theta_n}$ is non-zero.

In [BZ77; Zel80] the authors define a family of exact functors $\pi \mapsto \pi^{(i)}$ for $0 \leq i \leq n$ from the category of smooth representations of $\mathrm{GL}_n(F)$ to the category of smooth representations of $\mathrm{GL}_{n-i}(F)$. The representation $\pi^{(i)}$ is called the i-th derivative of π. We have $\pi^{(0)} = \pi$ and $\pi^{(n)} = \pi_{\theta_n}$. If π is irreducible and supercuspidal then $\pi^{(i)} = 0$ for $0 < i < n$ and $\pi^{(n)}$ is one dimensional, by [BZ77, Theorem 4.4]. By [BZ77, Corollary 4.6] we have that $(\pi_1 \times \cdots \times \pi_r)^{(n)} \cong (\pi_1)^{(n_1)} \otimes \cdots \otimes (\pi_r)^{(n_r)}$, which is one dimensional, since the π_i are supercuspidal representations of $\mathrm{GL}_{n_i}(F)$. It follows from [Zel80, Theorem 6.2] that $Z(\Delta_1, \ldots, \Delta_r)^{(n)} \neq 0$, as required. □

If π in Ω is irreducible, then the action of $\mathfrak{Z}_\Omega$ on π defines a $\overline{\mathbb{Q}}_p$-algebra morphism $\chi_\pi : \mathfrak{Z}_\Omega \to \mathrm{End}_G(\pi) \cong \overline{\mathbb{Q}}_p$. The following is a strengthening of Theorem 4.1 of [Dat99].

3.10 Proposition. *Let π be an irreducible smooth $\overline{\mathbb{Q}}_p$-representation of G, such that $\mathrm{rec}_p(\pi)|_{I_F} \sim \tau$ and $N = 0$ on $\mathrm{rec}_p(\pi)$. Then*

$$\text{c-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_\Omega, \chi_\pi} \overline{\mathbb{Q}}_p \cong \pi_1 \times \cdots \times \pi_r,$$

where π_i is a supercuspidal representation of $\mathrm{GL}_{n_i}(F)$, such that if $i < j$ then $\pi_j \not\cong \pi_i(1)$ (the condition being empty if $n_i \neq n_j$), and $\mathrm{rec}_p(\pi) \cong \oplus_{i=1}^r \mathrm{rec}_p(\pi_i)$. Moreover, the representation π is the unique irreducible quotient of $\text{c-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_\Omega, \chi_\pi} \overline{\mathbb{Q}}_p$.

Proof. Since $N = 0$ on $\mathrm{rec}_p(\pi)$, we may write it as a direct sum of irreducible representations of the Weil group W_F, and there is a partition $n = n_1 + \cdots + n_r$ and supercuspidal representations π_i of $\mathrm{GL}_{n_i}(F)$ such that $\mathrm{rec}_p(\pi) \cong \oplus_{i=1}^r \mathrm{rec}_p(\pi_i)$. After reordering we may assume that if $i < j$ then $\pi_j \not\cong \pi_i(1)$. It follows from Proposition 3.9 that $\pi_1 \times \cdots \times \pi_r$ has a unique irreducible quotient π'.

Then $\mathrm{rec}_p(\pi') \cong \oplus_{i=1}^r \mathrm{rec}_p(\pi_i)$ as representations of W_F, and $N = 0$ on $\mathrm{rec}_p(\pi')$ since all the segments have length one, as in the proof of Proposition 3.9. Thus $\mathrm{rec}_p(\pi') \cong \mathrm{rec}_p(\pi)$, which implies that $\pi \cong \pi'$. Since the socle of $\pi_1 \times \cdots \times \pi_r$ is irreducible and occurs as a subquotient with multiplicity one, the action of $\mathfrak{Z}_\Omega$ on $\pi_1 \times \cdots \times \pi_r$ factors through an algebra morphism $\chi : \mathfrak{Z}_\Omega \to \overline{\mathbb{Q}}_p$. Since the cosocle is isomorphic to π, we deduce that $\chi = \chi_\pi$.

Theorem 3.7 implies that π is a quotient of $\text{c-Ind}_K^G\sigma(\tau)$. Since $\text{c-Ind}_K^G\sigma(\tau)$ is projective there exists a G-equivariant map $\varphi : \text{c-Ind}_K^G\sigma(\tau) \to \pi_1\times\cdots\times\pi_r$ such that the composition with $\pi_1\times\cdots\times\pi_r \twoheadrightarrow \pi$ is surjective. The G-cosocle of the cokernel of φ is zero. Since $\pi_1\times\cdots\times\pi_r$ is of finite length, so is the cokernel of φ, and we deduce that φ is surjective. Since $\mathfrak{Z}_\Omega$ acts naturally on everything, φ induces a surjection $\bar\varphi : \text{c-Ind}_K^G\sigma(\tau)\otimes_{\mathfrak{Z}_\Omega,\chi_\pi}\overline{\mathbb{Q}}_p \twoheadrightarrow \pi_1\times\cdots\times\pi_r$.

Furthermore, [Dat99, Theorem 4.1] implies that the semisimplification of $\text{c-Ind}_K^G\sigma(\tau)\otimes_{\mathfrak{Z}_\Omega}\chi_\pi$ is isomorphic to the semisimplification of $\pi_1'\times\cdots\times\pi_s'$, where π_i' are supercuspidal representations of $\mathrm{GL}_{n_i'}(F)$ for some integers n_i', such that $n = n_1'+\cdots+n_s'$. Since π is an irreducible subquotient of both $\pi_1'\times\ldots\times\pi_s'$ and $\pi_1\times\cdots\times\pi_r$, [BZ77, Theorem 2.9] implies that $\pi_1'\times\cdots\times\pi_s'$ and $\pi_1\times\cdots\times\pi_r$ have the same semisimplification. Thus $\text{c-Ind}_K^G\sigma(\tau)\otimes_{\mathfrak{Z}_\Omega,\chi_\pi}\overline{\mathbb{Q}}_p$ and $\pi_1\times\cdots\times\pi_r$ have the same semisimplification, which implies that $\bar\varphi$ is an isomorphism, as required. □

3.11 Corollary. *Let Ω be a Bernstein component corresponding to an inertial type τ and let $\mathfrak{Z}_\Omega$ be the centre of Ω. Let $\chi : \mathfrak{Z}_\Omega \to \overline{\mathbb{Q}}_p$ be a $\overline{\mathbb{Q}}_p$-algebra morphism. Then the G-socle of* $\text{c-Ind}_K^G\sigma(\tau)\otimes_{\mathfrak{Z}_\Omega,\chi}\overline{\mathbb{Q}}_p$ *is irreducible and generic, and all the other irreducible subquotients are not generic.*

Conversely, if an irreducible representation π in Ω is generic then π is isomorphic to the G-socle of $\text{c-Ind}_K^G\sigma(\tau)\otimes_{\mathfrak{Z}_\Omega,\chi_\pi}\overline{\mathbb{Q}}_p$.

Proof. The first part follows from Propositions 3.9 and 3.10. The converse may be seen as follows. There exist supercuspidal representations π_i of $\mathrm{GL}_{n_i}(F)$ such that $n = n_1+\cdots+n_r$ and π is a subquotient of $\pi_1\times\cdots\times\pi_r$. If w is a permutation of $\{1,\ldots,r\}$ then $\pi_1\times\cdots\times\pi_r$ and $\pi_{w(1)}\times\cdots\times\pi_{w(r)}$ have the same semisimplification by [BZ77, Theorem 2.9], so we may assume that the π_i satisfy the conditions of Proposition 3.9. Since the socle of $\pi_1\times\cdots\times\pi_r$ is irreducible and occurs as a subquotient with multiplicity one, the action of $\mathfrak{Z}_\Omega$ on $\pi_1\times\cdots\times\pi_r$ factors through a maximal ideal, which is equal to χ_π, as π occurs as a subquotient. If we let π' be the G-cosocle of $\pi_1\times\cdots\times\pi_r$ then π' satisfies the conditions of Proposition 3.10, and we have $\chi_{\pi'} = \chi_\pi$. The assertion follows from Propositions 3.9 and 3.10. □

3.12 Corollary. *Let π be an irreducible smooth generic $\overline{\mathbb{Q}}_p$-representation of G, such that $\mathrm{rec}_p(\pi)|_{I_F}\sim\tau$ and $N = 0$ on $\mathrm{rec}_p(\pi)$. Then we have a natural isomorphism* $\text{c-Ind}_K^G\sigma(\tau)\otimes_{\mathfrak{Z}_\Omega,\chi_\pi}\overline{\mathbb{Q}}_p\cong\pi$.

Proof. By Corollary 3.11, we see that the G-socle of $\text{c-Ind}_K^G\sigma(\tau)\otimes_{\mathfrak{Z}_\Omega,\chi_\pi}\overline{\mathbb{Q}}_p$ is isomorphic to π and occurs with multiplicity one as a subquotient. Theorem 3.7 implies that π is a quotient of $\text{c-Ind}_K^G\sigma(\tau)\otimes_{\mathfrak{Z}_\Omega,\chi_\pi}\overline{\mathbb{Q}}_p$. This implies the assertion. □

3.13. Rationality

As a preparation for the next section we explain in this subsection how the results above remain true with a finite extension E of $\mathbb{Q}_p$ as coefficient field, as long as E is sufficiently large (depending on the Bernstein component Ω). We do this by following various parts of the construction of the Bernstein centre in [Ren10], working over E rather than over $\overline{\mathbb{Q}}_p$, and we then deduce the results from those over $\overline{\mathbb{Q}}_p$ by faithfully flat descent. Let (M, ω) be the supercuspidal support of some irreducible representation in Ω, let $\mathcal{X}(M)$ be the group of unramified characters $\chi : M \to \overline{\mathbb{Q}}_p^\times$, and let

$$\mathcal{X}(M)(\omega) := \{\chi \in \mathcal{X}(M) : \omega \cong \omega \otimes \chi\}.$$

Let M^0 be the intersection of the kernels of the characters $\chi \in \mathcal{X}(M)$, and let T be the intersection of the kernels of the $\chi \in \mathcal{X}(M)(\omega)$. The restriction of ω to M^0 is a finite direct sum of irreducible representations, see [Ren10, p. VI.3.2]. We fix one irreducible summand ρ. It follows from Lemme VI.4.4 of [Ren10] and its proof that ρ extends to a representation ρ_T of T, such that $\mathrm{Ind}_T^M \rho_T$ is isomorphic to a finite direct sum of copies of ω. Thus $\chi \in \mathcal{X}(M)$ lies in $\mathcal{X}(M)(\omega)$ if and only if the restriction of χ to T is trivial. Thus the restriction to T induces a bijection

$$\mathcal{X}(M)/\mathcal{X}(M)(\omega) \xrightarrow{\cong} \mathcal{X}(T), \tag{3.14}$$

where $\mathcal{X}(T)$ is the group of characters from T to $\overline{\mathbb{Q}}_p^\times$, which are trivial on M^0.

Let D be the Bernstein component (for M) containing ω, let $\mathfrak{z}_D$ be the centre of D and let $\Pi(D) := \text{c-Ind}_{M^0}^M \rho$. It is shown in [Ren10, p. VI.4.1] that $\Pi(D)$ is a projective generator for D. Thus we may identify $\mathfrak{z}_D$ with the centre of the ring $\mathrm{End}_M(\Pi(D))$. Since ρ is irreducible, $\Pi(D)$ is a finitely generated $\overline{\mathbb{Q}}_p[M]$-module.

Let $\mathfrak{z}_\Omega$ be the Bernstein centre of Ω and let $\Pi(\Omega) := i_P^G \Pi(D)$, where P is any parabolic subgroup with Levi subgroup M. It is shown in [Ren10, Thm.VI.10.1] that $\Pi(\Omega)$ is a projective generator of Ω, which is a finitely generated $\overline{\mathbb{Q}}_p[G]$-module. Thus we may identify $\mathfrak{z}_\Omega$ with the centre of the ring $\mathrm{End}_G(\Pi(\Omega))$.

It follows from Bushnell–Kutzko theory that $\omega \cong \text{c-Ind}_{\mathbf{J}}^M \Lambda$, and $\rho \cong \text{c-Ind}_J^{M^0} \lambda$, where $\mathbf{J}$ is an open compact-mod-centre subgroup of M, J is an open compact subgroup of M, and Λ, λ are (necessarily) finite-dimensional irreducible representations. We may realise both Λ and λ over a finite extension E of $\mathbb{Q}_p$. By compactly inducing these realisations, we deduce that both ω and ρ can be realised over E. We denote these representations by ω_E and ρ_E, respectively. It is shown in [Ren10, Lemme V.2.7] that $\mathcal{X}(M)(\omega)$ is a finite group. Let $W(D)$ be the subgroup of $N_G(M)/M$ stabilising D. For each $w \in W(D)$ there are precisely $|\mathcal{X}(M)(\omega)|$ unramified characters ξ such that $\omega^w \cong \omega \otimes \xi$. Since the group M/M^0 is finitely generated, by replacing E with a finite extension, we may assume that all the characters ξ are E-valued. By further enlarging E we may assume that $\sqrt{q}$, where q is the number of elements in the residue field of F, is contained in E. Then the modulus character of P is defined over E.

Let $\Pi(D)_E := \text{c-Ind}_{M^0}^M \rho_E$, and let $\mathfrak{Z}_{D,E}$ denote the centre of its endomorphism ring $\mathrm{End}_M\big(\Pi(D)_E\big)$. Since $\rho_E \otimes_E \overline{\mathbb{Q}}_p \cong \rho$, we have $\Pi(D)_E \otimes_E \overline{\mathbb{Q}}_p \cong \Pi(D)$. We may express the generators of $\Pi(D)$ (as a $\overline{\mathbb{Q}}_p[M]$-module) as a finite $\overline{\mathbb{Q}}_p$-linear combination of elements of $\Pi(D)_E$. The $E[M]$-submodule of $\Pi(D)_E$ generated by these elements has to equal $\Pi(D)_E$, as the quotient is zero once we extend the scalars to $\overline{\mathbb{Q}}_p$. In particular, $\Pi(D)_E$ is a finitely generated $E[M]$-module.

Let $\Pi(\Omega)_E := i_P^G \Pi(D)_E$ and let $\mathfrak{Z}_{\Omega,E}$ be the centre of $\mathrm{End}_G(\Pi(\Omega)_E)$. The smooth parabolic induction commutes with $\otimes_E \overline{\mathbb{Q}}_p$, as the set $P\backslash G/H$ is finite for every open subgroup H of G and tensor products commute with inductive limits, so $\Pi(\Omega)_E \otimes_E \overline{\mathbb{Q}}_p \cong \Pi(\Omega)$. Since $\Pi(\Omega)$ is a finitely generated $\overline{\mathbb{Q}}_p[G]$-module, arguing as in the previous paragraph we deduce that $\Pi(\Omega)_E$ is a finitely generated $E[G]$-module.

The following observation (see for example Lemma 5.1 of [Paš13]) is very useful. If π and π' are representations of some group G on E-vector spaces, such that π is a finitely generated $E[\mathrm{G}]$-module, then

$$\mathrm{Hom}_{E[\mathrm{G}]}(\pi, \pi') \otimes_E \overline{\mathbb{Q}}_p \cong \mathrm{Hom}_{\overline{\mathbb{Q}}_p[\mathrm{G}]}(\pi \otimes_E \overline{\mathbb{Q}}_p, \pi' \otimes_E \overline{\mathbb{Q}}_p). \tag{3.15}$$

It follows from (3.15) that

$$\mathrm{End}_M(\Pi(D)_E) \otimes_E \overline{\mathbb{Q}}_p \cong \mathrm{End}_M\big(\Pi(D)\big), \tag{3.16}$$

$$\mathrm{End}_G(\Pi(\Omega)_E) \otimes_E \overline{\mathbb{Q}}_p \cong \mathrm{End}_G\big(\Pi(\Omega)\big). \tag{3.17}$$

Let D_E be the full subcategory of smooth representation ω' of M on E-vector spaces, such that $\omega' \otimes_E \overline{\mathbb{Q}}_p$ is in D. It follows from (3.15) that $\mathrm{Hom}_M(\Pi(D)_E, \omega') \otimes_E \overline{\mathbb{Q}}_p \cong \mathrm{Hom}_M(\Pi(D), \omega' \otimes_E \overline{\mathbb{Q}}_p)$. Since $\Pi(D)$ is a projective generator of D, we deduce that $\Pi(D)_E$ is a projective generator of

D_E. (This follows from the fact that $\overline{\mathbb{Q}}_p$ is faithfully flat over E; we will repeatedly use this fact below without further comment.) In particular, $\mathfrak{Z}_{D,E}$ is naturally isomorphic to the centre of the category D_E.

Similarly we let Ω_E be the full subcategory of smooth representations π' of G on E-vector spaces, such that $\pi' \otimes_E \overline{\mathbb{Q}}_p$ is in Ω. The same argument as above gives that $\Pi(\Omega)_E$ is a projective generator of Ω_E and $\mathfrak{Z}_{\Omega,E}$ is naturally isomorphic to the centre of Ω_E.

3.18 Lemma. *Let $\mathcal{A}$ be an E-algebra and let $\mathcal{Z}$ be an E-subalgebra of $\mathcal{A}$. If $\mathcal{Z} \otimes_E \overline{\mathbb{Q}}_p$ is the centre of $\mathcal{A} \otimes_E \overline{\mathbb{Q}}_p$ then $\mathcal{Z}$ is the centre of $\mathcal{A}$.*

Proof. For each $z \in \mathcal{Z}$, we define an E-linear map $\phi_z : \mathcal{A} \to \mathcal{A}$, $a \mapsto az - za$. Since $z \otimes 1$ is central in $\mathcal{A} \otimes_E \overline{\mathbb{Q}}_p$ by assumption, we deduce that $(\operatorname{Im} \phi_z) \otimes_E \overline{\mathbb{Q}}_p = 0$, which implies that $\operatorname{Im} \phi_z = 0$. We deduce that $\mathcal{Z}$ is contained in $\mathcal{Z}(\mathcal{A})$, the centre of $\mathcal{A}$. If $z \in \mathcal{Z}(\mathcal{A})$ then $z \otimes 1$ is contained in the centre of $\mathcal{A} \otimes_E \overline{\mathbb{Q}}_p$ and thus in $\mathcal{Z} \otimes_E \overline{\mathbb{Q}}_p$ by assumption. Hence $(\mathcal{Z}(\mathcal{A})/\mathcal{Z}) \otimes_E \overline{\mathbb{Q}}_p = 0$, which implies that $\mathcal{Z} = \mathcal{Z}(\mathcal{A})$. □

3.19 Lemma. *The isomorphism* (3.16) *induces an isomorphism $\mathfrak{Z}_{D,E} \otimes_E \overline{\mathbb{Q}}_p \cong \mathfrak{Z}_D$.*

Proof. Since $\Pi(D) \cong \operatorname{Ind}_T^M(\text{c-Ind}_{M^0}^T \rho)$ and induction is a functor, we have an inclusion $\operatorname{End}_T(\text{c-Ind}_{M^0}^T \rho) \subset \operatorname{End}_M\big(\Pi(D)\big)$. Now [Ren10, Thm.VI.4.4] implies that this inclusion identifies $\operatorname{End}_T(\text{c-Ind}_{M^0}^T \rho)$ with the centre of $\operatorname{End}_M\big(\Pi(D)\big)$. The assertion follows from Lemma 3.18 applied to $\mathcal{Z} = \operatorname{End}_T(\text{c-Ind}_{M^0}^T \rho_E)$ and $\mathcal{A} = \operatorname{End}_M\big(\Pi(D)_E\big)$. □

Let $\operatorname{Irr}(D)$ be the set of irreducible representations in D. Every such irreducible representation is of the form $\omega \otimes \chi$ for some $\chi \in \mathcal{X}(M)$. We thus have a bijection $\mathcal{X}(M)/\mathcal{X}(M)(\omega) \xrightarrow{\cong} \operatorname{Irr}(D)$, $\chi \mapsto \omega \otimes \chi$. Composing this bijection with (3.14) we obtain a bijection

$$\operatorname{Irr}(D) \xrightarrow{\cong} \mathcal{X}(T). \tag{3.20}$$

Now $\mathcal{X}(T)$ is naturally isomorphic to the set of homomorphisms of $\overline{\mathbb{Q}}_p$-algebras from $\overline{\mathbb{Q}}_p[T/M^0]$ to $\overline{\mathbb{Q}}_p$. It is explained in [Ren10, p. VI.4.4] that we have identifications

$$\mathfrak{Z}_D \cong \operatorname{End}_T(\operatorname{Ind}_{M^0}^T \rho) \cong \overline{\mathbb{Q}}_p[T/M^0];$$

see Théorème VI.4.4 for the first isomorphism and Proposition VI.4.4 for the second, so that (3.20) induces a natural bijection between $\operatorname{Irr}(D)$ and $\operatorname{MaxSpec} \mathfrak{Z}_D$.

The group $W(D)$ acts on $\mathrm{Irr}(D)$ by conjugation. For each $w \in W(D)$ let $\xi \in \mathcal{X}(M)$ be any character such that $\omega^w \cong \omega \otimes \xi$, and let ξ_w be the restriction of ξ to T. If ξ_1 and ξ_2 are two such characters then $\omega \otimes \xi_1 \cong \omega^w \cong \omega \otimes \xi_2$, and hence $\xi_1 \xi_2^{-1}$ lies in $\mathcal{X}(M)(\omega)$. It follows from the definition of T that the restriction of $\xi_1 \xi_2^{-1}$ to T is trivial. Thus ξ_w depends only on w and not on the choice of ξ. If $\chi \in \mathcal{X}(M)$ then $(\omega \otimes \chi)^w \cong \omega \otimes \chi^w \xi$. Thus the action of $W(D)$ on $\mathcal{X}(T)$ via (3.20) is given by $w.\chi = \chi^w \xi_w$. If we identify $\mathcal{X}(T)$ with the maximal spectrum of $\overline{\mathbb{Q}}_p[T/M^0]$ then this action is induced by the action of $W(D)$ on $\overline{\mathbb{Q}}_p[T/M^0]$ by $\overline{\mathbb{Q}}_p$-linear automorphisms given on the basis elements by $w.(tM^0) = \xi_w^{-1}(t) t^w M^0$: if $\chi : \overline{\mathbb{Q}}_p[T/M^0] \to \overline{\mathbb{Q}}_p$ is a morphism of $\overline{\mathbb{Q}}_p$-algebras then $(w.\chi)(tM^0) = \chi(w^{-1}.(tM^0)) = \chi(\xi_{w^{-1}}^{-1}(t) w^{-1} t w M^0) = \xi_w(t) \chi^w(tM^0)$.

3.21 Lemma. *The action of $W(D)$ on $\mathfrak{Z}_D$ preserves $\mathfrak{Z}_{D,E}$.*

Proof. Since ω and ρ can both be defined over E, so can the representation ρ_H defined at the beginning of [Ren10, p. VI.4.4], and in particular so can ρ_T, its restriction to T. Hence, if we identify $\mathfrak{Z}_D$ with $\overline{\mathbb{Q}}_p[T/M^0]$ as in [Ren10, Prop.VI.4.4] then $\mathfrak{Z}_{D,E}$ is identified with $E[T/M^0]$. Since the characters ξ_w are E-valued by the choice of E, we get the assertion. □

3.22 Lemma. $\mathfrak{Z}_{\Omega,E} = \mathfrak{Z}_{D,E}^{W(D)}$.

Proof. Since $\Pi(\Omega) = i_P^G \Pi(D)$ and parabolic induction is a functor, we have an inclusion $\mathfrak{Z}_D \subset \mathrm{End}_G(\Pi(\Omega))$. It follows from the discussion immediately preceding the proof of Theorem VI.10.4 of [Ren10] that this inclusion identifies $\mathfrak{Z}_D^{W(D)}$ with the centre of $\mathrm{End}_G(\Pi(\Omega))$. The assertion follows from Lemma 3.18 applied to $\mathcal{Z} = \mathfrak{Z}_{D,E}^{W(D)}$ and $\mathcal{A} = \mathrm{End}_G(\Pi(\Omega)_E)$. □

3.23 Proposition. *The isomorphism (3.17) induces an isomorphism*

$$\mathfrak{Z}_{\Omega,E} \otimes_E \overline{\mathbb{Q}}_p \cong \mathfrak{Z}_\Omega.$$

Proof. Using Lemmas 3.19 and 3.22 we obtain

$$\mathfrak{Z}_{\Omega,E} \otimes_E \overline{\mathbb{Q}}_p \cong \mathfrak{Z}_{D,E}^{W(D)} \otimes_E \overline{\mathbb{Q}}_p \cong (\mathfrak{Z}_{D,E} \otimes_E \overline{\mathbb{Q}}_p)^{W(D)} \cong \mathfrak{Z}_D^{W(D)} \cong \mathfrak{Z}_\Omega,$$

where the last isomorphism follows from [Ren10, p. VI.10.4], as in the proof of Lemma 3.22. □

3.24 Lemma. *$\mathfrak{Z}_{\Omega,E}$ coincides with the ring $E[\mathcal{B}]$ constructed in* [Che09, Prop. 3.11].

Proof. Let Δ be the subgroup of $\mathcal{X}(M) \rtimes W(D)$ consisting of pairs (ξ, w), such that $\omega^w \cong \omega \otimes \xi$. This subgroup acts naturally on $E[M/M^0]$. The map $\xi \mapsto (\xi, 1)$ identifies $\mathcal{X}(M)(\omega)$ with a normal subgroup of Δ and the quotient is isomorphic to $W(D)$. We have

$$\overline{\mathbb{Q}}_p[M/M^0]^{\Delta} \cong (\overline{\mathbb{Q}}_p[M/M^0]^{\mathcal{X}(M)(\omega)})^{W(D)} \cong \overline{\mathbb{Q}}_p[T/M^0]^{W(D)} \cong \mathfrak{Z}_D^{W(D)},$$

see [Ren10, Rem.VI.4.4] for the second isomorphism. Chenevier defines $E[\mathcal{B}]$ to be $E[M/M^0]^{\Delta}$. This subring gets identified with $E[T/M^0]^{W(D)}$ inside $\overline{\mathbb{Q}}_p[T/M^0]^{W(D)}$, and with $\mathfrak{Z}_{D,E}^{W(D)}$ inside $\mathfrak{Z}_D^{W(D)}$, see the proof of Lemma 3.21. The assertion follows from Lemma 3.22. □

Let $\sigma(\tau)$ be the representation of K given by Theorem 3.7. After replacing E by a finite extension we may assume that there exists a representation $\sigma(\tau)_E$ of K on an E-vector space, such that $\sigma(\tau)_E \otimes_E \overline{\mathbb{Q}}_p \cong \sigma(\tau)$. Then $\text{c-Ind}_K^G \sigma(\tau)_E$ is an object in Ω_E. Since $\mathfrak{Z}_{\Omega,E}$ is the centre of Ω_E it acts on $\text{c-Ind}_K^G \sigma(\tau)_E$, thus inducing a homomorphism $\mathfrak{Z}_{\Omega,E} \to \mathrm{End}_G\big(\text{c-Ind}_K^G \sigma(\tau)_E\big)$.

3.25 Lemma. *The map $\mathfrak{Z}_{\Omega,E} \to \mathrm{End}_G\big(\text{c-Ind}_K^G \sigma(\tau)_E\big)$ is an isomorphism.*

Proof. It follows from Theorem 4.1 of [Dat99] and Proposition 3.23 above that the map is an isomorphism once we extend scalars to $\overline{\mathbb{Q}}_p$. This implies the assertion. □

Let $\mathcal{R} := \mathrm{End}_G\big(\Pi(\Omega)\big)$. Since $\Pi(\Omega)$ is a projective generator the functors $M \mapsto M \otimes_{\mathcal{R}} \Pi(\Omega)$ and $\pi \mapsto \mathrm{Hom}_G(\Pi(\Omega), \pi)$ induce an equivalence of categories between the category of right $\mathcal{R}$-modules and Ω. If π is irreducible, then the action of $\mathfrak{Z}_\Omega$ on π factors through $\chi_\pi : \mathfrak{Z}_\Omega \to \overline{\mathbb{Q}}_p$. It follows from [Ren10, Lem.VI.10.4] that $\mathcal{R}$ is a finitely generated $\mathfrak{Z}_\Omega$-module, which implies that the module corresponding to π is a finite dimensional $\overline{\mathbb{Q}}_p$-vector space. Since $\mathfrak{Z}_{\Omega,E}$ is a finitely generated algebra over E, $E(\chi_\pi) := \chi_\pi(\mathfrak{Z}_{\Omega,E})$ is a finite extension of E.

In the above E was only required to be sufficiently large. Thus if E' is a subfield of $\overline{\mathbb{Q}}_p$ containing E, then we let $\Omega_{E'}$, $\Pi(\Omega)_{E'}$ be the corresponding objects defined over E' instead of E. Then $\Pi(\Omega)_{E'}$ is a projective generator of $\Omega_{E'}$ and the functors $M \mapsto M \otimes_{\mathcal{R}_{E'}} \Pi(\Omega)_{E'}$ and $\pi \mapsto \mathrm{Hom}_G(\Pi(\Omega)_{E'}, \pi)$ induce an equivalence of categories between the category of right $\mathcal{R}_{E'}$-modules and $\Omega_{E'}$, where $\mathcal{R}_{E'} := \mathrm{End}_G(\Pi(\Omega)_{E'})$.

3.26 Lemma. *Every irreducible generic $\pi \in \Omega$ can be realised over $E(\chi_\pi)$.*

Proof. In order to ease the notation, we write $E' := E(\chi_\pi)$ and $\pi' := \text{c-Ind}_K^G \sigma(\tau)_E \otimes_{\mathfrak{z}_{\Omega,E}} E'$. Then

$$\pi' \otimes_{E'} \overline{\mathbb{Q}}_p \cong \text{c-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{z}_\Omega, \chi_\pi} \overline{\mathbb{Q}}_p \cong \pi_1 \times \ldots \times \pi_r,$$

where the last isomorphism is given by Proposition 3.10. Hence, $\pi' \otimes_{E'} \overline{\mathbb{Q}}_p$ is of finite length, which implies that $\mathrm{Hom}_G(\Pi(\Omega), \pi' \otimes_{E'} \overline{\mathbb{Q}}_p)$ is a finite dimensional $\overline{\mathbb{Q}}_p$-vector space, which implies that $M' := \mathrm{Hom}_G(\Pi(\Omega)_E, \pi')$ is a finite dimensional E'-vector space.

If π'' is an irreducible E'-subrepresentation of π', and if we define $M'' := \mathrm{Hom}_G(\Pi(\Omega)_{E'}, \pi'')$, then M'' is an irreducible $\mathcal{R}_{E'}$-module which is finite dimensional over E'. It follows from [BouAlg, Cor.12.7.1a)] that $M'' \otimes_{E'} \overline{\mathbb{Q}}_p$ is a semi-simple $\mathcal{R}$-module. Hence, $\pi'' \otimes_{E'} \overline{\mathbb{Q}}_p$ is a semi-simple G-representation. Proposition 3.9 implies that the G-socle of $\pi'' \otimes_{E'} \overline{\mathbb{Q}}_p$ is irreducible and is isomorphic to π. Thus $\pi'' \otimes_{E'} \overline{\mathbb{Q}}_p \cong \pi$. □

Henceforth for each Bernstein component Ω we will fix a sufficiently large E as above and work with it. Agreeing on this, we will *omit E from the notation* when there is no danger of confusion. For instance we will write $\mathfrak{z}_\Omega$, $\sigma(\tau)$, and so on, in place of $\mathfrak{z}_{\Omega,E}$, $\sigma(\tau)_E$ and so on. Note that we fixed a choice of E in Section 2; however, it is harmless to replace our patched module M_∞ with its base extension to the ring of integers in any larger choice of E, and we will do so without further comment.

4. Local-global compatibility

The goal of this section is to prove that the patched module M_∞ satisfies local-global compatibility, in the following sense: the G-action on M_∞ (obtained by patching global objects) will induce a tautological Hecke action on certain patched modules for particular K-types. On the other hand, we will define a second Hecke action via an interpolation of the classical local Langlands correspondence. We will then prove that these two Hecke actions coincide. The details are made explicit below.

Note that it is plausible that M_∞ should satisfy local-global compatibility, since it is patched together from spaces of algebraic modular forms; the difficulty in proving this is that the modules at finite level are all p-power torsion, while local-global compatibility is usually defined after inverting p, so that we need to establish some integral control over the compatibility. Some of our arguments were inspired by the treatment of the two-dimensional crystalline case in [Kis07, §3.6], and somewhat related considerations in the arguments of [Kis09b].

Let σ be a locally algebraic type for $G = \mathrm{GL}_n(F)$ defined over E. Then by definition σ is an absolutely irreducible representation of $K = \mathrm{GL}_n(\mathcal{O}_F)$ over E of the form $\sigma_{\mathrm{sm}} \otimes \sigma_{\mathrm{alg}}$, where σ_{sm} is a smooth type for K (i.e. $\sigma_{\mathrm{sm}} = \sigma(\tau)$ for some inertial type τ) and σ_{alg} is the restriction to K of an irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}_n$; we will sometimes also write σ_{alg} for the corresponding G-representation. (So, all of our locally algebraic types are "potentially crystalline", in the sense that they detect representations for which $N = 0$.) Set $\mathcal{H}(\sigma) := \mathrm{End}_G(\text{c-Ind}_K^G \sigma)$.

We say that a continuous representation $r : G_F \to \mathrm{GL}_n(E)$ has Hodge–Tate weights prescribed by σ_{alg} if r is regular of weight ξ and σ_{alg} is *dual* to (the restriction to K of) the representation with highest weight vector ξ. (Given such an r, the representation σ_{alg} is the restriction to K of the representation π_{alg} defined in Section 1.8). We will say that r is *potentially crystalline of type* σ if it is potentially crystalline with inertial type τ and Hodge–Tate weights prescribed by σ_{alg}. We also say that a global representation has type σ if it restricts to such an r. Let $R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)$ be the local universal lifting ring of type σ at $\widetilde{\mathfrak{p}}$ (i.e. the unique reduced and p-torsion free quotient of $R^{\square}_{\widetilde{\mathfrak{p}}}$ corresponding to potentially crystalline lifts of type σ).

Let $\mathcal{X} = \mathrm{Spf}\, R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)$, with ideal of definition taken to be the maximal ideal, and let $\mathcal{X}^{\mathrm{rig}}$ denote its rigid generic fibre (as constructed in [Jon95, §7]). Note that $\mathcal{X}^{\mathrm{rig}} = \cup_j U_j$ is an increasing union of affinoids, and in fact is a quasi-Stein rigid space, since it is a closed subspace of an open polydisc, which is an increasing union of closed polydiscs. By a standard abuse of notation, we will write $\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}}$ for the ring of rigid-analytic functions on $\mathcal{X}^{\mathrm{rig}}$. Then $\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}} = \varprojlim_j \Gamma(U_j, \mathcal{O}_{U_j})$ and we equip it with the inverse limit topology. We note that by [Jon95, Lemma 7.1.9], there is a bijection between the points of $\mathcal{X}^{\mathrm{rig}}$ and the closed points of $\mathrm{Spec}\, R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p]$. The universal lift over $R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)$ gives rise to a continuous family of representations $\rho^{\mathrm{rig}} : G_F \to \mathrm{GL}_n(\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}})$. (The continuity of ρ^{rig} is equivalent to that of each of the representations $G_F \to \mathrm{GL}_n\big(\Gamma(U_j, \mathcal{O}_{U_j})\big)$ obtained by restricting elements of $\mathrm{GL}_n(\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}})$ to U_j.) If x is a point of $\mathcal{X}^{\mathrm{rig}}$ with residue field E_x, we denote by $\rho_x : G_F \to \mathrm{GL}_n(E_x)$ the specialisation of ρ^{rig} at x. We define the locally algebraic G-representation $\pi_{\mathrm{l.alg},x} := \pi_{\mathrm{sm}}(\rho_x) \otimes_E \sigma_{\mathrm{alg}} = \pi_{\mathrm{sm}}(\rho_x) \otimes_{E_x} \pi_{\mathrm{alg}}(\rho_x)$. (Recall the notation $\pi_{\mathrm{sm}}(\rho_x)$ and $\pi_{\mathrm{alg}}(\rho_x)$ from §1.8; in particular, $\pi_{\mathrm{sm}}(\rho_x) = r_p^{-1}(\mathrm{WD}(\rho_x)^{F-\mathrm{ss}})$.) Note that $\mathcal{H}(\sigma)$ acts via a character on the space $\mathrm{Hom}_K(\sigma, \pi_{\mathrm{l.alg},x})$, the latter being one-dimensional (by Theorem 3.7 together with the argument of [ST06, Lemma 1.4]).

The following theorem, which may be of independent interest, gives our interpolation of the local Langlands correspondence. Its proof will occupy much of this section.

4.1 Theorem. *There is an E-algebra homomorphism*

$$\eta : \mathcal{H}(\sigma) \to R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p]$$

which interpolates the local Langlands correspondence r_p. More precisely, for any closed point x of $\operatorname{Spec} R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p]$, the $\mathcal{H}(\sigma)$-action on $\operatorname{Hom}_K(\sigma, \pi_{\mathrm{l.alg},x})$ factors as η composed with the evaluation map $R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p] \to E_x$.

We begin by proving the following weaker result, showing the existence of a rigid analytic local Langlands map.

4.2 Proposition. *There is an E-algebra homomorphism $\eta : \mathcal{H}(\sigma) \to \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}}$ which interpolates the local Langlands correspondence r_p. More precisely, for any point $x \in \mathcal{X}^{\mathrm{rig}}$, the action of $\mathcal{H}(\sigma)$ on $\operatorname{Hom}_K(\sigma, \pi_{\mathrm{l.alg},x})$ factors as η composed with the evaluation map $\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}} \to E_x$.*

Recall that $\mathcal{X}^{\mathrm{rig}} = \cup_j U_j$ can be written as an increasing union of rigid spaces associated to reduced affinoid algebras. Théorème C of [BC08] associates a family of Weil-Deligne representations to a family of Galois representations over the rigid space associated to a reduced affinoid algebra. Applying it to each U_j and to $\rho^{\mathrm{rig}}|_{U_j}$, we obtain a compatible family of Weil–Deligne representations $\rho^{\mathrm{rig}}_{\mathrm{WD}} : W_F \to \mathrm{GL}_n(\Gamma(U_j, \mathcal{O}_{U_j}))$ and thus a Weil–Deligne representation $\rho^{\mathrm{rig}}_{\mathrm{WD}} : W_F \to \mathrm{GL}_n(\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}})$. Note that $\rho^{\mathrm{rig}}_{\mathrm{WD}}$ has $N = 0$.

For a point x of $\mathcal{X}^{\mathrm{rig}}$, we denote by $\rho_{\mathrm{WD},x}$ the specialisation of $\rho^{\mathrm{rig}}_{\mathrm{WD}}$ at x. Then $\rho_{\mathrm{WD},x}|_{I_F} \simeq \tau$ for all points x of $\mathcal{X}^{\mathrm{rig}}$. Recall that $\mathfrak{Z}_\Omega$ is the Bernstein centre for the Bernstein component Ω corresponding to $\sigma(\tau)$.

4.3 Proposition. *There exists a unique E-algebra map $I : \mathfrak{Z}_\Omega \to \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}}$ such that for any point x of $\mathcal{X}^{\mathrm{rig}}$ with residue field E_x, the smooth G-representation π_x corresponding to $\rho_{\mathrm{WD},x}$ via the local Langlands correspondence rec_p determines via specialisation the map $x \circ I : \mathfrak{Z}_\Omega \to E_x$.*

Proof. Consider the following map, obtained by specialisation:

$$\gamma_G : \mathfrak{Z}_\Omega \to \prod_{x \in \mathcal{X}^{\mathrm{rig}}} E'_x,$$

where γ_G is defined on the factor corresponding to x by evaluating $\mathfrak{Z}_\Omega$ at the closed point in the Bernstein component Ω determined via local Langlands by x, and E'_x/E_x is a sufficiently large finite extension.

Consider as well the following map, also obtained by specialisation:

$$\gamma_{\mathrm{WD}} : \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}} \to \prod_{x \in \mathcal{X}^{\mathrm{rig}}} E'_x.$$

This is an injection since $\mathcal{X}^{\mathrm{rig}}$ is reduced and since each $\Gamma(U_j, \mathcal{O}_{U_j})$ is Jacobson. (The Jacobson property is true of any affinoid algebra. To see that $\mathcal{X}^{\mathrm{rig}}$ is reduced, it is enough to check it on completed local rings at closed points, but these are the same as the completed local rings of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma)\,[1/p]$ by [Jon95, Lemma 7.1.9]. The latter is reduced (being a localisation of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma)$, which is reduced by definition) and excellent, since $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma)$ is a complete, local, noetherian ring (and thus excellent by [EGAIV$_2$, Scholie 7.8.3(iii)]). The reducedness of the completed local rings now follows from [EGAIV$_2$, Scholie 7.8.3(v)].)

In order to define our map I, it suffices to show that the image of $\mathfrak{Z}_\Omega$ under γ_G is contained in the image of γ_{WD}. Let $T : W_F \to \mathfrak{Z}_\Omega$ be the pseudo-representation constructed in Proposition 3.11 of [Che09]. (Note that Chenevier's $E[\mathcal{B}]$ is our $\mathfrak{Z}_\Omega$ by Lemma 3.24.) By the construction of T, we have $\gamma_G \circ T = \gamma_{\mathrm{WD}} \circ \mathrm{tr}(\rho^{\mathrm{rig}}_{\mathrm{WD}})$. Therefore the proof of the proposition is reduced to Lemma 4.5 below. ☐

Write $v : W_F \twoheadrightarrow \mathbb{Z}$ for the valuation map assigning 1 to any lift of the geometric Frobenius. Let $\phi \in W_F$ be an element of valuation 1. For $w \in W_F$ and any I_F-representation r_0, let r_0^w be the twist $r_0^w(\gamma) := r_0(w^{-1}\gamma w)$.

4.4 Lemma. *Let r be an irreducible continuous representation of W_F over $\overline{\mathbb{Q}}_p$.*

1. *The restriction $r|_{I_F}$ decomposes as a direct sum of non-isomorphic irreducible I_F-representations $\oplus_{i=1}^f r_1^{\phi^i}$ for some integer $f \geq 1$. If $t \in \mathbb{Z}$ then $r(\phi^t)$ respects the decomposition (i.e. $r(\phi^t)$ sends $r_1^{\phi^i}$ into itself for $1 \leq i \leq f$) exactly when $f \mid t$.*
2. *We have $\mathrm{tr}\big(r(w)\big) \neq 0$ for some $w \in W_F$ of valuation t if and only if $f \mid t$.*
3. *The unramified characters χ of W_F satisfying $r \otimes \chi \simeq r$ are exactly the characters of order dividing f.*

Proof. (1) The representation $r|_{I_F}$ factors through a finite quotient I_F/H, so it decomposes as a direct sum of irreducible I_F-representations $\oplus_{i=1}^f r_i$, for some integer $f \geq 1$. The fact that r is irreducible as a W_F-representation implies that $r(\phi)$ acts transitively on (the representation spaces of) the r_i. Up to reordering the r_i, we may assume that it sends r_i to r_{i+1}, where $r_{f+1} := r_1$. Moreover, we also deduce that $r_{i+1} \simeq r_i^{\phi}$ and that $r_1 \simeq r_1^{\phi^f}$. Finally, all the representations r_i are non-isomorphic, since if there was an isomorphism between them, we could define a proper W_F-subrepresentation of r and thus contradict the irreducibility of r. (More precisely, if we had an isomorphism $r_1 \simeq r_{1+s}$ for some $1 \leq s < f$, then we could assume that $f = sf'$ for some integer f' and get I_F-isomorphisms

$$\alpha_{sk} : r_1 \oplus \cdots \oplus r_s \xrightarrow{\sim} r_{1+sk} \oplus \cdots \oplus r_{s(1+k)}$$

for each $1 \leq k < f'$. In that case, we could take the I_F-subrepresentation of r generated by $v + \alpha_s(v) + \cdots + \alpha_{s(f'-1)}(v)$ with $v \in r_1 \oplus \cdots \oplus r_s$; it is easy to check that this space is also stable under ϕ if we choose the α_{sk} appropriately.) The fact that $r(\phi)$ induces a cyclic permutation of the f irreducible constituents implies the statement about $r(\phi^t)$.

(2) Since $r(w)$ is not supported on the diagonal unless $f \mid t$ we get the only if part. For the if part, assume that $f \mid t$. By part 1, the matrix $r(\phi^t)$ has the same block decomposition as $r|_{I_F}$. Note that the group algebra of I_F/H surjects onto $\oplus_{i=1}^{f} \mathrm{End}_{\overline{\mathbb{Q}}_p}(r_i)$, since the r_i are non-isomorphic irreducible representations of the finite group I_F/H. Therefore, there is some linear combination of matrices $\sum_{h \in I_F} c_h \cdot r(h)$ which has non-zero trace against the non-zero matrix $r(\phi^t)$. This implies that $\mathrm{tr}\big(r(h \cdot \phi^t)\big) \neq 0$ for some $h \in I_F$.

(3) Observe that $r \otimes \chi \simeq r$ if and only if $\chi(w) \mathrm{tr}\, r(w) = \mathrm{tr}\, r(w)$, $\forall w \in W$. The latter condition is equivalent via part 2 to the condition that $\chi(w) = 1$ for all $w \in W_F$ such that $f | v(w)$, or equivalently that $\chi^f = 1$. Hence part 3 is verified. □

4.5 Lemma. *The image of T generates $\mathfrak{Z}_\Omega$ as an E-algebra.*

Proof. It suffices (by the faithful flatness of the extension $\overline{\mathbb{Q}}_p/E$) to prove the result after replacing E with $\overline{\mathbb{Q}}_p$. Since the inertial type τ factors through a finite quotient I_F/H, it decomposes as a direct sum $\oplus_{i=1}^{r} (\tau_i)^{d_i}$, where the τ_i are non-isomorphic inertial types such that $\sigma(\tau_i)$ is cuspidal. As in the proof of Proposition 3.11 of [Che09], the Bernstein component Ω decomposes as $\Omega_1 \times \cdots \times \Omega_r$ and $\mathfrak{Z}_\Omega = \otimes_{i=1}^{r} \mathfrak{Z}_{\Omega_i}$, where each Ω_i corresponds to the simple type $\sigma((\tau_i)^{d_i})$. If we let $T_i : W_F \to \mathfrak{Z}_{\Omega_i}$ be the pseudo-representation associated to Ω_i by Proposition 3.11 of [Che09], then by definition $T(g) :=$

$\sum_{i=1}^r T_i(g)$. It suffices to show that the image of T in $\mathfrak{Z}_\Omega$ generates each $\mathfrak{Z}_\Omega$-subalgebra $\mathfrak{Z}_{\Omega_i}$ for $i = 1, \dots, r$.

Let r_i be an irreducible W_F-representation such that $r_i|_{I_F} \simeq \tau_i$, and let f_i be the integer associated to r_i by Lemma 4.4. By choosing $\mathrm{rec}_p^{-1}(r_i)^{\otimes d_i}$ as a base point, each closed point of Spec $\mathfrak{Z}_{\Omega_i}$ may be represented by an unramified character $\chi_i = (\chi_{i,1}, \dots, \chi_{i,d_i})$ (or more precisely by $\otimes_{j=1}^{d_i}(\mathrm{rec}_p^{-1}(r_i) \otimes \chi_{i,j})$ up to a permutation of factors), where the $\chi_{i,j}$ are unramified characters of $F^\times$. Then each $T_i(g)$ is defined by

$$T_i(g)(\chi_i) := \mathrm{tr}(r_i)(g) \sum_{j=1}^{d_i} \chi_{i,j}\big(\mathrm{Art}_F(g)\big).$$

Consider elements $g \in W_F$ of the form $h \cdot \phi^{t_i}$, with $h \in I_F$ and $t_i \in f_i\mathbb{Z}$. By Lemma 4.4(1), the matrix $r_i(\phi^{t_i})$ is non-zero and consists of f_i blocks which match the block decomposition of τ_i. Because the constituents of τ_i are non-isomorphic for different i's, we may choose the c_h such that

$$\sum_{h \in I_F/H} c_h \cdot \mathrm{tr}\big(r_i(h) \cdot r_i(\phi^{t_i})\big) \neq 0$$

and $\sum_{h \in I_F/H} c_h \cdot r_{i'}(h) = 0$ for $i' \neq i$. In particular, this means that $\sum_{h \in I_F/H} c_h \cdot T(h\phi^{t_i}) \in \mathfrak{Z}_{\Omega_i}$.

We will now compute $\sum_{h \in I_F/H} c_h \cdot T(h\phi^{t_i})$, as an element of $\mathfrak{Z}_{\Omega_i}$. Since the terms for $i' \neq i$ vanish, we can identify this with $\sum_{h \in I_F/H} c_h \cdot T_i(h\phi^{t_i})$ and work inside the Bernstein centre $\mathfrak{Z}_{\Omega_i}$ for the simple type $\sigma((\tau_i)^{d_i})$. If we factor out the non-zero scalar $\sum_{h \in I_F/H} c_h \cdot \mathrm{tr}\big(r_i(h) \cdot r_i(\phi^{t_i})\big)$, we are left with $\sum_{j=1}^{d_i} \chi_{i,j}\big(\mathrm{Art}_F(\phi^{t_i})\big)$. We wish to identify this as a regular function on the Bernstein component $\mathfrak{Z}_{\Omega_i}$. Notice that $\mathrm{Art}_F(\phi^{t_i}) \in F^\times$ has valuation t_i, which by Lemma 4.4(3) and Remark 3.5 coincides with the valuation of $\det(\pi_{E_i})^{t_i/f_i}$, where E_i/F is the extension in Lemma 3.4 for the cuspidal type $\sigma(\tau_i)$.

By the proof of Lemma 3.4, the Hecke algebra $\mathcal{H}(\sigma(\tau_i))$ is generated by the Hecke operator (well-defined up to a non-zero scalar) supported on π_{E_i}. By the isomorphism between $\mathcal{H}(\sigma(\tau_i))$ and the Bernstein centre $\mathfrak{Z}_{\Omega(\sigma(\tau_i))}$ for the type $\sigma(\tau_i)$, the latter is generated by the regular function on unramified characters

$$\chi \mapsto \chi(\det(\pi_{E_i})).$$

Now, the Bernstein centre $\mathfrak{Z}_{\Omega_i}$ can be identified with the elements in the product $\prod_{j=1}^{d_i} \mathfrak{Z}_{\Omega(\sigma(\tau_i))}$ which are invariant under the action of the symmetric group S_{d_i} (see the proof of Proposition 3.11 of [Che09] or use the Satake isomorphism on the level of Hecke algebras). For $j = 1, \ldots, d_i$, let $X_{ij} \in \prod_{j=1}^{d_i} \mathfrak{Z}_{\Omega(\sigma(\tau_i))}$ correspond to the regular function defined above in the jth component. From the observation on the valuation of $\mathrm{Art}_F(\phi^{t_i})$, we see that the function

$$(\chi_{i,1}, \ldots, \chi_{i,d_i}) \mapsto \sum_{j=1}^{d_i} \chi_{i,j}\big(\mathrm{Art}_F(\phi^{t_i})\big)$$

matches $\sum_{j=1}^{d_i} X_{ij}^{t_i/f_i} \in \mathfrak{Z}_{\Omega_i}$ up to a non-zero scalar.

Note that we can ensure that t_i/f_i is any integer. Therefore, we can generate all elements in $\mathfrak{Z}_{\Omega_i}$ of the form $\sum_{j=1}^{d_i} X_{ij}^k$ for any $k \in \mathbb{Z}$. Since $\mathfrak{Z}_{\Omega_i}$ is obtained by taking invariants under S_{d_i} in $\bar{\mathbb{Q}}_p[X_{i1}^{\pm 1}, \ldots, X_{id_i}^{\pm 1}]$, it is generated as a $\overline{\mathbb{Q}}_p$-algebra by the elementary symmetric polynomials in X_{ij} together with $\prod_{j=1}^{d_i} X_{ij}^{-1}$. Over $\overline{\mathbb{Q}}_p$, which is a field of characteristic 0, we may take the sums of powers of d_i variables as generators for the elementary symmetric polynomials in those variables. We may also generate the product of the inverses of the variables from sums of powers with negative exponents. □

4.6 Remark. While the proof of Lemma 4.5 is slightly technical, the lemma itself is rather natural; it expresses the idea that local Langlands should make sense in families, and hence that the family of G-representations parameterised by $\mathfrak{Z}_\Omega$ — and thus the parameter ring $\mathfrak{Z}_\Omega$ itself — should be completely determined by the corresponding family of Weil group representations, which are encoded by the $\mathfrak{Z}_\Omega$-valued pseudo-representation T.

If we let $\mathfrak{A}_\Omega$ denote the E-subalgebra of $\mathfrak{Z}_\Omega$ generated by the image of T, then this is a finite type E-algebra, and we have a morphism $\operatorname{Spec} \mathfrak{Z}_\Omega \to \operatorname{Spec} \mathfrak{A}_\Omega$. It is not hard to see (e.g. by applying local Langlands over the fraction field of $\mathfrak{A}_\Omega$) that this is a birational map, which is in fact a bijection on points (as one sees by applying local Langlands at the closed points). Unfortunately, we were unable to find a completely conceptual proof in general that this morphism is an isomorphism of varieties over E.

In the case when $\mathfrak{Z}_\Omega$ parameterises supercuspidal representations, one can see this as follows: it suffices to check that one obtains an isomorphism after passing to the formal completion at each closed point $x \in \operatorname{Spec} \mathfrak{Z}_\Omega$. Let π_x be the supercuspidal G-representation corresponding to x, and let $T_x : W_F \to E_x$ the specialisation of T to the image of x in $\operatorname{Spec} \mathfrak{A}_\Omega$. Let R_x be the universal formal deformation ring of T_x, so that we have morphisms

$$\operatorname{Spf} \widehat{\mathfrak{Z}_{\Omega}}_x \to \operatorname{Spf} \widehat{\mathfrak{A}_\Omega} \to \operatorname{Spf} R_x,$$

the second being induced by T. Let $r_x : W_F \to \mathrm{GL}_n(E'_x)$ denote the (absolutely) irreducible representation attached to π_x via local Langlands, where E'_x/E_x is a finite extension, and let T'_x denote the composite $T_x : W_F \to E_x \to E'_x$. Then T'_x is the pseudo-representation attached to r_x. Since r_x is irreducible, the universal formal deformation rings of r_x and T'_x coincide ([Nys96, Théorème 3], [Rou96, Corollaire 6.2]), and are thus both given by $R_x \otimes_{E_x} E'_x$. A direct analysis, using that the source and target are both obtained simply by forming unramified twists, and that local Langlands gives a bijection on isomorphism classes that is compatible with twisting, shows that the composite of the base change to E'_x of the above morphisms is an isomorphism. Since the first of them is dominant, it is also an isomorphism. Thus the morphism $\operatorname{Spec} \mathfrak{Z}_\Omega \to \operatorname{Spec} \mathfrak{A}_\Omega$ is a bijection on closed points and induces isomorphisms after completing at each closed point. From the latter, we see that it is étale and radiciel, hence an open immersion by [EGAIV_4, Théorème 17.9.1] and, since it is also surjective, we see that it is in fact an isomorphism.

One could use a variant of the argument in first paragraph of the proof of Lemma 4.5 to reduce the general case of the lemma to the case when $\mathfrak{Z}_\Omega$ parameterises a family of supercuspidal representations, where the preceding argument then applies. In this way, one could give a slightly more conceptual proof of the Lemma.

Proof of Proposition 4.2. We adopt the notation of §3.13. In particular M is the Levi subgroup in the supercuspidal support of some (thus any) irreducible representation in Ω, and $\mathcal{X}(M)$ is the group of unramified characters of $M(F)$. The group automorphism $\mathcal{X}(M) \xrightarrow{\sim} \mathcal{X}(M)$ given by $\chi_M \mapsto \chi_M |\det|^{\frac{1-n}{2}}$ gives rise to an E-isomorphism $\operatorname{Spec} \mathfrak{Z}_D \xrightarrow{\sim} \operatorname{Spec} \mathfrak{Z}_D$. The latter map is invariant under the $W(D)$-action (the point is that $|\det|$ is invariant under G-conjugation) so it descends to an E-isomorphism $\operatorname{Spec} \mathfrak{Z}_\Omega \xrightarrow{\sim} \operatorname{Spec} \mathfrak{Z}_\Omega$ in view of Lemma 3.22. Let $\mathrm{tw} : \mathfrak{Z}_\Omega \to \mathfrak{Z}_\Omega$ denote the induced isomorphism.

We have a natural isomorphism $\iota_\sigma : \mathcal{H}(\sigma_{\mathrm{sm}}) \xrightarrow{\sim} \mathcal{H}(\sigma)$; viewing Hecke algebras as endomorphism-valued functions on G, this is given by $\psi \mapsto \psi \cdot \sigma_{\mathrm{alg}}$. (This is *a priori* only an injection, but in fact is an isomorphism by the proof of Lemma 1.4 of [ST06].) Now we construct η as the following composite map

$$\mathcal{H}(\sigma) \overset{\iota_\sigma^{-1}}{\simeq} \mathcal{H}(\sigma_{\mathrm{sm}}) \simeq \mathfrak{Z}_\Omega \xrightarrow{\mathrm{tw}} \mathfrak{Z}_\Omega \xrightarrow{I} \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}},$$

where the second map comes from Lemma 3.25. Note that η is already defined over E. To verify the desired interpolation property of η, we let

$x : \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}} \to E_x$ be an E-algebra map. Then $x \circ I \circ \mathrm{tw} : \mathfrak{Z}_\Omega \to E_x$ gives the supercuspidal support of $\pi_{\mathrm{sm}}(\rho_x) = r_p^{-1}(\mathrm{WD}(\rho_x)^{F-\mathrm{ss}})$ by Proposition 4.3; indeed, since I interpolates rec_p by that proposition, $I \circ \mathrm{tw}$ interpolates r_p. In order to complete the proof, we can and do base change to $\overline{\mathbb{Q}}_p$. Then Proposition 3.10 shows us that $\mathfrak{Z}_\Omega$ acts on $\mathrm{Hom}_K\big(\sigma_{\mathrm{sm}}, \pi_{\mathrm{sm}}(\rho_x)\big)$ through $x \circ I \circ \mathrm{tw}$.

To conclude, it is enough to observe that the action of $\mathcal{H}(\sigma_{\mathrm{sm}})$ on the space $\mathrm{Hom}_K\big(\sigma_{\mathrm{sm}}, \pi_{\mathrm{sm}}(\rho_x)\big)$ is compatible with the $\mathfrak{Z}_\Omega$-action on the same space via the isomorphism $\mathcal{H}(\sigma_{\mathrm{sm}}) \simeq \mathfrak{Z}_\Omega$, and also with the $\mathcal{H}(\sigma)$-action on $\mathrm{Hom}_K(\sigma, \pi_{\mathrm{l.alg},x})$ via the canonical isomorphisms between the algebras (via ι_σ) and the modules. These are readily checked. □

In order to deduce Theorem 4.1 from Proposition 4.2, we will now use the results of [Hu09] to show that the image of η is bounded, in the sense that for any $h \in \mathcal{H}(\sigma)$, the valuation of $\eta(h)$ at each point of $\mathcal{X}^{\mathrm{rig}}$ is uniformly bounded.

Recall that $\sigma = \sigma_{\mathrm{alg}} \otimes \sigma_{\mathrm{sm}}$, where $\sigma_{\mathrm{sm}} = \sigma(\tau)$ for some inertial type $\tau : I_F \to \mathrm{GL}_n(E)$. Let $x : R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p] \to E_x$ be a closed point, so that E_x is a finite extension field of E. Then x defines a local Galois representation $\rho_x : G_F \to \mathrm{GL}_n(E_x)$ which is potentially crystalline, and has Hodge–Tate weights determined by the highest weight ξ of σ_{alg}. Set $\pi_x := \pi_{\mathrm{sm}}(\rho_x)$. Recall that $\pi_{\mathrm{l.alg},x}$ is the locally algebraic representation defined over E_x corresponding to ρ_x (see §1.8), whose smooth part is π_x and which determines the character $\chi_{\pi_x} \circ \iota_\sigma^{-1} : \mathcal{H}(\sigma) \to E_x$ (via the action of $\mathcal{H}(\sigma)$ on $\mathrm{Hom}_K(\sigma, \pi_{\mathrm{l.alg},x})$.

Let $P = MN$ be a parabolic subgroup of G, with Levi M and unipotent radical N. Let $Z(M)$ be the centre of M, let $N_0 \subset N$ be a compact open subgroup and define $Z(M)^+ := \{t \in Z(M) | tN_0t^{-1} \subset N_0\}$. When P is a standard (upper) parabolic, the subgroup $Z(M)^+$ of $Z(M)$ consists of elements with non-decreasing p-adic valuations on the diagonal. Then [Eme06a] defines a Jacquet module functor J_P on locally analytic representations of G.

We will consider the following condition on a locally analytic representation V of G.

4.7 Condition. For every parabolic subgroup $P = MN$ as above, with modulus character δ_P, every $\chi : Z(M) \to E^\times$ such that $\mathrm{Hom}_{Z(M)}\big(\chi, J_P(V)\big) \neq 0$, and every $t \in Z(M)^+$, we have $|\chi(t)\delta_P(t)^{-1}|_p \leq 1$.

As above, we write $\tau = \oplus_{i=1}^r (\tau_i)^{d_i}$, where the τ_i are pairwise non-isomorphic I_F-representations corresponding via Theorem 3.7 (the inertial local Langlands correspondence) to cuspidal types $\sigma(\tau_i)$ of $\mathrm{GL}_{e_i}(\mathcal{O}_F)$. Let

$M = \prod_{i=1}^r \mathrm{GL}_{e_i}(F)^{d_i}$ be a standard Levi of G, with corresponding standard parabolic $P = MN$.

From now on until the end of this section, we will replace π_x (as well as χ_{π_x}, σ_{sm}, and so on) by its base extension to $\overline{\mathbb{Q}}_p$. Then as recalled in Section 3.1, π_x is the unique irreducible quotient of a normalised parabolic induction $\tilde{\pi}_x := i_P^G \pi_{x,M}$, where $\pi_{x,M} = \otimes_{i=1}^r (\otimes_{j=1}^{d_i} \pi_{x,i,j})$ such that each $\pi_{x,i,j}$ is a supercuspidal representation of $\mathrm{GL}_{e_i}(F)$ containing the type $\sigma(\tau_i)$ and where for each i, we have $\pi_{x,i,j} \not\simeq \pi_{x,i,j'}(1)$ for $1 \le j' < j \le d_i$.

Proposition 3.10 gives a G-equivariant homomorphism $\varphi : \text{c-Ind}_K^G \sigma_{\mathrm{sm}} \to \tilde{\pi}_x$, which identifies $\tilde{\pi}_x$ with $\text{c-Ind}_K^G \sigma_{\mathrm{sm}} \otimes_{\mathcal{H}(\sigma_{\mathrm{sm}}),\chi_{\pi_x}} \overline{\mathbb{Q}}_p$. We identify $W_{[M,\pi_{x,M}]}$ with $\prod_{i=1}^r S_{d_i}$ in the obvious way, where S_{d_i} is the symmetric group on $\{1, \dots, d_i\}$. Note that $W_{[M,\pi_{x,M}]}$ and the identification are independent of x.

4.8 Lemma. *Let $\chi(\pi_{x,i,j})$ denote the central character of $\pi_{x,i,j}$. For an element $w = \{w_i\}_{i=1}^r \in W_{[M,\pi_{x,M}]}$, define characters $\chi_{x,w} : Z(M) \to \overline{\mathbb{Q}}_p^\times$ by $\chi_{x,w} = \bigotimes_{i=1}^r \bigotimes_{j=1}^{d_i} \chi(\pi_{x,i,w_i(j)})$.*

For every $t \in Z(M)^+$, there exists a constant C_t such that $|\chi_{x,w}(t)|_p \le C_t$ for all points x of $\mathcal{X}^{\mathrm{rig}}$.

Proof. We know that $\sigma_{\mathrm{alg}} \otimes \tilde{\pi}_x$, after twisting by a unitary character (this twist is discussed at the beginning of §5 below), corresponds to the potentially crystalline Galois representation ρ_x with Hodge–Tate weights determined by σ_{alg}, in the sense that $\tilde{\pi}_x|\det|^{1-n} \leftrightarrow \mathrm{WD}(\rho_x)^{F-ss}$ via the modified local Langlands correspondence as in Section 4 of [BS07]. Moreover, note that by Lemma 4.2 of Section 4 of [BS07], $\sigma_{\mathrm{alg}} \otimes \tilde{\pi}_x$ actually has a model over a sufficiently large finite extension of $\mathbb{Q}_p$, so the characters $\chi_{x,w}$ then take values in some sufficiently large finite extension $E'_x/\mathbb{Q}_p$.

The equivalence between parts (ii) and (iv) of [Hu09, Thm. 1.2] (where our coefficient field is taken to be E'_x) shows that $\sigma_{\mathrm{alg}}(\det)^{1-n} \otimes \tilde{\pi}_x|\det|^{1-n}$ has a unitary central character and satisfies Condition 4.7 [Eme06a]. Therefore so does $\sigma_{\mathrm{alg}} \otimes \tilde{\pi}_x$.

Note that by Proposition 4.3.6 of [Eme06a] we have $J_P(\sigma_{\mathrm{alg}} \otimes \tilde{\pi}_x) \xrightarrow{\sim} \sigma_{\mathrm{alg}}^N \otimes r_P^G(\tilde{\pi}_x)\delta_P^{1/2}$, where r_P^G is the normalised Jacquet functor for smooth representations. Putting this formula together with Condition 4.7, we see that then we have $|\chi(t)|_p \le |\sigma_{\mathrm{alg}}^N(t) \cdot \delta_P(t)^{-1/2}|^{-1}$ for every χ occurring in $r_P^G(\tilde{\pi}_x)$.

Now, Proposition 3.2(2) of [Hu09] computes $r_P^G(\tilde{\pi}_x)$ (observe that in the notation of *loc. cit.*, all b_i are 1 in our case) and shows that the characters $\chi_{x,w} = \bigotimes_{i=1}^r \bigotimes_{j=1}^{d_i} \chi(\pi_{x,i,w_i(j)})$ of $Z(M)$ for all sets w of permutations w_i of $\{1, \dots, d_i\}$ occur in $r_P^G(\tilde{\pi}_x)$. The result follows. □

Let $Z(M)^{++} \subset Z(M)$ be the subgroup generated by elements with the property that the p-adic valuations are non-decreasing on the diagonal of each block $\mathrm{GL}_{e_i}^{d_i}$. Clearly, $Z(M)^{+} \subset Z(M)^{++}$.

4.9 Corollary. *The conclusion of Lemma 4.8 holds for all $t \in Z(M)^{++}$.*

Proof. There is a permutation of $\{1, \dots, r\}$ which induces a permutation on the factors $\mathrm{GL}_{e_i}^{d_i}$ of M such that the image of t under that permutation has non-decreasing p-adic valuations. Let M' be the Levi subgroup of G with the permuted blocks as factors. Abstractly, $M' \simeq M$ and by Proposition 6.4 of [Zel80], we know that the induction $i_P^G \pi_M$ is independent of the ordering of the τ_i. We conclude by applying Lemma 4.8 to the Levi M' instead of M. ☐

As discussed in Sections 3.3 and 3.6, we have a semisimple Bushnell–Kutzko type (J, λ) such that σ_{sm} is a direct summand of $\mathrm{Ind}_J^K(\lambda)$, and the natural map $s_P : \mathcal{H}(G, \lambda) \to \mathcal{H}(\sigma_{\mathrm{sm}})$ induces an isomorphism

$$Z\big(\mathcal{H}(G, \lambda)\big) \xrightarrow{\sim} \mathcal{H}(\sigma_{\mathrm{sm}}).$$

In particular, this means that $\tilde{\pi}_x|_J$ contains λ. Then in the notation of Section 3.3, $\pi_{x,M}$ contains the type $(J \cap M, \lambda_M)$. Let $\chi_{\pi_{x,M}}$ be the character by which $\mathcal{H}(M, \lambda_M)$ acts on $\mathrm{Hom}_{J\cap M}(\lambda_M, \pi_M)$.

4.10 Corollary. *Let $t \in Z(M)$ and let $\nu_t \in \mathcal{H}(M, \lambda_M)$ be an intertwiner supported on $t(J \cap M)$. Then there exists a constant C_t such that for all points x of $\mathcal{X}^{\mathrm{rig}}$ we have $|\chi_{\pi_{x,M}}(\nu_t)|_p \le C_t$.*

Proof. Assume $\nu_t \neq 0$. Note that since $t \in Z(M)$, $\nu_t(t)$ commutes with the action of $J \cap M$ on λ_M and since λ_M is irreducible we deduce that $\nu_t(t)$ is a nonzero scalar. Rescaling, we may assume that $\nu_t(t) := \mathrm{id}_{\lambda_M}$. Let $s \in Z(M)^{++}$ be such that $s = {}^w t$ for some $w \in W_{[M, \pi_{x,M}]}$. It follows from the definitions that $\chi_{\pi_{x,M}}(\nu_t) = \chi_{x,w}(s)$, and the corollary then follows from Corollary 4.9. ☐

4.11 Corollary. *Let $\nu \in \mathcal{H}(M, \lambda_M)$. Then there exists a constant C_ν such that $|\chi_{\pi_{x,M}}(\nu)|_p \le C_\nu$ for all points x of $\mathcal{X}^{\mathrm{rig}}$.*

Proof. Since ν has compact support and we only need some bounded constant C_ν, it suffices to prove the claim in the case that ν is an element of the basis of $\mathcal{H}(M, \lambda_M)$ given by Lemma 3.4. Let ν' be the e-fold convolution of ν with itself, where e is as in the statement of Lemma 3.4; then from Corollary 4.10 applied to ν', we see that there is some constant C_ν such that $|\chi_{\pi_{x,M}}(\nu')|_p \le C_\nu^e$ for all x. This implies that $|\chi_{\pi_{x,M}}(\nu)|_p \le C_\nu$, as required. ☐

4.12 Proposition. *For any $\nu \in \mathcal{H}(\sigma)$, there is a constant C_ν such that*

$$|\chi_{\pi_x}\big(\iota_\sigma^{-1}(\nu)\big)|_p \leq C_\nu$$

for all points x of $\mathcal{X}^{\mathrm{rig}}$.

Proof. Recall from Section 3.6 that we have an isomorphism

$$(s_P \circ t_P) : \mathcal{H}(M, \lambda_M)^{W_{[M,\pi_{x,M}]}} \to \mathcal{H}(\sigma_{\mathrm{sm}}).$$

Set $\nu_t := (s_P \circ t_P)^{-1}\big(\iota_\sigma^{-1}(\nu)\big)$. Corollary 4.11 implies that there is a constant C_ν such that $|\chi_{\pi_{x,M}}(\nu_t)|_p \leq C_\nu$ for all x.

Now, in order to conclude, we just need to relate $\chi_{\pi_{x,M}}(\nu_M)$ to $\chi_{\pi_x}\big((s_P \circ t_P)(\nu_M)\big)$ for any $\nu_M \in \mathcal{H}(M, \lambda_M)^{W_{[M,\pi_{x,M}]}}$. As recalled in Section 3.3, i_P^G corresponds on the level of Hecke modules to pushforward along the map t_P. More precisely, if we let $\mathcal{M} := \mathrm{Hom}_{J\cap M}(\lambda_M, \pi_{x,M})$ and $\mathcal{N} := \mathrm{Hom}_J(\lambda, \pi_x)$, then $\mathcal{N} \simeq \mathrm{Hom}_{\mathcal{H}(M,\lambda_M)}(\mathcal{H}(G,\lambda), \mathcal{M})$. Here we view $\mathcal{H}(G,\lambda)$ as a left $\mathcal{H}(M,\lambda_M)$-module via t_P and the action of $\mathcal{H}(G,\lambda)$ on the space of homomorphisms is via right translation. For $z \in Z\big(\mathcal{H}(G,\lambda)\big)$, we note that the right action is also a left action, so the eigenvalue of z on $\mathcal{N}$ is the same as the eigenvalue of $(t_P)^{-1}(z)$ on $\mathcal{M}$. On the other hand, $\mathrm{Hom}_K(\sigma, \pi_x) \simeq e_K\mathcal{N}$ for the idempotent e_K in $\mathcal{H}(G,\lambda)$ which defines σ. Therefore, any eigenvalue of $e_K * z$ on $e_K\mathcal{N}$ is an eigenvalue of z on $\mathcal{N}$. We deduce that $\chi_{\pi_{x,M}}(\nu_M) = \chi_{\pi_x}\big((s_P \circ t_P)(\nu_M)\big)$. ☐

Proof of Theorem 4.1. By [Kis08, Thm 3.3.8] we know that $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$ is a regular ring. Proposition 7.3.6 of [Jon95] (which is applicable because $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$ is in particular normal) then implies that the ring of rigid analytic functions on $\mathcal{X}^{\mathrm{rig}}$ whose absolute value is bounded by 1 coincides precisely with the normalisation of $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)$ in $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$, and so the ring of bounded rigid analytic functions on $\mathcal{X}^{\mathrm{rig}}$ is equal to $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$.

The result then follows immediately from Proposition 4.12 and the defining property of η. ☐

4.13 Remark. In order to deduce Theorem 4.1 from Proposition 4.2 in the *crystalline* case (that is, the case that σ_{sm} is the trivial representation), one could appeal directly to the inequalities in Proposition 3.2 of [BS07] (see also [ST06]). In this case, one can obtain a precise bound in terms of the Hodge–Tate weights (so in terms of $\sigma = \sigma_{\mathrm{alg}}$) on the power of p by which we need to scale the usual generators of the spherical Hecke algebra $\mathcal{H}(\sigma)$. Therefore, in the crystalline case, one can prove that the integral Hecke algebra $\mathcal{H}(\sigma^\circ)$ (with σ° the algebraic representation of K over $\mathcal{O}$ satisfying $\sigma^\circ \otimes_{\mathcal{O}} E \simeq \sigma_{\mathrm{alg}}$) maps to the normalisation of the local deformation ring of type σ. We expect a statement like this should hold true in the general case as well, see Remark 4.21 for more details.

We now return to the global setting. Let the notation be as in Section 2. Fix a K-stable $\mathcal{O}$-lattice σ° in σ. Set

$$M_\infty(\sigma^\circ) := \left(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ)^d)\right)^d,$$

where we are considering homomorphisms that are continuous for the profinite topology on M_∞ and the p-adic topology on $(\sigma^\circ)^d$, and where we equip $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ)^d)$ with the p-adic topology. Note that $M_\infty(\sigma^\circ)$ is an $\mathcal{O}$-torsion free, profinite, linear-topological $\mathcal{O}$-module.

4.14 Lemma. *There is a natural isomorphism of topological $\mathcal{O}$-modules*

$$M_\infty(\sigma^\circ) \cong \varprojlim_n \left(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ/\varpi^n)^\vee)\right)^\vee.$$

Proof. Let $H := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ)^d)$, so that $M_\infty(\sigma^\circ) = H^d$. Then $H^d \cong \varprojlim_n \mathrm{Hom}_{\mathcal{O}}(H, \mathcal{O}/\varpi^n) \cong \varprojlim_n \mathrm{Hom}_{\mathcal{O}}(H/\varpi^n, \mathcal{O}/\varpi^n) \cong \varprojlim_n (H/\varpi^n)^\vee$. Since M_∞ is a projective $\mathcal{O}[[K]]$-module, the short exact sequence

$$0 \to (\sigma^\circ)^d \overset{\varpi^n\cdot}{\to} (\sigma^\circ)^d \to (\sigma^\circ)^d/\varpi^n \to 0$$

yields an isomorphism $H/\varpi^n \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ)^d/\varpi^n)$. Finally,

$$(\sigma^\circ)^d/\varpi^n \cong \mathrm{Hom}_{\mathcal{O}}(\sigma^\circ/\varpi^n, \mathcal{O}/\varpi^n) \cong (\sigma^\circ/\varpi^n)^\vee. \qquad \square$$

4.15 Remark. One may modify the proof of Lemma 4.14 to show that $M_\infty(\sigma^\circ)$ is naturally isomorphic to $\left(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ)^\vee)\right)^\vee$.

4.16 Remark. $M_\infty(\sigma^\circ)$ is essentially the patched module constructed in Section 5.5 of [EG14], although as the conventions and constructions of the current paper differ slightly from those of [EG14] (see e.g. the difference in the choices of v_1, noted in the discussion of Subsection 2.3, as well as Remark 2.9) we will not make this precise.

Let $\mathcal{H}(\sigma^\circ) := \mathrm{End}_G(\text{c-Ind}_K^G \sigma^\circ)$; this is an $\mathcal{O}_E$-subalgebra of $\mathcal{H}(\sigma)$. Note that since σ° is a free $\mathcal{O}$-module of finite rank, it follows from the proof of Theorem 1.2 of [ST02] that Schikhof duality induces an isomorphism

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ)^d) \cong \mathrm{Hom}_K(\sigma^\circ, (M_\infty)^d).$$

Frobenius reciprocity gives $\mathrm{Hom}_K(\sigma^\circ, (M_\infty)^d) \cong \mathrm{Hom}_G(\text{c-Ind}_K^G \sigma^\circ, (M_\infty)^d)$. Thus $M_\infty(\sigma^\circ)$ is equipped with a tautological Hecke action of $\mathcal{H}(\sigma^\circ)$, which commutes with the action of R_∞.

Let $R_\infty(\sigma)$ be the quotient of R_∞ which acts faithfully on $M_\infty(\sigma^\circ)$. (It follows from Lemma 2.16 of [Paš15] that this is independent of the choice of lattice $\sigma^\circ \subset \sigma$.)

Set $R_\infty(\sigma)' := R_\infty \otimes_{R_{\widetilde{\mathfrak{p}}}^\square} R_{\widetilde{\mathfrak{p}}}^\square(\sigma)$.

4.17 Lemma. *1. $R_\infty(\sigma)$ is a reduced $\mathcal{O}$-torsion free quotient of $R_\infty(\sigma)'$.*

2. If $h \in \mathcal{H}(\sigma^\circ)$ is such that $\eta(h) \in R_{\widetilde{\mathfrak{p}}}^\square(\sigma)$, then the action of h on $M_\infty(\sigma^\circ)$ agrees with the action of $\eta(h)$ via the natural map $R_{\widetilde{\mathfrak{p}}}^\square(\sigma) \to R_\infty(\sigma)'$.

Proof. (1) That $R_\infty(\sigma)$ is $\mathcal{O}$-torsion free follows immediately from the fact that by definition it acts faithfully on the $\mathcal{O}$-torsion free module $M_\infty(\sigma^\circ)$. The fact that it is actually a quotient of $R_\infty(\sigma)'$ is then essentially an immediate consequence of classical local-global compatibility at $\widetilde{\mathfrak{p}}$, but to see this will require a little unraveling of the definitions. Note that if N is sufficiently large, then K_N acts trivially on $(\sigma^\circ)^d/\varpi^N$. Recall that Γ_N is defined to be $\mathrm{GL}_n(\mathcal{O}_F/\varpi_F^N\mathcal{O}_F)$. Using Lemma 4.14, we see that

$$\begin{aligned} M_\infty(\sigma^\circ) &= \varprojlim_N \mathrm{Hom}_{\mathcal{O}[\Gamma_{2N}]}\left((M_\infty/\mathfrak{b}_N)_{K_{2N}}, (\sigma^\circ)^d/\varpi^N\right)^\vee \\ &= \varprojlim_N \mathrm{Hom}_{\mathcal{O}[\Gamma_{2N}]}\left((M^\square_{1,Q_{N'(N)}}/\mathfrak{b}_N)_{K_{2N}}, (\sigma^\circ)^d/\varpi^N\right)^\vee, \end{aligned}$$

so it suffices to show that if $N \gg 0$ then the action of R_∞ on

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}\big(M^\square_{1,Q_{N'(N)}}, (\sigma^\circ)^d/\varpi^N\big)$$

factors through $R_\infty(\sigma)'$. Now, by definition we have

$$\begin{aligned} &M^\square_{1,Q_{N'(N)}} = \\ &\quad \mathrm{pr}^\vee\big(S_{\xi,\tau}(U_1(Q_{N'(N)})_{2N'(N)}, \mathcal{O}/\varpi^{N'(N)})^\vee_{\mathfrak{m}_{Q_{N'(N)}}}\big) \otimes_{R^{\mathrm{univ}}_{\mathcal{S}_{Q_{N'(N)}}}} R^{\square_T}_{\mathcal{S}_{Q_{N'(N)}}}, \end{aligned}$$

so it suffices to prove the same result for

$$\begin{aligned} \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}\Big(S_{\xi,\tau}(U_1(Q_{N'(N)})_{2N'(N)}, \mathcal{O}/\varpi^{N'(N)})^\vee_{\mathfrak{m}_{Q_{N'(N)}}} \otimes_{R^{\mathrm{univ}}_{\mathcal{S}_{Q_{N'(N)}}}} R^{\square_T}_{\mathcal{S}_{Q_{N'(N)}}}, \\ (\sigma^\circ)^d/\varpi^N\Big), \end{aligned}$$

which is equal to

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}\Big(S_{\xi,\tau}\big(U_1(Q_{N'(N)})_{2N'(N)}, \mathcal{O}/\varpi^{N'(N)}\big)^{\vee}_{\mathfrak{m}_{Q_{N'(N)}}}, (\sigma^\circ)^d/\varpi^N\Big) \otimes_{R^{\mathrm{univ}}_{\mathcal{S}_{Q_{N'(N)}}}} \big(R^{\square_T}_{\mathcal{S}_{Q_{N'(N)}}}\big)^{\vee},$$

which in turn equals

$$S_{\xi,\tau}\big(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d/\varpi^N\big)_{\mathfrak{m}_{Q_{N'(N)}}} \otimes_{R^{\mathrm{univ}}_{\mathcal{S}_{Q_{N'(N)}}}} \big(R^{\square_T}_{\mathcal{S}_{Q_{N'(N)}}}\big)^{\vee}.$$

Therefore it would suffice to prove the same result for

$$S_{\xi,\tau}\big(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d\big)_{\mathfrak{m}_{Q_{N'(N)}}} \otimes_{R^{\mathrm{univ}}_{\mathcal{S}_{Q_{N'(N)}}}} \big(R^{\square_T}_{\mathcal{S}_{Q_{N'(N)}}}\big)^{\vee}.$$

If $\mathbb{T}$ denotes the image of $\mathbb{T}^{S_p\cup Q_{N'(N)},\mathrm{univ}}$ in the endomorphism ring

$$\mathrm{End}_{\mathcal{O}}\Big(S_{\xi,\tau}\big(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d\big)_{\mathfrak{m}_{Q_{N'(N)}}}\Big),$$

then the action of $R^{\mathrm{univ}}_{\mathcal{S}_{Q_{N'(N)}}}$ on $S_{\xi,\tau}\big(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d\big)_{\mathfrak{m}_{Q_{N'(N)}}}$ is given by an $\mathcal{O}$-algebra homomorphism $R^{\mathrm{univ}}_{\mathcal{S}_{Q_{N'(N)}}} \to \mathbb{T}$. Since the space of automorphic forms $S_{\xi,\tau}\big(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d\big)_{\mathfrak{m}_{Q_{N'(N)}}}$ is $\mathcal{O}$-torsion free (by the choice of U_{m,v_1} in Section 2.3) and the algebra $\mathbb{T}$ is reduced (by the usual comparison between algebraic modular forms and classical automorphic forms, and the semisimplicity of the space of cuspidal automorphic forms; *cf.* [CHT08, Corollary 3.3.3 and §3.4]), then by the definition of $R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)$, we need to show that if $\mathbb{T} \to \overline{\mathbb{Q}}_p$ is a closed point, then the restriction to $G_{F_{\widetilde{\mathfrak{p}}}}$ of the corresponding Galois representation $G_{\widetilde{F}^+,T\cup Q_{N'(N)}} \to \mathcal{G}_n(\overline{\mathbb{Q}}_p)$ is potentially crystalline of type σ; but this is immediate from classical local-global compatibility (see e.g. Theorem 1.1 of [Car14]).

Finally, to see that $R_\infty(\sigma)$ is reduced, we note that by part (2) of Lemma 4.18 below, the ring $R_\infty(\sigma)[1/p]$ is a direct factor of the regular (by [Kis08, Thm 3.3.8]) ring $R_\infty(\sigma)'[1/p]$. Thus $R_\infty(\sigma)[1/p]$ is regular, and in particular reduced, and hence so is its subring $R_\infty(\sigma)$. (The reader can easily check that this reducedness is not used in the proof of Lemma 4.18, and hence no circularity is involved in this argument.)

(2) Again, this is essentially an immediate consequence of classical local-global compatibility at $\widetilde{\mathfrak{p}}$, but a little explanation is needed in order to make this plain.

Note first that the natural action of $\mathcal{H}(\sigma^\circ)$ on $M_\infty(\sigma^\circ)$ is induced via Frobenius reciprocity from the G-action on M_∞, which is patched from the partial G-actions defined by

$$\overline{\alpha}_{N'(N)} : (M_\infty/\mathfrak{b}_N)_{K_{2N}} \to \text{c-Ind}_{KZ}^{G_N}\big((M_\infty/\mathfrak{b}_N)_{K_N}\big)$$

and these in turn, after taking homomorphisms into $(\sigma^\circ)^d/\varpi^N$, induce partial $\mathcal{H}(\sigma^\circ)$-actions on spaces of algebraic modular forms of weight $(\sigma^\circ)^d$. (More precisely, the G-action on ϖ-adically completed cohomology

$$\tilde{S}_{\xi,\tau}\big(U_1^{\mathfrak{p}}(Q_{N'(N)}), \mathcal{O}\big) := \varprojlim_s \Big(\varinjlim_m S_{\xi,\tau}\big(U_1(Q_{N'(N)})_m, \mathcal{O}/\varpi^s\big)\Big)$$

gives rise, via Frobenius reciprocity and the identification

$$S_{\xi,\tau}\big(U_1^{\mathfrak{p}}(Q_{N'(N)})_0, (\sigma^\circ)^d\big) \simeq \mathrm{Hom}_K\Big(\sigma^\circ, \tilde{S}_{\xi,\tau}\big(U_1^{\mathfrak{p}}(Q_{N'(N)}), \mathcal{O}\big)\Big),$$

to a natural action of $\mathcal{H}(\sigma^\circ)$ on $S_{\xi,\tau}\big(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d\big)$.) We see therefore, as in part (1), that it is enough to consider the natural action of each $h \in \mathcal{H}(\sigma^\circ)$ on the spaces

$$S_{\xi,\tau}\big(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d\big)_{\mathfrak{m}_{Q_{N'(N)}}} \otimes_{R^{\mathrm{univ}}_{\mathcal{S}_{Q_{N'(N)}}}} \big(R^{\square_T}_{\mathcal{S}_{Q_{N'(N)}}}\big)^\vee$$

for $N \gg 0$. In addition to the natural action of $\mathcal{H}(\sigma^\circ)$, this is equipped with an action of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma)$ via the composite $R^{\square}_{\tilde{\mathfrak{p}}} \to R^{\mathrm{loc}} \to R^{\square_T}_{\mathcal{S}_{Q_{N'(N)}}}$, which factors through $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma)$ by part (1). By classical local-global compatibility and the defining property of the morphism η of Theorem 4.1, we see that, after inverting p, the action of h on this space agrees with the action of $\eta(h)$. The desired result now follows from the fact that $S_{\xi,\tau}\big(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d\big)_{\mathfrak{m}_{Q_{N'(N)}}}$ is $\mathcal{O}$-torsion free. □

We now use the usual commutative algebra arguments underlying the Taylor–Wiles–Kisin method to study the support of $M_\infty(\sigma^\circ)$.

4.18 Lemma. *1. The module $M_\infty(\sigma^\circ)$ is finitely generated over $R_\infty(\sigma)$ and Cohen–Macaulay, and moreover $M_\infty(\sigma^\circ)[1/p]$ is locally free of rank one over $R_\infty(\sigma)[1/p]$. The topology on $M_\infty(\sigma^\circ)$ coincides with its $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ denotes the maximal ideal of $R_\infty(\sigma)$.*

2. The support of $M_\infty(\sigma^\circ)$ in $\mathrm{Spec}\, R_\infty(\sigma)'$ is a union of irreducible components of $R_\infty(\sigma)'$.

3. *Let $\overline{R}_\infty(\sigma)$ be the normalisation of $R_\infty(\sigma)$ inside $R_\infty(\sigma)[1/p]$. Then the action of $\mathcal{H}(\sigma^\circ)$ on $M_\infty(\sigma^\circ)$ induces an $\mathcal{O}$-algebra map $\alpha : \mathcal{H}(\sigma^\circ) \to \overline{R}_\infty(\sigma)$.*

Proof. Since M_∞ is a finite projective $S_\infty[[K]]$-module, the module $M_\infty(\sigma^\circ)$ is finite and projective (equivalently, free) over S_∞. Indeed, we may write M_∞ as a direct summand of $S_\infty[[K]]^r$ for some $r \geq 0$, and so the space $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}\big(M_\infty, (\sigma^\circ)^d\big)^d$ is a direct summand of

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}\big(S_\infty[[K]]^r, (\sigma^\circ)^d\big)^d \cong \Big(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}\big(S_\infty[[K]], (\sigma^\circ)^d\big)^d\Big)^r.$$

Thus it suffices to note that since $S_\infty[[K]] \cong S_\infty \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[K]]$ as an $\mathcal{O}[[K]]$-module, there is a natural isomorphism

$$\begin{aligned}\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}\big(S_\infty[[K]], (\sigma^\circ)^d\big)^d &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}\big(\mathcal{O}[[K]], \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(S_\infty, (\sigma^\circ)^d)\big)^d \\ &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(S_\infty \otimes_{\mathcal{O}} \sigma^\circ, \mathcal{O})^d \cong S_\infty \otimes_{\mathcal{O}} \sigma^\circ.\end{aligned}$$

Since $M_\infty(\sigma^\circ)$ is free of finite rank over the formal power series ring S_∞, it is Cohen–Macaulay. Since the S_∞-action on $M_\infty(\sigma^\circ)$ factors through the action of R_∞, which in turn factors through $R_\infty(\sigma)$ by definition, we also conclude that $M_\infty(\sigma^\circ)$ is finitely generated over $R_\infty(\sigma)$.

Since the identification of M_∞ as a direct summand of $S_\infty[[K]]$ is compatible with the natural topologies on each of M_∞ and $S_\infty[[K]]$, one easily verifies that the topology on $M_\infty(\sigma^\circ)$ coincides with its $\mathfrak{n}$-adic topology, where $\mathfrak{n}$ denotes the maximal ideal of S_∞. Furthermore, since by definition $R_\infty(\sigma)$ embeds into $\mathrm{End}_{S_\infty}\big(M_\infty(\sigma^\circ)\big)$, we find that $R_\infty(\sigma)$ is finite as an S_∞-algebra, and so in particular the $\mathfrak{n}$-adic topology and $\mathfrak{m}$-adic topology on $M_\infty(\sigma^\circ)$ coincide (where, as in the statement of the lemma, $\mathfrak{m}$ denotes the maximal ideal of $R_\infty(\sigma)$). Thus the topology on $M_\infty(\sigma^\circ)$ coincides with its $\mathfrak{m}$-adic topology.

By Lemma 3.3 of [BLGHT11], Lemma 2.4.19 of [CHT08], and Theorem 3.3.8 of [Kis08], we see that the ring $R_\infty(\sigma)'$ is equidimensional of the same Krull dimension as S_∞. Since $M_\infty(\sigma^\circ)$ is free of finite rank over S_∞, and the image of $R_\infty(\sigma)'$ in $\mathrm{End}(M_\infty(\sigma^\circ))$ is an S_∞-algebra, we see that the depth of $M_\infty(\sigma^\circ)$ as an $R_\infty(\sigma)'$-module is at least the Krull dimension of S_∞. Since this is equal to the Krull dimension of $R_\infty(\sigma)'$, it follows immediately from Lemma 2.3 of [Tay08] that the support of $M_\infty(\sigma^\circ)$ is a union of irreducible components of $R_\infty(\sigma)'$. (Of course, conjecturally $R_\infty(\sigma)$ is actually equal to $R_\infty(\sigma)'$.)

That $M_\infty(\sigma^\circ)[1/p]$ is locally free over $R_\infty(\sigma)[1/p]$ follows by an argument of Diamond (*cf.* [Dia97]). More precisely, $R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p]$ and each of the rings $R^{\sigma,\xi,\tau}_{\widetilde{v}}[1/p]$ for places $v \mid p$, $v \neq \mathfrak{p}$ are regular by Theorem 3.3.8 of [Kis08], and $R^{\square}_{\widetilde{v}_1}$ is formally smooth by Lemma 2.5, so $R_\infty(\sigma)'[1/p]$ is regular by Corollary A.2. Therefore $R_\infty(\sigma)[1/p]$ is also regular, so $M_\infty(\sigma^\circ)[1/p]$ is locally free over $R_\infty(\sigma)[1/p]$ by Lemma 3.3.4 of [Kis09a] (or rather by its proof, which goes over unchanged to our setting, where we do not assume that $R_\infty(\sigma)[1/p]$ is a domain).

That it is actually locally free of rank one can be checked at finite level, where it follows from the multiplicity one assertion in Theorem 3.7, the choice of v_1 (and the fact that we have fixed the action mod p of the Hecke operators at $\widetilde{v}_1$), and the irreducibility of $\overline{\rho}$, together with [Lab11, Thms. 5.4 and 5.9].

This completes the proof of parts (1) and (2), and so we turn to proving (3). To this end, let $\mathcal{A}$ be the R_∞-subalgebra of the endomorphism algebra of $M_\infty(\sigma^\circ)$ generated by $\mathcal{H}(\sigma^\circ)$. Since $M_\infty(\sigma^\circ)$ is a finite type $R_\infty(\sigma)$-module, we see that $\mathcal{A}$ is a finite $R_\infty(\sigma)$-algebra. Since $M_\infty(\sigma^\circ)[1/p]$ is in fact locally free of rank one over $R_\infty(\sigma)[1/p]$ we have the equality $\mathrm{End}_{R_\infty(\sigma)[1/p]}\big(M_\infty(\sigma^\circ)[1/p]\big) = R_\infty(\sigma)[1/p]$, so that $\mathcal{A}[1/p] = R_\infty(\sigma)[1/p]$. So the natural map $\mathcal{H}(\sigma^\circ) \to \mathcal{A}$ lands inside $\overline{R}_\infty(\sigma)$. □

The morphism α of Lemma 4.18 induces an E-algebra morphism $\alpha : \mathcal{H}(\sigma) \to R_\infty(\sigma)[1/p]$.

4.19 Theorem. *The map α coincides with the composition*

$$\mathcal{H}(\sigma) \xrightarrow{\eta} R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p] \to R_\infty(\sigma)[1/p].$$

Proof. Note firstly that if $h \in \mathcal{H}(\sigma^\circ)$ is such that $\eta(h) \in R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)$, then the two maps agree on h by Lemma 4.17 (2). Since $R_\infty(\sigma)$ is p-torsion free by Lemma 4.17 (1), it is therefore enough to show that $\mathcal{H}(\sigma)$ is spanned over E by such elements.

Now, $\mathcal{H}(\sigma^\circ)$ certainly spans $\mathcal{H}(\sigma)$ over E, so it is enough to show that for any element $h' \in \mathcal{H}(\sigma^\circ)$, we have $\eta(p^C h') \in R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)$ for some $C \geq 0$; but this is obvious. □

4.20 Remark. It follows from Lemma 4.18 (2) that the locus of closed points of $\operatorname{Spec} R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p]$ which come from closed points of $\operatorname{Spec} R_\infty(\sigma)[1/p]$ is a union of irreducible components, which we call the set of *automorphic components* of $\operatorname{Spec} R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p]$. (Note that we do not know *a priori* that this notion is independent of the choice of global setting, although of course we expect that in fact every component of $\operatorname{Spec} R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p]$ is an automorphic component.)

4.21 Remark. We expect that $\eta(\mathcal{H}(\sigma^\circ))$ is contained in the normalisation of $R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)$ in $R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p]$; it may well be possible to prove this via our methods, but as we do not need this result, we have not pursued it. It is easy to see that the analogous result holds for the quotient of $R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)$ corresponding to the automorphic components in the sense of Remark 4.20.

4.22. The action of the centre of G

We next prove a structural result (Proposition 4.23 below) which describes the action of the centre Z of G on M_∞.

As usual, we identify Z with $F^\times$, by associating to each element of $F^\times$ the corresponding scalar matrix. Local class field theory then gives an embedding $Z \cong F^\times \xrightarrow{\mathrm{Art}_F} G_F^{ab}$, which we again denote by Art_F.

If $r^{\mathrm{univ}} : G_F \to \mathrm{GL}_n(R^{\square}_{\widetilde{\mathfrak{p}}})$ denotes the universal lift of $\bar{r}$, then its determinant is a character $\det r^{\mathrm{univ}} : G_F^{\mathrm{ab}} \to (R^{\square}_{\widetilde{\mathfrak{p}}})^\times$, which, when composed with Art_F, induces a character $\det r^{\mathrm{univ}} \circ \mathrm{Art}_F : Z \to (R^{\square}_{\widetilde{\mathfrak{p}}})^\times$. If we let Λ_Z denote the completion of the group algebra $\mathcal{O}[Z]$ at the maximal ideal generated by ϖ and the elements $z - (\epsilon^{n(n-1)/2} \det r^{\mathrm{univ}}) \circ \mathrm{Art}_F(z)$, then this character induces a homomorphism $\Lambda_Z \to R^{\square}_{\widetilde{\mathfrak{p}}}$; the corresponding morphism of schemes $\operatorname{Spec} R^{\square}_{\widetilde{\mathfrak{p}}} \to \operatorname{Spec} \Lambda_Z$ simply associates to each deformation r of $\bar{r}$ the character $(\epsilon^{n(n-1)/2} \det r^{\mathrm{univ}}) \circ \mathrm{Art}_F$ of Z; in this optic, the complete local ring Λ_Z is identified with the universal deformation ring of the character $\bar{\epsilon}^{n(n-1)/2} \det \bar{r}$.

By local-global compatibility, Z acts on $S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathbb{F})[\mathfrak{m}_{Q_N}]$ via the character $(\bar{\epsilon}^{n(n-1)/2} \det \bar{r}) \circ \mathrm{Art}_F$. This implies the elements of the form $z - (\epsilon^{n(n-1)/2} \det r^{\mathrm{univ}}) \circ \mathrm{Art}_F(z)$ act nilpotently on $M_\infty^\vee[\mathfrak{m}_\infty^n]$, for all $n \geq 1$ and for all $z \in Z$. Hence the action of $\mathcal{O}[Z]$ on $M_\infty^\vee$ extends to a continuous action of Λ_Z. Pontryagin duality induces an isomorphism $\mathrm{End}^{\mathrm{cont}}_{\mathcal{O}}(M_\infty^\vee) \cong (\mathrm{End}^{\mathrm{cont}}_{\mathcal{O}}(M_\infty))^{\mathrm{op}}$, which makes M_∞ into a continuous Λ_Z^{op}-module. Since Λ_Z is commutative, $\Lambda_Z^{\mathrm{op}} = \Lambda_Z$.

Let ϖ_F be a choice of uniformiser of F and let $z = \mathrm{diag}(\varpi_F, \ldots, \varpi_F) \in Z$, so that $Z = (Z \cap K) z^{\mathbb{Z}}$. Write $S = z - [\mu] \in \mathcal{O}[Z]$, where we set $\mu = (\bar{\epsilon}^{n(n-1)/2} \det \bar{r}) \circ \mathrm{Art}_F(\varpi_F)$. Let Λ be the closure in Λ_Z of the subring $\mathcal{O}[S]$. Since S lies in the maximal ideal of Λ_Z, we see that Λ is isomorphic to $\mathcal{O}[[S]]$. We now make use of the category of pseudo-compact $\Lambda[[K]]$-modules; see [Gab62, §IV.3], [Bru66] for the definition and properties of this category.

4.23 Proposition. *M_∞ is projective in the category of pseudo-compact $\Lambda[[K]]$-modules.*

Proof. Let P be a pro-p Sylow subgroup of K. Since the index $(K : P)$ is finite and is not divisible by p, it is enough to show that M_∞ is projective in the category of pseudo-compact $\Lambda[[P]]$-modules. We will in fact show that M_∞ is a pro-free $\Lambda[[P]]$-module, i.e. it is isomorphic to a product of copies of $\Lambda[[P]]$.

Since P is a pro-p group, $\Lambda[[P]]$ is a local ring with residue field $\mathbb{F}$. It follows from the topological Nakayama's lemma for pseudo-compact $\Lambda[[P]]$-modules, that it is enough to show that the first right derived functor of $-\widehat{\otimes}_{\Lambda[[P]]}\mathbb{F}$ vanishes, see [Bru66, Proposition 3.1]. We denote this derived functor by $\widehat{\mathrm{Tor}}_1^{\Lambda[[P]]}(\mathbb{F}, M_\infty)$.

Note that the functor $-\widehat{\otimes}_{\Lambda[[P]]}\mathbb{F}$ is the composite of the two functors $-\widehat{\otimes}_{\mathcal{O}[[P]}\mathbb{F}$ and $-\widehat{\otimes}_{\Lambda/\varpi}\mathbb{F}$. Considering the corresponding spectral sequence, we see that it is enough to show that both $\widehat{\mathrm{Tor}}_1^{\mathcal{O}[[P]]}(\mathbb{F}, M_\infty)$ and $\widehat{\mathrm{Tor}}_1^{\Lambda/\varpi}(\mathbb{F}, M_\infty\widehat{\otimes}_{\mathcal{O}[[P]]}\mathbb{F})$ vanish.

We know by Proposition 2.10 that M_∞ is a finitely generated projective $S_\infty[[K]]$-module, thus a free, finitely generated $S_\infty[[P]]$-module, and thus projective in the category of pseudo-compact $\mathcal{O}[[P]]$-modules. This implies that $\widehat{\mathrm{Tor}}_1^{\mathcal{O}[[P]]}(\mathbb{F}, M_\infty)$ vanishes.

Since $\Lambda/\varpi \cong \mathbb{F}[[S]]$ is a DVR, to show that $\widehat{\mathrm{Tor}}_1^{\Lambda/\varpi}(\mathbb{F}, M_\infty\widehat{\otimes}_{\mathcal{O}[[P]]}\mathbb{F})$ vanishes it suffices to show that $M_\infty\widehat{\otimes}_{\mathcal{O}[[P]]}\mathbb{F}$ is S-torsion free. Since M_∞ is a free finitely generated $S_\infty[[P]]$-module, $M_\infty\widehat{\otimes}_{\mathcal{O}[[P]]}\mathbb{F}$ is a free finitely generated S_∞/ϖ-module. Since S_∞/ϖ is a domain, it is enough to show that there is a polynomial $q \in \mathbb{F}[X]$ with zero constant term, such that the action of $q(S)$ on $M_\infty\widehat{\otimes}_{\mathcal{O}[[P]]}\mathbb{F}$ is given by a non-zero element of S_∞/ϖ.

We will now construct such a polynomial q, thus finishing the proof. Let $\widetilde{Z}$ denote the centre of $\widetilde{G}$, so that in particular $\widetilde{Z}(\mathbb{A}^\infty_{\tilde{F}^+})$ contains Z; since the group $\widetilde{Z}(\mathbb{A}^\infty_{\tilde{F}^+})/\big(\widetilde{Z}(\mathbb{A}^\infty_{\tilde{F}^+})\cap U_0\big)\widetilde{Z}(\tilde{F}^+)$ is finite, we can choose some $a > 0$ for which we can write $z^a = u\gamma$, with $u \in \widetilde{Z}(\mathbb{A}^\infty_{\tilde{F}^+})\cap U_0$ and $\gamma \in \widetilde{Z}(\tilde{F}^+)$. We may write $u = u_p u^p$ with $u_p \in Z \cap K$, $u^p \in U_0^{\mathfrak{p}} \cap \widetilde{Z}(\mathbb{A}^\infty_{\tilde{F}^+})$. By replacing a by a multiple we may assume that $u_p \in Z \cap P$.

We claim that $q(X) = (X+\mu)^a - \mu^a$ does the job, and note that $q(S) = z^a - \mu^a = \gamma u^p u_p - \mu^a$. Since γ acts trivially on each M_{1,Q_N}, it acts trivially on M_∞, and hence the action of z on M_∞ coincides with the action of $u_p u^p$. Since $u_p \in Z\cap P$, the action of $q(S)$ on $M_\infty\widehat{\otimes}_{\mathcal{O}[[P]]}\mathbb{F}$ coincides with the action of $u^p - \mu^a$. The action of u^p on each M_{1,Q_N} factors through that of Δ_{Q_N}.

(Indeed, for each $v \in Q_N$, the component of u^p at v lies in $U_0(Q_N)_v$; since $U_1(Q_n)_v$ acts trivially on M_{1,Q_N}, the claim is immediate from the definition of Δ_{Q_N}.) Thus the action of $q(S)$ on $M_\infty \widehat{\otimes}_{\mathcal{O}[[P]]} \mathbb{F}$ coincides with the action of an element of S_∞/ϖ, and this action is zero if and only if u^p acts trivially on M_∞ and $\mu^a = 1$. In this case, z^a would act trivially on $M_\infty/\mathfrak{I}M_\infty$, where $\mathfrak{I}$ is the maximal ideal of $\mathcal{O}[[Z \cap P]]$.

Suppose for the sake of contradiction that this happens. We choose a locally algebraic type σ such that $M_\infty(\sigma^\circ) \neq 0$. It follows from [Paš15, Lemma 2.14] and the fact that $Z \cap P$ acts trivially on σ°/ϖ that

$$\begin{aligned} M_\infty(\sigma^\circ)/\varpi M_\infty(\sigma^\circ) &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty/\varpi M_\infty, (\sigma^\circ/\varpi)^\vee)^\vee \\ &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty/\mathfrak{I}M_\infty, (\sigma^\circ/\varpi)^\vee)^\vee. \end{aligned}$$

Thus to see that $(z^a - \mu^a)$ does not act by zero on $M_\infty/\mathfrak{I}M_\infty$, it is enough to show that it does not act by zero on $M_\infty(\sigma^\circ)/\varpi M_\infty(\sigma^\circ)$.

Theorem 4.19 implies that the action of z on $M_\infty(\sigma^\circ)$ is the same as the action of z under the map $\mathcal{H}(\sigma^\circ) \to R^\square_{\tilde{\mathfrak{p}}}(\sigma) \to R_\infty(\sigma)$, which can be checked to be compatible with the map $\Lambda_Z \to R^\square_{\tilde{\mathfrak{p}}} \to R_\infty$. Explicitly, if Frob_p is the element of G_F^{ab} corresponding to ϖ_F by local class field theory, then the action of z on $M_\infty(\sigma^\circ)$ matches the determinant of Frob_p. Then by Theorem 4.19, $M_\infty(\sigma^\circ)/\varpi_E M_\infty(\sigma^\circ)$ would be supported on a quotient of $R^\square_{\tilde{\mathfrak{p}}}(\sigma)$ corresponding to representations where the determinant of Frob_p^a is fixed; that is to say (again by local-global compatibility), for any representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ arising from a $\overline{\mathbb{Q}}_p$-valued point of $\operatorname{Spec} R(\sigma)[1/p]$, the value of $\det r$ on Frob_p^a would be determined modulo ϖ.

However, we know that $\operatorname{Spec} R(\sigma)$ is a union of irreducible components of $\operatorname{Spec} R(\sigma)' := \operatorname{Spec} R_\infty \otimes_{R^\square_{\tilde{\mathfrak{p}}}} R^\square_{\tilde{\mathfrak{p}}}(\sigma)$, and hence any twist of r by an unramified character which is trivial modulo $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$ also arises from a $\overline{\mathbb{Q}}_p$-valued point of $\operatorname{Spec} R(\sigma)$. Making an unramified twist by an appropriate character (e.g. by a character which takes Frob_p to $1 + \varpi_{E'}$, for some sufficiently ramified extension E' of E), shows that $\det r(\mathrm{Frob}_p^a)$ is not constant mod ϖ_E, as required. □

4.24 Remark. Since Pontrjagin duality induces an anti-equivalence between pseudo-compact and discrete $\Lambda[[K]]$-modules [Bru66, Prop. 2.3], Proposition 4.23 may be reformulated as the statement that $M_\infty^\vee$ is injective in the category of discrete $\Lambda[[K]]$-modules.

4.25 Remark. Following [Eme10, Definition 2.3.1], we say that a smooth representation V of ZK is locally Z-finite, if for every $v \in V$ the $\mathcal{O}[Z]$-

submodule generated by v is finitely generated as an $\mathcal{O}$-module. Such representations form a full subcategory $\mathrm{Mod}_{ZK}^{Z-\mathrm{fin}}(\mathcal{O})$ of the category $\mathrm{Mod}_{ZK}^{\mathrm{sm}}(\mathcal{O})$ of smooth representations of ZK on $\mathcal{O}$-torsion modules. The category of discrete $\Lambda[[K]]$-modules coincides with the full subcategory of $\mathrm{Mod}_{ZK}^{Z-\mathrm{fin}}(\mathcal{O})$ consisting of representations V, such that every $v \in V$ is annihilated by a power of the maximal ideal of Λ.

It follows from the Chinese remainder theorem that the category of discrete $\Lambda[[K]]$-modules is a direct summand of $\mathrm{Mod}_{ZK}^{Z-\mathrm{fin}}(\mathcal{O})$, so by Remark 4.24, $M_\infty^\vee$ is injective in $\mathrm{Mod}_{ZK}^{Z-\mathrm{fin}}(\mathcal{O})$. Since K is compact, every smooth representation of K is locally admissible. Combining this observation with [Eme10, Lemma 2.3.4] we deduce that $\mathrm{Mod}_{ZK}^{Z-\mathrm{fin}}(\mathcal{O})$ coincides with the category of locally admissible representation of ZK on $\mathcal{O}$-torsion modules, $\mathrm{Mod}_{ZK}^{\mathrm{l.adm}}(\mathcal{O})$. Thus $M_\infty^\vee$ is injective in $\mathrm{Mod}_{ZK}^{\mathrm{l.adm}}(\mathcal{O})$.

In some situations it can be useful to consider quotients of M_∞ having a fixed central character. (This corresponds, on the Galois side, to considering deformations of $\bar{r}$ having a fixed determinant.) To this end, we state and prove Corollary 4.26 below.

Let $x : \Lambda_Z \to \mathcal{O}$ be an $\mathcal{O}$-algebra homomorphism. Let ξ be the composition $Z \to \Lambda_Z^\times \xrightarrow{x} \mathcal{O}^\times$. Let $\mathrm{Mod}_{ZK}^{\mathrm{sm},\xi}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_{ZK}^{\mathrm{sm}}(\mathcal{O})$ consisting of those representations on which Z acts by the character ξ^{-1}. Let $\mathrm{Mod}_{ZK}^{\mathrm{pro.aug}}(\mathcal{O})$ be the category of profinite augmented representations of ZK over $\mathcal{O}$, as defined in [Eme10, Definition 2.1.6]. Pontrjagin duality induces an anti-equivalence of categories between $\mathrm{Mod}_{ZK}^{\mathrm{sm}}(\mathcal{O})$ and $\mathrm{Mod}_{ZK}^{\mathrm{pro.aug}}(\mathcal{O})$, [Eme10, (2.2.8)]. Let $\mathfrak{C}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_{ZK}^{\mathrm{pro.aug}}(\mathcal{O})$ consisting of those representations on which Z acts by the character ξ, so that $\mathfrak{C}(\mathcal{O})$ is anti-equivalent to $\mathrm{Mod}_{ZK}^{\mathrm{sm},\xi}(\mathcal{O})$ via Pontrjagin duality.

4.26 Corollary. *$M_\infty \widehat{\otimes}_{\Lambda_Z,x} \mathcal{O}$ is a non-zero, projective object in $\mathfrak{C}(\mathcal{O})$.*

Proof. Note that $\mathfrak{C}(\mathcal{O})$ is naturally a full subcategory of the category of pseudo-compact $\Lambda[[K]]$-modules. The projectivity of $M_\infty \widehat{\otimes}_{\Lambda_Z,x} \mathcal{O}$ follows from the fact that the functor $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(M_\infty \widehat{\otimes}_{\Lambda_Z,x} \mathcal{O}, -)$ coincides with the restriction of the functor $\mathrm{Hom}_{\Lambda[[K]]}^{\mathrm{cont}}(M_\infty, -)$ to $\mathfrak{C}(\mathcal{O})$; the exactness of the latter follows from Proposition 4.23. The reduction of $M_\infty \widehat{\otimes}_{\Lambda_Z,x} \mathcal{O}$ modulo ϖ is isomorphic to $M_\infty \widehat{\otimes}_{\Lambda_Z} \mathbb{F}$, which is non-zero, as otherwise the topological Nakayama's lemma for pseudo-compact Λ_Z-modules would imply that M_∞ is zero. □

4.27 Remark. The same argument as in Remark 4.24 shows that the dual $(M_\infty \widehat{\otimes}_{\Lambda_Z,x} \mathcal{O})^\vee$ is injective in $\mathrm{Mod}_{ZK}^{\mathrm{sm},\xi}(\mathcal{O})$.

4.28. Locally algebraic vectors

We conclude this section by computing the locally algebraic vectors in $V(r)$, in the case when r is a generic, potentially crystalline point of some $R_\infty(\sigma)$. (In other words, we identify the locally algebraic vectors at such points on automorphic components of type σ.)

Let Ω be a Bernstein component corresponding to the inertial type τ and let $(J, \lambda(\tau))$ be a type for this component, as in §3.3. The representation $\sigma(\tau)$ in Theorem 3.7 is a quotient of $\operatorname{Ind}_J^K \lambda(\tau)$.

4.29 Remark. To orientate the reader not familiar with types, the example to keep in mind is the following: if τ is a direct sum of copies of the trivial representation, then J is the Iwahori subgroup, $\lambda(\tau)$ is the trivial representation of J, and $\sigma(\tau)$ is the trivial representation of K. The Steinberg representation St lies in Ω, but $\operatorname{Hom}_K(\sigma(\tau), \mathrm{St}) = 0$. If we worked only with $\sigma(\tau)$, we would not be able to control copies of St tensored with an algebraic representation inside the locally algebraic vectors of our patched modules. This explains the need to work with $\lambda(\tau)$ instead of $\sigma(\tau)$.

We will redo some of the lemmas in §4 with λ instead of σ. We denote by λ_{alg} the representation denoted by σ_{alg} in §4. We let $\lambda := \lambda(\tau) \otimes \lambda_{\mathrm{alg}}$ and fix a J-stable $\mathcal{O}$-lattice λ° in λ. Set

$$M_\infty(\lambda^\circ) := \left(\operatorname{Hom}^{\mathrm{cont}}_{\mathcal{O}[[J]]}(M_\infty, (\lambda^\circ)^d)\right)^d.$$

4.30 Lemma. *$M_\infty(\lambda^\circ)$ is a free S_∞-module of finite rank.*

Proof. This follows from the fact, proved in Proposition 2.10, that M_∞ is projective as an $S_\infty[[K]]$-module; see the proof of Lemma 4.18. □

Let $R_\infty(\lambda)$ be the quotient of R_∞ which acts faithfully on $M_\infty(\lambda^\circ)$, and set $R_\infty(\lambda)' := R_\infty \otimes_{R^\square_{\tilde{\mathfrak{p}}}} R^\square_{\tilde{\mathfrak{p}}}(\lambda)$, where $R^\square_{\tilde{\mathfrak{p}}}(\lambda)$ is the unique reduced and p-torsion free quotient of $R^\square_{\tilde{\mathfrak{p}}}$ corresponding to potentially semi-stable lifts of weight λ_{alg} and inertial type τ.

4.31 Lemma. *$R_\infty(\lambda)$ is a p-torsion free quotient of $R_\infty(\lambda)'$.*

Proof. The proof is the same as the proof of part (1) of Lemma 4.17. □

4.32 Lemma. *The support of $M_\infty(\lambda^\circ)$ in* $\operatorname{Spec} R_\infty(\lambda)'$ *is a union of irreducible components of $R_\infty(\lambda)'$. In particular, $R_\infty(\lambda)$ is reduced.*

Proof. The first assertion follows from Lemma 4.30 and the fact that S_∞ and $R_\infty(\lambda)'$ have the same Krull dimension. Since $R_\infty(\lambda)'$ is reduced, any non-reduced quotient of the same dimension will have an associated prime, which is not minimal. It follows from Lemma 4.30 that $M_\infty(\lambda^\circ)$ is a faithful Cohen–Macaulay module over $R_\infty(\lambda)$, thus this cannot happen, and so $R_\infty(\lambda)$ is reduced. ☐

4.33 Proposition. *Let x be a closed E-valued point of* $\operatorname{Spec} R_\infty(\lambda)[1/p]$, *let r_x be the corresponding Galois representation and let $V(r_x)$ be the unitary Banach space representation defined in §2.12, and let $V(r_x)^{\mathrm{l.alg}}$ be the subspace of locally algebraic vectors in $V(r_x)$. Then $V(r_x)^{\mathrm{l.alg}} \cong \pi \otimes \pi_{\mathrm{alg}}(r_x)$, where π is a smooth admissible representation which lies in Ω.*

Proof. Let $\Pi^{\mathrm{l.alg}}$ be any locally algebraic representation of G. Let W be an irreducible algebraic representation of G. We assume that E is large enough, so that any such W is absolutely irreducible. Then W is also an absolutely irreducible representation of the Lie algebra of G and this category is semi-simple, as E has characteristic 0. This induces an isomorphism

$$\Pi^{\mathrm{l.alg}} \cong \bigoplus_W \operatorname{Hom}_E(W, \Pi^{\mathrm{l.alg}})^{\mathrm{sm}} \otimes_E W,$$

where the sum is taken over all irreducible algebraic representations W of G and $\operatorname{Hom}_E(W, \Pi^{\mathrm{l.alg}})^{\mathrm{sm}}$ denotes the smooth vectors for the conjugation action of G on $\operatorname{Hom}_E(W, \Pi^{\mathrm{l.alg}})$. The theory of the Bernstein centre asserts that any smooth representation π of G decomposes as

$$\pi \cong \bigoplus_\Omega \pi[\Omega],$$

where the sum is taken over all the Bernstein components and $\pi[\Omega]$ is the maximal subquotient of π lying in Ω. Thus

$$\Pi^{\mathrm{l.alg}} \cong \bigoplus_{W,\Omega} \operatorname{Hom}_E(W, \Pi^{\mathrm{l.alg}})^{\mathrm{sm}}[\Omega] \otimes_E W.$$

We claim that $V(r_x)^{\mathrm{l.alg}} \cong \pi \otimes \pi_{\mathrm{alg}}(r_x)$ with π in Ω. If this was not the case then by the above there would be $\lambda' = \lambda_{\mathrm{sm}}(\tau') \otimes \lambda'_{\mathrm{alg}}$, such that either $\tau \not\cong \tau'$ or $\lambda'_{\mathrm{alg}} \not\cong \lambda_{\mathrm{alg}}$ and $\operatorname{Hom}_{J'}(\lambda', V(r_x)^{\mathrm{l.alg}}) \neq 0$. But Lemma 4.31 implies that the inertial type of r_x is τ' and the Hodge–Tate weights correspond to the highest weight of λ'_{alg}. This is a contradiction. Since π lies in Ω, π is admissible if and only if $\mathrm{Hom}_J(\lambda(\tau),\pi)$ is finite dimensional. We have

$$\dim_E \mathrm{Hom}_J(\lambda(\tau),\pi) = \dim_E \mathrm{Hom}_J(\lambda, V(r_x)) = \dim_E M_\infty(\lambda^\circ)\otimes_{R_\infty,x} E,$$

where the last equality follows from [Paš15, Prop. 2.20]. Since $M_\infty(\lambda^\circ)$ is a finitely generated R_∞-module, we deduce that the above dimensions are finite and hence π is admissible. □

4.34 Proposition. *Let x, y be closed, E-valued points of* $\mathrm{Spec}\, R_\infty(\lambda)[1/p]$, *lying on the same irreducible component. If x is smooth then*

$$\dim_E \mathrm{Hom}_J(\lambda, V(r_x)^{\mathrm{l.alg}}) \le \dim_E \mathrm{Hom}_J(\lambda, V(r_y)^{\mathrm{l.alg}}).$$

Proof. Since λ is locally algebraic we have

$$\mathrm{Hom}_J(\lambda, V(r_y)^{\mathrm{l.alg}}) \cong \mathrm{Hom}_J(\lambda, V(r_y)).$$

It follows from Proposition 2.20 of [Paš15] that

$$\dim_E \mathrm{Hom}_J(\lambda, V(r_y)) = \dim_E M_\infty(\lambda^\circ)\otimes_{R_\infty,y} E.$$

If x is a smooth closed point of $\mathrm{Spec}\, R_\infty(\lambda^\circ)[1/p]$ then the localisation $R_\infty(\lambda^\circ)_{\mathfrak{m}_x}$ at x is a regular ring. Since $M_\infty(\lambda^\circ)$ is a Cohen–Macaulay module, so is its localisation $M_\infty(\lambda^\circ)_{\mathfrak{m}_x}$ at x. Since $R_\infty(\lambda^\circ)_{\mathfrak{m}_x}$ is regular, the standard argument using the Auslander–Buchsbaum theorem allows us to conclude $M_\infty(\lambda^\circ)_{\mathfrak{m}_x}$ is a free $R_\infty(\lambda^\circ)_{\mathfrak{m}_x}$-module of rank equal to $\dim_E M_\infty(\lambda^\circ)\otimes_{R_\infty,x} E$. Let $V(\mathfrak{q})$ be the irreducible component of $\mathrm{Spec}\, R_\infty(\lambda^\circ)$ containing x. By further localising $M_\infty(\lambda^\circ)_{\mathfrak{m}_x}$ at $\mathfrak{q}$ we deduce that

$$\dim_E M_\infty(\lambda^\circ)\otimes_{R_\infty,x} E = \dim_{\kappa(\mathfrak{q})} M_\infty(\lambda^\circ)\otimes_{R_\infty} \kappa(\mathfrak{q}).$$

Since the function $\mathfrak{p}\mapsto \dim_{\kappa(\mathfrak{p})} M_\infty(\lambda^\circ)\otimes_{R_\infty}\kappa(\mathfrak{p})$ is upper semi-continuous on $\mathrm{Spec}\, R_\infty$, we conclude that for any E-valued point $y\in V(\mathfrak{q})$ we have

$$\dim_{\kappa(\mathfrak{q})} M_\infty(\lambda^\circ)\otimes_{R_\infty}\kappa(\mathfrak{q}) \le \dim_E M_\infty(\lambda^\circ)\otimes_{R_\infty,y} E.$$ □

4.35 Theorem. *Let x be a closed E-valued point of* $\mathrm{Spec}\, R_\infty(\sigma)[1/p]$, *such that $\pi_{\mathrm{sm}}(r_x)$ is generic. Then*

$$V(r_x)^{\mathrm{l.alg}} \cong \pi_{\mathrm{sm}}(r_x)\otimes \pi_{\mathrm{l.alg}}(r_x).$$

Proof. We claim that x is a smooth point of $\operatorname{Spec} R_\infty(\lambda)$. Lemma 4.32 implies that it is enough to check that x is a smooth point of $\operatorname{Spec} R_\infty(\lambda)'$. It follows from [Kis08, Thm 3.3.8] that $R_{\tilde{v}}^{\square,\xi,\tau}[1/p]$ is a regular ring for all $v \in S_p \setminus \{\mathfrak{p}\}$. Moreover, both $R_{\tilde{v}_1}^{\square}$ and $\mathcal{O}[[x_1,\dots,x_{q-[\tilde{F}^+:\mathbb{Q}]n(n-1)/2}]]$ are regular rings (the former by Lemma 2.5). We let

$$B := \left(\widehat{\otimes}_{v\in S_p\setminus\{\mathfrak{p}\}} R_{\tilde{v}}^{\square,\xi,\tau}\right) \widehat{\otimes}\, R_{\tilde{v}_1}^{\square} \,\widehat{\otimes}\, \mathcal{O}[[x_1,\dots,x_{q-[\tilde{F}^+:\mathbb{Q}]n(n-1)/2}]].$$

Corollary A.2 implies that $B[1/p]$ is a regular ring. It follows from [All14, Thm. D] that the restriction of r_x to G_F defines a smooth point $x_{\tilde{\mathfrak{p}}}$ of $\operatorname{Spec} R_{\tilde{\mathfrak{p}}}^{\square}(\lambda)[1/p]$. Since complete local noetherian rings are excellent and the localisations of excellent rings are excellent, the smooth locus is open in $\operatorname{Spec} R_{\tilde{\mathfrak{p}}}^{\square}(\lambda)[1/p]$. Thus there is $f \in R_{\tilde{\mathfrak{p}}}^{\square}(\lambda)$, such that $\operatorname{Spec} R_{\tilde{\mathfrak{p}}}^{\square}(\lambda)[1/pf]$ is an open neighborhood of $x_{\tilde{\mathfrak{p}}}$ contained in the smooth locus. It follows from Corollary A.2 that $(R_{\tilde{\mathfrak{p}}}^{\square}(\lambda)\,\widehat{\otimes}_{\mathcal{O}} B)[1/pf]$ is a regular ring. Since $R_\infty(\lambda)' = R_{\tilde{\mathfrak{p}}}^{\square}(\lambda)\,\widehat{\otimes}_{\mathcal{O}} B$, this proves the claim.

Let y be any closed point in $V(\mathfrak{a}) \cap \operatorname{Spec} R_\infty(\sigma)[1/p]$, where

$$\mathfrak{a} = (y_1,\dots,y_h) \subset S_\infty.$$

It follows from the Corollary 2.11 via Proposition 2.20 of [Paš15], that $V(r_y)$ is identified with the closed subspace of the completed cohomology $\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} E$, consisting of vectors annihilated by the maximal ideal $\mathfrak{m}_y$ corresponding to y. Thus

$$V(r_y)^{\mathrm{l.alg}} \cong (\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} E)^{\mathrm{l.alg}}[\mathfrak{m}_y] \cong \pi_{\mathrm{sm}}(r_y)\otimes\pi_{\mathrm{alg}}(r_y).$$

The last isomorphism follows from Prop. 3.2.4 of [Eme06b], which shows that locally algebraic vectors of any given weight are precisely the algebraic automorphic forms of that weight, together with classical local-global compatibility (Thm 1.1 of [Car14]). A priori, $\pi_{\mathrm{sm}}(r_y)\otimes\pi_{\mathrm{alg}}(r_y)$ may appear with some multiplicity, but this multiplicity is seen to be 1 by our choice of $U^{\mathfrak{p}}$ (and the fact that we have fixed the action mod p of the Hecke operators at $\tilde{v}_1$), and the irreducibility of $\overline{\rho}$, together with [Lab11, Thms. 5.4 and 5.9].

Proposition 4.33 implies that $V(r_x)^{\mathrm{l.alg}} \cong \pi\otimes\pi_{\mathrm{alg}}(r_x)$, where π is a smooth representation lying in Ω. Since x lies in the support of $M_\infty(\sigma)$, $\operatorname{Hom}_K(\sigma(\tau),\pi)\neq 0$. It follows from Theorems 4.1, 4.19 and Corollary 3.12 that $\pi_{\mathrm{sm}}(r_x)$ is a subrepresentation of π. Since $\lambda(\tau)$ is a type for Ω, it is enough to show that

$$\dim_E \operatorname{Hom}_J(\lambda(\tau), \pi) \leq \dim_E \operatorname{Hom}_J(\lambda(\tau), \pi_{\mathrm{sm}}(r_x)).$$

Since r_x and r_y have the same Hodge–Tate weights, $\pi_{\mathrm{alg}}(r_y) = \pi_{\mathrm{alg}}(r_x)$, and the restriction of these representations to J is equal to λ_{alg}. Since λ_{alg} is an irreducible representation of the Lie algebra of G, we have isomorphisms

$$\operatorname{Hom}_J(\lambda(\tau), \pi) \cong \operatorname{Hom}_J(\lambda, V(r_x)^{\mathrm{l.alg}})$$

and

$$\operatorname{Hom}_J(\lambda(\tau), \pi_{\mathrm{sm}}(r_y)) \cong \operatorname{Hom}_J(\lambda, V(r_y)^{\mathrm{l.alg}}).$$

Proposition 4.34 implies that

$$\dim_E \operatorname{Hom}_J(\lambda(\tau), \pi) \leq \dim_E \operatorname{Hom}_J(\lambda(\tau), \pi_{\mathrm{sm}}(r_y)).$$

Thus it is enough to show that

$$\dim_E \operatorname{Hom}_J(\lambda(\tau), \pi_{\mathrm{sm}}(r_y)) \leq \dim_E \operatorname{Hom}_J(\lambda(\tau), \pi_{\mathrm{sm}}(r_x)).$$

Since both r_x and r_y are potentially crystalline Theorem 3.7 together with Proposition 3.10 implies that there is a surjection

$$\pi'_1 \times \ldots \times \pi'_r \twoheadrightarrow \pi_{\mathrm{sm}}(r_y).$$

Since $\pi_{\mathrm{sm}}(r_x)$ is assumed to be generic, the same argument together with Corollary 3.12 gives an isomorphism

$$\pi_1 \times \ldots \times \pi_r \cong \pi_{\mathrm{sm}}(r_x).$$

Moreover, in this case we have $\pi_1 \times \ldots \times \pi_r \cong \pi_{\alpha(1)} \times \ldots \times \pi_{\alpha(r)}$ for any permutation $\alpha \in S_r$. Since $\pi_{\mathrm{sm}}(r_x)$ and $\pi_{\mathrm{sm}}(r_y)$ lie in the same Bernstein component, they have the same inertial support. Thus we may assume that each π'_i is a twist of π_i by an unramified character. This implies that there is a J-equivariant surjection $\pi_{\mathrm{sm}}(r_x)|_J \twoheadrightarrow \pi_{\mathrm{sm}}(r_y)|_J$, which implies the desired inequality. □

4.36 Remark. When r is ordinary (more precisely when r satisfies the assumptions on r_w in Theorem 4.4.8 of [BH15]), it should be possible to prove that $V(r)$ contains the locally-defined representation $\Pi(r)^{\mathrm{ord}}$ of [BH15]. This should follow precisely the same strategy of proof as Theorem 4.4.8 of *op.*

cit., which roughly shows that, when r is the restriction of a global automorphic representation, the representation $\Pi(r)^{\mathrm{ord}}$ occurs in completed cohomology. The global ingredients used for this are the computation of locally algebraic vectors in completed cohomology and the fact that the reduction mod ϖ of completed cohomology is an injective object in the category of smooth K-representations. In our case, $V(r)$ is obtained by taking the fibre of $M_\infty^d[1/p]$ at a point corresponding to r, and $M_\infty^d[1/p]$ can be thought of as a patched version of completed cohomology. The analogous ingredients are the computation of locally algebraic vectors in Theorem 4.35 and the projectivity of M_∞ in Proposition 2.10.

4.37 Remark. The computation of $V(r_x)^{\mathrm{l.alg}}$, when x is a closed point of $R_\infty(\lambda)[1/p]$ for which the corresponding representation r_x is not necessarily potentially crystalline, and related questions connected to the Breuil–Schneider conjecture, will be discussed in the forthcoming thesis of Alexandre Pyvovarov.

5. The Breuil–Schneider conjecture

Continue to assume that $p \nmid 2n$, and that F is a finite extension of $\mathbb{Q}_p$. If $r : G_F \to \mathrm{GL}_n(E)$ is a de Rham representation of regular weight then we say that r is generic if $\pi_{\mathrm{sm}}(r)$ is generic. In this case, we set

$$\mathrm{BS}(r) := \pi_{\mathrm{alg}}(r) \otimes \pi_{\mathrm{sm}}(r).$$

(In fact, our $\mathrm{BS}(r)$ differs from the definition made in [BS07] in that $\pi_{\mathrm{alg}}(r)$ and $\pi_{\mathrm{sm}}(r)$ are their analogues in [BS07] times the characters $\det^{n-1}$ and $|\det|^{n-1}$, respectively. Since $(\det|\det|)^{n-1}$ is a unitary character, this makes no difference to the following conjecture. See also Section 2.4 of [Sor15] for a discussion of the difference between these conventions.) The following is [BS07, Conjecture 4.3] (in the open direction, in the generic case).

5.1 Conjecture. *If $r : G_F \to \mathrm{GL}_n(E)$ is de Rham and has regular weight, then* $\mathrm{BS}(r)$ *admits a nonzero unitary Banach completion.*

5.2 Remark. In fact, it seems reasonable (particularly in the light of Corollary 5.4 below) to conjecture that there is even a nonzero *admissible* completion. (We recalled the definition of admissibility in Section 2.12.) Indeed, completed cohomology always gives rise to admissible Banach space representations, so this is a reasonable expectation from the point of view of the global p-adic Langlands correspondence. Further motivation for focussing on admissible completions is provided by the functor constructed in [Sch15], which takes as input admissible $\mathbb{F}$-representation of G.

Fix a representation $r : G_F \to \mathrm{GL}_n(E)$, and assume from now on that r is potentially crystalline of regular weight, and that r is generic. By Remark 2.15 (and possibly replacing E with a finite extension if necessary), we may replace r with a conjugate representation so that $r : G_F \to \mathrm{GL}_n(\mathcal{O})$, and $\bar{r}$ satisfies the hypotheses of Section 2. We can therefore carry out the construction of Section 2, obtaining the patched module M_∞. Recall that r is induced from an $\mathcal{O}$-algebra homomorphism $x : R_{\tilde{\mathfrak{p}}}^{\square} \to \mathcal{O}$, which we extended to an $\mathcal{O}$-algebra homomorphism $y : R_\infty \to \mathcal{O}$. Then $V(r)$ is obtained from the fibre of $(M_\infty)^d[1/p]$ above the closed point of $R_\infty[1/p]$ determined by y.

Write $\sigma_{\mathrm{sm}}(r)$ for $\sigma(\tau)$, $\sigma_{\mathrm{alg}}(r)$ for $\pi_{\mathrm{alg}}(r)|_K$, and let $\sigma := \sigma_{\mathrm{alg}}(r) \otimes \sigma_{\mathrm{sm}}(r)$, keeping in mind the convention at the end of Section 3.13. (Also enlarge E to another finite extension if necessary as explained in that section.) As above, we write $\mathcal{H}(\sigma)$ for $\mathrm{End}_G(\text{c-Ind}_K^G \sigma)$, which is isomorphic to $\mathrm{End}_G\big(\text{c-Ind}_K^G \sigma(\tau)\big)$ via ι_σ, so that $\pi_{\mathrm{sm}}(r)$ determines a character $\chi_{\pi_{\mathrm{sm}}(r)} : \mathcal{H}(\sigma) \to E$. Since r is generic, we see from Corollary 3.12 that $\pi_{\mathrm{sm}}(r) \cong \big(\text{c-Ind}_K^G \sigma_{\mathrm{sm}}(r)\big) \otimes_{\mathcal{H}(\sigma), \chi_{\pi_{\mathrm{sm}}(r)}} E$. Tensoring with $\pi_{\mathrm{alg}}(r)$, we have

$$\mathrm{BS}(r) \cong (\text{c-Ind}_K^G \sigma) \otimes_{\mathcal{H}(\sigma), \chi_{\pi_{\mathrm{sm}}(r)}} E.$$

Since our representations $V(r)$ are unitary Banach representations, and since $\mathrm{BS}(r)$ is irreducible by [ST01, Appendix], in order to prove Conjecture 5.1 it would be enough to check that $\mathrm{Hom}_G(\mathrm{BS}(r), V(r)) \neq 0$. While we cannot at present do this in general, we are able to reinterpret the problem in terms of automorphy lifting theorems, and deduce new cases of Conjecture 5.1. In particular, Corollary 5.5 below gives the first general results in the principal series case.

5.3 Theorem. *Suppose that $p \nmid 2n$, and that $r : G_F \to \mathrm{GL}_n(E)$ is a generic potentially crystalline representation of regular weight. If r corresponds to a closed point on an automorphic component of $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$ (in the sense of Remark 4.20), then $\mathrm{BS}(r)$ admits a non-zero unitary admissible Banach completion.*

Proof of Theorem 5.3. As remarked above, since $\mathrm{BS}(r)$ is irreducible it suffices to show that $\mathrm{Hom}_G\big(\mathrm{BS}(r), V(r)\big) \neq 0$. This follows immediately from Theorem 4.35. We also observe that it admits a simpler direct proof. Proposition 2.20 of [Paš15] implies that

$$\dim_E \mathrm{Hom}_K\big(\sigma, V(r)\big) = \dim_E M_\infty(\sigma^\circ) \otimes_{R_\infty, y} E.$$

(More specifically, in the notation of that paper we take $R = R_\infty$, $\Theta = \sigma^\circ$, $V = \sigma$, $N = M_\infty$, and $\mathrm{m}^\circ = \mathcal{O}$, regarded as an R_∞-module via y. Note that [Paš15] assumes that N is finitely generated as an $R[[K]]$-module, which is satisfied in our case: M_∞ is a finitely generated $R_\infty[[K]]$-module, since it is finite over $S_\infty[[K]]$ and the S_∞-action on M_∞ factors through a map $S_\infty \to R_\infty$.)

Together with Frobenius reciprocity, this shows that

$$\mathrm{Hom}_G\big(\text{c-Ind}_K^G\sigma, V(r)\big) = \mathrm{Hom}_K\big(\sigma, V(r)\big) \neq 0,$$

as y is in the support of $M_\infty(\sigma^\circ)[1/p]$ by assumption. Since $\mathrm{BS}(r)$ is isomorphic to $(\text{c-Ind}_K^G\sigma) \otimes_{\mathcal{H}(\sigma),\chi_{\pi_{\mathrm{sm}}(r)}} E$, we need only show that the action of $\mathcal{H}(\sigma^\circ)$ on $M_\infty(\sigma^\circ) \otimes_{R_\infty,y} \mathcal{O}$ factors through the character $\chi_{\pi_{\mathrm{sm}}(r)}$; but this is immediate from Theorem 4.19. □

5.4 Corollary. *Suppose that $p \nmid 2n$ and that $r : G_F \to \mathrm{GL}_n(E)$ is de Rham of regular weight and potentially diagonalisable in the sense of* [BLGGT14]*. Suppose also that r is generic. Then* $\mathrm{BS}(r)$ *admits a nonzero unitary admissible Banach completion.*

Proof. By Theorem 5.3, we need only prove that r corresponds to a point on an automorphic component of $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$. Recalling that y was chosen to correspond to the potentially diagonalisable representation $r_{\mathrm{pot.diag}}$ at the places $v \mid p$, $v \neq \mathfrak{p}$, this follows from Theorem A.4.1 of [BLGG13], which constructs a global automorphic Galois representation corresponding to a point on the same component of $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$ as r (*cf.* the proof of Corollary 4.4.3 of [GK14]).

Indeed, the existence of a global automorphic Galois representation corresponding to a point on $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$ shows that $S_{\xi,\tau}(U_0,(\sigma^\circ)^d)_{\mathfrak{m}}[1/p]$ is supported at this point. We have $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$-equivariant isomorphisms

$$S_{\xi,\tau}(U_0,(\sigma^\circ)^d)_{\mathfrak{m}}[1/p] \simeq \mathrm{Hom}_K\big(\sigma^\circ, \tilde{S}_{\xi,\tau}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}\big)[1/p]$$

$$\simeq \mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}}\big(M_\infty/\mathfrak{a}M_\infty, (\sigma^\circ)^d\big)[1/p],$$

where the former comes by identifying locally algebraic vectors in completed cohomology and the latter follows from Schikhof duality and Corollary 2.11. These isomorphisms imply that $\big(M_\infty(\sigma^\circ)/\mathfrak{a}M_\infty(\sigma^\circ)\big)^d[1/p]$ is supported at the same point of $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$ coming from a global automorphic Galois representation. Therefore, $M_\infty(\sigma^\circ)^d[1/p]$ is supported at a point of $R_\infty(\sigma)[1/p]$ on the same component as r. Finally, we conclude that the component of $R^{\square}_{\widetilde{\mathfrak{p}}}(\sigma)[1/p]$ corresponding to r is automorphic in the sense of Remark 4.20. $\square$

5.5 Corollary. *Suppose that $p > 2$, that $r : G_F \to \mathrm{GL}_n(E)$ is de Rham of regular weight, and that r is generic. Suppose further that either*

1. *$n = 2$, and r is potentially Barsotti–Tate, or*
2. *$F/\mathbb{Q}_p$ is unramified, r is crystalline, $n \neq p$ and r has Hodge–Tate weights in the extended Fontaine–Laffaille range; that is, for each $\kappa : F \hookrightarrow E$, any two elements of $\mathrm{HT}_\kappa(r)$ differ by at most $p - 1$.*

Then $\mathrm{BS}(r)$ admits a nonzero unitary admissible Banach completion.

Proof. Note that in case (2), the hypothesis on the Hodge–Tate weights implies that $p \geq n$, so as $p > 2$ and we are assuming that $n \neq p$, we certainly have $p \nmid 2n$. By Corollary 5.4, it is enough to check that our hypotheses imply that r is potentially diagonalisable; in case (1), this is Lemma 4.4.1 of [GK14], and in case (2), it is the main result of [GL14]. $\square$

5.6 Remark. The attentive reader will have noticed that since throughout the paper we assumed that E is sufficiently large (and allowed it to be enlarged in the course of making our argument), we have not proved cases of Conjecture 5.1 as it is written, but rather an apparently weaker version, which allows a finite extension of scalars. However, Conjecture 5.1 is an immediate consequence of this version, in the following way: given an (admissible) unitary Banach completion of $\mathrm{BS}(r) \otimes_E E'$, where E'/E is a finite extension, we may regard this completion as being a representation over E, and then the closure of $\mathrm{BS}(r)$ inside it gives the required representation.

6. Relationship with a hypothetical *p*-adic local Langlands correspondence

In this section we describe the relationship between our construction and a hypothetical p-adic local Langlands correspondence.

6.1. A hypothetical formulation of the *p*-adic local Langlands correspondence

Perhaps the strongest hypothesis one might make regarding a p-adic local Langlands correspondence is the following: that given $\bar{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$

with trivial endomorphisms[5] and with associated deformation ring $R_{\bar r}$, there exists a finitely generated $R_{\bar r}[[K]]$-module L_∞ which is $\mathcal{O}$-torsion free, and is equipped with an $R_{\bar r}$-linear G-action which extends the K-action arising from its $R_{\bar r}[[K]]$-module structure.

6.2 Remark. In the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, let $\bar\pi$ be the mod p representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to $\bar r$ by the mod p local Langlands correspondence of [Col10]. Then L_∞ can be taken to be the projective envelope of $\bar\pi^\vee$ in an appropriate category $\mathfrak{C}(\mathcal{O})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. See the discussion in Section 1.2 of [Paš13] for more details. As it follows from the discussion on page 10 of *op. cit.*, in the case when $\bar r$, (and thus $\bar\pi$), has trivial endomorphisms, the local deformation ring $R_{\bar r}$ can be identified with the endomorphism ring of the projective envelope L_∞ in $\mathfrak{C}(\mathcal{O})$.

Given such an object L_∞, then for any $r : G_F \to \mathrm{GL}_n(E)$ arising from an $\mathcal{O}_E$-valued point x of $\operatorname{Spec} R_{\bar r}$, we may associate a unitary Banach space representation $B(r) := (L_\infty \otimes_{R_{\bar r},x} \mathcal{O}_E)^d[1/p]$ of G, which should be "the" representation of G associated to r via the p-adic local Langlands correspondence.

One might conjecture that such a structure should exist and satisfy the following properties:

6.2.1. Relationship with classical local Langlands. For any potentially semistable lift $r : G_F \to \mathrm{GL}_n(E)$ of $\bar r$ with regular Hodge–Tate weights, the locally algebraic vectors of $B(r)$ are isomorphic to $\mathrm{BS}(r)$.

6.2.2. Local-global compatibility. Using the notation for completed cohomology and the Hecke algebra that acts on it introduced in the discussion preceding Corollary 2.11, there is an isomorphism of $\mathbb{T}^{S_p}_{\xi,\tau}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m}[G]$-modules

$$\widetilde{S}_{\xi,\tau}(U^\mathfrak{p},\mathcal{O})^d_\mathfrak{m} \cong \left(\mathbb{T}^{S_p}_{\xi,\tau}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m} \otimes_{R_{\bar r}} L_\infty\right)^{\oplus m(U^\mathfrak{p})}$$

(here $\mathbb{T}^{S_p}_{\xi,\tau}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m}$ is regarded as an $R_{\bar r}$-algebra via the natural maps $R_{\bar r} \to R^{\mathrm{univ}}_S \to \mathbb{T}^{S_p}_{\xi,\tau}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m}$, where the first morphism corresponds to restricting global Galois representations to the decomposition group at $\mathfrak{p}$, and the second morphism is induced by the universal automorphic deformation of

[5] We make this assumption in what follows for simplicity, since the discussion is purely hypothetical in any case. If $\bar r$ admits non-trivial endomorphisms, then we would instead work with the lifting ring $R^\square_{\bar r}$ in everything that follows, and the representation L_∞ would be endowed with a further equivariant structure for the group GL_n acting by "change of basis".

$\overline{\rho}$), as well as analogous isomorphisms when we add the auxiliary Q_N-level structure, as in the patching process.[6]

6.3. The relationship between L_∞ and M_∞.

If Condition 6.2.2 held, then we would find that our patched $R_\infty[G]$-module M_∞ is obtained by restricting $R_\infty \otimes_{R_{\bar{r}}} L_\infty$ to the smallest closed subscheme of $\operatorname{Spec} R_\infty$ which contains the support of all the modules $(M_1, Q^{\square}_{N'(N)} \otimes_{S_\infty} S_\infty/\mathfrak{b}_N)_{K_{2N}}$ that arise in the patching process. Optimistic conjectures (of "big R equals big $\mathbb{T}$"-type) might suggest that this support is all of R_∞, and thus that M_∞ is obtained from L_∞ simply by pulling it back along the natural map $\operatorname{Spec} R_\infty \to \operatorname{Spec} R_{\bar{r}}$. For this reason, we are hopeful that our patched representation M_∞ *is* a good candidate for (the pull-back to R_∞ of) p-adic local Langlands. (We note that, in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, we can prove that L_∞, constructed as in Remark 6.2, and M_∞ have the desired relationship.)

6.4. The relationship with the Fontaine–Mazur conjecture

Note that if both Conditions 6.2.1 and 6.2.2 held, together with an appropriate "big R equals big $\mathbb{T}$" result, then we would find that if ρ is a de Rham deformation of $\overline{\rho}$ corresponding to a point of R_S^{univ}, then it would contribute to the locally algebraic vectors of completed cohomology. However, locally algebraic vectors in the completed cohomology arise precisely from algebraic automorphic forms (see Prop. 3.2.4 of [Eme06b]), and hence we would conclude that ρ would be an automorphic Galois representation. (This is an abstraction of the strategy used to deduce the Fontaine–Mazur conjecture for most odd two-dimensional representations of $G_{\mathbb{Q}}$ in [Eme11].)

6.5. Concluding remarks

The preceding discussion shows that the question of whether one can in fact relate M_∞ to a purely local correspondence which satisfies the above two

[6]We remark that in the local-global compatibility isomorphism above there is a multiplicity $m(U^{\mathfrak{p}})$, which depends on the level away from $\mathfrak{p}$. However, it should be possible to impose certain global conditions, as we do in Section 2.1, which will ensure that this multiplicity can be taken to be 1. This multiplicity should not increase when we add auxiliary Q_N-level, since we also apply the projection operators defined in [Tho12] at primes in Q_N.

conditions is closely related to the Fontaine–Mazur conjecture for deformations of $\overline{\rho}$. Given this, we expect it to be a difficult question to precisely determine the support of M_∞ and the related modules $M_\infty(\sigma^\circ)$ in general, and we likewise expect it to be difficult to analyse the extent to which M_∞ arises from a purely local construction over $\operatorname{Spec} R_{\bar{r}}$.

Furthermore, it seems quite possible that our hypothetical formulation of the p-adic local Langlands correspondence is too naive; even if a local correspondence of some kind exists, it may be of a more subtle nature. In this case, we would still expect it to have a strong relationship to our patched modules M_∞, but perhaps not as direct as the one considered in the above discussion.

In any case, whatever the eventual truth might be, the preceding discussion suggests that the further investigation of the patched representation M_∞ is a problem of substantial interest, which we hope to return to in future work.

Appendix A. Completed tensor product and Serre's conditions

Let L be a finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}$ and residue field k. Let $\mathcal{C}$ be the category of complete local noetherian $\mathcal{O}$-algebras, which are $\mathcal{O}$-flat and have residue field k. If A and B are objects in $\mathcal{C}$ then the completed tensor product over $\mathcal{O}$ is defined as

$$A \widehat{\otimes}_{\mathcal{O}} B := \varprojlim_n A/\mathfrak{m}_A^n \otimes_{\mathcal{O}} B/\mathfrak{m}_B^n.$$

It is easy to see that $A \widehat{\otimes}_{\mathcal{O}} B$ is again in $\mathcal{C}$. For example, if $A = \mathcal{O}[[x_1, \dots, x_n]]$ and $B = \mathcal{O}[[y_1, \dots, y_m]]$, then $A \widehat{\otimes}_{\mathcal{O}} B \cong \mathcal{O}[[x_1, \dots, x_n, y_1, \dots, y_m]]$, and every ring in $\mathcal{C}$ can be obtained as a quotient of such rings. The aim of this note is the following Proposition.

A.1 Proposition. *Let A, B be objects in $\mathcal{C}$, let $x \in A$ and let $y \in B$. If $A[1/px]$ and $B[1/py]$ satisfy Serre's condition (R_i) (resp. (S_i)) then so does $(A \widehat{\otimes}_{\mathcal{O}} B)[1/pxy]$.*

We note that the completed tensor product does not commute with localisation. Indeed, $A[1/p]$ is not even a local ring. A standard application of Serre's conditions (R_i) and (S_i), see [Mat89, §23], yields the following result.

A.2 Corollary. *Let* $\boldsymbol{P}$ *be one of the following properties: reduced, regular, Cohen–Macaulay, normal. If* $A[1/px]$ *and* $B[1/py]$ *have* $\boldsymbol{P}$ *then so does* $(A \widehat{\otimes}_{\mathcal{O}} B)[1/pxy]$.

We will split the proof of Proposition A.1 into several steps.

A.3 Lemma. *Let* K *be a field,* A *a noetherian* K*-algebra and* K' *a finite separable field extension of* K*. If* A *satisfies* (R_i) *(resp.* (S_i)*) then so does* $K' \otimes_K A$*.*

Proof. Let $B = K' \otimes_K A$, let $\mathfrak{P}$ be a prime ideal of B and let $\mathfrak{p} = A \cap \mathfrak{P}$. Then B is a free A-module of finite rank. This implies that $B_{\mathfrak{P}}$ is flat over $A_{\mathfrak{p}}$. Since the conditions (R_i) and (S_i) are local, $A_{\mathfrak{p}}$ satisfies (R_i) (resp. (S_i)). It is enough to show that for every prime ideal $\mathfrak{q}$ of $A_{\mathfrak{p}}$ the fibre ring $B_{\mathfrak{P}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{q})$ satisfies (R_i) (resp. (S_i)), [Mat89, Thm.23.9]. The fibre ring is a localisation at $\mathfrak{P}$ of $B \otimes_A \kappa(\mathfrak{q}) \cong K' \otimes_K \kappa(\mathfrak{q})$. Since K' is a finite separable extension of K, this ring is isomorphic to a finite product of fields, and hence is regular. □

A.4 Lemma. *Let* $A, B \in \mathcal{C}$ *be integral domains and let* $K(A)$ *and* $K(B)$ *be the quotient fields of* A *and* B*, respectively. Then* $K(A) \otimes_A (A \widehat{\otimes}_{\mathcal{O}} B) \otimes_B K(B)$ *is a regular ring.*

Proof. We first note

$$K(A) \otimes_A (A \widehat{\otimes}_{\mathcal{O}} B) \otimes_B K(B) \cong S_A^{-1}(S_B^{-1}(A \widehat{\otimes}_{\mathcal{O}} B)), \tag{1.4}$$

where S_A and S_B denote the multiplicative sets $A \setminus \{0\}$ and $B \setminus \{0\}$, respectively.

If both A and B are formally smooth, then $A \widehat{\otimes}_{\mathcal{O}} B$ is formally smooth, as explained above, and hence regular. Since a localisation of a regular ring is again regular, [Mat89, Thm. 19.3], we deduce from (1.4) that the assertion holds if both A and B are formally smooth.

In general, by Cohen's structure theorem for complete local rings there are subrings $A' \subset A$, $B' \subset B$, such that A' and B' are formally smooth objects of $\mathcal{C}$ and A is a finite A'-module, B is a finite B'-module. The last property implies that

$$A \widehat{\otimes}_{\mathcal{O}} B \cong A \otimes_{A'} (A' \widehat{\otimes}_{\mathcal{O}} B') \otimes_{B'} B.$$

This induces an isomorphism between $K(A) \otimes_A (A \widehat{\otimes}_{\mathcal{O}} B) \otimes_B K(B)$ and

$$K(A) \otimes_{K(A')} (K(A') \otimes_{A'} (A' \widehat{\otimes}_{\mathcal{O}} B') \otimes_{B'} K(B')) \otimes_{K(B')} K(B).$$

Since A' and B' are formally smooth, and $K(A')$, $K(B')$ are of characteristic 0, Lemma A.3 implies the assertion. □

A.5 Lemma. *Let $A, B \in \mathcal{C}$ with A an integral domain with quotient field $K(A)$ let $y \in B$. If $B[1/py]$ satisfies (R_i) (resp. (S_i)), then so does $K(A) \otimes_A (A \widehat{\otimes}_{\mathcal{O}} B)[1/y]$.*

Proof. Since A is $\mathcal{O}$-flat, $A \widehat{\otimes}_{\mathcal{O}} B$ is B-flat. Moreover,

$$C := K(A) \otimes_A (A \widehat{\otimes}_{\mathcal{O}} B)[1/y]$$

is $A \widehat{\otimes}_{\mathcal{O}} B$-flat, since it is a localisation of $A \widehat{\otimes}_{\mathcal{O}} B$ at S_A and $\{1, y, y^2, \ldots\}$. Thus C is B-flat. Let $\mathfrak{P}$ be a prime ideal of C and let $\mathfrak{p} = B \cap \mathfrak{P}$. Then $C_{\mathfrak{P}}$ is flat over $B_{\mathfrak{p}}$. We note that $p, y \notin \mathfrak{p}$. Since $B[1/py]$ is assumed to satisfy (R_i) (resp. (S_i)) and these conditions are local, $B_{\mathfrak{p}}$ satisfies (R_i) (resp. (S_i)). It follows from [Mat89, Thm. 23.9] that it is enough to show that for every prime $\mathfrak{q}$ of $B_{\mathfrak{p}}$ the fibre ring $C_{\mathfrak{P}} \otimes_{B_{\mathfrak{p}}} \kappa(\mathfrak{q})$ satisfies (R_i) (resp. (S_i)). We claim that $C_{\mathfrak{P}} \otimes_{B_{\mathfrak{p}}} \kappa(\mathfrak{q})$ is regular, so that these conditions are satisfied. Since it is the localisation of $C \otimes_B \kappa(\mathfrak{q})$ at $\mathfrak{P}$, it is enough to show that $C \otimes_B \kappa(\mathfrak{q})$ is regular. Since

$$C \otimes_B \kappa(\mathfrak{q}) \cong K(A) \otimes_A (A \widehat{\otimes}_{\mathcal{O}} B/\mathfrak{q}) \otimes_{B/\mathfrak{q}} K(B/\mathfrak{q})$$

the assertion follows from Lemma A.4. □

Proof of Proposition A.1. The idea is the same as in the proof of Lemma A.5. Let $C := A \widehat{\otimes}_{\mathcal{O}} B$ and let $\mathfrak{P}$ be a prime ideal of C not containing p, x and y. Let $\mathfrak{p} := A \cap \mathfrak{P}$. Then $A_{\mathfrak{p}}$ satisfies (R_i) (resp. (S_i)) and $C_{\mathfrak{P}}$ is flat over $A_{\mathfrak{p}}$. It is enough to show that for all prime ideals $\mathfrak{q}$ of $A_{\mathfrak{p}}$ the fibre ring $C_{\mathfrak{P}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{q})$ satisfies (R_i) (resp. (S_i)). The ring $K(A/\mathfrak{q}) \otimes_{A/\mathfrak{q}} (A/\mathfrak{q} \widehat{\otimes}_{\mathcal{O}} B)[1/y]$ satisfies (R_i) (resp. (S_i)) by Lemma A.5. Since the fibre ring is the localisation of this ring at $\mathfrak{P}$, the assertion follows. □

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