Cambridge Journal of Mathematics
Volume 3, Number 4, 619–631, 2015

Corrigendum to "$PSL(2;\mathbb{C})$ connections on 3-manifolds with L^2 bounds on curvature" [Cambridge Journal of Mathematics 1 (2013) 239–397]

Clifford Henry Taubes

The proof of Theorem 1.1 in the published version of $PSL(2;\mathbb{C})$ *connections on 3-manifolds with L^2 bounds on curvature* (Cambridge Journal of Mathematics **1** (2013) 239–397, denoted subsequently by [T1]) has an error that was discovered by Thomas Walpulski shortly after the publication of the article. As a result of this error, certain assertions in Theorem 1.1 from [T1] can not be said to be proved.

By way of background, this theorem considers sequences of connections on a given $PSL(2;\mathbb{C})$ bundle over a compact, Riemannian 3-manifold with an apriori bound on the L^2 norm of the associated sequence of curvature 2-forms. Theorem 1.1 in [T1] makes assertions about the limits of such sequences. This theorem can be viewed as having two parts. The first part makes an assertion to the effect that if no subsequence converges directly in a certain Sobolev topology (the bundle analog of a function and its derivatives being square integrable), then, after suitable renormalization, the sequence has subsequences that converge in this Sobolev topology on compact sets in the complement of a nowhere dense closed set in the manifold (henceforth denoted by Z). This first part of Theorem 1.1 also observes that the limiting data in this case is characterized by a non-zero, harmonic 1-form with values in a real line bundle that is defined on the complement of Z; and that the norm of this 1-form extends over Z as a Hölder continuous function with Z being a component of its zero locus. The error found by Thomas Walpulski does not concern this first part of Theorem 1.1 in [T1]. (The proof of this first part of Theorem 1.1 builds upon techniques that were pioneered many years ago by Karen Uhlenbeck.)

The second part of Theorem 1.1 in [T1] says more about the fine structure of the set Z. In particular, the second part asserts that Z is contained in a countable union of Lipshitz curves and that there is a dense open set in Z that has the structure of a C^1 submanifold of the ambient 3-manifold.

This second part of Theorem 1.1 is affected by the error found by Thomas Walpulski. In particular, the assertion that Z is contained in a countable union of Lipshitz curves can not be said now to be proved. What can still be said is that Z has Hausdorff dimension 1 and that it has a dense open set with the structure of a C^1 submanifold. (One can conjecture that Z is a C^1 submanifold on the complement of a finite set; or that such is the case when the metric is suitably generic.)

1. Theorems 1.1a and 1.1b

Theorems 1.1a and 1.1b which are stated momentarily assert those parts of Theorem 1.1 of [T1] that are proved. Theorem 1.1a makes a formal restatement of what was just described as being the first part of Theorem 1.1 in [T1]. Theorem 1.1b makes a formal statement of what can be proved with regards the second part of Theorem 1.1 in [T1]. The mistake found by Thomas Walpulski is in the proof of [T1]'s Lemma 8.8; and the assertions in Theorem 1.1a constitute parts of Theorem 1.1 of [T1] that are not affected by the error in Lemma 8.8. Meanwhile, Theorem 1.1b revises and elaborates on the assertions of Theorem 1.1 in [T1] that depended on [T1]'s Lemma 8.8.

The notation used below and the context is the same as that for Theorem 1.1 in [T1]. By way of a very brief review of the context and notation, M in what follows denotes a compact, oriented 3-dimensional manifold and $P \to M$ denotes a given principal $SO(3)$ bundle. The Lie algebra of the group $SU(2)$ (and hence $SO(3)$) is denoted by $\mathfrak{su}(2)$. Supposing that A is a connection on P, its curvature is denoted by F_A. If $\mathfrak{a}$ is a section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ then its A-covariant exterior derivative is denoted by d_A. Note that if $(A, \mathfrak{a})$ is a pair as just described, then $\mathbb{A} = A + i\mathfrak{a}$ defines a connection on the associated principal bundle $P \times_{SO(3)} PSL(2;\mathbb{C})$. Conversely, any connection on this bundle can be written as $A + i\mathfrak{a}$ with A and $\mathfrak{a}$ as above. Given such a connection $\mathbb{A} = A + i\mathfrak{a}$, which is to say a pair $(A, \mathfrak{a})$, define

$$\mathfrak{F}(\mathbb{A}) = \int_M (|F_A - \mathfrak{a} \wedge \mathfrak{a}|^2 + |d_A \mathfrak{a}|^2 + |d_A * \mathfrak{a}|^2)$$

with the norms defined by the Riemannian metric and the trace inner product on $\mathfrak{su}(2)$. What is denoted by $*$ is the metric's Hodge star operator.

Theorem 1.1a. Suppose that $\{\mathbb{A}_n = A_n + i\mathfrak{a}_n\}_{n=1,2\ldots}$ is a sequence of connections on $P \times_{SO(3)} PSL(2;\mathbb{C})$ with the corresponding sequence $\{\mathfrak{F}(\mathbb{A}_n)\}_{n=1,2,\ldots}$ being bounded. For each $n \in \{1, 2, \ldots\}$, use r_n to denote the L^2 norm of $\mathfrak{a}_n$.

- If the sequence $\{r_n\}_{n=1,2,\dots}$ has a bounded subsequence, then there exists a subsequence of $\{\mathbb{A}_n\}_{n=1,2,\dots}$, hence renumbered consecutively from 1, and a corresponding sequence of automorphisms of P, this denoted by $\{g_n\}_{n=1,2,\dots}$, such that $\{g_n{}^*\mathbb{A}_n\}_{n=1,2,\dots}$ converges weakly in the L^2_1 topology to an L^2_1 connection on $P \times_{SO(3)} PSL(2;\mathbb{C})$.
- If the sequence $\{r_n\}_{n=1,2,\dots}$ has no bounded subsequence, then there exists a subsequence of $\{\mathbb{A}_n\}_{n=1,2,\dots}$, hence renumbered consecutively from 1, a corresponding sequence of automorphisms of P, this denoted by $\{g_n\}_{n=1,2,\dots}$, plus the following extra data: A closed set, nowhere dense set $Z \subset M$, a real line bundle $\mathcal{I} \to M{-}Z$, and a harmonic $\mathcal{I}$ valued 1-form on $M{-}Z$. The latter is denoted by ν. These are such that

    1. The norm $|\nu|$ of ν extends to the whole of M as a Hölder continuous, L^2_1 function with its zero locus being the set Z.
    2. The sequence $\{g_n{}^*A_n\}_{n=1,2,\dots}$ converges weakly in the $L^2_{1;\mathrm{loc}}$ topology on $M{-}Z$ to an $L^2_{1;\mathrm{loc}}$ connection on $P|_{M-Z}$, this denoted by A.
    3. The sequence $\{r_n^{-1} g_n{}^*\mathfrak{a}_n\}n = 1, 2, \dots$ converges weakly in the $L^2_{1;\mathrm{loc}}$ topology on $M{-}Z$ to $\nu\sigma$ with σ being a unit length, A-covariantly constant homorphism over $M{-}Z$ from $\mathcal{I}$ into $P\times_{SO(3)} \mathfrak{su}(2)$. Meanwhile, $\{r_n^{-1}|\mathfrak{a}_n|\}_{n=1,2,\dots}$ converges to $|\nu|$ in the weak L^2_1 topology and the C^0 topology on the whole of M.

This theorem differs from Theorem 1.1 in [T1] to the extent that it makes no claim about the set Z being contained in a countable union of 1-dimensional Lipshitz curves. The now unproved Lemma 8.8 in [T1] was used in [T1] to prove the latter claim. A part of the upcoming Theorem 1.1b makes a slightly weaker assertion about the structure of Z. (Lemma 8.8 in [T1] makes an assertion to the effect that Z at each of its points has a unique tangent cone. This is likely to be true, and perhaps it can be proved using techniques from [KW].)

To set the notation for Theorem 1.1b, suppose that $(Z, \mathcal{I}, \nu)$ is a data set that comes from the second bullet of Theorem 1.1a. Theorem 1.1b introduces the notion of a *point of discontinuity* for the bundle $\mathcal{I}$. A point $p \in Z$ is a point of discontinuity for $\mathcal{I}$ if $\mathcal{I}$ is not isomorphic to the product $\mathbb{R}$ bundle on the complement of Z in any neighborhood of p. These are the interesting points in Z because if p is not a point of discontinuity for $\mathcal{I}$, then ν near p can be viewed as an honest $\mathbb{R}$-valued harmonic 1-form.

Theorem 1.1b also introduces the notion of a *geodesic arc* in a given ball in M. Supposing that B is the ball, a geodesic arc in B is a properly embedded geodesic segment in B through B's center point.

Theorem 1.1b introduces one final notion, this being the notion of a *1-dimensional, Lipshitz graph* in M. For the present such a set is characterized as follows: Let p denote any given point in the graph. Let $x = (x_1, x_2, x_3)$ denote Euclidean coordinates for $\mathbb{R}^3$. There is a coordinate chart for M centered at p that writes the graph near p as the small $|x|$ part of the map $t \to (x_1 = t, x_2 = \varphi_2(t), x_3 = \varphi_3(t))$ with $\varphi = (\varphi_1, \varphi_2)$ being a Lipshitz map from $\mathbb{R}$ to $\mathbb{R}^2$. By way of a reminder, a continuous map from an interval $\mathbf{I} \subset \mathbb{R}$ into a Riemannian manifold is said to be Lipshitz under the following circumstances: Let γ denote the map in question. Then γ is Lipshitz when $\sup_{t,t' \in I} \operatorname{dist}(\gamma(t), \gamma(t')) \leq c_\gamma |t - t'|$ with c_γ being a constant.

THEOREM 1.1b. *Let $(Z, \mathcal{I}, \nu)$ denote a data set that comes via the second bullet of Theorem 1.1a from a sequence of $PSL(2; \mathbb{C})$ connections on M.*

- *Z has Hausdorff dimension at most 1.*
- *Given $\varepsilon > 0$ and $\theta > 0$, there are finite sets of balls $\mathfrak{U}$ and $\mathfrak{V}$ with the following properties: Their union contains Z, the balls in $\mathfrak{U}$ have pairwise disjoint closure, and $\sum_{B \in \mathfrak{U}} (\operatorname{radius}(B))^\theta < \varepsilon$. Meanwhile, if $B \in \mathfrak{V}$ and if r is the radius of B, then $B \cap \mathfrak{V}$ is contained in the radius $r\varepsilon$ tubular neighborhood of a geodesic arc in B.*
- *An open dense subset of Z is contained in a countable union of Lipshitz graphs.*
- *The points of discontinuity for $\mathcal{I}$ are the closure of an open set in Z that is an embedded C^1 curve in M.*

Theorem 1.1b constitutes special cases of Theorems 1.2, 1.3 and 1.4 in a new paper, [T2], titled *The zero locus of $\mathbb{Z}/2$ harmonic spinors in dimensions 2, 3 and 4* which can be found at http://arxiv.org/abs/1407.6206. The input from this paper needed to invoke the results in [T2] consists of a data of the following sort:

- A continuous non-negative function on M to be denoted by f obeying
  - i) $f > 0$ somewhere.
  - ii) There exists $\varepsilon > 0$ such that if $p \in M$ and $f(p) = 0$, and if r is sufficiently small but positive, then $\int_{\operatorname{dist}(p,\cdot)<r} f^2 \leq r^{3+\varepsilon}$.
- Let Z denote $f^{-1}(0)$. A real line bundle $\mathcal{I} \to M - Z$.

- A section ν of $T^*M \otimes \mathcal{I}$ over $M{-}Z$ that obeys
  - i) $d\nu = 0$ and $d * \nu = 0$.
  - ii) $|\nu| = f$.
  - iii) The function $|\nabla\nu|^2$ is integrable on $M{-}Z$.

The data set $(Z, \mathcal{I}, \nu)$ from Proposition 7.1 in [T1] with $f = |\nu|$ obeys the conditions in (1.2); the condition in Item ii) of the first bullet being a consequence of Items b) and c) of the fourth bullet of Proposition 7.1 in [T1] and what is said by the second bullet of the Lemma 7.7 in [T1]. The existence of the desired ε also follows from the identification $|\nu| = |\hat{a}_\diamond|$ in Proposition 7.1 of [T1] and from what is said in Proposition 6.1 of this corrigendum to the effect that $|\hat{a}_\diamond|$ is uniformly Hölder continuous along its zero locus.

The remainder of this corrigendum contains corrections/revisions for some minor errors and omissions in Sections 3, 6 and 7 of [T1]. An uninterrupted, corrected version of the proof of Theorem 1.1a can be found at http://arxiv.org/abs/1205.0514. Note that this latest arXiv version has given more detail for some of the arguments in Section 2 of [T1] and greatly simplified and shortened the arguments in Section 4b)–4d) of [T1].

2. Corrections for Section 3 of [T1]

In the definition of κ_U in (3.1), the lower bound $\kappa_U > 1$ should be $\kappa_U > c_0$. (The precise value of κ_U is ultimately determined by Sobolev embedding constants.)

Equation (3.17) in [T1] has some (inexplicable) sign errors. The corrected equation reads

- $*(d\mathfrak{a} - me \wedge [\tau, \mathfrak{b}]) + d\mathfrak{a}_0 + me[\tau, \mathfrak{b}_0]$ and $-*(d * \mathfrak{a} + me \wedge [\tau, \mathfrak{b}])$,
- $-*(d\mathfrak{b} + me \wedge [\tau, \mathfrak{a}]) - d\mathfrak{b}_0 + me[\tau\mathfrak{a}_0]$ and $*(d * \mathfrak{b} - me \wedge [\tau, *\mathfrak{a}])$,

(3.17)

There are analogous changes to Equation (3.22):

- $*\mathrm{S} = m(e_\Delta - e) \wedge [\tau, \mathfrak{b}] - \langle \tau \hat{a}_{A_\Delta} \rangle \wedge [\tau, \mathfrak{a}] + \mathfrak{s}$,
- $*\mathrm{R} = -m(e_\Delta - e) \wedge [\tau, \mathfrak{a}] + \langle \tau \hat{a}_{A_\Delta} \rangle \wedge [\tau, \mathfrak{b}] + \mathfrak{r}$,
- $*\mathrm{R}_0 = *(d(\pi_\phi \mathfrak{b}) - me \wedge [\tau, \pi_\phi \mathfrak{a}]) + m(e_\Delta - e) \wedge [\tau, *\phi\mathfrak{a}] + \langle \tau \hat{a}_{A_\Delta} \rangle \wedge [\tau, *\phi\mathfrak{b}] + \mathfrak{r}_0$,

(3.22)

And, there is the analogous change to Equation (3.23).

- $*(d\mathfrak{a} - me \wedge [\tau, \mathfrak{b}]) = \mathrm{S}$ and $-*(d * \mathfrak{a} + me \wedge [\tau, *\mathfrak{b}]) = \mathrm{S}_0$.
- $-*(d\mathfrak{b} - me \wedge [\tau, \mathfrak{a}]) = \mathrm{R}$ and $*(d * \mathfrak{b} - me \wedge [\tau, *\mathfrak{a}]) = \mathrm{R}_0$.

(3.23)

3. Corrections for Section 4 of [T1]

Equation (4.16) in [T1] should read:

$$|(g^*\tilde{a}_*)|_x - (g^*\tilde{a}_*)|_y| \leq c_{E,\mu}(\delta + r_\Delta)^{1/4} \tag{4.16}$$

Since [T1] has less than a sentence by way of justification for (4.16), a derivation of this inequality is given in the next few paragraphs.

To start, note that the fifth bullet in Proposition 4.1 with μ replaced by $\frac{1}{4}\mu$ and the third bullet of (3.1) lead to a $c_{E,\mu}$ a priori bound for the L^2 norm of $\nabla(\nabla(g^*\tilde{a}_*))$ on the $|x| \leq 1 - \frac{1}{4}\mu$ ball centered at the origin. Here and in what follows, ∇ denotes the covariant derivative that is defined by the product connection θ. Granted this bound, then the top bullet of (3.10) can be invoked to see that

$$\int_{|x|\leq 1-\mu/2} |\nabla(g^*\tilde{a}_*)|^4 \leq c_{E,\mu} \left(\int_{|x|\leq 1-\mu/2} |\nabla(g^*\tilde{a}_*)|^2 \right)^{1/2}.$$

Granted this bound, then the bound asserted by (4.16) follows from the third bullet of (3.10) given a $c_{E,\mu}(\delta + r_\Delta)$ bound for the L^2 norm of $\nabla(g^*\tilde{a}_*)$ on the $|x| \leq 1 - \frac{1}{2}\mu$ ball in $\mathbb{R}^3$.

To obtain the desired bound, let α denote for the moment the component of $g^*\tilde{a}_*$ along a given coordinate direction as defined by the Gaussian normal coordinates. The $\mathfrak{su}(2)$ valued 1-form $\nabla_{A_\diamond}\alpha$ can be written as

$$\nabla_{A_\diamond}\alpha = \nabla\alpha + [\hat{a}_{A_\diamond}, \alpha] + \Gamma \cdot (g^*\tilde{a}_*)$$

with Γ denoting a linear map with C^1 norm bounded by $c_0 r_\Delta^2$. (This Γ is a linear combination of the metric's Christoffel symbols.) Take the inner product of both sides of the preceding identity with $\nabla\alpha$ and then integrate the result over the $|x| \leq 1$ ball. This results in the following inequality:

$$\int_{|x|\leq 1} \nabla\alpha|^2 \leq c_0(\delta^2 + r_\Delta^2) + 2\,|\int_{|x|\leq 1} \langle \nabla\alpha, [\hat{a}_{A_\Delta}, \alpha]\rangle|.$$

The notation here uses $\langle , \rangle$ to indicate the combination of the metric inner product and the inner product on $\mathfrak{su}(2)$.

Let $\underline{\alpha}$ denote the average of α over the $|x| \leq 1$ ball. Since $\nabla\underline{\alpha} = 0$, the integrand for the integral on the right hand side of the preceding inequality does not change if $\nabla\alpha$ is replaced by $\nabla(\alpha - \underline{\alpha})$. Make this change and then integrate by parts. Because $d^\dagger\hat{a}_{A_\Delta} = 0$ and because the radial component of $\hat{a}_{A_\Delta}$ is zero on the $|x| = 1$ sphere, the integral on the right hand side is equal to the following integral:

$$\int_{|x|\le 1} \langle [(\alpha - \underline{\alpha}), \nabla\alpha], \hat{a}_{A_\Delta} \rangle.$$

Meanwhile, the norm of the latter is no greater than

$$2\|\nabla\alpha\|_2 \|\alpha - \underline{\alpha}\|_4 \|\hat{a}_{A_\Delta}\|_4,$$

which in turn is no greater than $\mathfrak{c}\frac{1}{5}\kappa_U^{-1/2}\|\nabla\alpha\|_2^2$ with $\mathfrak{c} \le c_0$. This bound on $\mathfrak{c}$ is a consequence of the top bullet of (3.10) and the fact that the L^2 norm of $\alpha - \underline{\alpha}$ is less than c_0 times the L^2 norm of $\nabla(\alpha - \underline{\alpha})$, thus c_0 times the L^2 norm of $\nabla\alpha$. With these bounds in hand, one is led to conclude that

$$\int_{|x|\le 1} |\nabla\alpha|^2 \le c_0(\delta^2 + r_\Delta^2) + \mathfrak{c}\kappa_U^{-1/2} \int_{|x|\le 1} |\nabla\alpha|^2.$$

Since $\kappa_U \ge c_0$ was assumed, one can assume that the factor in front of the integral of $|\nabla\alpha|^2$ on the right hand side of this last inequality is less than $\frac{1}{2}$; and so the preceding inequality implies the desired bound $\|\nabla\alpha\|_2 \le c_0(\delta^2 + r_\Delta^2)^{1/2}$.

4. Corrections for Section 6 of [T1]

The wording of Proposition 6.1 in [T1] suggests that the Hölder exponent at a zero of the function $|\hat{a}_\Diamond|$, this denoted by $1/\kappa$, has no a priori lower bound. In fact, such a lower bound exists and such a bound is implicitly used subsequently in [T1]. The correct wording should be as follows:

Proposition 6.1. *Fix a subsequence $\Lambda \subset \{1, 2, \ldots\}$ so that $\{(A_n, \hat{a}_n)\}_{n\in\Lambda}$ is described by Proposition 2.2. The limit function $|\hat{a}_\Diamond|$ given by the second bullet of Proposition 2.2 is continuous. This function is also Hölder continuous with exponent $\frac{1}{4}$ on compact sets where it is bounded away from zero. Meanwhile, there exists $\kappa > 1$ such that if $p \in M$ and if $|\hat{a}_\Diamond|(p) = 0$, then $|\hat{a}_\Diamond| \le \operatorname{dist}(p, \cdot)^{1/\kappa}$ on a sufficiently small radius ball centered at p.*

The proof of this slightly stronger version of Proposition 6.1 is identical to that of the original except for what is said in Section 6c). Moreover, the changes in Section 6c) amount to little more than keeping track of what determines the size of κ. In this regard, the statement of Lemma 6.4 should be amended with an assertion to the effect that its version of κ depends only on the given constant c:

LEMMA 6.4. Fix $p \in M$ and $c > 2$. Suppose there exists a subsequence in Proposition 6.1's sequence Λ, this denoted by Λ_p, such that $\lim_{n \in \Lambda_p} h_n(r) \leq c\, r^{1/c} r^2$ when $r \leq c^{-1}$. Then $|\hat{a}_\diamondsuit|(p) = 0$. Moreover, $|\hat{a}_\diamondsuit|$ is continuous at p and there exists a constant $\kappa > 1$ depending only on c such that $|\hat{a}_\diamondsuit(\cdot)| \leq \kappa \mathrm{dist}(p, \cdot)^{1/\kappa}$ on the ball of radius κ^{-1} centered at p.

The proof in Part 1 of Section 6c in [T1] of the [T1] version of Lemma 6.4 proves this stronger version also.

As noted above, changes in Part 2 of Section 6c to prove the corrected version of Proposition 6.1 are needed only to the extent that one must keep track of the factors that determine the Hölder exponent. Although this is essentially straightforward, the full revision is given below.

Part 2: The assertion in Proposition 6.1 to the effect that $|\hat{a}_\diamondsuit|$ is continuous across its zero locus follows from the assertion in the proposition to the effect that $|\hat{a}_\diamondsuit|$ is Hölder continuous at each of its zeros for a fixed Hölder exponent. The proof of the uniform Hölder bound along the zero section of $|\hat{a}_\diamondsuit|$ is given in five steps. These steps employ the following terminology: The *local Hölder property* is said to hold at a given point p if there exist numbers $\kappa > 1$ and $\rho > 0$ such that $|\hat{a}_\diamondsuit(q)| < \mathrm{dist}(p, q)^{1/\kappa}$ for all $q \in M$ with $\mathrm{dist}(p, q) < \rho$. The Hölder assertion in Proposition 6.1 follows with a proof that each point in Z has the local Hölder property with κ being independent of the given point.

By way of a heads-up, the following observation is used implicitly in what follows: Suppose that $p \in Z$ and that there exist $\kappa > 1$ and $\rho' > 0$ and $x_p > 1$ such that $|\hat{a}_\diamondsuit(q)| \leq x_p \mathrm{dist}(p, q)^{2/\kappa}$ when $\mathrm{dist}(p, q) \leq \rho'$. Then $|\hat{a}_\diamondsuit|(q) \leq \mathrm{dist}(p, q)^{1/\kappa}$ when $\mathrm{dist}(p, q)$ is less than the minimum of ρ' and x_p^κ.

Step 1. Fix $p \in M$ with $|\hat{a}_\diamondsuit|(p) = 0$. Since $|\hat{a}_\diamondsuit|(p) = 0$, so $\lim_{n\to\infty} |\hat{a}_n|(p) = 0$ also, this a consequence of the second bullet in Proposition 2.2. There are now two cases to consider, the first being where p's version of the sequence $\{r_{\diamondsuit n}\}_{n\in\Lambda}$ has a subsequence that is bounded away from zero. If such is the case, let $r_* > 0$ denote a lower bound for this subsequence. The corresponding subsequence of $\{\hat{a}_n\}_{n\in\Lambda}$ is bounded in the L^2_2 topology on the radius r_*

ball in M centered at p, and so it has a subsequence that converges strongly in the exponent $\frac{1}{4}$ Hölder topology on this ball. Let $\Lambda_p \subset \Lambda$ denote the indexing set for the latter subsequence. The Hölder convergence of $\{\hat{a}_n\}_{n\in\Lambda_p}$ on the radius r_* ball centered at p has the following consequence: Given $\varepsilon > 0$, there exists N_ε such that if $n \in \Lambda_p$ and $n > N_\epsilon$, then

$$|\hat{a}_n| \leq \varepsilon + \operatorname{dist}(p,\cdot)^{1/4} \text{ on the radius } r_* \text{ ball centered at } p. \tag{6.18}$$

Fix $n \in \Lambda_p$ with $n > N_\varepsilon$. The bound in (6.18) implies that the p and $(A_n, \hat{a}_n)$ version of the function $h_n(\cdot)$ obeys $h_n(r) \leq c_0(\varepsilon + r^{1/4})r^2$ when $r \in (0, r_*)$. Granted this last bound, invoke Lemma 6.4 to see that the local Hölder property holds at p with Hölder exponent $\frac{1}{4}$.

Step 2. Assume here and in the subsequent steps that $\{(A_n, \hat{a}_n)\}_{n\varepsilon\Lambda}$ is such that $\lim_{n\varepsilon\Lambda} |\hat{a}_n|(p) = 0$ and $\lim_{n\varepsilon\Lambda} r_{\Diamond n} = 0$. Granted this assumption, then at least one of the three cases in the subsequent list describes $\{(A_n, \hat{a}_n)\}_{n\varepsilon\Lambda}$. Step 3 contains the proof that the list is inclusive.

Case 1. This case occurs if there is a subsequence $\Lambda_p \subset \Lambda$ with two properties, the first being the following: If $n \in \Lambda_p$, then there exists $r_{\ddagger n} \in [\frac{1}{2}r_{\Diamond n}, c_0^{-1}]$ which is such that $h_n(r_{\ddagger n}) \leq r_{\ddagger n}^{2+1/16}$. The second property requires that $\lim_{n\in\Lambda_p} r_{\ddagger n} = 0$. Fix $n \in \Lambda_p$. Let $r_{1n} \in [r_{\ddagger n}, c_0^{-1}]$ denote the maximal value for r such that $h_n(s) \leq s^{2+1/16}$ for all $s \in [r_{\ddagger_n}, r_{1n}]$. It follows from the fourth bullet of Proposition 5.1 that $h_n(r) \leq c_0 r^{2+1/16}$ for all $r \in [r_{1n}, c_0^{-1}]$; it follows from the definitions of r_{1n} and $r_{\ddagger n}$ that $h_n(r) < r^{2+1/16}$ for all $r \in [r_{\ddagger n}, r_{1n}]$. Since $\lim_{n\in\Lambda_p} r_{\ddagger n} = 0$, the subsequence Λ_p with any $c > c_0$ can be used as input to Lemma 6.4 to prove that the local Hölder property holds at p with Hölder exponent greater than c_0^{-1}.

Case 2. This case occurs if there exist $\delta > 0$ and a subsequence $\Lambda' \subset \Lambda$ with the following property: If $n \in \Lambda'$, then $h_n(r) > r^{2+1/16}$ for all $r \in [\frac{1}{2}r_{\Diamond n}, 9r_{\Diamond n}]$ and there exists $r \in [\frac{1}{2}r_{\Diamond n}, 9r_{\Diamond n}]$ with $\mathrm{N}_n(r) \geq \delta$. Let $r_{1n} \in (9r_{\Diamond n}, c_0^{-1}]$ denote the maximal r which is such that $h_n(s) \geq s^{2+1/16}$ for all $s \in [r_{\Diamond n}, r_{1n}]$.

Suppose first that $\liminf_{n\in\Lambda'} r_{1n} = 0$. Fix $n \in \Lambda'$. The fourth bullet of Proposition 5.1 implies that $h_n(r) \leq c_0 r^{2+1/16}$ for all $r \in [r_{1n}, c_0^{-1}]$. Fix a subsequence $\Lambda_p \subset \Lambda'$ such that $\lim_{n\in\Lambda_p} r_{1n} = 0$. The fact that $\lim_{n\in\Lambda_p} r_{1n} =$

0 implies that Λ_p and any $c > c_0$ version of Lemma 6.4 can again be used to prove that the local Hölder property holds at p.

Suppose on the other hand that there exists $r_0 < c_0^{-1}$ such that $\liminf_{n\in\Lambda'} r_{1n} > 2r_0$. Fix $n \in \Lambda'$ such that $r_{1n} > r_0$. Then $h_n(r) > r^{2+1/16}$ on $[r_{\Diamond n}, r_0]$. This being the case, the second bullet of Proposition 5.1 can be invoked to see that $\mathrm{N}_n(r) \geq \frac{1}{2}\delta$ if $r \in [\frac{1}{2}r_{\Diamond n}, c_0^{-1}r_0]$. With this understood, invoke the first bullet of Proposition 5.1 to obtain the inequality

$$\frac{d}{dr}h_n \geq 2r^{-1}(1+\frac{1}{2}\delta)h_n - c_0\delta^{-1}\varkappa_n^{-1}r \tag{6.19}$$

where $r \in [\frac{1}{2}r_{\Diamond n}, c_0^{-1}r_0]$. Fix r in this range, and integrate (6.19) from r to $c_0^{-1}r_0$ and use the fact that $h_n(c_0^{-1}r_0) \leq c_0 r_0^2$ to conclude that $h_n(r) \leq c_0(r_0^{-1/(2\delta)}r^{2+1/(2\delta)} + \varkappa_n^{-1}r_0^2)$. Let $r_{\ddagger n}$ denote the number $\varkappa_n^{-1/(2+1/(2\delta))}r_0$ and let x denote $c_0(\delta^{-1} + c_0 r_0^{-1/2\delta})$. Then the preceding bound on $h_n(r)$ implies that $h_n(r) \leq xr^{2+1/x}$ when $r \in [r_{\ddagger_n}, c_0^{-1}r_0]$. Noting that $\lim_{n\in\Lambda'} r_{\ddagger n} = 0$, Lemma 6.4 can be invoked using as input $\Lambda_p = \Lambda'$ and x to prove that the local Hölder property assertion holds at p with Hölder exponent greater than $c_0^{-1}\delta$.

The statement of the third case reintroduces notation from Part 3 of Section 6b.

Case 3. This case occurs when three conditions are met. The first condition requires that $h_n(r) > r^{2+1/16}$ for all $r \in [\frac{1}{2}r_{\Diamond_n}, 9r_{\Diamond n}]$ when $n \in \Lambda$ is sufficiently large; and the second condition requires that $\lim_{n\in\Lambda}\sup_{r\in[\frac{1}{2}r_{\Diamond n}, 9r_{\Diamond n}]} \mathrm{N}_n(r) < \delta$. The third condition requires there be a subsequence $\Lambda' \in \{1, 2, \ldots\}$ and an associated sequence $\{p_{*n}\}_{n\in\Lambda'} \subset M$ with the following properties.

- Each $n \in \Lambda'$ version of p_{*n} has distance less than $3r_{\Diamond n}$ from p.
- Either or both of the following statements are true.
  - i) If $n \in \Lambda'$, then there exists $r_{*\ddagger n} \in [\frac{1}{2}r_{*\Diamond n}, 9r_{*\Diamond n}]$ such that $h_{*n}(r_{*\ddagger n}) \leq r_{*\ddagger n}^{2+1/16}$.
  - ii) $\sup_{r\in[\frac{1}{2}r_{*\Diamond n}, 9r_{*\Diamond n}]} \mathrm{N}_{*n}(r) \geq \delta$.

(6.20)

Suppose there is a subsequence $\Lambda'' \subset \Lambda'$ such that Item i) in the second bullet of (6.20) holds for all $n \in \Lambda''$. Fix $n \in \Lambda''$ and let h_{*n} denote the p_{*n} version of h. But for cosmetic changes, the argument in Case 1 can be used with p_{*n} replacing p to see that $h_{*n}(s) \leq c_0 s^{2+1/16}$ for all $s \in [9r_{*\diamond n}, c_0^{-1}]$. Keeping this in mind, use an integration by parts with the fact that $\|\nabla_{A_n}\hat{a}_n\|^2 \leq c_0$ and $|\hat{a}_n| \leq c_0$ to see that

$$h_n(s) \leq h_{*n}(s + 4r_{\diamond n}) + c_0 r_{\diamond n}^{3/2}, \tag{6.21}$$

when $s > 10r_{\diamond n}$. Fix $r > 0$ and (6.21) implies that $\lim_{n\in\Lambda'} h_n(r) \leq c_0 r^{2+1/16}$. This being the case, then Lemma 6.4 can be invoked using as input $\Lambda_p = \Lambda''$ and any $c \geq c_0$ to prove that p has the local Hölder property.

Suppose next that Item i) of (6.20) is not true if $n \in \Lambda'$ is large. This understood, throw out the finite set of integers where Item i) is true and use Λ' now to denote the remaining set. Each $n \in \Lambda'$ obeys the condition in Item ii) of (6.20). But for cosmetic changes, the argument in Case 2 can be used with each $n \in \Lambda'$ version of p_{*n} replacing p to obtain the following data: A number $c > 1$ that depends only on δ and a sequence $\{r_{*\ddagger n}\}_{n\in\Lambda'} \subset (0, c^{-1]})$ with limit zero and with the following additional property: If $n \in \Lambda'$, then $r_{*\ddagger n}$ is such that $h_{*n}(s) \leq cs^{2+1/c}$ when $s \in [r_{*\ddagger n}, c^{-1}]$. Granted this data, use (6.21) to conclude that $\lim_{n\in\Lambda'} h_n(r) \leq c_0 c r^{2+1/c}$ for each $r \in (0, c^{-1})$. It follows from the latter bound that the sequence $\Lambda_p = \Lambda'$ and the given value of c can be used as input to Lemma 6.4 to prove that p has the local Hölder property with Hölder exponent greater than $c_0^{-1}\delta$.

Step 3. Assume that $\{(A_n, \hat{a}_n)\}_{n\in\Lambda}$ is such that $\lim_{n\in\Lambda} |\hat{a}_n|(p) = 0$ and $\lim_{n\to\infty} r_{\diamond n} = 0$. The paragraphs that follow prove that at least one of the three cases in Step 2 applies with δ being greater than c_0^{-1}. To this end, assume to the contrary that none of these cases apply for a given δ. The existence of such a sequence is shown below to lead to nonsense when δ is smaller than c_0^{-1}.

After discarding a finite set of terms and then relabeling the result as Λ, then any given $n \in \Lambda$ pair from the sequence $\{(A_n, \hat{a}_n)\}_{n\in\Lambda}$ must have the following properties:

- $h_n(r) \geq r^{2+1/16}$ for all $r \in [\frac{1}{2}r_{\diamond n}, 4r_{\diamond n}]$.
- $\mathrm{N}_n(r) < \delta$ for all $r \in [\frac{1}{2}r_{\diamond n}, 4r_{\diamond n}]$.
- Supposing that p_{*n} has distance less than $3r_{\diamond n}$ from p and that $r \in [\frac{1}{2}r_{*\diamond n}, 9r_{*\diamond n}]$, then $h_{*n}(r) \leq r^{2+1/16}$ and $\mathrm{N}_{*n}(r) < \delta$.

(6.22)

Indeed, the first bullet of (6.22) must be obeyed to avoid a Case 1 label, the second bullet of (6.22) must be obeyed to avoid a Case 2 label, and the third bullet of (6.22) must be obeyed to avoid a Case 3 label.

Step 4. Fix $n \in \Lambda$ and a point $p_{*n} \in M$ with $\operatorname{dist}(p, p_{*n}) \leq 3r_{\Diamond n}$. The constructions in Part 4 of Section 6b can be repeated to construct what is denoted there by $\hat{a}_{**n}$. The L_2 norm of $\hat{a}_{**n}$ on the $|x| \leq 1$ ball in $\mathbb{R}^3$ is equal to 1. Moreover, the condition on N_{*n} in the third bullet of (6.22) implies that

$$\int_{|x|\leq 1} |\nabla_{A_{*\Diamond n}} \hat{a}_{**n}|^2 \leq c_0\delta. \tag{6.23}$$

With (6.23) understood, fix $\varepsilon \in (0, 1]$ and let κ_ε denote the $E \leq c_0$ and $\mu = \frac{1}{4}$ version of what is denoted by $\kappa_{E,\mu,\varepsilon}$ in Proposition 3.2 and (3.8). It follows from (6.23) that if $\delta \leq c_0^{-1}\kappa^{-1}$ and if n is large so that $r_{\Diamond n} \leq c_0^{-1}\kappa_\varepsilon^{-1}$, then the square of the L^2 norm of $F_{A_{*\Diamond n}}$ on the $|x| \leq \frac{3}{4}$ ball is bounded by ε.

Step 5. Having fixed $n \in \Lambda$, repeat the iteration procedure in Part 6 of Section 6b to construct an iteration sequence $\{p, p_{*n,1}, \ldots, p_{*n,k}\}$. The final paragraph of Step 4 implies that what is said about $p_{*n,k}$ in Part 7 in Section 6b holds in this case also. In particular, (6.15) holds. The latter inequality is nonsensical if $\varepsilon < c_0^{-1}$ and n is large for the same reason it is nonsensical in Section 6b: It runs afoul of the definition of $r_{*\Diamond n,k}$.

5. Corrections for Section 7 of [T1]

Lemma 7.7's second bullet makes the assertion that there is a strictly positive lower bound for the value of the function $p \to \mathrm{N}_{(p)}(0)$ on Z. The argument given for this in Part 5 of Section 7c) is circular and so does not prove the claim. The argument that follows gives a proof of Lemma 7.7's assertion that there is $\kappa > 1$ such that $\mathrm{N}_{(p)}(0) > \kappa^{-1}$ on Z. The proof starts with the identity $|v| = |\hat{a}_\Diamond|$ from Proposition 7.1. Granted this identity, then Proposition 6.1 supplies $\kappa > 1$ such that if $p \in Z$ and q is a point in M in a small radius ball centered at p, then $|v|(q) \leq \operatorname{dist}(p, q)^{1/\kappa}$. What follows is a consequence: If r is positive but small, then $h_{(p)}(r) \leq c_0 x r^{2+2/\kappa}$. Meanwhile, Items b) and c) of the fourth bullet of Proposition 7.1 that $h_{(p)}(r) \geq x' r^{2+2N_{(p)}(0)}$ if r is small with x' being independent of r if r is small. These two bounds are not compatible if $\mathrm{N}_{(p)}(0) < \kappa$.

References

[KW] B. Krummel and N. Wickramasekera, *Fine properties of branch point singularities: Two-valued harmonic functions*; arXiv 1311.0923.

[T1] C. H. Taubes, *PSL(2;C) connections on 3-manifolds with L^2 bounds on curvature*; Cambridge Journal of Mathematics 1 (2013) 239–397.

[T2] C. H. Taubes, *The zero locus of Z/2 harmonic spinors in dimensions 2, 3 and 4*; arXiv 1407.6206.

Clifford Henry Taubes
Department of Mathematics
Harvard University
Cambridge, MA 02138
USA
E-mail address: chtaubes@math.harvard.edu

Received October 22, 2015