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Compact generation of the category of D-modules on the stack of G-bundles on a curve

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Let G be a reductive group. Let Bun_G denote the stack of G-bundles on a smooth complete curve over a field of characteristic 0, and let $\text{D-mod}(\mathrm{Bun}_G)$ denote the DG category of D-modules on Bun_G. The main goal of the paper is to show that $\text{D-mod}(\mathrm{Bun}_G)$ is compactly generated (this is not automatic because Bun_G is not quasi-compact). The proof is based on the following observation: Bun_G can be written as a union of quasi-compact open substacks $j : U \hookrightarrow \mathrm{Bun}_G$, which are "co-truncative", i.e., the functor $j_!$ is defined on the entire category $\text{D-mod}(U)$.

Introduction

0.1. The main result

Let k be an algebraically closed field of characteristic 0. Let X be a smooth complete connected curve over k and let Bun_G denote the moduli stack of principal G-bundles on X, where G is a connected reductive group.

0.1.1. The object of study of this paper is the DG category $\text{D-mod}(\mathrm{Bun}_G)$ of D-modules on Bun_G. Our main goal is to prove the following theorem:

Theorem 0.1.2. *The DG category* $\text{D-mod}(\mathrm{Bun}_G)$ *is compactly generated.*

For the reader's convenience we will review the theory of DG categories, and the notion of compact generation in Sect. 1.

Essentially, the property of compact generation is what makes a DG category manageable.

0.1.3. The above theorem is somewhat surprising for the following reason.

It is known that if an algebraic stack $\mathcal{Y}$ is quasi-compact and the automorphism group of every field-valued point of $\mathcal{Y}$ is affine, then the DG category $\text{D-mod}(\mathcal{Y})$ is compactly generated. This result is established in [DrGa1, Theorem 0.2.2]. In fact, the compact generation of $\text{D-mod}(\mathcal{Y})$ for most stacks $\mathcal{Y}$ that one encounters in practice is much easier than the above-mentioned theorem of [DrGa1]: it is nearly obvious for stacks of the form Z/H, where Z is a quasi-compact scheme and H an algebraic group acting on it.

However, if $\mathcal{Y}$ is not quasi-compact then $\text{D-mod}(\mathcal{Y})$ does not have to be compactly generated. We will exhibit two such examples in Sect. 12.1; in both of them $\mathcal{Y}$ will actually be a smooth non quasi-compact scheme (non-separated in the first example, and separated in the second one).

0.1.4. So the compact generation of $\text{D-mod}(\mathcal{Y})$ encodes a certain geometric property of the stack $\mathcal{Y}$. We do not know how to formulate a necessary and sufficient condition for $\text{D-mod}(\mathcal{Y})$ to be compactly generated.

But we do formulate a sufficient condition, which we call "truncatibility" (see Sect. 0.2.3 or Definition 4.1.1). The idea is that $\mathcal{Y}$ is truncatable if it can be represented as a union of quasi-compact open substacks U so that for each of them direct image functor

$$j_* : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$$

has a particularly nice property explained below.

0.2. Truncativeness, co-truncativeness and truncatability

0.2.1. Let $\mathcal{Y}$ be a quasi-compact algebraic stack with affine automorphism groups of points, and let $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ be a closed embedding. By [DrGa1, Theorem 0.2.2], both categories $\text{D-mod}(\mathcal{Z})$ and $\text{D-mod}(\mathcal{Y})$ are compactly generated.

We have a pair of adjoint functors

$$i_{\mathrm{dR},*} : \text{D-mod}(\mathcal{Z}) \rightleftarrows \text{D-mod}(\mathcal{Y}) : i^!.$$

Being a left adjoint, the functor $i_{\mathrm{dR},*}$ preserves compactness. But there is no reason for $i^!$ to have this property. We will say that $\mathcal{Z}$ is *truncative* in $\mathcal{Y}$ if $i^!$ does preserve compactness.

Truncativeness is a purely "stacky" phenomenon. In Sect. 3.2.1 we will show that it never occurs for schemes, unless $\mathcal{Z}$ is a union of connected components of $\mathcal{Y}$.

Let $U \overset{j}{\hookrightarrow} \mathcal{Y}$ be the embedding of the complementary open substack. We say that U is *co-truncative* in $\mathcal{Y}$ if $\mathcal{Z}$ is truncative. This property can be reformulated as saying that the functor

$$j_* : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$$

preserves compactness. We show that the property of co-truncativeness can be also reformulated as the existence of the functor $j_! : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$, *left* adjoint to the restriction functor j^*. (A priori, $j_!$ is only defined on the holonomic subcategory.)

Remark 0.2.2. The property of being compact for an object in $\text{D-mod}(\mathcal{Y})$ is somewhat subtle (e.g., it is not local in the smooth topology). In Sect. 3.5 we reformulate the notion of truncativeness and co-truncativeness in terms of the more accessible property of *coherence* instead of compactness.

0.2.3. Let us now drop the assumption that $\mathcal{Y}$ be quasi-compact. We say that a closed substack $\mathcal{Z}$ (resp., open substack U) is truncative (resp., co-truncative), if for every quasi-compact open $\overset{\circ}{\mathcal{Y}} \subset \mathcal{Y}$, the intersection $\mathcal{Z} \cap \overset{\circ}{\mathcal{Y}}$

(resp., $U \cap \overset{\circ}{\mathcal{Y}}$) is truncative (resp., co-truncative) in $\overset{\circ}{\mathcal{Y}}$.

We say that $\mathcal{Y}$ is *truncatable* if it equals the union of its quasi-compact co-truncative open substacks. We will show that a union of two co-truncative open substacks is co-truncative. So $\mathcal{Y}$ is truncatable if and only if every open quasi-compact substack of $\mathcal{Y}$ is contained in one which is co-truncative.

We will show (see Proposition 4.1.6) that *if $\mathcal{Y}$ is truncatable, then* $\text{D-mod}(\mathcal{Y})$ *is compactly generated.* (This is an easy consequence of [DrGa1, Theorem 0.2.2].)

0.2.4. Thus Theorem 0.1.2, follows from the next statement, which is the main technical result of this paper.

Theorem 0.2.5. *The stack* Bun_G *is truncatable.*

Let us explain how to cover Bun_G by quasi-compact open co-truncative substacks. For every dominant rational coweight θ let $\mathrm{Bun}_G^{(\leq\theta)} \subset \mathrm{Bun}_G$ denote the open substack parameterizing G-bundles whose Harder-Narasimhan coweight[1] is $\underset{G}{\leq} \theta$ (the partial ordering $\underset{G}{\leq}$ on coweights is defined as usual: $\lambda_1 \underset{G}{\leq} \lambda_2$ if $\lambda_2 - \lambda_1$ is a linear combination of simple coroots with non-negative coefficients). Equivalently, $\mathrm{Bun}_G^{(\leq\theta)}$ parameterizes those G-bundles $\mathcal{P}_G$ that have the following property: for every reduction $\mathcal{P}_B$ to the Borel, the degree of $\mathcal{P}_B$ (which is a coweight of G) is $\underset{G}{\leq} \theta$.

The substacks $\mathrm{Bun}_G^{(\leq\theta)}$ are quasi-compact and cover Bun_G. So Theorem 0.2.5 is a consequence of the following fact proved in Sect. 9:

The substack $\mathrm{Bun}_G^{(\leq\theta)}$ *is co-truncative if for every simple root $\check{\alpha}_i$ one has*

$$\langle \theta \,, \check{\alpha}_i \rangle \geq 2g-2, \tag{0.1}$$

where g is the genus of X.

E.g., if $G = GL(2)$ this means that the open substack

$$\mathrm{Bun}_{GL_2}^{(\leq m)} \cap \mathrm{Bun}_{GL_2}^{n} \subset \mathrm{Bun}_{GL_2}$$

that parameterizes rank 2 vector bundles of degree n all of whose line subbundles have degree $\leq m$, is co-truncative provided that $2m - n \geq 2g - 2$.

Condition (0.1) means that θ is "deep enough" inside the dominant chamber (of course, if $g \leq 1$ then the condition holds for any dominant θ).

[1] This rational coweight was defined by Harder-Narasimhan [HN] in the case $G = GL(n)$ and by A. Ramanathan [R1] for any G.

0.2.6. Establishing truncativeness. To prove Theorem 0.2.5, we will have to show that certain explicitly defined locally closed substacks of Bun_G are truncative.

We will do this by using a "contraction principle", see Proposition 5.1.2. In its simplest form, it says that the substack $\{0\}/\mathbb{G}_m \hookrightarrow \mathbb{A}^n/\mathbb{G}_m$ is truncative (here $\mathbb{G}_m$ acts on $\mathbb{A}^n$ by homotheties).

0.3. Duality

0.3.1. Recall the notion of dualizability of a DG category in the sense of Lurie (see Sect. 1.5.1). Any compactly generated DG category is automatically dualizable. In particular, such is $\text{D-mod}(\mathcal{Y})$ when $\mathcal{Y}$ is a truncatable algebraic stack.

0.3.2. However, more is true. As we recall in Sect. 2.2.14, if $\mathcal{Y}$ is quasi-compact, not only is the category $\text{D-mod}(\mathcal{Y})$ dualizable, but Verdier duality defines an equivalence

$$\text{D-mod}(\mathcal{Y})^\vee \simeq \text{D-mod}(\mathcal{Y}).$$

It is natural to ask for a description of the dual category $\text{D-mod}(\mathcal{Y})^\vee$ when $\mathcal{Y}$ is no longer quasi-compact, but just truncatable.

0.3.3. As we will see in Sect. 4.2–4.4, the category $\text{D-mod}(\mathcal{Y})^\vee$ can be described explicitly, but it is a priori *different* from $\text{D-mod}(\mathcal{Y})$.

There exists a naturally defined functor

$$\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y},\mathrm{naive}} : \text{D-mod}(\mathcal{Y})^\vee \to \text{D-mod}(\mathcal{Y}),$$

but we show (see Proposition 4.4.5) that this functor is *not* an equivalence unless the closure of every quasi-compact open in $\mathcal{Y}$ is again quasi-compact.

0.3.4. However, in Sect. 4.4.8 we define a less obvious functor

$$\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y},!} : \text{D-mod}(\mathcal{Y})^\vee \to \text{D-mod}(\mathcal{Y}),$$

which may differ from $\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y},\mathrm{naive}}$ even for $\mathcal{Y}$ quasi-compact.

In general, $\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y},!}$ is not an equivalence, but there are important and nontrivial examples of quasi-compact and non-quasi-compact stacks $\mathcal{Y}$ for which $\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y},!}$ is an equivalence.

In particular, in a subsequent publication[2] it will be shown that *the functor* $\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y},!}$ *is an equivalence if* $\mathcal{Y} = \mathrm{Bun}_G$*, where* G *is any reductive group.*

Thus, for any reductive G, *the DG category* D-mod(Bun_G) *identifies with its dual* (in a non-trivial way and for non-trivial reasons).

0.4. Generalizations and open questions

Let us return to the main result of this paper, namely, Theorem 0.1.2.

0.4.1. In the situation of Quantum Geometric Langlands, one needs to consider the categories of twisted D-modules on Bun_G. The corresponding analog of Theorem 0.1.2, with the same proof, holds in this more general context.

0.4.2. Let $x_1, \ldots, x_n \in X$. Instead of Bun_G, consider the stack of G-bundles on X with a reduction to a parabolic P_i at x_i, $1 \leq i \leq n$. Most probably, in this situation an analog of Theorem 0.1.2 holds and can be proved in a similar way.

0.4.3. Suppose now that instead of reductions to parabolics (as in Sect. 0.4.2), one considers deeper level structures at $x_1, \ldots, x_n$ (the simplest case being reduction to the unipotent radical of the Borel).

We do not know whether an analog of Theorem 0.1.2 holds in this case, and we do not know what to expect. In any case, our strategy of the proof of Theorem 0.1.2 fails in this context.

0.4.4. Here are some more questions:

Question 0.4.5. *Does the assertion of Theorem 0.1.2 (and its strengthening, Theorem 0.2.5) hold for* $\mathcal{Y}$ *being one of the stacks* $\overline{\mathrm{Bun}}_B$, Bun_P, $\overline{\mathrm{Bun}}_P$ *and* $\widetilde{\mathrm{Bun}}_P$*, where* B *is the Borel, and* P *a general parabolic?*

We are quite confident that the answer is "yes" for $\overline{\mathrm{Bun}}_B$, but are less sure in other cases.

Question 0.4.6. *Does the assertion of Theorem 0.1.2 hold for an* arbitrary *connected affine algebraic group* G *(i.e., without the assumption that* G *be reductive)?*

0.5. Organization of the paper

0.5.1. In Sect. 1 we review some basic facts regarding DG categories.

[2] For a draft see [Ga2].

0.5.2. In Sect. 2 we review some general facts about the category of D-modules on an algebraic stack $\mathcal{Y}$. We first consider the case when $\mathcal{Y}$ is quasi-compact and make a summary of the relevant results from [DrGa1]. We then consider the case when $\mathcal{Y}$ is not quasi-compact and characterize the subcategory of D-mod($\mathcal{Y}$) formed by compact objects.

0.5.3. In Sect. 3, we introduce some of the main definitions for this paper: the notions of truncativeness (for a locally closed substack) and co-truncativeness (for an open substack). We study the behavior of these notions under morphisms, base change, refinement of stratification, etc. We also discuss the "non-standard" functors associated to a truncative closed (or locally closed) substack (see Sects. 3.3 and Remark 3.4.5), in particular, the very unusual functors $i_?$ and $j^?$.

0.5.4. Sect. 4 is, philosophically, the heart of this article.

In Sect. 4.1 we introduce the notion of truncatable stack. We show that if $\mathcal{Y}$ is truncatable then the category D-mod($\mathcal{Y}$) is compactly generated. In particular, we obtain that Theorem 0.2.5 implies Theorem 0.1.2.

In Sects. 4.2–4.5 we discuss the behavior of Verdier duality on truncatable stacks and the relation beween the category D-mod($\mathcal{Y}$) and its dual.

0.5.5. In Sect. 5 we formulate a *contraction principle*, see Proposition 5.1.2. It shows that a closed substack with the property that we call *contractiveness* is truncative.

In Sect. 5.3 we explicitly describe the non-standard functors i^* and $i_?$ in the setting of Proposition 5.1.2.

0.5.6. In Sect. 6 we prove Theorem 0.2.5 in the particular case of $G = SL_2$. The proof in the general case follows the same idea, but is more involved combinatorially.

0.5.7. In Sect. 7 we recall the stratification of Bun_G according to the Harder-Narasihman coweight of the G-bundle. We briefly indicate a way to establish the existence of such a stratification using the relative compactification of the map $\mathrm{Bun}_P \to \mathrm{Bun}_G$.

0.5.8. In Sect. 8 we introduce a book-keeping device that allows to produce locally closed substacks of Bun_G from locally closed substacks of Bun_M, where M is a Levi subgroup of G. Certain locally closed substacks of Bun_G obtained in this way, will turn out to be *contractive*, and hence *truncative*, and as such will play a crucial role in the proof of Theorem 0.2.5.

0.5.9. In Sect. 9–11 we finally prove Theorem 0.2.5. The proof amounts to combining the Harder-Narasimhan-Shatz strata of Bun_G (i.e., the strata corresponding to a fixed value of the Harder-Narasihman coweight) into certain larger locally closed substacks and applying the contraction principle. A more detailed explanation of the idea of the proof can be found in Sect. 9.1.

In Sect. 9 we prove Theorem 0.2.5 modulo a key Proposition 9.2.2. The latter is proved in Sect. 10–11.

0.5.10. In Sect. 12 we prove the existence of non quasi-compact stacks $\mathcal{Y}$ such that the category $\text{D-mod}(\mathcal{Y})$ is not compactly generated.

Namely, we show that if $\mathcal{Y} = Y$ is a smooth scheme containing a non quasi-compact divisor, then the category $\text{D-mod}(\mathcal{Y})$ is not generated by compact objects. More precisely, we show that (locally) coherent D-modules on Y that belong to the full subcategory generated by compact objects cannot have all of $T^*(Y)$ as their singular support. In particular, the D-module $\mathcal{D}_Y$ does not belong to the subcategory.

0.5.11. In Appendix A we recall an explicit description of open, closed, and locally closed subsets of a (pre)-ordered set equipped with its natural topology. We use this description (combined with the Harder-Narasihman map) to explicitly construct some locally closed substacks of Bun_G, see Sect. 7.4.10 and Corollary 7.4.11.

0.5.12. In Appendix B we give a variant of the proof of Theorem 9.1.2 that has some advantages compared with the one from Sect. 9.3. The method is to define a coarsening of the Harder-Narasimhan-Shatz stratification such that each stratum is contractive (and therefore truncative). This is done using the *Langlands retraction* of the space of rational coweights onto the dominant cone.

0.5.13. In Appendix C we prove a "stacky" generalization of the contraction principle from Sect. 5.1 and of the adjunction from Proposition 5.3.2.

0.6. Conventions and notation

0.6.1. Our conventions on ∞-categories follow those of [DrGa1, Sect. 0.6.1]. Whenever we say "category", by default we mean an $(\infty, 1)$-category. We denote by $\infty\text{-Cat}$ the $(\infty, 1)$-category of ∞-categories.

We denote by $\infty\text{-Grpd} \subset \infty\text{-Cat}$ in the $(\infty, 1)$-subcategory spanned by ∞-groupoids, a.k.a., spaces. We denote by $\mathbf{C} \mapsto \mathbf{C}^{\mathrm{grpd}}$ the functor $\infty\text{-Cat} \to$

∞-Grpd right adjoint to the above embedding. Explicitly, $\mathbf{C}^{\mathrm{grpd}}$ is obtained from $\mathbf{C}$ by discarding non-invertible 1-morphisms.

For $\mathbf{C} \in \infty$-Cat and objects $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$ we denote by $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \infty$-Grpd the corresponding space of maps. We let $\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ denote the set $\pi_0(\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2))$.

0.6.2. Schemes and stacks. This paper deals with categorical aspects of the category of D-modules, i.e., we do not need derived algebraic geometry for this paper. Therefore, by a *scheme* we shall understand a *classical scheme*. We let Sch (resp., $\mathrm{Sch}^{\mathrm{aff}}$) denote the category of schemes (resp., affine schemes) over k, and $\mathrm{Sch}_{\mathrm{lft}}$ (resp., $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$) its full subcategory consisting of affine schemes locally of finite type (resp., affine schemes of finite type).

By a prestack we shall mean an arbitrary functor $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty$-Grpd.

By a stack we shall mean a prestack that satisfies the fppf descent condition. For the general notion of Artin stack we refer the reader to [GL:Stacks, Sect. 4.2]. However, neither general stacks nor Artin stacks are necessary for this paper. What we need is the more restricted (and standard) notion of *algebraic stack*. We adopt the following conventions: a stack $\mathcal{Y}$ is said to be an algebraic stack if:

- The diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is schematic, quasi-compact and quasi-separated;
- There exists a scheme Z equipped with a morphism $f : Z \to \mathcal{Y}$ (this morphism is automatically schematic, by the previous condition) such that f is smooth and surjective.

The pair (Z, f) is called a *presentation* or *atlas* of $\mathcal{Y}$.

We note that this definition is slightly more restrictive than the one in [GL:Stacks, Sect. 4.2.8].

0.6.3. Finite type(ness). All schemes, algebraic stacks and prestacks considered in this paper will be *locally of finite type* over k.

We recall that a classical prestack, i.e., a functor $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty$-Grpd, is said to be locally of finite type if it takes limits in $\mathrm{Sch}^{\mathrm{aff}}$ to colimits in ∞-Grpd. Equivalently, a classical prestack is locally of finite type if it is the left Kan extension from the full subcategory $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \subset \mathrm{Sch}^{\mathrm{aff}}$. The upshot is that when considering prestacks locally of finite type, one can forget about all affine schemes altogether and restrict one's attention to $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

An algebraic stack is said to be locally of finite type if it is such when considered as a prestack. This is equivalent to requiring that it admit an atlas (Z, f) with Z being locally of finite type. Or, still equivalently, that for any $Z \in \mathrm{Sch}$ equipped with a smooth map to $\mathcal{Y}$, the scheme Z is of finite type. The equivalence of these conditions is established, e.g., in [GL:Stacks, Proposition 4.9.2].

0.6.4. D-modules. We refer the reader to the paper [GR] for the theory of D-modules (a.k.a. crystals) on prestacks locally of finite type.

For a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ of prestacks we have a tautologically defined functor

$$f^! : \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1).$$

This functor may or may not have a left adjoint, which we denote by $f_!$.

If f is schematic[3] and quasi-compact, we also have a functor of direct image

$$f_{\mathrm{dR},*} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2).$$

However, when f is an open embedding, we will use the notation j_* instead of $j_{\mathrm{dR},*}$, and j^* instead of $j^!$, for reasons of tradition. This is not supposed to cause confusion, as the above functors go to the same-named functors for the underlying $\mathcal{O}$-modules.

0.7. Acknowledgements

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1. DG categories

Sects. 1.1–1.6 are devoted to recollections and conventions regarding DG categories. In Sects. 1.7–1.9 we provide a categorical framework for Sects. 4.2–4.3; this material can definitely be skipped until it is used.

1.1. The setting

1.1.1. Throughout this paper we will work with DG categories over the ground field k. We refer the reader to [GL:DG] for a survey.[4]

[3] Recall that f is said to be schematic if $\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} S$ is a scheme for any scheme S equipped with a morphism $S \to \mathcal{Y}_2$.

[4] Whenever we talk about a DG category $\mathbf{C}$, we will always assume that it is *pre-triangulated*, which by definition means that $\mathrm{Ho}(\mathbf{C})$ is *triangulated*.

We let Vect denote the DG category of chain complexes of k-vector spaces.

We let DGCat denote the ∞-category of all DG categories.[5]

1.1.2. Cocomplete DG categories. Our basic object of study is the $(\infty, 1)$-category $\mathrm{DGCat}_{\mathrm{cont}}$ whose objects are cocomplete DG categories (i.e., ones that contain arbitrary direct sums, or equivalently, colimits), and where 1-morphisms are continuous functors (i.e., exact functors that commute with arbitrary direct sums, or equivalently all colimits).

The construction of $\mathrm{DGCat}_{\mathrm{cont}}$ as an $(\infty, 1)$-category has not been fully documented. A pedantic reader can replace $\mathrm{DGCat}_{\mathrm{cont}}$ by the equivalent $(\infty, 1)$-category of stable ∞-categories tensored over k, whose construction is a consequence of [Lu2, Sects. 4.2 and 6.3].

We have a forgetful functor $\mathrm{DGCat}_{\mathrm{cont}} \to \mathrm{DGCat}$ that induces an isomorphism on 2-morphisms and higher.

1.1.3. Terminological deviation *(i)*. We will sometimes encounter non-cocomplete DG categories (e.g., the subcategory of compact objects in a given DG category). Every time that this happens, we will say so explicitly.

1.1.4. The category $\mathrm{DGCat}_{\mathrm{cont}}$ has a natural symmetric monoidal structure given by Lurie's tensor product, denoted by $\otimes$ (see [Lu2, Sect. 6.3] or [GL:DG, Sect. 1.4] for a brief review).

Its unit object is the category Vect of chain complexes of k-vector spaces.

1.1.5. Functors. For $\mathbf{C}_1, \mathbf{C}_2 \in \mathrm{DGCat}_{\mathrm{cont}}$ we will denote by $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_1, \mathbf{C}_2)$ their internal Hom in $\mathrm{DGCat}_{\mathrm{cont}}$, which is therefore another DG category.

1.1.6. Terminological deviation *(ii)*. For two DG categories $\mathbf{C}_1$ and $\mathbf{C}_2$ we will sometimes encounter functors $\mathbf{C}_1 \to \mathbf{C}_2$ that are not continuous (but still exact). For example, for a non-compact object $\mathbf{c} \in \mathbf{C}$, such is the functor $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \mathrm{Vect}$ (see below for the notation).

Every time when we encounter a non-continuous functor, we will say so explicitly.

All exact functors $\mathbf{C}_1 \to \mathbf{C}_2$ also form a DG category, which we denote by $\mathrm{Funct}(\mathbf{C}_1, \mathbf{C}_2)$.

[5] We will ignore set-theoretic issues; however, the reader can assume that all DG categories and functors are *accessible* in the sense of [Lu1, Sect. 5.4.2].

1.1.7. Mapping spaces. Any DG category $\mathbf{C}$ can be thought of as an ∞-category enriched over Vect with the same set of objects. For two objects $\mathbf{c}_1, \mathbf{c}_2$, we will denote by $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \text{Vect}$ the corresponding Hom object.

We let $\text{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \infty\text{-Grpd}$ denote the Hom-space, when we consider $\mathbf{C}$ as a plain ∞-category. The object $\text{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ equals the image of $\tau^{\leq 0}(\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2))$ under the Dold-Kan functor

$$\text{Vect}^{\leq 0} \to \infty\text{-Grpd}\,.$$

We denote by $\text{Hom}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ the object $H^0(\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)) \in \text{Vect}^{\heartsuit}$. Its underlying set identifies with $\pi_0(\text{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2))$.

1.1.8. t-structures. Whenever a DG category $\mathbf{C}$ has a t-structure, we let $\mathbf{C}^{\leq 0}$ (resp., $\mathbf{C}^{\geq 0}$) denote the full subcategory of connective (resp., coconnective) objects. We denote by $\mathbf{C}^{\heartsuit}$ the heart of the t-structure.

1.2. Compactness and compact generation

1.2.1. Recall that an object $\mathbf{c}$ in a (cocomplete) DG category $\mathbf{C}$ is called *compact* if the functor

$$\text{Hom}_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \text{Vect}^{\heartsuit}$$

commutes with arbitrary direct sums. This is equivalent to the (a priori non-continuous) functor

$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \text{Vect}$$

being continuous, or the functor of ∞-categories

$$\text{Maps}_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \infty\text{-Grpd}$$

commuting with filtered colimits.

For a DG category $\mathbf{C}$, we let $\mathbf{C}^c$ denote the full (but not cocomplete) DG subcategory that consists of compact objects.

1.2.2. Compact generation. Let $\mathbf{C}$ be a cocomplete DG category. We say that a set of objects $\mathbf{c}_\alpha \in \mathbf{C}$ *generates* $\mathbf{C}$ if for every $\mathbf{c} \in \mathbf{C}$ the following implication holds:

$$\text{Hom}_{\mathbf{C}}(\mathbf{c}_\alpha, \mathbf{c}) = 0, \ \forall \alpha \ \Rightarrow \ \mathbf{c} = 0. \tag{1.1}$$

This is known to be equivalent to the following condition: $\mathbf{C}$ does not contain a proper full cocomplete DG subcategory that contains all the objects $\mathbf{c}_\alpha$.

A cocomplete DG category $\mathbf{C}$ is called *compactly generated* if there exists a set of compact objects $\mathbf{c}_\alpha$ that generates $\mathbf{C}$ in the above sense.

1.2.3. The following observations will be used repeatedly throughout the paper:

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be a pair of DG categories, and let $\mathsf{G} : \mathbf{C}_2 \to \mathbf{C}_1$ be a (not necessarily continuous) functor. If G admits a left adjoint functor $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ then F is automatically continuous.

Let F, G be as above and suppose, in addition, that $\mathbf{C}_1$ is compactly generated. Then G is continuous if and only if F preserves compactness (i.e., $\mathsf{F}(\mathbf{C}_1^c) \subset \mathbf{C}_2^c$). This implies the "only if" part of the following well-known proposition.

Proposition 1.2.4. *Let $\mathbf{C}_1$ be a compactly generated DG category and $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ a continuous DG functor. Then F has a continuous right adjoint if and only if $\mathsf{F}(\mathbf{C}_1^c) \subset \mathbf{C}_2^c$.*

Proof of the "if" statement. The existence of the *not necessarily continuous right adjoint* G follows from the Adjoint Functor Theorem, see [Lu1, Corollary 5.5.2.9]. To test that G is continuous, it is enough to show that the functors

$$\mathcal{M}aps_{\mathbf{C}_1}(\mathbf{c}_1, \mathsf{G}(-)) : \mathbf{C}_2 \to \mathrm{Vect}$$

are continuous for $\mathbf{c}_1 \in \mathbf{C}_1^c$. The required continuity follows from the assumption on F. $\square$

1.3. Ind-completions

1.3.1. Let $\mathbf{C}^0$ be an essentially small (but not cocomplete) DG category. We can functorially assign to it a cocomplete DG category, denoted $\mathrm{Ind}(\mathbf{C}^0)$ (and called the *ind-completion* of $\mathbf{C}^0$), equipped with a functor $\mathbf{C}^0 \to \mathrm{Ind}(\mathbf{C}^0)$ and characterized by the property that restriction defines an equivalence

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Ind}(\mathbf{C}^0), \mathbf{D}) \to \mathrm{Funct}(\mathbf{C}^0, \mathbf{D}) \tag{1.2}$$

for a cocomplete category $\mathbf{D}$ (see [Lu1, Sect. 5.3.5] for the corresponding construction for general ∞-categories).

The category $\mathrm{Ind}(\mathbf{C}^0)$ can be explicitly constructed as $\mathrm{Funct}((\mathbf{C}^0)^{\mathrm{op}}, \mathrm{Vect})$.

It is known that the functor $\mathbf{C}^0 \to \mathrm{Ind}(\mathbf{C}^0)$ is fully faithful, and that its essential image belongs to the subcategory $\mathrm{Ind}(\mathbf{C}^0)^c$. It follows formally from (1.2) that the essential image of $\mathbf{C}^0$ generates $\mathrm{Ind}(\mathbf{C}^0)$.

1.3.2. Thus, the assignment $\mathbf{C}^0 \rightsquigarrow \mathrm{Ind}(\mathbf{C}^0)$ is a way to obtain compactly generated categories. In fact, all cocomplete compactly generated DG categories arise in this way. Namely, we have the following assertion (see [Lu1, Proposition 5.3.5.11]):

Lemma 1.3.3. *Let* $\mathbf{C}$ *be a cocomplete compactly generated DG category. Let* $\mathsf{F}^0 : \mathbf{C}^0 \to \mathbf{C}^c$ *be a fully faithful functor, such that its essential image* generates $\mathbf{C}$. *Then the resulting functor* $\mathsf{F} : \mathrm{Ind}(\mathbf{C}^0) \to \mathbf{C}$, *obtained from* F^0 *via* (1.2), *is an equivalence.*

As a consequence, we obtain:

Corollary 1.3.4. *Let* $\mathbf{C}$ *be a cocomplete compactly generated DG category. Then the tautological functor* $\mathrm{Ind}(\mathbf{C}^c) \to \mathbf{C}$ *is an equivalence.*

1.4. Karoubi-completions

1.4.1. Let $\mathbf{C}^0$ be an essentially small (but non-cocomplete) DG category. We say that $\mathbf{C}^0$ is *Karoubian* if its homotopy category is idempotent-complete.

For example, for a cocomplete compactly generated DG category $\mathbf{C}$, the corresponding subcategory $\mathbf{C}^c$ is Karoubian.

1.4.2. Let $\mathbf{C}^0 \to \mathbf{C}^0_{\mathrm{Kar}}$ be a functor between essentially small (but non-cocomplete) DG categories.

We say that the above functor realizes $\mathbf{C}^0_{\mathrm{Kar}}$ as a *Karoubi-completion* of $\mathbf{C}^0$ if restriction defines an equivalence

$$\mathrm{Funct}(\mathbf{C}^0_{\mathrm{Kar}}, {}'\mathbf{C}_0) \to \mathrm{Funct}(\mathbf{C}^0, {}'\mathbf{C}_0)$$

for any Karoubian ${}'\mathbf{C}_0$. Clearly, $\mathbf{C}^0_{\mathrm{Kar}}$, if it exists, is defined up to a canonical equivalence.

The following is a reformulation of the Thomason-Trobaugh-Neeman localization theorem (see [N, Theorem 2.1] or [BeV, Proposition 1.4.2]):

Lemma 1.4.3. (a) *Let* $\mathbf{C}^0$ *be an essentially small (but not cocomplete) DG category. The canonical functor* $\mathbf{C}^0 \to \mathrm{Ind}(\mathbf{C}^0)^c$ *realizes* $\mathrm{Ind}(\mathbf{C}^0)^c$ *as a Karoubi-completion of* $\mathbf{C}^0$.

(b) *Every object of* $\mathrm{Ind}(\mathbf{C}^0)^c$ *can be realized as a direct summand of one in* $\mathbf{C}^0 \subset \mathrm{Ind}(\mathbf{C}^0)^c$.

Lemma 1.4.3 implies that the functor $\mathrm{Ho}(\mathbf{C}^0) \to \mathrm{Ho}(\mathbf{C}^0_{\mathrm{Kar}})$ identifies $\mathrm{Ho}(\mathbf{C}^0_{\mathrm{Kar}})$ with the idempotent completion of $\mathrm{Ho}(\mathbf{C}^0)$.

1.4.4. We obtain that the assignments

$$\mathbf{C}^0 \rightsquigarrow \mathrm{Ind}(\mathbf{C}^0) \text{ and } \mathbf{C} \rightsquigarrow \mathbf{C}^c$$

define mutually inverse equivalences between the appropriate ∞-categories.

The two ∞-categories are as follows. One is $\mathrm{DGCat}_{\mathrm{Kar}}$, whose objects are essentially small Karoubian DG categories and morphisms are exact functors. The other is $\mathrm{DGCat}^{\mathrm{comp.gen.}}_{\mathrm{cont,pr.comp.}}$, whose objects are cocomplete compactly generated categories and morphisms are continuous functors preserving compactness.

1.4.5. Let $\mathbf{C}^0$ be an essentially small (but not cocomplete) DG category. Let $\mathbf{S}$ be a subset of its objects.

We say that $\mathbf{S}$ *Karoubi-generates* $\mathbf{C}^0$ if every object in the homotopy category of $\mathbf{C}^0$ can be obtained from objects in $\mathbf{S}$ by a finite iteration of operations of taking the cone of a morphism, and passing to a direct summand of an object.

By combining Lemmas 1.4.3 with 1.3.3 we obtain:

Corollary 1.4.6. *Let* $\mathbf{C}$ *be a cocomplete DG category. Let* $\mathbf{S} \subset \mathbf{C}^c$ *be a subset of objects that generates* $\mathbf{C}$*. Then* $\mathbf{S}$ Karoubi-generates $\mathbf{C}^c$*.*

1.5. Symmetric monoidal structure and duality

1.5.1. The notion of dual of a DG category. A DG category $\mathbf{C}$ is called *dualizable* if it is such as an object of the symmetric monoidal category $(\mathrm{DGCat}_{\mathrm{cont}}, \otimes)$. We refer the reader to [DrGa1, Sect. 4.1] for a review of some of the properties of this notion. The most important ones are listed below.

For a dualizable category $\mathbf{C}$ we denote by $\mathbf{C}^\vee$ its dual. One constructs $\mathbf{C}^\vee$ explicitly as

$$\mathbf{C}^\vee \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}, \mathrm{Vect}). \tag{1.3}$$

In addition, for any $\mathbf{D} \in \mathrm{DGCat}_{\mathrm{cont}}$, the natural functor

$$\mathbf{C}^\vee \otimes \mathbf{D} \to \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}, \mathbf{D})$$

is an equivalence.

1.5.2. If $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ is a (continuous) functor between dualizable categories, there exists a canonically defined dual functor $\mathsf{F}^\vee : \mathbf{C}_2^\vee \to \mathbf{C}_1^\vee$ (the construction follows, e.g., from (1.3)). The assignment $\mathsf{F} \mapsto \mathsf{F}^\vee$ is functorial in F. One has $(\mathsf{F}^\vee)^\vee = \mathsf{F}$, $(\mathsf{G} \circ \mathsf{F})^\vee = \mathsf{F}^\vee \circ \mathsf{G}^\vee$.

From here we obtain that if the functors

$$\mathsf{F} : \mathbf{C}_1 \leftrightarrows \mathbf{C}_2 : \mathsf{G}$$

are mutually adjoint, then so are the functors

$$\mathsf{G}^\vee : \mathbf{C}_2^\vee \leftrightarrows \mathbf{C}_1^\vee : \mathsf{F}^\vee.$$

1.5.3. If $\mathbf{C}$ is compactly generated, then it is dualizable. We have a canonical identification

$$(\mathbf{C}^\vee)^c \simeq (\mathbf{C}^c)^{\mathrm{op}}.$$

Vice versa, if $\mathbf{C}_1$ and $\mathbf{C}_2$ are two compactly generated categories, then an identification

$$\mathbf{C}_1^c \simeq (\mathbf{C}_2^c)^{\mathrm{op}}$$

gives rise to an identification

$$\mathbf{C}_1^\vee \simeq \mathbf{C}_2.$$

1.6. Limits of DG categories

The reason that we work with DG categories rather than with triangulated ones is that the limit (i.e., projective limit) of DG categories is well-defined as a DG category (while the corresponding fact for triangulated categories is false).

More precisely, the $(\infty, 1)$-categories $\mathrm{DGCat}_{\mathrm{cont}}$ and DGCat admit limits and the forgetful functor $\mathrm{DGCat}_{\mathrm{cont}} \to \mathrm{DGCat}$ commutes with limits (this is essentially [Lu1, Proposition 5.5.3.13]) This is important for us because the DG category of D-modules on an algebraic stack is defined as a limit (see Sect. 2.1.1 below).

1.6.1. Let

$$i \mapsto \mathbf{C}_i, \ (i \to j) \mapsto (\phi_{i,j} \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_i, \mathbf{C}_j))$$

be a diagram of DG categories, parameterized by an index category I. The limit

$$\mathbf{C} := \underset{\underset{i \in I}{\longleftarrow}}{lim}\, \mathbf{C}_i$$

is a priori defined by a universal property in $\mathrm{DGCat}_{\mathrm{cont}}$: for a DG category $\mathbf{D}$ we have a functorial isomorphism

$$(\mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}))^{\mathrm{grpd}} \simeq \underset{\underset{i \in I}{\longleftarrow}}{lim}\, (\mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}_i))^{\mathrm{grpd}}$$

where the in the left-hand side the limit is taken in the $(\infty,1)$-category ∞-Grpd. We remind that the superscript "grpd" means that we are taking the maximal ∞-subgroupoid in the corresponding ∞-category.

1.6.2. Note that [Lu1, Corollary 3.3.3.2] provides a more explicit description of $\mathbf{C}$. Namely, objects of $\mathbf{C}$ are *Cartesian sections,* i.e., assignments

$$i \rightsquigarrow (\mathbf{c}_i \in \mathbf{C}_i),\ \phi_{i,j}(\mathbf{c}_i) \overset{\alpha_{\phi_{i,j}}}{\simeq} \mathbf{c}_j,$$

equipped with data making $\alpha_{\phi_{i,j}}$ coherently associative. In fact, this description follows easily from the above functorial description, by taking $\mathbf{D} = \mathrm{Vect}$, and using the fact $\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Vect}, \mathbf{C}) \simeq \mathbf{C}$ as DG categories.

If $\mathbf{c} := (\mathbf{c}_i, \alpha_{\phi_{i,j}})$ and $\widetilde{\mathbf{c}} := (\widetilde{\mathbf{c}}_i, \widetilde{\alpha}_{\phi_{i,j}})$ are two such objects, then one can upgrade the assignment

$$i \mapsto \mathcal{M}aps_{\mathbf{C}_i}(\mathbf{c}_i, \widetilde{\mathbf{c}}_i)$$

into a homotopy I-diagram in Vect, and

$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}, \widetilde{\mathbf{c}}) \simeq \underset{\underset{i\in I}{\longleftarrow}}{lim}\, \mathcal{M}aps_{\mathbf{C}_i}(\mathbf{c}_i, \widetilde{\mathbf{c}}_i)$$

as objects of Vect.

1.6.3. The following observation will be useful in the sequel. Let $\mathbf{C} = \underset{\underset{i\in I}{\longleftarrow}}{lim}\, \mathbf{C}_i$ be as above, and let

$$(\alpha \in A) \mapsto (\mathbf{c}_\alpha \in \mathbf{C})$$

be a collection of objects of $\mathbf{C}$ parameterized by some category A. In particular, for every $i \in I$ we obtain a functor

$$(\alpha \in A) \mapsto (\mathbf{c}_{i,\alpha} \in \mathbf{C}_i).$$

We have:

Lemma 1.6.4. *For every i, the map from $\underset{\underset{\alpha\in A}{\longrightarrow}}{colim}\, \mathbf{c}_{i,\alpha} \in \mathbf{C}_i$ to the i-th component of the object $\underset{\underset{\alpha\in A}{\longrightarrow}}{colim}\, \mathbf{c}_\alpha$ is an isomorphism.*

In other words, colimits in a limit of DG categories can be computed component-wise.

Remark 1.6.5. The assertion of Lemma 1.6.4 can be reformulated as saying that the evaluation functors $\mathrm{ev}_i : \mathbf{C} \to \mathbf{C}_i$ commute with colimits, i.e., are continuous. This is tautological from the definition of $\mathbf{C}$ as a limit in the category $\mathrm{DGCat}_{\mathrm{cont}}$.

1.7. Colimits in $\mathbf{DGCat}_{\mathbf{cont}}$

The goal of the remaining part of Sect. 1 is to provide a categorical framework for Sects. 4.2–4.3. This material is not used in other parts of the article.

1.7.1. As was mentioned in Sect. 1.6, limits in $\mathrm{DGCat}_{\mathrm{cont}}$ are the same as limits in DGCat. However, *colimits are different* (for example, the colimit taken in DGCat does not have to be cocomplete).

It is known to experts that under suitable set-theoretical conditions, colimits in $\mathrm{DGCat}_{\mathrm{cont}}$ always exist. We are unable to find a really satisfactory reference for this fact.

On the other hand, in this paper we work only with colimits of those functors

$$\Psi : I \to \mathrm{DGCat}_{\mathrm{cont}}$$

that satisfy the following condition: for every arrow $i \to j$ in I the corresponding functor $\psi_{i,j} : \Psi(i) \to \Psi(j)$ *admits a continuous right adjoint.* In this case existence of the colimit of Ψ is provided by Proposition 1.7.5 below.

1.7.2. The setting. Let I be a small category, and let $\Psi : I \to \mathrm{DGCat}_{\mathrm{cont}}$ be a functor

$$i \rightsquigarrow \mathbf{C}_i, \quad (i \to j) \in I \rightsquigarrow \psi_{i,j} \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_i, \mathbf{C}_j).$$

Assume that for every arrow $i \to j$ in I, the above functor $\psi_{i,j}$ admits a continuous right adjoint, $\phi_{j,i}$.

We can then view the assignment

$$i \rightsquigarrow \mathbf{C}_i, \quad (i \to j) \in I \rightsquigarrow \phi_{j,i} \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_j, \mathbf{C}_i).$$

as a functor $\Phi : I^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$.

Remark 1.7.3. Some readers may prefer to assume, in addition, that each DG category $\mathbf{C}_i$ is compactly generated. As explained in Sect. 1.9 below, this special case of the situation of Sect. 1.7.2 is very easy (Propositions 1.7.5 and 1.8.3 formulated below become obvious, and it is easy to understand "who is who"). Moreover, this case is enough for the applications in Sects. 4.2–4.3.

1.7.4. The following proposition is a variant of [Lu1, Corollary 5.5.3.4]; a digest of the proof is given in [GL:DG, Lemma 1.3.3].

Proposition 1.7.5. *In the situation of Sect. 1.7.2, the colimit*

$$\underset{i\in I}{\underrightarrow{colim}}\, \mathbf{C}_i := \underset{I}{\underrightarrow{colim}}\, \Psi \in \mathrm{DGCat}_{\mathrm{cont}}$$

exists and is canonically equivalent to the limit

$$\underset{i\in I^{\mathrm{op}}}{\underleftarrow{lim}}\, \mathbf{C}_i := \underset{I^{\mathrm{op}}}{\underleftarrow{lim}}\, \Phi \in \mathrm{DGCat}_{\mathrm{cont}}\,;$$

the equivalence is uniquely characterized by the condition that for $i_0 \in I$, the evaluation functor

$$\mathrm{ev}_{i_0} : \underset{i\in I^{\mathrm{op}}}{\underleftarrow{lim}}\, \mathbf{C}_i \to \mathbf{C}_{i_0}$$

is right adjoint to the tautological functor

$$\mathrm{ins}_{i_0} : \mathbf{C}_{i_0} \to \underset{i\in I}{\underrightarrow{colim}}\, \mathbf{C}_i\,,$$

in a way compatible with arrows in I.

The above proposition can be reformulated as follows. Let $\mathbf{C}$ denote the limit of the DG categories $\mathbf{C}_i$. The claim is that each functor $\mathrm{ev}_i : \mathbf{C} \to \mathbf{C}_i$ admits a left adjoint functor $'\mathrm{ins}_i : \mathbf{C}_i \to \mathbf{C}$, and that the functors $'\mathrm{ins}_i : \mathbf{C}_i \to \mathbf{C}$ together with the isomorphisms

$$'\mathrm{ins}_j \circ \psi_{i,j} \simeq {}'\mathrm{ins}_i\,, \quad (i \to j) \in I,$$

that one obtains by adjunction, make $\mathbf{C}$ into a colimit of the DG categories $\mathbf{C}_i$.

Remark 1.7.6. Let I be filtered. In this case one can show (see [GL:DG, Lemma 1.3.6]) that if an index $i_0 \in I$ is such that for every arrow $i_0 \to i$ the functor $\psi_{i_0,i} : \mathbf{C}_{i_0} \to \mathbf{C}_i$ is fully faithful then the functor ins_{i_0} is fully faithful. If $\mathbf{C}$ is compactly generated this follows from Lemma 1.9.5(ii) below.

1.8. Colimits and duals

1.8.1. Assume now that the categories $\mathbf{C}_i$ are dualizable. Then we can produce yet another functor

$$\Phi^\vee : I \to \mathrm{DGCat}_{\mathrm{cont}}$$

that sends

$$i \rightsquigarrow \mathbf{C}_i^\vee, \quad (i \to j) \in I \rightsquigarrow (\phi_{j,i})^\vee \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_i^\vee, \mathbf{C}_j^\vee).$$

1.8.2. In this case we have the following result ([GL:DG, Lemma 2.2.2]):

Proposition 1.8.3. *The category*

$$\underset{i \in I^{\mathrm{op}}}{\underleftarrow{lim}} \mathbf{C}_i := \underset{I^{\mathrm{op}}}{\underleftarrow{lim}} \Phi$$

is dualizable, and its dual is given by

$$\underset{i \in I}{\underrightarrow{colim}} \mathbf{C}_i^\vee := \underset{I}{\underrightarrow{colim}} \Phi^\vee.$$

This identification is uniquely characterized by the property that for $i_0 \in I$, we have

$$(\mathrm{ins}_{i_0,\Phi^\vee})^\vee \simeq \mathrm{ev}_{i_0,\Phi}\,, \tag{1.4}$$

in a way compatible with arrows in I.

In formula (1.4), the notation $\mathrm{ins}_{i_0,\Psi^\vee}$ means the functor

$$\mathrm{ins}_{i_0} : \mathbf{C}_{i_0}^\vee \to \underset{I}{\underrightarrow{colim}} \Phi^\vee,$$

and the notation $\mathrm{ev}_{i_0,\Phi}$ means the functor

$$\mathrm{ev}_{i_0} : \underset{I^{\mathrm{op}}}{\underleftarrow{lim}} \Phi \to \mathbf{C}_{i_0}.$$

Remark 1.8.4. By adjunction between ins and ev (see Proposition 1.7.5), one gets from (1.4) a similar isomorphism $(\mathrm{ins}_{i_0,\Psi})^\vee \simeq \mathrm{ev}_{i_0,\Psi^\vee}$.

1.9. Colimits of compactly generated categories

The main goal of this subsection is to demonstrate that the results of Sects. 1.7–1.8 are very easy under the additional assumption that each DG category $\mathbf{C}_i$ is compactly generated.

1.9.1. Who is who. Suppose that in the situation of Sect. 1.7.2 each of the categories $\mathbf{C}_i$ is compactly generated, so

$$\mathbf{C}_i \simeq \mathrm{Ind}(\mathbf{C}_i^c)$$

or equivalently,

$$\mathbf{C}_i \simeq \mathrm{Funct}((\mathbf{C}_i^c)^{\mathrm{op}}, \mathrm{Vect}). \tag{1.5}$$

By Proposition 1.2.4, the assumption that the functor $\psi_{i,j} : \mathbf{C}_i \to \mathbf{C}_j$ has a continuous right adjoint just means that $\psi_{i,j}(\mathbf{C}_i^c) \subset \mathbf{C}_j^c$ (so $\psi_{i,j}$ is the ind-extension of a functor $\psi_{i,j}^c : \mathbf{C}_i^c \to \mathbf{C}_j^c$).

Moreover, the right adjoint functor $\phi_{j,i} : \mathbf{C}_j \to \mathbf{C}_i$ is just the restriction functor

$$\mathrm{Funct}((\mathbf{C}_j^c)^{\mathrm{op}}, \mathrm{Vect}) \to \mathrm{Funct}((\mathbf{C}_i^c)^{\mathrm{op}}, \mathrm{Vect})$$

corresponding to $\psi_{i,j}^c : \mathbf{C}_i^c \to \mathbf{C}_j^c$.

1.9.2. On Propositions 1.7.5 and 1.8.3 in the compactly generated case.

Set $\mathbf{C} := \underset{i\in I}{\underrightarrow{colim}}\, \mathbf{C}_i$. In our situation the existence of this colimit is clear: in fact,

$$\mathbf{C} \simeq \mathrm{Ind}(\underset{i\in I}{\underrightarrow{colim}}\, \mathbf{C}_i^c), \tag{1.6}$$

where the colimit in the right hand side is computed in DGCat.

Just as in Sect. 1.9.1, we can rewrite (1.6) as

$$\mathbf{C} \simeq \mathrm{Funct}(\underset{i\in I}{\underrightarrow{colim}}\, (\mathbf{C}_i^c)^{\mathrm{op}}, \mathrm{Vect}). \tag{1.7}$$

Now the canonical equivalence

$$\mathbf{C} \simeq \underset{i\in I^{\mathrm{op}}}{\underleftarrow{lim}}\, \mathbf{C}_i$$

from Proposition 1.7.5 becomes obvious: this is just the composition

$$\mathbf{C} \simeq \mathrm{Funct}(\underset{i\in I}{\underrightarrow{colim}}\, (\mathbf{C}_i^c)^{\mathrm{op}}, \mathrm{Vect}) \simeq \underset{i\in I^{\mathrm{op}}}{\underleftarrow{lim}}\, \mathrm{Funct}((\mathbf{C}_i^c)^{\mathrm{op}}, \mathrm{Vect}) \simeq \underset{i\in I^{\mathrm{op}}}{\underleftarrow{lim}}\, \mathbf{C}_i\,,$$

where the first equivalence is (1.7) and the third one comes from (1.5).

Proposition 1.8.3 says that $\mathbf{C}$ is dualizable and

$$\mathbf{C}^\vee \simeq \underset{i\in I}{\underrightarrow{colim}}\, \mathbf{C}_i^\vee\,. \tag{1.8}$$

This is clear because by formula (1.6) and Sect. 1.5.3, both sides of (1.8) canonically identify with

$$\operatorname{Ind}(\underset{i\in I}{\underrightarrow{colim}}\,(\mathbf{C}_i^c)^{\mathrm{op}})$$

(the colimit in this formula is computed in DGCat).

1.9.3. As a consequence of (1.6), we obtain the following

Corollary 1.9.4. *In the situation of Sect. 1.9.1 the category $\mathbf{C}$ is compactly generated. More precisely, objects of $\mathbf{C}$ of the form*

$$\operatorname{ins}_i(\mathbf{c}), \qquad i \in I,\, \mathbf{c} \in \mathbf{C}_i^c \tag{1.9}$$

are compact and generate $\mathbf{C}$.

In addition, one has the following lemma.

Lemma 1.9.5. *Suppose that in the situation of Sect. 1.9.1 the category I is filtered. Then*

(i) *every compact object of $\mathbf{C}$ is of the form (1.9);*
(ii) *for any $i, i' \in I$, $\mathbf{c} \in \mathbf{C}_i^c$, and $\mathbf{c}' \in \mathbf{C}_{i'}^c$ the canonical map*

$$\underset{j,\,\alpha:i\to j,\,\beta:i'\to j}{colim} \mathcal{M}aps_{\mathbf{C}_j}(\psi_{i,j}(\mathbf{c}), \psi_{i',j}(\mathbf{c}')) \to \mathcal{M}aps_{\mathbf{C}}(\operatorname{ins}_i(\mathbf{c}), \operatorname{ins}_{i'}(\mathbf{c}'))$$

is an isomorphism.

Proof. For statement (ii), see [Roz].

Using (ii) and the assumption that I is filtered, it is easy to see that the class of objects of the form (1.9) is closed under cones and direct summands. So (i) follows from (ii) and Corollary 1.4.6. □

2. Preliminaries on the DG category of D-modules on an algebraic stack

In this section we recall some definitions and results from [DrGa1].

2.1. D-modules on prestacks and algebraic stacks

2.1.1. Let $\mathcal{Y}$ be a prestack (always assumed locally of finite type). Recall following [DrGa1, Sect. 6.1] that the category $\text{D-mod}(\mathcal{Y})$ is defined as the limit

$$\underset{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}}}{\underset{\longleftarrow}{lim}} \text{D-mod}(S), \tag{2.1}$$

where the limit is taken in the $(\infty,1)$-category $\mathrm{DGCat}_{\mathrm{cont}}$. Here

$$S \mapsto \text{D-mod}(S)$$

is the functor

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

were for $f : S' \to S$ the corresponding map $\text{D-mod}(S) \to \text{D-mod}(S')$ is $f^!$.

I.e., as was explained in Sect. 1.6.2, informally, an object $\mathcal{F} \in \text{D-mod}(\mathrm{Bun}_G)$ is an assignment for every $S \to \mathcal{Y}$ of an object $\mathcal{F}_S \in \text{D-mod}(S)$, and for every $f : S' \to S$ over $\mathcal{Y}$ of an isomorphism $f^!(\mathcal{F}_S) \simeq \mathcal{F}_{S'}$.

In particular, for $\mathcal{F}_1, \mathcal{F}_2 \in \text{D-mod}(\mathrm{Bun}_G)$, the complex $\mathcal{M}aps(\mathcal{F}_1, \mathcal{F}_2)$ is calculated as

$$\underset{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}}}{\underset{\longleftarrow}{lim}} \mathcal{M}aps_{\text{D-mod}(S)}((\mathcal{F}_1)_S, (\mathcal{F}_2)_S).$$

This definition has several variants. For example, we can replace the category of affine schemes by that of quasi-compact schemes, or all schemes.

2.1.2. Assume now that $\mathcal{Y}$ is an Artin stack (see [GL:Stacks, Sect. 4] for our conventions regarding Artin stacks).

In this case, as in [Ga1, Corollary 11.2.3], in the formation of the limit in (2.1), we can replace the category $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}}$ by its non-full subcategory $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y},\mathrm{smooth}}$, where we restrict objects to be those pairs $(S, g : S \to \mathcal{Y})$ for which the map g is smooth, and 1-morphisms to smooth maps between affine schemes.

As before, we can replace the word "affine" by "quasi-compact", or just consider all schemes.

2.2. D-modules on a quasi-compact algebraic stack

2.2.1. QCA and locally QCA stacks. QCA is shorthand for "quasi-compact and with affine automorphism groups".

Definition 2.2.2. *We say that an algebraic stack* $\mathcal{Y}$ *is* locally QCA *if the automorphism groups of its field-valued points are affine. We say that* $\mathcal{Y}$ *is* QCA *if it is quasi-compact and locally QCA.*

Convention: *in this article all stacks will be assumed to be locally QCA.* The reason is clear from Theorem 2.2.4 below.

2.2.3. A property of QCA stacks. The following result is established in [DrGa1, Theorem 8.1.1].

Theorem 2.2.4. *Let* $\mathcal{Y}$ *be a QCA stack. Then the category* $\text{D-mod}(\mathcal{Y})$ *is compactly generated.*

Remark 2.2.5. In fact, [DrGa1, Theorem 8.1.1] produces an explicit set of compact generators of $\text{D-mod}(\mathcal{Y})$. These are objects induced from coherent sheaves on $\mathcal{Y}$.

Remark 2.2.6. Before [DrGa1], the above result was known for algebraic stacks that can be represented as Z/G, where Z is a quasi-compact scheme and G is an affine algebraic group acting on S. Most quasi-compact Artin stacks that appear in practice (e.g., all quasi-compact open substacks of Bun_G) admit such a representation. More generally, it was known for algebraic stacks that are perfect in the sense of [BFN].

2.2.7. Cartesian products. The following result is established in [DrGa1, Corollary 8.3.4].

Proposition 2.2.8. *Let* $\mathcal{Y}$ *and* $\mathcal{Y}'$ *be QCA stacks. Then the natural functor*

$$\text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}') \to \text{D-mod}(\mathcal{Y} \times \mathcal{Y}')$$

is an equivalence.

Remark 2.2.9. In fact, as is remarked in the proof of [DrGa1, Corollary 8.3.4], the assertion of Proposition 2.2.8 is valid for any pair of prestacks $\mathcal{Y}$ and $\mathcal{Y}'$ as long as either $\text{D-mod}(\mathcal{Y})$ or $\text{D-mod}(\mathcal{Y}')$ is dualizable (see Sect. 1.5.1).

2.2.10. Compactness and coherence. Let Z be a quasi-compact scheme. An object of $\text{D-mod}(Z)$ is said to be *coherent* if it is a bounded complex whose cohomology sheaves are coherent D-modules.

It is known that the (non cocomplete) subcategory $\text{D-mod}_{\mathrm{coh}}(Z)$ that consists of coherent objects coincides with $\text{D-mod}(Z)^c$ (see [DrGa1, Sect. 5.1.17]). Recall from Sect. 1.2.1 that for a DG category $\mathbf{C}$ we denote by $\mathbf{C}^c$ the full subcategory of compact objects.

For an algebraic stack $\mathcal{Y}$, an object $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ is said to be *coherent* if $f^!(\mathcal{F})$ (or equivalently, $f^*_{\mathrm{dR}}(\mathcal{F})$) is coherent for any smooth map $f : Z \to \mathcal{Y}$, where Z is a quasi-compact scheme. So by definition, the property of coherence is local for the smooth topology. The full (but non-cocomplete) subcategory of coherent objects of $\text{D-mod}(\mathcal{Y})$ is denoted by $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$.

Theorem 2.2.11. *Let $\mathcal{Y}$ be a QCA stack.*

(i) *We have the inclusion* $\text{D-mod}(\mathcal{Y})^c \subset \text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$.
(ii) *The above inclusion is an equality if and only if for every geometric point y of $\mathcal{Y}$, the quotient of the automorphism group* $\mathrm{Aut}(y)$ *by its unipotent radical is finite.*

This theorem is proved in [DrGa1, Lemma 7.3.3 and Corollary 10.2.7].

Remark 2.2.12. One may wonder how far coherence is from compactness. The answer is provided by the notion of *safety*, introduced in [DrGa1, Sect. 9.2]. In [DrGa1, Proposition 9.2.3] it is shown that an object of $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ is compact if and only if it is safe.

Remark 2.2.13. Note that the notion of coherence of D-modules makes sense for *any* algebraic stack $\mathcal{Y}$, i.e., it does not have to be quasi-compact: we test it by smooth maps $Z \to \mathcal{Y}$, where Z is a quasi-compact scheme. The inclusion of point (i) of Theorem 2.2.11 remains valid in this context. The proof is very easy: for a map $f : Z \to \mathcal{Y}$, the functor f^*_{dR} sends compacts to compacts because it admits a continuous right adjoint, namely $f_{\mathrm{dR},*}$.

2.2.14. Verdier duality. Let $\mathcal{Y}$ be a QCA stack. Accoding to [DrGa1, Sect. 7.3.4], the (non-cocomplete) DG category $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ carries a natural anti-involution

$$\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}} : (\text{D-mod}_{\mathrm{coh}}(\mathcal{Y}))^{\mathrm{op}} \to \text{D-mod}_{\mathrm{coh}}(\mathcal{Y}),$$

which we refer to as *Verdier duality*.

The following key feature of this functor is established in [DrGa1, Corollary 8.4.2]:

Theorem 2.2.15. *The functor* $\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}}$ *sends the subcategory*

$$(\text{D-mod}(\mathcal{Y})^c)^{\mathrm{op}} \subset (\text{D-mod}_{\mathrm{coh}}(\mathcal{Y}))^{\mathrm{op}}$$

to $\text{D-mod}(\mathcal{Y})^c \subset \text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$.

2.2.16. By Sect. 1.5.3, we obtain that the resulting functor

$$\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}} : (\text{D-mod}(\mathcal{Y})^c)^{\mathrm{op}} \to \text{D-mod}(\mathcal{Y})^c$$

uniquely extends to an equivalence

$$\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y}} : \text{D-mod}(\mathcal{Y})^{\vee} \simeq \text{D-mod}(\mathcal{Y}). \tag{2.2}$$

Alternatively, we can view the Verdier duality functor as follows. By Sect. 1.5.1, the DG category $\mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y}), \text{D-mod}(\mathcal{Y}))$ identifies tautologically with

$$\text{D-mod}(\mathcal{Y})^{\vee} \otimes \text{D-mod}(\mathcal{Y}).$$

The equivalence (2.2) is characterized by the property that the identity functor on $\text{D-mod}(\mathcal{Y})$ corresponds to the object of $\text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y})$ that identifies via Proposition 2.2.8 with

$$(\Delta_{\mathcal{Y}})_{\mathrm{dR},*}(\omega_{\mathcal{Y}}) \in \text{D-mod}(\mathcal{Y} \times \mathcal{Y}).$$

Here $\omega_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y})$ is the dualizing object and $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is the diagonal.

Let $\mathcal{Y}_1, \mathcal{Y}_2$ be QCA stacks. If $\mathsf{F} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2)$ is a continuous functor then the dual functor $\mathsf{F}^{\vee} : \text{D-mod}(\mathcal{Y}_2)^{\vee} \to \text{D-mod}(\mathcal{Y}_1)^{\vee}$ (see Sect. 1.5.2) will be considered, via (2.2), as a functor $\text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1)$.

We will use the following fact [DrGa1, Proposition 8.4.8].

Proposition 2.2.17. *For any schematic quasi-compact morphism* $f : \mathcal{Y}_1 \longrightarrow \mathcal{Y}_2$ *, the functors*

$$f_{\mathrm{dR},*} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2), \qquad f^! : \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1)$$

are dual to each other in the sense of Sect. 1.5.2.

2.3. Non quasi-compact algebraic stacks

Let $\mathcal{Y}$ be now a stack, which is only assumed to be *locally* QCA. Then every quasi-compact open substack $U \subset \mathcal{Y}$ is QCA, so the category $\text{D-mod}(U)$ is compactly generated by Theorem 2.2.4. However, it is not true, in general, that the category $\text{D-mod}(\mathcal{Y})$ is compactly generated. For a counterexample, see Sect. 12.

In this subsection we give a description of the subcategory of compact objects

$$\text{D-mod}(\mathcal{Y})^c \subset \text{D-mod}(\mathcal{Y}),$$

see Proposition 2.3.7 below.

2.3.1. The category D-mod($\mathcal{Y}$) as a limit. The following statement immediately follows from the definition of $\text{D-mod}(\mathcal{Y})$, see Sect. 2.1.1.

Lemma 2.3.2. *The restriction functor*

$$\text{D-mod}(\mathcal{Y}) \to \underset{U \subset \mathcal{Y}}{\underleftarrow{lim}}\, \text{D-mod}(U),$$

is an equivalence, where the limit is taken over the poset of open quasi-compact substacks of $\mathcal{Y}$.

In particular, we obtain that for $\mathcal{F}_1, \mathcal{F}_2 \in \text{D-mod}(\mathcal{Y})$, the natural map

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}_1, \mathcal{F}_2) \to \underset{U \subset \mathcal{Y}}{\underleftarrow{lim}}\, \mathcal{M}aps_{\text{D-mod}(U)}(\mathcal{F}_1|_U, \mathcal{F}_2|_U) \tag{2.3}$$

is an isomorphism.

The following observation will be useful in the sequel:

Corollary 2.3.3. *Suppose that a family of objects $\mathcal{F}_\alpha \in \text{D-mod}(\mathcal{Y})$ is locally finite, i.e., for every quasi-compact open $U \subset \mathcal{Y}$ the set of α's such that $\mathcal{F}_\alpha|_U \neq 0$ is finite. Then the map*

$$\underset{\alpha}{\oplus} \mathcal{F}_\alpha \to \underset{\alpha}{\Pi} \mathcal{F}_\alpha$$

is an isomorphism.

Proof. Follows immediately from (2.3) and Lemma 1.6.4. $\square$

2.3.4. The functors j^* and $j_!$. Let $U \overset{j}{\hookrightarrow} \mathcal{Y}$ be an open substack. We have a pair of (continuous) adjoint functors

$$j^* : \text{D-mod}(\mathcal{Y}) \rightleftarrows \text{D-mod}(U) : j_*.$$

In particular, the functor j^* sends $\text{D-mod}(\mathcal{Y})^c$ to $\text{D-mod}(U)^c$.

Now, the functor j^* has a *partially defined* left adjoint, denoted $j_!$. It again follows automatically that if for $\mathcal{F}_U \in \text{D-mod}(U)^c$, the object $j_!(\mathcal{F}_U) \in \text{D-mod}(\mathcal{Y})$ is defined, then it is compact.

We claim:

Lemma 2.3.5. *Let $\mathcal{F}_U \in \text{D-mod}(U)$ be such that $j_!(\mathcal{F}_U)$ is defined.*

(a) *The canonical map*

$$\mathcal{F}_U \to j^*(j_!(\mathcal{F}_U)) \tag{2.4}$$

is an isomorphism.

(b) *If $j' : U' \hookrightarrow \mathcal{Y}$ is another open substack, then*

$$(j')^*(j_!(\mathcal{F}_U)) \simeq \widetilde{j}_!(\mathcal{F}_U|_{U \cap U'}),$$

(where $\widetilde{j} : U \cap U' \hookrightarrow U'$). In particular, $\widetilde{j}_!(\mathcal{F}_U|_{U \cap U'})$ is defined.

Proof. The functor $j^* \circ j_!$ is the partially defined left adjoint of $j^* \circ j_*$, and the natural transformation $\text{Id} \to j^* \circ j_!$ is obtained by adjunction from the co-unit $j^* \circ j_* \to \text{Id}$. However, the latter is an isomorphism since j_* is fully faithful.

Statement (b) follows similarly. □

2.3.6. A description of the subcategory $\text{D-mod}(\mathcal{Y})^c \subset \text{D-mod}(\mathcal{Y})$.

Proposition 2.3.7. *An object $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ is compact if and only if*

$$\mathcal{F} = j_!(\mathcal{F}_U) \tag{2.5}$$

for some open quasi-compact $U \overset{j}{\hookrightarrow} \mathcal{Y}$ and some $\mathcal{F}_U \in \text{D-mod}(U)^c$.

Formula (2.5) should be understood as follows: the partially defined functor $j_!$ is defined on $\mathcal{F}_U$, and the resulting object is isomorphic to $\mathcal{F}$.

Remark 2.3.8. By Lemma 2.3.5(a), the object $\mathcal{F}_U$ can be recovered from $\mathcal{F}$ as $\mathcal{F}|_U := j^*(\mathcal{F})$.

2.3.9. Proof of Proposition 2.3.7. First, let us give two more reformulations of condition (2.5):

Lemma 2.3.10. *For $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ the following conditions are equivalent:*

(1) $\mathcal{F} = j_!(\mathcal{F}_U)$ *for some* $\mathcal{F}_U \in \text{D-mod}(U)$.
(2) *For any* $\mathcal{F}_1 \in \text{D-mod}(\mathcal{Y})$, *supported on* $\mathcal{Y} - U$, *we have* $\mathrm{Hom}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}_1) = 0$.
(3) *For any* $U \overset{\widetilde{j}}{\hookrightarrow} U' \overset{j'}{\hookrightarrow} \mathcal{Y}$, *where U' is another open quasi-compact substack of $\mathcal{Y}$, we have:*

$$\mathcal{F}|_{U'} \simeq \widetilde{j}_!(\mathcal{F}_U),$$

in particular, the object $\widetilde{j}_!(\mathcal{F}_U)$ is defined.

Proof. By adjunction, (1)⇔(2). The implication (1)⇒(3) follows from Lemma 2.3.5(b).

Let us show that (3) implies (2). By formula (2.3), for any $\mathcal{F}, \mathcal{F}_1 \in \text{D-mod}(\mathcal{Y})$ one has

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}_1) \simeq \underset{\underset{U'}{\longleftarrow}}{lim}\, \mathcal{M}aps_{\text{D-mod}(U')}(\mathcal{F}|_{U'}, \mathcal{F}_1|_{U'}). \tag{2.6}$$

If $\mathcal{F}_1$ is supported on $\mathcal{Y} - U$ then all the terms in the RHS are zero, so the LHS is zero. □

Let us now prove Proposition 2.3.7.

Proof. As was remarked in Sect. 2.3.4, if (2.5) holds then the compactness of $\mathcal{F}$ follows by adjunction.

Conversely, suppose $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ is compact. Then by Sect. 2.3.4, for every open $U \subset \mathcal{Y}$ the object $\mathcal{F}|_U \in \text{D-mod}(U)$ is compact. So it remains to show that (2.5) holds for some quasi-compact open $U \overset{j}{\hookrightarrow} \mathcal{Y}$.

Assume the contrary. Using the equivalence (1)⇔(3) of Lemma 2.3.10, we obtain that for every quasi-compact open $U \subset \mathcal{Y}$ there is a quasi-compact open $U' \subset \mathcal{Y}$ containing U such that $(j_{U,U'})_!(\mathcal{F}|_U) \neq (\mathcal{F}|_{U'})$ (here $j_{U,U'} : U \hookrightarrow U'$).

Thus, we obtain an increasing sequence of open quasi-compact substacks $U_i \subset \mathcal{Y}$ such that $(j_{U_i,U_{i+1}})_!(\mathcal{F}|_{U_i}) \neq \mathcal{F}|_{U_{i+1}}$. Therefore, by Lemma 2.3.10, for each i there exists $\mathcal{E}_i \in \text{D-mod}(U_{i+1})$ such that $\mathcal{E}_i|_{U_i} = 0$ but $\mathrm{Hom}(\mathcal{F}|_{U_{i+1}}, \mathcal{E}_i) \neq 0$.

Let V be the union of the U_i's and let $\tilde{\mathcal{E}}_i \in \text{D-mod}(V)$ be the direct image of $\mathcal{E}_i$ under $U_i \hookrightarrow V$. Then

$$\mathrm{Hom}(\mathcal{F}|_V, \tilde{\mathcal{E}}_i) = \mathrm{Hom}(\mathcal{F}|_{U_{i+1}}, \mathcal{E}_i) \neq 0. \tag{2.7}$$

By Corollary 2.3.3,

$$\operatorname{Hom}(\mathcal{F}|_V, \underset{i}{\oplus}\, \tilde{\mathcal{E}}_i) \simeq \prod_i \operatorname{Hom}(\mathcal{F}|_V, \tilde{\mathcal{E}}_i). \tag{2.8}$$

On the other hand, by Sect. 2.3.4, $\mathcal{F}|_V$ is compact, so $\operatorname{Hom}(\mathcal{F}|_V, \underset{i}{\oplus}\, \tilde{\mathcal{E}}_i) \simeq \underset{i}{\oplus} \operatorname{Hom}(\mathcal{F}|_V, \tilde{\mathcal{E}}_i)$. This contradicts (2.8) because of (2.7). □

3. Truncativeness and co-truncativeness

Until the last subsection of this section we let $\mathcal{Y}$ be a QCA stack.

3.1. The notion of truncative substack

3.1.1. Let $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ be a closed substack, and let $\mathcal{Y} \overset{j}{\hookleftarrow} U$ be the complementary open. Consider the corresponding pairs of adjoint functors

$$i_{\mathrm{dR},*} : \text{D-mod}(\mathcal{Z}) \rightleftarrows \text{D-mod}(\mathcal{Y}) : i^!, \qquad j^* : \text{D-mod}(\mathcal{Y}) \rightleftarrows \text{D-mod}(U) : j_*\,.$$

Recall that by Theorem 2.2.4, all the categories involved are compactly generated.

Proposition 3.1.2. *The following conditions are equivalent:*

- (i) *The functor $i^!$ sends* $\text{D-mod}(\mathcal{Y})^c$ *to* $\text{D-mod}(\mathcal{Z})^c$.
- (i′) *The functor $i^!$ admits a continuous right adjoint.*
- (ii) *The functor j_* sends* $\text{D-mod}(U)^c$ *to* $\text{D-mod}(\mathcal{Y})^c$.
- (ii′) *The functor j_* admits a continuous right adjoint.*
- (iii) *The functor $j_!$, left adjoint to j^*, is defined on all of* $\text{D-mod}(U)$.
- (iii′) *The functor $j_!$, left adjoint to j^*, is defined on* $\text{D-mod}(U)^c$.
- (iv) *The functor i^*_{dR}, left adjoint to $i_{\mathrm{dR},*}$, is defined on all of* $\text{D-mod}(\mathcal{Y})$.
- (iv′) *The functor i^*_{dR}, left adjoint to $i_{\mathrm{dR},*}$, is defined on* $\text{D-mod}(\mathcal{Y})^c$.

Note that in the situation of (iii) and (iv) if the functors $j_!$ and i^*_{dR} are defined they are automatically continuous by adjunction.

To prove the proposition, we need the following lemma.

Lemma 3.1.3. *The essential image of* $\text{D-mod}(\mathcal{Y})^c$ *under* $j^* : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(U)$ *Karoubi-generates* $\text{D-mod}(U)^c$.

Proof. Since j^* has a continuous right adjoint j_*, we have $j^*(\text{D-mod}(\mathcal{Y})^c) \subset \text{D-mod}(U)^c$. Since the functor j_* is conservative $j^*(\text{D-mod}(\mathcal{Y})^c)$ generates $\text{D-mod}(U)$. By Corollary 1.4.6, this implies that $j^*(\text{D-mod}(\mathcal{Y})^c)$ Karoubi-generates $\text{D-mod}(U)^c$. □

Proof of Proposition 3.1.2. Since j^* preserves compactness and $i_{\mathrm{dR},*}$ is fully faithful and continuous, the fact that (ii) implies (i) follows from the exact triangle

$$i_{\mathrm{dR},*}(i^!(\mathcal{F})) \to \mathcal{F} \to j_* \circ (j^*(\mathcal{F})).$$

The implication (i)$\Rightarrow$(ii) follows from Lemma 3.1.3 and the same exact triangle.

The equivalences (i)$\Leftrightarrow$(i′) and (ii)$\Leftrightarrow$(ii′) follow from (the tautological) Proposition 1.2.4.

Let us show that (iii′)$\Leftrightarrow$(iii)$\Leftrightarrow$(ii′). The full subcategory of objects of $\mathrm{D\text{-}mod}(U)$ on which $j_!$ is defined is closed under colimits. Since $\mathrm{D\text{-}mod}(U)$ is generated by $\mathrm{D\text{-}mod}(U)^c$ we see that (iii′)$\Leftrightarrow$(iii). By Proposition 2.2.17, the dual of the functor $j^* = j^! : \mathrm{D\text{-}mod}(\mathcal{Y}) \to \mathrm{D\text{-}mod}(U)$ identifies, via the self-duality equivalences

$$\mathbf{D}_U^{\mathrm{Verdier}} : \mathrm{D\text{-}mod}(U)^\vee \simeq \mathrm{D\text{-}mod}(U), \qquad \mathbf{D}_{\mathcal{Y}}^{\mathrm{Verdier}} : \mathrm{D\text{-}mod}(\mathcal{Y})^\vee \simeq \mathrm{D\text{-}mod}(\mathcal{Y}),$$

with $j_* : \mathrm{D\text{-}mod}(U) \to \mathrm{D\text{-}mod}(\mathcal{Y})$. By duality (see Sect. 1.5.2), the existence of a continuous right adjoint to j_* is equivalent to the existence of (the automatically continuous) left adjoint of $j_*^\vee \simeq j^*$. I.e., (iii)$\Leftrightarrow$(ii′).

Similarly to the above proof of (iii′)$\Leftrightarrow$(iii)$\Leftrightarrow$(ii′), one shows that (iv′)$\Leftrightarrow$(iv)$\Leftrightarrow$(i′). $\square$

3.1.4. The following definition is crucial for this paper.

Definition 3.1.5. *A closed substack $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ is called* truncative *(resp., an open substack $U \overset{j}{\hookrightarrow} \mathcal{Y}$ is called* co-truncative*) if it satisfies the equivalent conditions of Proposition 3.1.2.*

3.1.6. Let us reformulate Definition 3.1.5 in terms of the non-cocomplete DG categories $\mathrm{D\text{-}mod}(\mathcal{Y})^c$, $\mathrm{D\text{-}mod}(\mathcal{Z})^c$, $\mathrm{D\text{-}mod}(U)^c$.

First, let $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ be *any* closed substack, and let $U \overset{j}{\hookrightarrow} \mathcal{Y}$ be the complementary open, then we have an exact sequence[6] of Karoubian (non-cocomplete) DG categories

$$0 \to \mathrm{D\text{-}mod}(\mathcal{Z})^c \overset{(i_{\mathrm{dR},*})^c}{\to} \mathrm{D\text{-}mod}(\mathcal{Y})^c \overset{(j^*)^c}{\to} \mathrm{D\text{-}mod}(U)^c \to 0. \tag{3.1}$$

[6]By definition, exactness means that i_*^c identifies $\mathrm{D\text{-}mod}(\mathcal{Z})^c$ with a full subcategory of $\mathrm{D\text{-}mod}(\mathcal{Y})^c$, and $(j^*)^c$ identifies the Karoubi-completion of the quotient $\mathrm{D\text{-}mod}(\mathcal{Y})/\,\mathrm{D\text{-}mod}(\mathcal{Z})^c$ with $\mathrm{D\text{-}mod}(U)^c$.

The exactness of (3.1) follows from the fact that the corresponding sequence of the ind-completions

$$0 \to \text{D-mod}(\mathcal{Z}) \overset{i_{\mathrm{dR},*}}{\to} \text{D-mod}(\mathcal{Y}) \overset{j^*}{\to} \text{D-mod}(U) \to 0$$

is exact, see Sect. 1.4.4.

Each of the conditions (i)–(ii) from Proposition 3.1.2 says that the subcategory

$$\text{D-mod}(\mathcal{Z})^c \subset \text{D-mod}(\mathcal{Y})^c$$

is *right-admissible*[7], which by definition means that the functor

$$(i_{\mathrm{dR},*})^c = (i_!)^c : \text{D-mod}(\mathcal{Z})^c \to \text{D-mod}(\mathcal{Y})^c$$

admits a right adjoint $(i^!)^c : \text{D-mod}(\mathcal{Y})^c \to \text{D-mod}(\mathcal{Z})^c$, or equivalently, the functor

$$(j^*)^c : \text{D-mod}(\mathcal{Y})^c \to \text{D-mod}(U)^c$$

has a right adjoint $(j_*)^c : \text{D-mod}(U)^c \to \text{D-mod}(\mathcal{Y})^c$.

Similarly, conditions (iii)–(iv) from Proposition 3.1.2 say that the subcategory

$$\text{D-mod}(\mathcal{Z})^c \subset \text{D-mod}(\mathcal{Y})^c$$

is *left-admissible*, which by definition means that the functor

$$(i_{\mathrm{dR},*})^c : \text{D-mod}(\mathcal{Z})^c \to \text{D-mod}(\mathcal{Y})^c$$

admits a left adjoint $(i^*_{\mathrm{dR}})^c : \text{D-mod}(\mathcal{Y})^c \to \text{D-mod}(\mathcal{Z})^c$, or, equivalently, the functor

$$(j^*)^c : \text{D-mod}(\mathcal{Y})^c \to \text{D-mod}(\mathcal{Z})^c$$

has a left adjoint $(j_!)^c : \text{D-mod}(\mathcal{Z})^c \to \text{D-mod}(\mathcal{Y})^c$.

In our situation left admissibility is equivalent to right admissibility by Verdier duality.

Thus if $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$ is truncative then in addition to (3.1) one has the exact sequences

$$0 \to \text{D-mod}(U)^c \overset{(j_!)^c}{\to} \text{D-mod}(\mathcal{Y})^c \overset{(i^*_{\mathrm{dR}})^c}{\to} \text{D-mod}(\mathcal{Z})^c \to 0, \tag{3.2}$$

$$0 \to \text{D-mod}(U)^c \overset{(j_*)^c}{\to} \text{D-mod}(\mathcal{Y})^c \overset{(i^!)^c}{\to} \text{D-mod}(\mathcal{Z})^c \to 0. \tag{3.3}$$

[7] Synonyms: right-admissible = coreflective, left-admissible = reflective.

It is convenient to arrange the functors between $\text{D-mod}(\mathcal{Z})^c$ and $\text{D-mod}(\mathcal{Y})^c$ into a sequence

$$(i^*_{\mathrm{dR}})^c\,, (i_{\mathrm{dR},*})^c\,, (i^!)^c \tag{3.4}$$

and the functors between $\text{D-mod}(U)$ and $\text{D-mod}(\mathcal{Y})$ into a sequence

$$(j_!)^c, (j^*)^c, (j_*)^c\,. \tag{3.5}$$

In each of the sequences, each neighboring pair forms an adjoint pair of functors.

3.2. Some examples of (co)-truncative substacks

3.2.1. (Co)-truncativeness is a purely "stacky" phenomenon, i.e., it almost never happens for schemes.

More precisely, it is easy to see that if $j : U \hookrightarrow Y$ is an open embedding of schemes which is not a closed embedding then U cannot be co-truncative. Indeed, choose $\mathcal{M} \in \mathrm{Coh}(U)$ such that $j_*(\mathcal{M})$ is not coherent. Then

$$j_*(\mathbf{ind}_{\text{D-mod}(U)}(\mathcal{M})) \simeq \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(j_*(\mathcal{M}))$$

is not in $\text{D-mod}(Y)^c$. Here $\mathbf{ind}_{\text{D-mod}(-)}$ denotes the induction functor from $\mathrm{IndCoh}(-)$ to $\text{D-mod}(-)$, see [DrGa1, Sect. 5.1.3].

3.2.2. Example. The following example of a co-truncative substack is most important for us:

Take $\mathcal{Y} = \mathbb{A}^n/\mathbb{G}_m$, where $\mathbb{G}_m$ acts on $\mathbb{A}^n$ by dilations. Take $U = (\mathbb{A}^n - \{0\})/\mathbb{G}_m \simeq \mathbb{P}^{n-1}$. In Sect. 5 we will see that $U \hookrightarrow \mathcal{Y}$ is co-truncative.

3.2.3. The most basic case of the above example is when $n = 1$. In this case, the co-truncativeness assertion is particularly evident. Namely, let us check that condition (iii′) of Proposition 3.1.2 holds. Indeed, $\text{D-mod}(U) \simeq \mathrm{Vect}$, so it is sufficient to show that $j_!(k)$ is defined, where k is the generator of Vect. This is clear since we are dealing with holonomic D-modules.

3.2.4. Here is a generalization of the example of Sect. 3.2.3 in a direction different from Sect. 3.2.2: if $\mathcal{Y}$ is any QCA stack that has only finitely many isomorphism classes of k-points then every open substack $U \subset \mathcal{Y}$ is co-truncative. Indeed, condition (iii′) of Proposition 3.1.2 is verified because every object of $\text{D-mod}(U)^c$ is holonomic.

Examples of such $\mathcal{Y}$ include $N\backslash G/B$, or any quasi-compact open of Bun_G for X of genus 0.

3.3. The non-standard functors

Let $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ be a truncative closed substack and $U \overset{j}{\hookrightarrow} \mathcal{Y}$ the corresponding co-truncative open substack.

Definition 3.3.1. *The functors right adjoint to $i^!$ and j_* are denoted by*

$$i_? : \text{D-mod}(\mathcal{Z}) \to \text{D-mod}(\mathcal{Y}), \quad j^? : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(U).$$

Remark 3.3.2. The proof of the equivalences (iii)⇔(ii′) and (iv)⇔(i′) from Proposition 3.1.2 shows that $i_?$ is the dual to $i^*_{\mathrm{dR}} : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Z})$ and $j^?$ is the dual to $j_! : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$ in the sense of Sect. 1.5.2. Recall that these dualities follow from the duality between $i^!$ and $i_{\mathrm{dR},*}$, and between j_* and j^*.

Remark 3.3.3. Recall that the existence of $i^?$ and/or $j^?$ as a continuous functor is among the equivalent definitions of truncativeness, see Definition 3.1.5 and Proposition 3.1.2(i′, ii′).

The existence of i^*_{dR} and/or $j_!$ as an everywhere defined (and automatically continuous) functor is also among the equivalent definitions of truncativeness, see Proposition 3.1.2(iii, iv).

The functors $i_?, j^?, i^*_{\mathrm{dR}}, j_!$ are called *the non-standard functors* associated to $\mathcal{Z} \subset \mathcal{Y}$ (or to $U \subset \mathcal{Y}$).

The functors i^* and $j_!$ are at least, familiar as *partially* defined functors (e.g., they are always defined on the holonomic subcategory), but $i_?$ and $j^?$ are quite unfamiliar. On the other hand, in some situations the non-standard functors identify with certain standard functors, see Example 3.3.9 and Remark 3.3.10 below.

3.3.4. Inventory. It is convenient to arrange the functors between D-mod($\mathcal{Z}$) and D-mod($\mathcal{Y}$) into a sequence

$$i^*_{\mathrm{dR}}, i_{\mathrm{dR},*}, i^!, i_? \tag{3.6}$$

and the functors between D-mod(U) and D-mod($\mathcal{Y}$) into a sequence

$$j_!, j^*, j_*, j^?. \tag{3.7}$$

In each of the sequences, each neighboring pair forms an adjoint pair of functors. The first and last functors in (3.6) and in (3.7) are non-standard, the other functors are standard. By Remark 3.3.2, each of the sequences (3.6)–(3.7) is self-dual in the sense of Sect. 1.5.2.

3.3.5. We know that the functors $i_{\mathrm{dR},*}$ and j_* are fully faithful; equivalently, the adjunctions

$$i^*_{\mathrm{dR}} \circ i_{\mathrm{dR},*} \to \mathrm{Id}_{\text{D-mod}(\mathcal{Z})}, \quad \mathrm{Id}_{\text{D-mod}(U)} \to j^? \circ j_* \tag{3.8}$$

are isomorphisms (just as are the adjunctions $j^* \circ j_* \to \mathrm{Id}_{\text{D-mod}(U)}$ and $\mathrm{Id}_{\text{D-mod}(\mathcal{Z})} \to i^! \circ i_{\mathrm{dR},*}$, which involve only the standard functors).

Proposition 3.3.6. (i) *The functors $i_?$ and $j_!$ are fully faithful.*

(ii) *The adjunctions $i^! \circ i_? \to \mathrm{Id}_{\text{D-mod}(\mathcal{Z})}$ and $\mathrm{Id}_{\text{D-mod}(U)} \to j^* \circ j_!$ are isomorphisms.*

Although this proposition is extremely simple, we will give two proofs.

Proof 1. Statements (i) and (ii) are clearly equivalent, so it suffices to prove (ii).

Recall that the adjoint pairs $(i^!, i_?)$ and $(j_*, j^?)$ are dual to the adjoint pairs $(i^*_{\mathrm{dR}}, i_{\mathrm{dR},*})$ and $(j_!, j^*)$. So statement (ii) follows from the fact that the adjunctions (3.8) are isomorphisms. □

Proof 2. We will deduce statement (i) from the following general lemma, which is part of the categorical folklore.[8]

Lemma 3.3.7. *Let F be a functor between ∞-categories that admits a left adjoint F^L and a right adjoint F^R. Then F^L is fully faithful if and only if F^R is.*

Let us apply Lemma 3.3.7 to $\mathsf{F} := j^*$. Since $(j^*)^R = j_*$ is fully faithful, we obtain that $(j^*)^L = j_!$ is fully faithful.

Let us apply Lemma 3.3.7 to $\mathsf{F} := i^!$. Since $(i^!)^L = i_{\mathrm{dR},*}$ is fully faithful, we obtain that $(i^!)^R = i_?$ is fully faithful. □

3.3.8. Regardless of whether the substack $\mathcal{Z} \subset \mathcal{Y}$ is truncative, one has canonical exact sequences of DG categories

$$0 \to \text{D-mod}(\mathcal{Z}) \overset{i_{\mathrm{dR},*}}{\to} \text{D-mod}(\mathcal{Y}) \overset{j^*}{\to} \text{D-mod}(U) \to 0 \tag{3.9}$$

and

[8]Lemma 3.3.7 for ∞-categories immediately follows from the same statement for usual categories. For proofs in the setting of usual categories, see [DT, Lemma 1.3], [KeLa, Proposition 2.3], and the article on adjoint triples from [nLab] (on the other hand, the reader can easily reconstruct the argument because we essentially used it in the proof of Lemma 2.3.5(a)). Note that in the case of triangulated categories and functors (which is enough for our purpose) Lemma 3.3.7 is well known.

$$0 \to \text{D-mod}(U) \xrightarrow{j_*} \text{D-mod}(\mathcal{Y}) \xrightarrow{i^!} \text{D-mod}(\mathcal{Z}) \to 0, \tag{3.10}$$

where the latter is obtained from the former by passing to right adjoints.

If $\mathcal{Z}$ is truncative one also has exact sequences

$$0 \to \text{D-mod}(\mathcal{Z}) \xrightarrow{i_?} \text{D-mod}(\mathcal{Y}) \xrightarrow{j^?} \text{D-mod}(U) \to 0 \tag{3.11}$$

and

$$0 \to \text{D-mod}(U) \xrightarrow{j_!} \text{D-mod}(\mathcal{Y}) \xrightarrow{i^*_{\text{dR}}} \text{D-mod}(\mathcal{Z}) \to 0, \tag{3.12}$$

where (3.11) is obtained by passing to right adjoints from (3.10), and (3.12) is obtained by passing to left adjoints from (3.9).

In addition, (3.9) and (3.10) are obtained from one another by passing to the dual categories and functors. Similarly, (3.12) and (3.11) are obtained from one another by passing to the dual categories and functors.

3.3.9. Example. Consider the situation of Sect. 3.2.3, i.e., the embedding $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$, where $\mathcal{Y} = \mathbb{A}^1/\mathbb{G}_m$, $\mathcal{Z} = \{0\}/\mathbb{G}_m$. Let $\pi : \mathcal{Y} \to \mathcal{Z}$ be the morphism induced by the map $\mathbb{A}^1 \to \{0\}$. Let us show that *the non-standard functors*

$$i^*_{\text{dR}} : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Z}) \ \textit{and} \ i_? : \text{D-mod}(\mathcal{Z}) \to \text{D-mod}(\mathcal{Y})$$

identify with the following standard functors:

$$i^*_{\text{dR}} \simeq \pi_{\text{dR},*}\,, \quad i_? \simeq \pi^!\,;$$

in other words, $(\pi_{\text{dR},*}, i_{\text{dR},*})$ and $(i^!, \pi^!)$ are adjoint pairs. By Proposition 2.2.17, $\pi_{\text{dR},*}$ is dual to $\pi^!$ and $i_{\text{dR},*}$ is dual to $i^!$, so it suffices to show that $(\pi_{\text{dR},*}, i_{\text{dR},*})$ is an adjoint pair. Let us prove that for any $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$, $\mathcal{N} \in \text{D-mod}(\mathcal{Z})$ the map

$$\begin{aligned}\pi_{\text{dR},*} : \operatorname{Hom}(\mathcal{M}, i_{\text{dR},*}(\mathcal{N})) &\to \operatorname{Hom}(\pi_{\text{dR},*}(\mathcal{M}), \pi_{\text{dR},*} \circ i_{\text{dR},*}(\mathcal{N})) \\ &= \operatorname{Hom}(\pi_{\text{dR},*}(\mathcal{M}), \mathcal{N})\end{aligned}$$

is an isomorphism.

This is clear if $\mathcal{M} \in \text{D-mod}(\mathcal{Z}) \subset \text{D-mod}(\mathcal{Y})$. The DG category $\text{D-mod}(\mathcal{Y})$ is generated by $\text{D-mod}(\mathcal{Z})$ and $j_!(k)$, where $j : \text{pt} = \mathcal{Y} - \mathcal{Z} \hookrightarrow \mathcal{Y}$ is the open embedding. So it remains to consider the case $\mathcal{M} = j_!(k)$. Then $\operatorname{Hom}(\mathcal{M}, i_{\text{dR},*}(\mathcal{N})) = 0$ and $\pi_{\text{dR},*}(\mathcal{M}) = 0$ (the latter follows from the fact the de Rham cohomology of $\mathbb{A}^1$ equals k).

Remark 3.3.10. Example 3.3.9 is a "baby case" of Proposition 5.3.2.

3.4. Truncativeness of locally closed substacks

Let $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ be a locally closed substack. This means that i becomes a locally closed embedding after any base change $Y \to \mathcal{Y}$, where Y is a scheme (in fact, it suffices to verify this condition for just one smooth or flat covering $Y \to \mathcal{Y}$).

Definition 3.4.1. *A locally closed substack $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ is said to be* truncative *if the functor $i^!$ preserves compactness (or equivalently, has a continuous right adjoint functor $i_?$).*

For instance, any open substack is truncative.

3.4.2. Definition 3.4.1 immediately implies that truncativeness is transitive:

Lemma 3.4.3. *Let $\mathcal{Y}_1 \hookrightarrow \mathcal{Y}_2 \hookrightarrow \mathcal{Y}_3$ be locally closed embeddings. If $\mathcal{Y}_1$ is truncative in $\mathcal{Y}_2$ and $\mathcal{Y}_2$ is truncative in $\mathcal{Y}_3$, then $\mathcal{Y}_1$ is truncative in $\mathcal{Y}_3$.*

As in the case of schemes, every locally closed embedding $\mathcal{Z} \hookrightarrow \mathcal{Y}$ can be factored (and even canonically so) as

$$\mathcal{Z} \overset{i'}{\hookrightarrow} \mathcal{Y}' \overset{j}{\hookrightarrow} \mathcal{Y}, \tag{3.13}$$

where i' is a closed embedding, and j is an open embedding. Namely, $\mathcal{Y}' := \mathcal{Y} - (\bar{\mathcal{Z}} - \mathcal{Z})$, where $\bar{\mathcal{Z}}$ is the closure of $\mathcal{Z}$ in $\mathcal{Y}$ (so that $\mathcal{Z}$ is open in $\bar{\mathcal{Z}}$).

Lemma 3.4.4. *A locally closed substack $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ is truncative if and only if for some/any factorization (3.13) with i' being closed and j open, $\mathcal{Z}$ in truncative in $\mathcal{Y}'$.*

Proof. The "if" statement follows from Lemma 3.4.3. It remains to show that if the composition (3.13) is truncative then so is $\mathcal{Z} \overset{i'}{\hookrightarrow} \mathcal{Y}'$. This follows from the fact that the essential image of $\text{D-mod}(\mathcal{Y})^c$ under j^* Karoubi-generates $\text{D-mod}(\mathcal{Y}')^c$, see Lemma 3.1.3. $\square$

Remark 3.4.5. In the case of locally closed substacks the situation with the non-standard functors is as follows. By duality (in the sense of Sect. 1.5.2), a locally closed substack $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ is truncative if and only if the functor $i_{\mathrm{dR},*}$ has a left adjoint functor i^*_{dR} (which is automatically continuous).

Thus for a truncative locally closed substack we have adjoint pairs of continuous functors $(i^*_{\mathrm{dR}}, i_{\mathrm{dR},*})$ and $(i^!, i_?)$ dual to each other. Just as in the case of closed embeddings (see Proposition 3.3.6), the functors $i_{\mathrm{dR},*}$ and $i_?$ are fully faithful; equivalently, the adjunctions $i^! \circ i_? \to \mathrm{Id}_{\text{D-mod}(\mathcal{Z})}$ and $\mathrm{Id}_{\text{D-mod}(\mathcal{Z})} \to i_{\mathrm{dR},*} \circ i^*_{\mathrm{dR}}$ are isomorphisms. But if the substack $\mathcal{Z}$ is not closed then $i_{\mathrm{dR},*} \neq i_!$, so the functors $i_{\mathrm{dR},*}$ and $i^!$ do not form an adjoint pair.

3.5. Truncativeness via coherence

3.5.1. As was mentioned in Sect. 2.2.10, the property of compactness of a D-module on a stack is subtle. For example, it is not local in the smooth topology. We are going to reformulate the notion of truncativeness via a more accessible property, namely, coherence.

Proposition 3.5.2. (a) *A locally closed substack $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ is truncative if and only if the functor $i^!$ sends* $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ *to* $\text{D-mod}_{\mathrm{coh}}(\mathcal{Z})$*.*

(b) *An open substack $U \overset{j}{\hookrightarrow} \mathcal{Y}$ is co-truncative if and only if j_* sends* $\text{D-mod}_{\mathrm{coh}}(U)$ *to* $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$*.*

Proof. To prove the "if" implications in both (a) and (b) we will use the notion of *safety* from [DrGa1, Sect. 9.2], and the fact that for a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ between QCA stacks, the functor $f_{\mathrm{dR},*}$ always preserves safety, and $f^!$ preserves safety if f itself is safe (in particular, when f is schematic); see [DrGa1, Lemma 10.4.2].

Thus, the "if" implications follow from the fact that "compactness=coherence+safety", see [DrGa1, Proposition 9.2.3].

To prove the "only if" implication in (a), we will use the following result (see [DrGa1, Lemma 9.4.7(a)]):

Lemma 3.5.3. *For a QCA stack $\mathcal{Y}$, an object $\mathcal{F} \in \text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ and an integer n, there exists $\mathcal{F}' \in \text{D-mod}(\mathcal{Y})^c$ and a map $\mathcal{F}' \to \mathcal{F}$, such that its cone lies in* $\text{D-mod}(\mathcal{Y})^{<-n}$*.*

Note that the functor $i^!$ is left t-exact, and has a finite cohomological amplitude, say k. For $\mathcal{F} \in \text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$, which lies in $\text{D-mod}(\mathcal{Y})^{\geq -m}$, choose $\mathcal{F}'$ as in Lemma 3.5.3 with $n > k + m$. Consider the exact triangle

$$i^!(\mathcal{F}') \to i^!(\mathcal{F}) \to i^!(\mathcal{F}''),$$

where $\mathcal{F}'' := \mathrm{Cone}(\mathcal{F}' \to \mathcal{F})$. By construction, the maps

$$\tau^{\geq -m}(i^!(\mathcal{F}')) \to \tau^{\geq -m}(i^!(\mathcal{F})) \to i^!(\mathcal{F}) \tag{3.14}$$

are isomorphisms.

By assumption, $i^!(\mathcal{F}') \in \text{D-mod}(\mathcal{Z})^c \subset \text{D-mod}_{\mathrm{coh}}(\mathcal{Z})$. Note also that the truncation functors preserve the subcategory $\text{D-mod}_{\mathrm{coh}}(-)$.

Hence $\tau^{\geq -m}(i^!(\mathcal{F}')) \in \text{D-mod}_{\mathrm{coh}}(\mathcal{Z})$. Hence, (3.14) implies that $i^!(\mathcal{F}) \in \text{D-mod}_{\mathrm{coh}}(\mathcal{Z})$, as desired.

The "only if" implication in (b) is proved similarly. $\square$

3.6. Stability of truncativeness

In this subsection $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$ denotes a locally closed embedding.

3.6.1. Cartesian products.

Lemma 3.6.2. *Suppose that a substack $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ is truncative. Then for any QCA stack $\mathcal{X}$, the substack $\mathcal{Z} \times \mathcal{X} \hookrightarrow \mathcal{Y} \times \mathcal{X}$ is also truncative.*

Proof. By [DrGa1, Corollary 8.3.4], for a pair of QCA stacks $\mathcal{X}_1$ and $\mathcal{X}_2$, the natural functor

$$\text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2) \to \text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2)$$

is an equivalence. So the functor $(i \times \mathrm{id}_{\mathcal{X}})^! : \text{D-mod}(\mathcal{Y} \times \mathcal{X}) \to \text{D-mod}(\mathcal{Z} \times \mathcal{X})$ identifies with the functor $i^! \otimes \mathrm{Id}_{\text{D-mod}(\mathcal{X})}$, which clearly preserves compactness. $\square$

3.6.3. Descent.

Proposition 3.6.4. *Let $\mathcal{Z} \subset \mathcal{Y}$ be a locally closed substack, $f : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ a smooth morphism, and $\widetilde{\mathcal{Z}} \subset \mathcal{Z} \underset{\mathcal{Y}}{\times} \widetilde{\mathcal{Y}}$ an open substack such that the resulting morphism $f' : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ is surjective. If the locally closed embedding $\widetilde{i} : \widetilde{\mathcal{Z}} \hookrightarrow \widetilde{\mathcal{Y}}$ is truncative then so is $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$.*

Proof. By Proposition 3.5.2(a), it suffices to show that $i^!$ sends $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ to $\text{D-mod}_{\mathrm{coh}}(\mathcal{Z})$. The morphism f' is smooth and surjective, so it suffices to show that the functor $f'^! \circ i^!$ preserves coherence. But $f'^! \circ i^! \simeq \widetilde{i}^! \circ f^!$, and each of the functors $\widetilde{i}^!$ and $f^!$ preserves coherence. $\square$

Corollary 3.6.5. *Let $\mathcal{Z} \subset \mathcal{Y}$ be a locally closed substack. Suppose that each $z \in \mathcal{Z}$ has a Zariski neighborhood $U \subset \mathcal{Y}$ such that $\mathcal{Z} \cap U$ is truncative in U. Then $\mathcal{Z}$ is truncative in $\mathcal{Y}$.*

Remark 3.6.6. The converse to Proposition 3.6.4 is false: truncativeness downstairs does *not* imply truncativeness upstairs (e.g., consider the embedding $\mathrm{pt}/\mathbb{G}_m \hookrightarrow \mathbb{A}^1/\mathbb{G}_m$ smoothly covered by $\mathrm{pt} \hookrightarrow \mathbb{A}^1$). However, the converse to Proposition 3.6.4 does hold for étale schematic morphisms; this follows from Lemma 3.6.9 below.

Lemma 3.6.7. *Suppose that in a Cartesian diagram*

$$\begin{tikzcd} \widetilde{\mathcal{Z}} \arrow[r, "\widetilde{i}"] \arrow[d, "f'"'] & \widetilde{\mathcal{Y}} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "i"] & \mathcal{Y} \end{tikzcd}$$

f is schematic, proper and surjective, and i a locally closed embedding. If $\widetilde{\mathcal{Z}}$ is truncative in $\widetilde{\mathcal{Y}}$ then $\mathcal{Z}$ is truncative in $\mathcal{Y}$.

Proof. First, by [DrGa1, Lemma 5.1.6], the functor $f^!$ is conservative. Hence, the essential image of $f_{\mathrm{dR},*}$ generates $\text{D-mod}(\mathcal{Y})$. Hence, by Corollary 1.4.6, the essential image of $\text{D-mod}(\widetilde{\mathcal{Y}})^c$ under $f_{\mathrm{dR},*}$ Karoubi-generates $\text{D-mod}(\mathcal{Y})^c$. Therefore, it is sufficient to show that the functor $i^! \circ f_{\mathrm{dR},*}$ preserves compactness. But $i^! \circ f_{\mathrm{dR},*} \simeq f'_{\mathrm{dR},*} \circ \widetilde{i}^!$, the functor $\widetilde{i}^!$ preserves compactness by assumption, and $f'_{\mathrm{dR},*}$ preserves compactness by properness (it has a continuous right adjoint given by $(f')^!$). $\square$

3.6.8. Quasi-finite base change.

Lemma 3.6.9. *Suppose that $f : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ is étale and schematic. If a locally closed embedding $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$ is truncative then so is $\widetilde{i} : \mathcal{Z} \underset{\mathcal{Y}}{\times} \widetilde{\mathcal{Y}} \hookrightarrow \widetilde{\mathcal{Y}}$.*

Proof. The functor $f_{\mathrm{dR},*} : \text{D-mod}(\widetilde{\mathcal{Y}}) \to \text{D-mod}(\mathcal{Y})$ is conservative. So by Corollary 1.4.6, the essential image of $\text{D-mod}(\mathcal{Y})^c$ under $f^*_{\mathrm{dR}} \simeq f^!$ Karoubi-generates $\text{D-mod}(\widetilde{\mathcal{Y}})^c$. So it is enough to show that $\widetilde{i}^! \circ f^!$ preserves compactness. However, $\widetilde{i}^! \circ f^! \simeq f'^! \circ i^!$. Now, $i^!$ preserves compactness by assumption, and $f'^!$ preserves compactness because it is isomorphic to $(f')^*_{\mathrm{dR}}$, which is the left adjoint of a continuous functor, namely, $f'_{\mathrm{dR},*}$. $\square$

Lemma 3.6.10. *If $f : \widetilde{\mathcal{Y}} \hookrightarrow \mathcal{Y}$ is a locally closed embedding and a locally closed substack $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is truncative then so is $\mathcal{Z} \underset{\mathcal{Y}}{\times} \widetilde{\mathcal{Y}} \hookrightarrow \widetilde{\mathcal{Y}}$.*

Proof. If f is an open embedding the statement holds by Lemma 3.6.9. If f is a closed embedding use the fact that an object $\mathcal{F} \in \text{D-mod}(\widetilde{\mathcal{Y}})$ is compact if and only if $f_{\mathrm{dR},*}(\mathcal{F}) \in \text{D-mod}(\mathcal{Y})$ is; this follows from the fact that the functor $f_{\mathrm{dR},*}$ is fully faithful and continuous. $\square$

Lemma 3.6.10 for a closed embedding f admits the following generalization.

Proposition 3.6.11. *Let $f : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be a finite schematic morphism. If a locally closed embedding $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$ is truncative then so is $\widetilde{i} : \mathcal{Z} \underset{\mathcal{Y}}{\times} \widetilde{\mathcal{Y}} \hookrightarrow \widetilde{\mathcal{Y}}$.*

To prove the proposition, we need the following lemma.

Lemma 3.6.12. *Let $g : \mathcal{X}' \to \mathcal{X}$ be a finite schematic morphism. If $\mathcal{F}' \in \mathrm{D\text{-}mod}(\mathcal{X}')$ is such that $g_{\mathrm{dR},*}(\mathcal{F}') \in \mathrm{D\text{-}mod}(\mathcal{X})$ is coherent then $\mathcal{F}'$ is coherent.*

Proof. Follows immediately from the fact that the functor $g_{\mathrm{dR},*}$ is t-exact and conservative. $\square$

Proof of Proposition 3.6.11. We have to show that the functor $\widetilde{i}^!$ preserves coherence. Applying Lemma 3.6.12 to the morphism $f' : \mathcal{Z} \underset{\mathcal{Y}}{\times} \widetilde{\mathcal{Y}} \to \mathcal{Z}$, we see that it suffices to prove that the composition $f'_{\mathrm{dR},*} \circ \widetilde{i}^!$ preserves coherence. But $f'_{\mathrm{dR},*} \circ \widetilde{i}^! \simeq i^! \circ f_{\mathrm{dR},*}$ and each of the functors $i^!$ and $f_{\mathrm{dR},*}$ preserves coherence. $\square$

Remark 3.6.13. One can combine Lemma 3.6.9 and Proposition 3.6.11 to the following statement: the assertion of Proposition 3.6.11 continues to hold when f is a quasi-finite compactifiable morphism.

3.7. Intersections and unions of truncative substacks

Lemma 3.7.1. *If $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are locally closed truncative substacks of $\mathcal{Y}$, then so is $\mathcal{Z}_1 \cap \mathcal{Z}_2$.*

Proof. By Lemma 3.6.10, $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is truncative in $\mathcal{Z}_1$. Now, the assertion follows from Lemma 3.4.3. $\square$

Proposition 3.7.2. *Suppose that a locally closed substack $\mathcal{Z} \subset \mathcal{Y}$ is equal to the union of (possibly intersecting) locally closed substacks $\mathcal{Z}_i$, $i = 1, \dots, n$. If each $\mathcal{Z}_i$ is truncative in $\mathcal{Y}$, then so is $\mathcal{Z}$.*

First, let us prove the following particular case of Proposition 3.7.2.

Lemma 3.7.3. *Let $\mathcal{Z}' \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{Y}$ be closed embeddings. If $\mathcal{Z}'$ and $\mathcal{Z} - \mathcal{Z}'$ are truncative in $\mathcal{Y}$ then so is $\mathcal{Z}$.*

Proof. Consider the open substacks $\mathcal{Y} - \mathcal{Z} \subset \mathcal{Y} - \mathcal{Z}' \subset \mathcal{Y}$. The fact that $\mathcal{Z}'$ is truncative in $\mathcal{Y}$ means, by definition, that $\mathcal{Y} - \mathcal{Z}'$ is co-truncative in $\mathcal{Y}$. By Lemma 3.4.4, the fact that $\mathcal{Z} - \mathcal{Z}'$ is truncative in $\mathcal{Y}$ implies that $\mathcal{Z} - \mathcal{Z}'$ is truncative in $\mathcal{Y} - \mathcal{Z}'$, i.e., that $\mathcal{Y} - \mathcal{Z}$ is co-truncative in $\mathcal{Y} - \mathcal{Z}'$. But the relation of co-truncativeness is transitive: this is clear if one uses property (ii) from Proposition 3.1.2 as a definition of co-truncativeness. So $\mathcal{Y} - \mathcal{Z}$ is co-truncative in $\mathcal{Y}$, i.e., $\mathcal{Z}$ is truncative in $\mathcal{Y}$. $\square$

Proof of Proposition 3.7.2. We proceed by induction on n.

By Corollary 3.6.5, it suffices to show that each $z \in \mathcal{Z}$ has a Zariski neighborhood $U \subset \mathcal{Y}$ such that $\mathcal{Z} \cap U$ is truncative in U. Choose i so that $z \in \mathcal{Z}_i$. After replacing $\mathcal{Z}$ by an open neighborhood of z, one can assume that $\mathcal{Z}_i$ and $\mathcal{Z}$ are closed in $\mathcal{Y}$.

Writing $\mathcal{Z} - \mathcal{Z}_i$ as a union of the substacks $\mathcal{Z}_j - (\mathcal{Z}_i \cap \mathcal{Z}_j)$, $j \neq i$, and applying the induction assumption, we see that $\mathcal{Z} - \mathcal{Z}_i$ is truncative in $\mathcal{Y} - \mathcal{Z}_i$ and therefore in $\mathcal{Y}$. It remains to apply Lemma 3.7.3 to $\mathcal{Z}_i \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{Y}$. □

3.8. Truncativeness and co-truncativeness for non quasi-compact stacks

Now suppose that $\mathcal{Y}$ is *locally* QCA (but not necessarily quasi-compact).

3.8.1. We give the following definitions:

Definition 3.8.2. (i) *A locally closed substack $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is said to be truncative if for every open quasi-compact substack $\overset{\circ}{\mathcal{Y}} \subset \mathcal{Y}$ the intersection $\mathcal{Z} \cap \overset{\circ}{\mathcal{Y}}$ is truncative in $\overset{\circ}{\mathcal{Y}}$.*

(ii) *An open substack $U \subset \mathcal{Y}$ is said to be co-truncative if for every open quasi-compact substack $\overset{\circ}{\mathcal{Y}} \subset \mathcal{Y}$ the intersection $U \cap \overset{\circ}{\mathcal{Y}}$ is co-truncative in $\overset{\circ}{\mathcal{Y}}$.*

3.8.3. Clearly, a closed substack $\mathcal{Z}$ is truncative if and only if its complementary open is co-truncative.

In addition:

Lemma 3.8.4. *If open substacks $U_1, U_2 \subset \mathcal{Y}$ are co-truncative then so is $U_1 \cup U_2$.*

Proof. This immediately follows from Lemma 3.7.1. □

3.8.5. As in Lemma 2.3.10, it is easy to see that U is co-truncative if and only if the functor $j_!$, left adjoint to j^*, is defined.

This formally implies that if $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$ is truncative, then the functor i^*_{dR}, left adjoint to $i_{\mathrm{dR},*}$, is also defined.

3.8.6. Finally, we note:

Lemma 3.8.7. *For a co-truncative open quasi-compact substack $U \overset{j}{\hookrightarrow} \mathcal{Y}$ the functor*

$$j_! : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$$

is fully faithful.

Proof. Follows from Lemma 2.3.5(a). □

4. Truncatable stacks

Let $\mathcal{Y}$ be an algebraic stack which is locally QCA. In this setting the notions of truncativeness and co-truncativeness were introduced in Sect. 3.8.

4.1. The notion of truncatibility

We will now formulate a condition on $\mathcal{Y}$ called “truncatibility”. According to Proposition 4.1.6 below, it implies that the category $\text{D-mod}(\mathcal{Y})$ is compactly generated.

Definition 4.1.1. *The stack $\mathcal{Y}$ is said to be truncatable if it can be covered by open quasi-compact substacks that are co-truncative.*

4.1.2. By Lemma 3.8.4, we can rephrase Definition 4.1.1 as follows:

Lemma 4.1.3. *A stack $\mathcal{Y}$ is truncatable if and only if every open quasi-compact substack is contained in one which is co-truncative. Equivalently, $\mathcal{Y}$ is truncatable if and only if the sub-poset of co-truncative open quasi-compact substacks in $\mathcal{Y}$ is cofinal among all open quasi-compact substacks.*

4.1.4. Notation. The poset of co-truncative open quasi-compact substacks $U \subset \mathcal{Y}$ is denoted by $\mathrm{Ctrnk}(\mathcal{Y})$; we will often consider this poset as a category. Let $\mathrm{Ctrnk}(\mathcal{Y})^{\mathrm{op}}$ denote the opposite poset (or category). Lemma 3.8.4 implies that $\mathrm{Ctrnk}(\mathcal{Y})$ is *filtered*.

The next statement immediately follows from Lemma 4.1.3.

Corollary 4.1.5. *If $\mathcal{Y}$ is truncatable then the natural restriction functor*

$$\text{D-mod}(\mathcal{Y}) \to \underset{U\in \mathrm{Ctrnk}(\mathcal{Y})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \text{D-mod}(U)$$

is an equivalence.

Proposition 4.1.6. *If $\mathcal{Y}$ is truncatable then the category* $\text{D-mod}(\mathcal{Y})$ *is compactly generated.*

Proof. Let $U \overset{j}{\hookrightarrow} \mathcal{Y}$ be a co-truncative open quasi-compact substack and $\mathcal{F}_U \in \text{D-mod}(U)^c$. By Proposition 2.3.7, the object $j_!(\mathcal{F}_U) \in \text{D-mod}(\mathcal{Y})$ (which is well-defined by the co-truncativeness assumption) is compact. It suffices to show that such objects generate $\text{D-mod}(\mathcal{Y})$. In other words, we have to show that if $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ is right-orthogonal to all such objects, then $\mathcal{F} = 0$.

For a given U, the fact that $\mathcal{F}$ is right-orthogonal to all $j_!(\mathcal{F}_U)$ as above is equivalent, by adjunction, to the fact that $j^*(\mathcal{F})$ is right-orthogonal to D-mod$(U)^c$. Since D-mod(U) is compactly generated, this implies that $j^*(\mathcal{F}) = 0$. By Corollary 4.1.5, this implies that $\mathcal{F} = 0$. $\square$

4.1.7. As was mentioned in the introduction, we use Proposition 4.1.6 to deduce the main result of this paper (namely, the compact generation of D-mod(Bun_G)) from the following result:

Theorem 4.1.8. *Let G be a connected reductive group and X a smooth complete connected curve over k. Let* Bun_G *denote the stack of G-bundles on X. Then* Bun_G *is truncatable.*

The proof for any connected reductive group G will be given in Sect. 9. But its main idea is the same as in the easy case $G = SL_2$, which is considered separately in Sect. 6.

4.1.9. In Sects. 4.2–4.5 below we discuss some general properties of the category D-mod$(\mathcal{Y})$ for a truncatable stack $\mathcal{Y}$.

4.2. Presentation as a colimit

In this subsection we fix $\mathcal{Y}$ to be a truncatable locally QCA stack. We will use the notation $\mathrm{Ctrnk}(\mathcal{Y})$ from Sect. 4.1.4.

4.2.1. Note that for a morphism $U_1 \overset{j_{1,2}}{\hookrightarrow} U_2$ in $\mathrm{Ctrnk}(\mathcal{Y})$, the pullback functor

$$\phi_{U_2,U_1} := j_{1,2}^* : \text{D-mod}(U_2) \to \text{D-mod}(U_1)$$

admits a left adjoint, $\psi_{U_1,U_2} := (j_{1,2})_! : \text{D-mod}(U_1) \to \text{D-mod}(U_2)$.

Hence, we are in the situation of Sect. 1.7.2 with $I = \mathrm{Ctrnk}(\mathcal{Y})$. In fact, we are in the more restrictive (and possibly more understandable) situation of Sects. 1.7.3 and 1.9.

4.2.2. Combining the assertion of Corollary 4.1.5 with that of Proposition 1.7.5, we obtain:

Corollary 4.2.3. *The category* D-mod$(\mathcal{Y})$ *is canonically equivalent to*

$$\underset{U \in \mathrm{Ctrnk}(\mathcal{Y})}{\underset{\longrightarrow}{colim}} \text{D-mod}(U),$$

where the functor $\mathrm{Ctrnk}(\mathcal{Y}) \to \mathrm{DGCat}_{\mathrm{cont}}$ *is*

$$U \mapsto \text{D-mod}(U), \quad (U_1 \overset{j_{1,2}}{\hookrightarrow} U_2) \mapsto (j_{1,2})_!.$$

Under this equivalence, for a co-truncative open quasi-compact substack $U_0 \overset{j_0}{\hookrightarrow} \mathcal{Y}$, *the functor*

$$\mathrm{ins}_{U_0} : \text{D-mod}(U_0) \to \underset{U \in \mathrm{Ctrnk}(\mathcal{Y})}{\underset{\longrightarrow}{colim}} \text{D-mod}(U) \simeq \text{D-mod}(\mathcal{Y}),$$

is $(j_0)_!$.

Remark 4.2.4. Note that the assertion of Proposition 2.3.7 for a truncative QCA stack $\mathcal{Y}$ follows also from Lemma 1.9.5(i). Note also that the assertion of Lemma 2.3.5 for U (resp., U and U') co-truncative is a particular case of Remark 1.7.6.

4.3. Description of the dual category

4.3.1. Combining Corollary 4.1.5 with Proposition 1.8.3 we obtain:

Corollary 4.3.2. *The category* D-mod$(\mathcal{Y})$ *is dualizable. Its dual category is canonically equivalent to*

$$\underset{U \in \mathrm{Ctrnk}(\mathcal{Y})}{\underset{\longrightarrow}{colim}} \text{D-mod}(U), \tag{4.1}$$

where the functor $\mathrm{Ctrnk}(\mathcal{Y}) \to \mathrm{DGCat}_{\mathrm{cont}}$ *is*

$$U \mapsto \text{D-mod}(U), \quad (U_1 \overset{j_{1,2}}{\hookrightarrow} U_2) \mapsto (j_{1,2})_*. \tag{4.2}$$

Under this equivalence, for a co-truncative open quasi-compact substack $U_0 \overset{j_0}{\hookrightarrow} \mathcal{Y}$, *the functor*

$$\mathrm{ins}_{U_0} : \text{D-mod}(U_0) \to \underset{U \in \mathrm{Ctrnk}(\mathcal{Y})}{\underset{\longrightarrow}{colim}} \text{D-mod}(U)$$

is the dual of restriction functor $j_0^* : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(U_0)$.

Proof. Follows from Proposition 2.2.17. $\square$

4.3.3. Notation. The category (4.1) that appears in Corollary 4.3.2 will be denoted by

$$\text{D-mod}(\mathcal{Y})_{\mathrm{co}}\,.$$

The equivalence of Corollary 4.3.2 will be denoted by

$$\mathbf{D}_{\mathcal{Y}}^{\mathrm{Verdier}} : \text{D-mod}(\mathcal{Y})^{\vee} \simeq \text{D-mod}(\mathcal{Y})_{\mathrm{co}}\,. \tag{4.3}$$

Note that when $\mathcal{Y}$ is quasi-compact, this is the same as the equivalence of (2.2).

4.3.4. Combining Corollary 4.3.2 with Proposition 1.7.5, we can rewrite $\text{D-mod}(\mathcal{Y})_{\text{co}}$ also as a limit:

Corollary 4.3.5. *The category* $\text{D-mod}(\mathcal{Y})_{\text{co}}$ *is canonically equivalent to*

$$\underset{U\in \text{Ctrnk}(\mathcal{Y})^{\text{op}}}{\underset{\longleftarrow}{lim}} \text{D-mod}(U),$$

where the functor $\text{Ctrnk}(\mathcal{Y})^{\text{op}} \to \text{DGCat}_{\text{cont}}$ *is*

$$U \mapsto \text{D-mod}(U), \quad (U_1 \overset{j_{1,2}}{\hookrightarrow} U_2) \mapsto j^?_{1,2}.$$

4.3.6. By construction, for every co-truncative quasi-compact open substack $U \overset{j}{\hookrightarrow} \mathcal{Y}$, we have a canonically defined functor

$$\text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})_{\text{co}}.$$

We denote this functor by $j_{\text{co},*}$. By construction, in terms of the identifications

$$\mathbf{D}^{\text{Verdier}}_U : \text{D-mod}(U)^\vee \simeq \text{D-mod}(U) \text{ and}$$
$$\mathbf{D}^{\text{Verdier}}_{\mathcal{Y}} : \text{D-mod}(\mathcal{Y})^\vee \simeq \text{D-mod}(\mathcal{Y})_{\text{co}},$$

we have

$$(j_{\text{co},*})^\vee \simeq j^*.$$

Similarly, from Corollary 4.3.5, we have a canonically defined functor

$$j^? : \text{D-mod}(\mathcal{Y})_{\text{co}} \to \text{D-mod}(U),$$

which is the dual of $j_! : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$, and the right adjoint of $j_{\text{co},*}$.

4.3.7. We claim:

Lemma 4.3.8. *The functor* $j_{\text{co},*}$ *is fully faithful.*

Proof. We need to show that the unit of the adjunction $\text{Id}_{\text{D-mod}(U)} \to j^? \circ j_{\text{co},*}$ is an isomorphism. This is obtained by passing to dual functors (see Sect. 1.5.2) in the map

$$\text{Id}_{\text{D-mod}(U)} \to j^* \circ j_!,$$

which is an isomorphism by Lemma 3.8.7. □

Remark 4.3.9. Note that Lemma 4.3.8 follows more abstractly from Remark 1.7.6. However, this way to deduce Lemma 4.3.8 is equivalent to the proof given above in view of Remark 4.2.4.

4.3.10. We claim that the category $\text{D-mod}(\mathcal{Y})_{\text{co}}$ is compactly generated and that its compact objects are ones of the form $j_{\text{co},*}(\mathcal{F}_U)$ for $\mathcal{F}_U \in \text{D-mod}(U)^c$, where U is a co-truncative quasi-compact open substack of $\mathcal{Y}$.

This follows from Proposition 2.3.7 and Sect. 1.5.3.

Alternatively, this follows from Corollary 1.9.4 and Lemma 1.9.5(i).

4.4. Relation between the category and its dual

In this subsection we continue to assume that $\mathcal{Y}$ is a truncatable locally QCA stack.

4.4.1. By construction and Sect. 1.5.1, the DG category $\text{Funct}_{\text{cont}}(\text{D-mod}(\mathcal{Y})_{\text{co}}, \text{D-mod}(\mathcal{Y}))$ identifies canonically with

$$(\text{D-mod}(\mathcal{Y})_{\text{co}})^{\vee} \otimes \text{D-mod}(\mathcal{Y}) \simeq \text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}).$$

In addition, by Proposition 2.2.8 and Remark 2.2.9, we have

$$\text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}) \simeq \text{D-mod}(\mathcal{Y} \times \mathcal{Y}).$$

Thus, every object $\mathcal{Q} \in \text{D-mod}(\mathcal{Y} \times \mathcal{Y})$ defines a functor

$$\mathsf{F}_{\mathcal{Q}} : \text{D-mod}(\mathcal{Y})_{\text{co}} \to \text{D-mod}(\mathcal{Y}).$$

4.4.2. The naive functor. Note that if $\mathcal{Y}$ is quasi-compact we have a tautological equivalence

$$\text{D-mod}(\mathcal{Y})_{\text{co}} \simeq \text{D-mod}(\mathcal{Y}).$$

Recall from Sect. 2.2.14 that the corresponding object in $\text{D-mod}(\mathcal{Y} \times \mathcal{Y})$ is $(\Delta_{\mathcal{Y}})_{\text{dR},*}(\omega_{\mathcal{Y}})$.

For *any* truncatable $\mathcal{Y}$ the functor $\text{D-mod}(\mathcal{Y})_{\text{co}} \to \text{D-mod}(\mathcal{Y})$ corresponding to

$$(\Delta_{\mathcal{Y}})_{\text{dR},*}(\omega_{\mathcal{Y}}) \in \text{D-mod}(\mathcal{Y} \times \mathcal{Y})$$

will be denoted by

$$\text{Ps-Id}_{\mathcal{Y},\text{naive}} : \text{D-mod}(\mathcal{Y})_{\text{co}} \to \text{D-mod}(\mathcal{Y})$$

(here Ps-Id stands for "pseudo-identity").

Let $\mathbf{D}^{\text{Verdier}}_{\mathcal{Y},\text{naive}} : \text{D-mod}(\mathcal{Y})^\vee \to \text{D-mod}(\mathcal{Y})$ denote the composition

$$\text{D-mod}(\mathcal{Y})^\vee \overset{\mathbf{D}^{\text{Verdier}}_{\mathcal{Y}}}{\simeq} \text{D-mod}(\mathcal{Y})_{\text{co}} \overset{\text{Ps-Id}_{\mathcal{Y},\text{naive}}}{\longrightarrow} \text{D-mod}(\mathcal{Y}).$$

4.4.3. An alternative description. Here is a tautologically equivalent description of the functor $\text{Ps-Id}_{\mathcal{Y},\text{naive}} : \text{D-mod}(\mathcal{Y})_{\text{co}} \to \text{D-mod}(\mathcal{Y})$.

By definition, to specify a continuous functor F from $\text{D-mod}(\mathcal{Y})_{\text{co}}$ to an arbitrary DG category $\mathbf{C}$, is equivalent to specifying a compatible collection of functors $\mathsf{F}_U : \text{D-mod}(U) \to \mathbf{C}$ for co-truncative quasi-compact open substacks $U \subset \mathcal{Y}$. The compatibility condition reads that for $U_1 \overset{j_{1,2}}{\hookrightarrow} U_2$, we must be given a (homotopy-coherent) system of isomorphism

$$\mathsf{F}_{U_1} \simeq \mathsf{F}_{U_2} \circ (j_{1,2})_*.$$

Taking $\mathbf{C} = \text{D-mod}(\mathcal{Y})$, the corresponding functors $(\text{Ps-Id}_{\mathcal{Y},\text{naive}})_U$ are

$$j_* : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$$

for $U \overset{j}{\hookrightarrow} \mathcal{Y}$.

4.4.4. Warning. For a general truncatable stack $\mathcal{Y}$, the functor $\text{Ps-Id}_{\mathcal{Y},\text{naive}}$ is *not* an equivalence. In particular, it is *not* an equivalence for $\mathcal{Y} = \text{Bun}_G$ unless G is solvable.

In fact, we have the following assertion:

Proposition 4.4.5. *If the functor* $\text{Ps-Id}_{\mathcal{Y},\text{naive}} : \text{D-mod}(\mathcal{Y})_{\text{co}} \to \text{D-mod}(\mathcal{Y})$ *is an equivalence then the closure of any quasi-compact open substack of* $\mathcal{Y}$ *is quasi-compact.*

The converse statement is also true (for tautological reasons).

The proof of Proposition 4.4.5 given below is based on the following lemma.

Lemma 4.4.6. *Let Z be a quasi-compact scheme, U a QCA stack, and $f : Z \to U$ a morphism. Then for any holonomic D-module $\mathcal{F}$ on Z the object* $f_{\text{dR},*}(\mathcal{F}) \in \text{D-mod}(U)$ *is compact.*

Let us give two proofs:

Proof 1. This follows from the following general observation:

Lemma 4.4.7. *Let $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ be a continuous functor between cocomplete DG categories. Let $\mathbf{c}_2 \in \mathbf{C}_2^c$ be such that the partially defined left adjoint F^L to F is defined on $\mathbf{c}_2$. Then $\mathsf{F}^L(\mathbf{c}_2) \in \mathbf{C}_1$ is compact.*

The functor $f_!$, left adjoint to $f^!$ is defined on holonomic objects. Hence, by the above lemma, $f_!(\mathbb{D}_Z^{\mathrm{Verdier}}(\mathcal{F})) \in \text{D-mod}_{\mathrm{coh}}(U)$ is compact. By Theorem 2.2.15,

$$\mathbb{D}_U^{\mathrm{Verdier}}(f_!(\mathbb{D}_Z^{\mathrm{Verdier}}(\mathcal{F}))) \simeq f_{\mathrm{dR},*}(\mathcal{F})$$

is compact, as required. □

Proof 2. The object $f_{\mathrm{dR},*}(\mathcal{F})$ is holonomic and therefore coherent. Since Z is a scheme, by Theorem 2.2.11(ii), $\mathcal{F}$ is safe. By [DrGa1, Lemma 9.4.2] we obtain that $f_{\mathrm{dR},*}(\mathcal{F})$ is also safe. Thus, $f_{\mathrm{dR},*}(\mathcal{F})$ is coherent and safe = compact. □

Proof of Proposition 4.4.5. Suppose that $\text{Ps-Id}_{\mathcal{Y},\mathrm{naive}}$ is an equivalence. Since $\mathcal{Y}$ is truncatable, it is enough to show that the closure of every co-truncative open quasi-compact substack is quasi-compact.

By assumption, the functor $\text{Ps-Id}_{\mathcal{Y},\mathrm{naive}}$ preserves compactness. From Sect. 4.4.3, we obtain that $\text{Ps-Id}_{\mathcal{Y},\mathrm{naive}}$ sends a compact object $j_{\mathrm{op},*}(\mathcal{F}_U) \in \text{D-mod}(\mathcal{Y})_{\mathrm{op}}$, $\mathcal{F}_U \in \text{D-mod}(U)^c$ with $U \overset{j}{\hookrightarrow} \mathcal{Y}$ co-truncative and quasi-compact, to $j_*(\mathcal{F}_U) \in \text{D-mod}(\mathcal{Y})$. Thus, we obtain that $j_*(\mathcal{F}_U)$ needs to be compact for any $\mathcal{F}_U \in \text{D-mod}(U)^c$ whenever U is co-truncative.

Take $\mathcal{F}_U = f_{\mathrm{dR},*}(k_Z)$, where Z is any quasi-compact scheme equipped with a morphism $f : Z \to U$ and k_Z is the "constant sheaf" on Z. By Proposition 2.3.7, there exists a quasi-compact open substack $V \subset \mathcal{Y}$ such that the $*$-stalk of $j_{\mathrm{dR},*}(\mathcal{F}_U) = (j \circ f)_{\mathrm{dR},*}(k_Z)$ over any point of $\mathcal{Y} - V$ is zero. This means that the closure of the image of $j \circ f : Z \to \mathcal{Y}$ is contained in V and therefore quasi-compact. Taking f surjective we see that the closure of U is quasi-compact. □

4.4.8. A better functor. Following [Ga3, Sect. 6], we define

$$\text{Ps-Id}_{\mathcal{Y},!} : \text{D-mod}(\mathcal{Y})_{\mathrm{co}} \to \text{D-mod}(\mathcal{Y})$$

to be the functor corresponding in terms of Sect. 4.4.1 to the object

$$(\Delta_{\mathcal{Y}})_!(k_{\mathcal{Y}}) \in \text{D-mod}(\mathcal{Y} \times \mathcal{Y}),$$

where $k_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y})$ is the "constant sheaf" on $\mathcal{Y}$. (The above object is well-defined because $k_{\mathcal{Y}}$ is holonomic.)

Let $\mathbf{D}_{\mathcal{Y},!}^{\mathrm{Verdier}} : \text{D-mod}(\mathcal{Y})^\vee \to \text{D-mod}(\mathcal{Y})$ denote the composition

$$\text{D-mod}(\mathcal{Y})^\vee \overset{\mathbf{D}_{\mathcal{Y}}^{\mathrm{Verdier}}}{\simeq} \text{D-mod}(\mathcal{Y})_{\mathrm{co}} \xrightarrow{\text{Ps-Id}_{\mathcal{Y},!}} \text{D-mod}(\mathcal{Y}).$$

4.4.9. Suppose for a moment that $\mathcal{Y}$ is smooth of dimension n, and that the diagonal map

$$\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$$

is separated. In this case we have an isomorphism

$$k_{\mathcal{Y}} \simeq \omega_{\mathcal{Y}}[-2n],$$

and a natural transformation

$$(\Delta_{\mathcal{Y}})_! \to (\Delta_{\mathcal{Y}})_{\mathrm{dR},*}\,,$$

which together define a natural transformation

$$\text{Ps-Id}_{\mathcal{Y},!} \to \text{Ps-Id}_{\mathcal{Y},\mathrm{naive}}[-2n]. \tag{4.4}$$

Remark 4.4.10. If $\mathcal{Y}$ is separated (i.e., if $\Delta_{\mathcal{Y}}$ is proper) then (4.4) is an isomorphism. However, most stacks are not separated.[9] Thus $\text{Ps-Id}_{\mathcal{Y},!}$ is usually *different* from $\text{Ps-Id}_{\mathcal{Y},\mathrm{naive}}$ (even for $\mathcal{Y}$ smooth and quasi-compact).

4.4.11. Here is a basic feature of the functor $\text{Ps-Id}_{\mathcal{Y},!} : \text{D-mod}(\mathcal{Y})_{\mathrm{co}} \to \text{D-mod}(\mathcal{Y})$.

Lemma 4.4.12. *Let $U \overset{j}{\hookrightarrow} \mathcal{Y}$ be a co-truncative quasi-compact open substack. Then there exists a canonical isomorphism of functors* $\text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$*:*

$$\text{Ps-Id}_{\mathcal{Y},!} \circ j_{\mathrm{co},*} \simeq j_! \circ \text{Ps-Id}_{U,!}\,.$$

Proof. Define $\Delta_{U,\mathcal{Y}} : U \to U \times \mathcal{Y}$ by $\Delta_{U,\mathcal{Y}}(u) := (u, j(u))$. It is easy to check that both functors $\text{Ps-Id}_{\mathcal{Y},!} \circ j_{\mathrm{co},*}$ and $j_! \circ \text{Ps-Id}_{U,!}$ correspond to the object $(\Delta_{U,\mathcal{Y}})_!(k_U) \in \text{D-mod}(U \times \mathcal{Y})$ via the equivalence

$$\begin{aligned}\text{D-mod}(U \times \mathcal{Y}) &\simeq \text{D-mod}(U) \otimes \text{D-mod}(\mathcal{Y}) \\ &\simeq \text{D-mod}(U)^\vee \otimes \text{D-mod}(\mathcal{Y}) \simeq \mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(U), \text{D-mod}(\mathcal{Y})). \qquad \square\end{aligned}$$

The meaning of this lemma is that *the functor* $\text{Ps-Id}_{\mathcal{Y},!}$ *sends objects that are $*$-extensions from a co-truncative quasi-compact open substack in* $\text{D-mod}(\mathcal{Y})_{\mathrm{op}}$ *to objects in* $\text{D-mod}(\mathcal{Y})$ *that are !-extensions* (from the same open).

[9] A separated locally QCA stack has to be a Deligne-Mumford stack. Indeed, if $\Delta_{\mathcal{Y}}$ is proper and affine then it is finite, and in characteristic 0 this means that $\mathcal{Y}$ is Deligne-Mumford.

4.4.13. Self-duality. Both functors

$$\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y},\mathrm{naive}} : \mathrm{D\text{-}mod}(\mathcal{Y})^\vee \to \mathrm{D\text{-}mod}(\mathcal{Y}), \quad \mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y},!} : \mathrm{D\text{-}mod}(\mathcal{Y})^\vee \to \mathrm{D\text{-}mod}(\mathcal{Y})$$

are canonically self-dual because the corresponding objects

$$(\Delta_{\mathcal{Y}})_{\mathrm{dR},*}(\omega_{\mathcal{Y}}),\ (\Delta_{\mathcal{Y}})_!(k_{\mathcal{Y}}) \in \mathrm{D\text{-}mod}(\mathcal{Y} \times \mathcal{Y})$$

are equivariant with respect to the action of the symmetric group S_2 on $\mathcal{Y} \times \mathcal{Y}$.

4.5. Miraculous stacks

4.5.1. Now let us give the following definition.

Definition 4.5.2. *A truncatable stack $\mathcal{Y}$ is called* miraculous *if the functor*

$$\mathrm{Ps\text{-}Id}_{\mathcal{Y},!} : \mathrm{D\text{-}mod}(\mathcal{Y})_{\mathrm{co}} \to \mathrm{D\text{-}mod}(\mathcal{Y})$$

is an equivalence.

Clearly this happens if and only if the functor $\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y},!} : \mathrm{D\text{-}mod}(\mathcal{Y})^\vee \to \mathrm{D\text{-}mod}(\mathcal{Y})$ is an equivalence.

4.5.3. The following easy lemma shows that not every algebraic stack is miraculous.

Lemma 4.5.4. *A separated quasi-compact scheme Z is a miraculous stack if and only if Z has the following "cohomological smoothness" property: k_Z and ω_Z are locally isomorphic up to a shift.*

Proof. In our situation $\mathrm{Ps\text{-}Id}_{Z,!} : \mathrm{D\text{-}mod}(Z) \to \mathrm{D\text{-}mod}(Z)$ is the functor $M \mapsto M \overset{!}{\otimes} k_Z$. Applying the functor to skyscrapers, we see that if $\mathrm{Ps\text{-}Id}_{Z,!}$ is an equivalence then each !-stalk of k_Z is isomorphic to k up to a shift. It is well known that this implies that k_Z and ω_Z are locally isomorphic up to a shift. □

It is also easy to produce an example of a *smooth* quasi-compact algebraic stack $\mathcal{Y}$ which is not miraculous: it suffices to take $\mathcal{Y}$ to be the non-separated scheme equal to $\mathbb{A}^1$ with a double point 0. We refer the reader to [Ga3, Sect. 5.3.5], where this example is analyzed (one easily shows that in this case the functor $\mathrm{Ps\text{-}Id}_{\mathcal{Y},!}$ does not preserve compactness).

4.5.5. A basic example of a miraculous stack is $\mathcal{Y} := \mathbb{A}^n/\mathbb{G}_m$; see [Ga3, Corollary 5.3.4].

In addition, the following theorem is proved in [Ga2]:

Theorem 4.5.6. *Let G be a reductive group. Then the stack* Bun_G *is miraculous.*

This theorem is equivalent to each quasi-compact co-truncative substack of Bun_G being miraculous. The equivalence follows from the next lemma.

Lemma 4.5.7. *A truncatable stack $\mathcal{Y}$ is miraculous if and only if every quasi-compact co-truncative open substack $U \subset \mathcal{Y}$ is.*

Proof. The "if" statement follows from Lemma 4.4.12 and the descriptions of $\text{D-mod}(\mathcal{Y})$ and $\text{D-mod}(\mathcal{Y})_{\mathrm{co}}$ as colimits (see Corollary 4.2.3 and Sect. 4.3.3).

Let us prove the "only if" statement. Suppose that $\mathcal{Y}$ is miraculous and $j : U \hookrightarrow \mathcal{Y}$ is a quasi-compact co-truncative open substack. The functor $j_!$ has a left inverse (namely, $j^! = j^*$). The functor $j_{\mathrm{co},*} : \text{D-mod}(U)_{\mathrm{co}} \to \text{D-mod}(\mathcal{Y})_{\mathrm{co}}$ also has a left inverse (see the first proof of Lemma 4.3.8). So Lemma 4.4.12 implies that $\mathrm{Ps\text{-}Id}_{U,!}$ has a left inverse.

We obtain that the functor $\mathbf{D}^{\mathrm{Verdier}}_{U,!} : \text{D-mod}(U)^\vee \to \text{D-mod}(U)$ has a left inverse. By self-duality of $\mathbf{D}^{\mathrm{Verdier}}_{U,!}$ (see Sect. 4.4.13), this implies that it has a right inverse as well. So $\mathbf{D}^{\mathrm{Verdier}}_{U,!}$ is an equivalence. □

5. Contractive substacks

In its simplest form, the contraction principle says that the substack $\{0\}/\mathbb{G}_m \hookrightarrow \mathbb{A}^n/\mathbb{G}_m$ is truncative (here $\mathbb{G}_m$ acts on $\mathbb{A}^n$ by homotheties). In this section we will prove a generalization of this fact, see Proposition 5.1.2.

In Sect. 5.3 we explicitly describe the non-standard functors i^*_{dR} and $i_?$ in the setting of Proposition 5.1.2.

We say that a substack of a stack is *contractive* if it locally satisfies the conditions of Proposition 5.1.2; for a precise definition, see Sect. 5.2.1.

The upshot of this section is that "contractiveness" ⇒ "truncativeness."

5.1. The contraction principle

5.1.1. Consider the following set-up. Suppose we have an affine morphism $p : W \to S$ between schemes. Assume that the monoid $\mathbb{A}^1$ (with respect to multiplication) acts on W over S (so that the action of $\mathbb{A}^1$ on S is trivial). Assume also that the endomorphism of W corresponding to $0 \in \mathbb{A}^1$ admits a factorization

$$W \xrightarrow{p} S \xrightarrow{\iota} W,$$

where ι is a section of $p : W \to S$. (Informally, we can say that the action of $\mathbb{G}_m \subset \mathbb{A}^1$ "contracts" W onto the closed subscheme $\iota(S)$.)

Set $\mathcal{Y} := W/\mathbb{G}_m$, $\mathcal{Z} := S/\mathbb{G}_m = S \times (\mathrm{pt}/\mathbb{G}_m)$.

Proposition 5.1.2. *Under the above circumstances, the closed substack* $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ *is truncative.*

The rest of this subsection is devoted to the proof of Proposition 5.1.2.

5.1.3. Without loss of generality, we can assume that S is quasi-compact. We have

$$W = \mathrm{Spec}_S(\mathcal{A}),$$

where $\mathcal{A} = \bigoplus_n \mathcal{A}_n$ is a quasi-coherent sheaf of non-negatively graded $\mathcal{O}_S$-algebras with $\mathcal{A}_0 = \mathcal{O}_S$. The section ι corresponds to the projection $\mathcal{A} \to \mathcal{A}_0 = \mathcal{O}_S$.

For $n \in \mathbb{N}$, let $\mathcal{A}^{(n)} \subset \mathcal{A}$ be the $\mathcal{O}_S$-subalgebra generated by $\mathcal{A}_n$. Choose n so that $\mathcal{A}$ is finite over $\mathcal{A}^{(n)}$ (if $\mathcal{A}$ is generated by $\mathcal{A}_{m_1}, \ldots, \mathcal{A}_{m_r}$ then one can take n to be the least common multiple of $m_1, \ldots, m_r$). Set $W' := \mathrm{Spec}(\mathcal{A}^{(n)})$, then the morphism $f : W \to W'$ is finite. Moreover, the embedding $\iota(S) \hookrightarrow f^{-1}(f(\iota(S)))$ induces an isomorphism between the corresponding reduced schemes. So by Proposition 3.6.11, it suffices to prove the proposition for W' instead of W.

5.1.4. Thus, we can assume that $\mathcal{A}$ is generated by $\mathcal{A}_n$. Moreover, since the proposition to be proved is local with respect to S (see Corollary 3.6.5), we can assume that $\mathcal{A}_n$ is a quotient of a locally free $\mathcal{O}_S$-module $\mathcal{E}$. Let V denote the vector bundle over S corresponding to $\mathcal{E}^*$ (in other words, V is the spectrum of the symmetric algebra of $\mathcal{E}$). Then $W = \mathrm{Spec}(\mathcal{A})$ identifies with a closed conical subscheme in V.

5.1.5. Thus by Lemma 3.6.10 (for closed embeddings) it suffices to consider the case where W is a vector bundle V over S equipped with an $\mathbb{A}^1$-action obtained from the standard one by composing it with the homomorphism

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^1, \quad \lambda \mapsto \lambda^n. \tag{5.1}$$

(here n is some positive integer). In this situation we have to prove that $\mathbf{0}/\mathbb{G}_m \subset V/\mathbb{G}_m$ is truncative, where $\mathbf{0} \subset V$ is the zero section.

5.1.6. Let $\widetilde{V} \overset{f}{\to} V$ be the blow-up of V along $\mathbf{0}$. Set $\widetilde{\mathbf{0}} := f^{-1}(\mathbf{0})$. Since f is proper and surjective, by Lemma 3.6.7, in order to prove the truncativeness of $\mathbf{0}/\mathbb{G}_m \subset V/\mathbb{G}_m$, it suffices to show that $\widetilde{\mathbf{0}}/\mathbb{G}_m$ is truncative in $\widetilde{V}/\mathbb{G}_m$.

Note that $\widetilde{V}$ is a line bundle over $\mathbb{P}(V)$ and $\widetilde{\mathbf{0}}$ is its zero section. So we see that it suffices to prove the statement from Sect. 5.1.5 for line bundles over arbitrary bases. Moreover, since the statement is local, it suffices to consider the trivial line bundle over an arbitrary quasi-compact scheme.

5.1.7. Thus it remains to prove that for any quasi-compact scheme S the substack

$$S \times (\{0\}/\mathbb{G}_m) \subset S \times (\mathbb{A}^1/\mathbb{G}_m)$$

is truncative (here we assume that $\lambda \in \mathbb{G}_m$ acts on $\mathbb{A}^1$ as multiplication by λ^n for some $n \in \mathbb{N}$). By Lemma 3.6.2, we can assume that $S = \mathrm{Spec}(k)$. In this case the statement follows from the fact that the number of $\mathbb{G}_m$-orbits in $\mathbb{A}^1$ is finite (see Sect. 3.2.4).

5.2. Contractiveness

5.2.1. We say that a locally closed substack $\mathcal{Z}'$ of a stack $\mathcal{Y}'$ is *contractive* if there exists a commutative diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathcal{Z}' \arrow[r] & \mathcal{Y}' \end{tikzcd} \tag{5.2}$$

such that

(i) the upper row of (5.2) is as in Proposition 5.1.2;
(ii) the morphism $\mathcal{Z} \to \mathcal{Z}' \underset{\mathcal{Y}'}{\times} \mathcal{Y}$ is an open embedding;
(iii) the vertical arrows of (5.2) are smooth and the left one is surjective.

In other words, a substack is contractive if it locally satisfies the conditions of Proposition 5.1.2.

5.2.2. We obtain:

Corollary 5.2.3. *A contractive substack is truncative.*

Proof. With no loss of generality, we can assume that $\mathcal{Y}$ is quasi-compact. Now combine Proposition 5.1.2 and 3.6.4. □

Note that the above definition of contractive substack makes sense without the characteristic 0 assumption.

Remark 5.2.4. We are not sure that the notion of contractiveness is really good. But it is convenient for the purposes of this article.

5.3. An adjointness result

5.3.1. Let $W \to S \overset{\iota}{\to} W$ be as in Proposition 5.1.2 (in particular, $\mathbb{A}^1$ acts on W).

Consider the corresponding morphisms $i : S/\mathbb{G}_m \hookrightarrow W/\mathbb{G}_m$ and $\pi : W/\mathbb{G}_m \to S/\mathbb{G}_m$. By Proposition 5.1.2, the functor $i_{\mathrm{dR},*}$ has a continuous left adjoint (denoted by i^*_{dR}) and $i^!$ has a continuous right adjoint (denoted by $i_?$).

The next proposition identifies the non-standard functors

$$i^*_{\mathrm{dR}} : \text{D-mod}(W/\mathbb{G}_m) \to \text{D-mod}(S/\mathbb{G}_m) \text{ and}$$
$$i_? : \text{D-mod}(S/\mathbb{G}_m) \to \text{D-mod}(W/\mathbb{G}_m)$$

with certain standard functors. Namely,

$$i^*_{\mathrm{dR}} \simeq \pi_{\mathrm{dR},*}, \quad i_? = \pi^!.$$

Proposition 5.3.2. *The functors*

$$\pi_{\mathrm{dR},*} : \text{D-mod}(W/\mathbb{G}_m) \rightleftarrows \text{D-mod}(S/\mathbb{G}_m) : i_{\mathrm{dR},*} \;\; \textit{and}$$
$$i^! : \text{D-mod}(W/\mathbb{G}_m) \rightleftarrows \text{D-mod}(S/\mathbb{G}_m) : \pi^!$$

form adjoint pairs with the adjunctions

$$\pi_{\mathrm{dR},*} \circ i_{\mathrm{dR},*} \xrightarrow{\sim} \mathrm{Id}_{\text{D-mod}(S/\mathbb{G}_m)} \;\; \textit{and} \;\; i^! \circ \pi^! \xrightarrow{\sim} \mathrm{Id}_{\text{D-mod}(S/\mathbb{G}_m)}$$

coming from the isomorphism $\pi \circ i \xrightarrow{\sim} \mathrm{Id}_{S/\mathbb{G}_m}$.

Note that a simple particular case of Proposition 5.3.2 was proved in Sect. 3.3.9.

Remark 5.3.3. Proposition 5.3.2 clearly implies Proposition 5.1.2.

Proposition 5.3.2 is well known (at least, in the setting of constructible sheaves instead of D-modules). It goes back to the works by Verdier [Ve, Lemma 6.1] and Springer [Sp, Proposition 1]; see also [KL, Lemma A.7] and [Br, Lemma 6].

The reader can easily prove Proposition 5.3.2 by slightly modifying the argument from Sect. 5.1 (which is based on blowing up and properness).

On the other hand, in Appendix C we give a complete proof of a "stacky" generalization of Proposition 5.3.2 (see Theorem C.5.3 and Corollary C.5.4). The approach from Appendix C is close to [Sp] (there are no blow-ups, no properness arguments, and we work with the monoid $\mathbb{A}^1$ rather than with the scheme or stack on which $\mathbb{A}^1$ acts).

5.4. A general lemma on contractiveness

The reader may prefer to skip this subsection on the first pass. Its main result (Lemma 5.4.3) will not be used until Proposition 11.3.7(b).

5.4.1. The notion of contractive substack was defined in Sect. 5.2.1. This notion is clearly local in the following sense:

Let $f : \mathcal{Y}' \to \mathcal{Y}$ be a smooth surjective morphism of algebraic stacks and $\mathcal{Z} \subset \mathcal{Y}$ a locally closed substack. If $f^{-1}(\mathcal{Z})$ is contractive in $\mathcal{Y}'$ then $\mathcal{Z}$ is contractive in $\mathcal{Y}$.

5.4.2. As before, we consider $\mathbb{A}^1$ as a monoid with respect to multiplication. It contains $\mathbb{G}_m$ as a subgroup.

We have:

Lemma 5.4.3. *Let $\pi : \mathcal{W} \to \mathcal{S}$ be an affine schematic morphism of algebraic stacks. Suppose that the monoid $\mathbb{A}^1$ acts on $\mathcal{W}$ by $\mathcal{S}$-endomorphisms (i.e., over $\mathcal{S}$, with the action of $\mathbb{A}^1$ on $\mathcal{S}$ being trivial). Assume that*

(i) *The $\mathcal{S}$-endomorphism of $\mathcal{W}$ corresponding to $0 \in \mathbb{A}^1$ equals $i \circ \pi$ for some section $i : \mathcal{S} \to \mathcal{W}$;*
(ii) *The action of $\mathbb{G}_m$ on $\mathcal{W}$ viewed as a stack over* pt *(rather than over $\mathcal{S}$) is isomorphic to the trivial action.*

Then the substack $\mathcal{S} \overset{i}{\hookrightarrow} \mathcal{W}$ is contractive.

Remark 5.4.4. Condition (i) implies that the action of $\mathbb{G}_m$ on $\mathcal{W}$ viewed as a stack over $\mathcal{S}$ is nontrivial unless $\pi : \mathcal{W} \to \mathcal{S}$ is an isomorphism. This does not contradict (ii): we are dealing with stacks, and the functor from the groupoid of $\mathcal{S}$-endomorphisms of $\mathcal{W}$ to that of k-endomorphisms of $\mathcal{W}$ is not fully faithful.

Before proving Lemma 5.4.3, let us consider two examples.

Example 5.4.5. Let $p : W \to S$ be as in Sect. 5.1.1. Set

$$\mathcal{W} := W/\mathbb{G}_m, \quad \mathcal{S} := S/\mathbb{G}_m = S \times (\mathrm{pt}/\mathbb{G}_m).$$

Then the conditions of Lemma 5.4.3 hold for the morphism $\pi : \mathcal{W} \to \mathcal{S}$. The conclusion of Lemma 5.4.3 holds tautologically.

Example 5.4.6. Let $\pi : \mathcal{W} \to \mathcal{S}$ be an affine schematic morphism of algebraic stacks. Suppose that an action of $\mathbb{A}^1$ on $\mathcal{W}$ by $\mathcal{S}$-endomorphisms satisfies condition (i) of Lemma 5.4.3. Set $\mathcal{W}' := \mathcal{W}/\mathbb{G}_m$ and $\mathcal{S}' := \mathcal{S}/\mathbb{G}_m = \mathcal{S} \times (\mathrm{pt}/\mathbb{G}_m)$. Then the morphism $\pi' : \mathcal{W}' \to \mathcal{S}'$ satisfies both conditions of

Lemma 5.4.3. The conclusion of Lemma 5.4.3 is clear because after a smooth surjective base change $S \to \mathcal{S}$ we get the situation of Example 5.4.5.

5.4.7. To prove Lemma 5.4.3, we need the following assertion:

Lemma 5.4.8. *Let $\varphi : \mathcal{Y} \to \mathcal{Y}'$ and $\psi : \mathcal{Y}' \to \mathcal{Y}$ be morphisms between algebraic stacks such that $\psi \circ \varphi \simeq \mathrm{Id}_{\mathcal{Y}}$. Suppose that φ is smooth and surjective. Then:*

(a) *The maps*

$$\begin{aligned}\{\textit{locally closed substacks of } \mathcal{Y}'\} &\to \{\textit{locally closed substacks of } \mathcal{Y}\},\\ &\quad \mathcal{Z}' \mapsto \varphi^{-1}(\mathcal{Z}'),\\ \{\textit{locally closed substacks of } \mathcal{Y}\} &\to \{\textit{locally closed substacks of } \mathcal{Y}'\},\\ &\quad \mathcal{Z} \mapsto \psi^{-1}(\mathcal{Z})\end{aligned}$$

are mutually inverse bijections;

(b) *A locally closed substack $\mathcal{Z} \subset \mathcal{Y}$ is contractive if and only if the corresponding substack $\mathcal{Z}' \subset \mathcal{Y}'$ is.*

Remark 5.4.9. Since φ and $\psi \circ \varphi$ are smooth and surjective ψ has the same properties.[10]

Proof. The maps from statement (a) are clearly injective. Since $\psi \circ \varphi \simeq \mathrm{Id}_{\mathcal{Y}}$ one has $\varphi^{-1}(\psi^{-1}(\mathcal{Z})) = \mathcal{Z}$. Statement (a) follows. To prove (b), use statement (a), Remark 5.4.9, and the locality of strong contractiveness, see Sect. 5.4.1. □

Proof of Lemma 5.4.3. Define $\pi' : \mathcal{W}' \to \mathcal{S}'$ as in Example 5.4.6, then the corresponding embedding $i' : \mathcal{S}' \hookrightarrow \mathcal{W}'$ is contractive. We have a Cartesian square

$$\begin{tikzcd} \mathcal{S} \arrow[r, "i"] \arrow[d] & \mathcal{W} \arrow[d, "\varphi"'] \\ \mathcal{S}' \arrow[r, "i'"] & \mathcal{W}' \end{tikzcd}$$

So by Lemma 5.4.8, it remains to show that the morphism $\varphi : \mathcal{W} \to \mathcal{W}'$ admits a left inverse $\psi : \mathcal{W}' \to \mathcal{W}$. But $\mathcal{W}'$ is the quotient of $\mathcal{W}$ by a *trivial* action of $\mathbb{G}_m$. Choosing a trivialization of this action we can identify $\mathcal{W}'$ with $\mathcal{W} \times (\mathrm{pt}/\mathbb{G}_m)$ and take ψ to be the projection

$$\mathcal{W} \times (\mathrm{pt}/\mathbb{G}_m) \to \mathcal{W}. \qquad \square$$

[10] By the way, this implies that $\mathcal{Y}'$ is the classifying space of a group over $\mathcal{Y}$.

6. The case of SL_2

In this section we will give a proof of Theorem 4.1.8 in the case $G = SL_2$, which will be the prototype of the argument in general.

6.1. The substack $\mathrm{Bun}_G^{(\leq n)}$

6.1.1. For an integer $n \geq 0$, let $\mathrm{Bun}_G^{(\leq n)} \subset \mathrm{Bun}_G$ be the open substack consisting of vector bundles that *do not admit* line sub-bundles of degree $> n$. It is easy to see that the substacks $\mathrm{Bun}_G^{(\leq n)}$ are quasi-compact and that their union is all of Bun_G.

Let g be the genus of X. We will show that for $n \geq \max(g-1, 0)$, the open substack $\mathrm{Bun}_G^{(\leq n)}$ is co-truncative.

6.1.2. Let $\mathrm{Bun}_G^{(n)}$ be the locally closed substack

$$\mathrm{Bun}_G^{(n)} := \mathrm{Bun}_G^{(\leq n)} - \mathrm{Bun}_G^{(\leq n-1)},$$

endowed, say, with the reduced structure.

By Proposition 3.7.2, it suffices to show that if $n > \max(g-1, 0)$ then $\mathrm{Bun}_G^{(n)}$ is a truncative substack of $\mathrm{Bun}_G^{(\leq n)}$. We will do this by combining Propositions 3.6.4 and 5.1.2.

6.1.3. Note, however, that if n is small relative to the genus of X, then the stratum $\mathrm{Bun}_G^{(n)}$ is *not* truncative. Indeed, one can choose X and n so that $\mathrm{Bun}_G^{(n)}$ has a non-empty intersection with an open substack Bun_G that is actually a *scheme*; then apply Sect. 3.2.1.

6.2. Reducing to a contracting situation

6.2.1. For an integer n, let Bun_B^n be the stack classifying short exact sequences

$$0 \to \mathcal{L}^{-1} \to \mathcal{M} \to \mathcal{L} \to 0, \tag{6.1}$$

where $\mathcal{M} \in \mathrm{Bun}_{SL_2}$, and $\mathcal{L}$ is a line bundle of degree $-n$. Let $\mathsf{p}^n : \mathrm{Bun}_B^n \to \mathrm{Bun}_G$ denote the natural projection. If $n > 0$ then the image of p^n equals $\mathrm{Bun}_G^{(n)}$.

Lemma 6.2.2. *Suppose that $n > \max(g-1,0)$. Then the morphism p^{-n} : $\mathrm{Bun}_B^{-n} \to \mathrm{Bun}_G$ is smooth and its image contains $\mathrm{Bun}_G^{(n)}$.*

Proof. A point $x \in \mathrm{Bun}_B^{-n}$ corresponds to an exact sequence (6.1) with $\deg \mathcal{L} = n$. The cokernel of the differential of p^{-n} at x equals $H^1(X, \mathcal{L}^{\otimes 2})$, which is zero because $\deg \mathcal{L}^{\otimes 2} = 2n > 2g-2$. So p^{-n} is smooth.

A point $y \in \mathrm{Bun}_G^{(n)}$ corresponds to an SL_2-bundle $\mathcal{M}$ that can be represented as an extension (6.1) with $\deg \mathcal{L} = -n$. Such an extension splits because $2n > 2g-2$. So $\mathcal{M}$ is also an extension of $\mathcal{L}^{-1}$ by $\mathcal{L}$. Hence, y is in the image of p^{-n}. $\square$

6.2.3. By Lemma 6.2.2 and Proposition 3.6.4, it suffices to show that if $n > \max(g-1,0)$ then the substack

$$\mathrm{Bun}_G^{(n)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B^{-n} \subset \mathrm{Bun}_B^{-n} \tag{6.2}$$

is truncative.

6.3. Applying the contraction principle

6.3.1. Let $\mathrm{Bun}_{GL(1)}^n$ denote the stack of line bundles on X of degree n. Note that we have a canonical isomorphism

$$\mathrm{Bun}_{GL(1)}^n \simeq \mathrm{Bun}_G^{(n)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B^{-n}$$

that sends a line bundle $\mathcal{L} \in \mathrm{Bun}_{GL(1)}^n$ to

$$0 \to \mathcal{L}^{-1} \to \mathcal{L}^{-1} \oplus \mathcal{L} \to \mathcal{L} \to 0.$$

6.3.2. Let Pic^n denote the coarse moduli scheme corresponding to $\mathrm{Bun}_{GL(1)}^n$. We have a vector bundle $\mathcal{V}$ on $\mathrm{Bun}_{GL(1)}^n$ whose fiber over $\mathcal{L} \in \mathrm{Bun}_{GL(1)}^n$ equals $\mathrm{Ext}(\mathcal{L}, \mathcal{L}^{-1})$.

Choose a section $s : \mathrm{Pic}^n \to \mathrm{Bun}_{GL(1)}^n$ of the morphism $\mathrm{Bun}^n \to \mathrm{Pic}^n$ (e.g., choose $x_0 \in X$ and identify Pic^n with the stack of line bundles of degree n trivialized over x_0).

Set $\mathcal{V}' = s^*(\mathcal{V})$. Let $\mathbf{0} \subset \mathcal{V}'$ denote the zero section. Then Bun_B^{-n} identifies with the quotient stack $\mathcal{V}'/\mathbb{G}_m$ and the substack

$$\mathrm{Bun}_G^{(n)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B^{-n} \simeq \mathrm{Bun}_{GL(1)}^n \hookrightarrow \mathrm{Bun}_B^{-n}$$

identifies with $\mathbf{0}/\mathbb{G}_m$. Hence, the substack (6.2) is truncative by Proposition 5.1.2.

7. Recollections from reduction theory

The goal of this section is to prepare for the proof of Theorem 4.1.8 by recalling the Harder-Narasimhan-Shatz stratification of Bun_G.

With future applications in mind, when defining these open substacks, we will remove the assumption that our ground field is of characteristic 0, unless we explicitly specify otherwise. Thus, we let G be a connected reductive group over any algebraically closed field k.

7.1. Notation related to G

7.1.1. To simplify the discussion, we will work with a fixed choice of a Borel subgroup $B \subset G$.

Conjugacy classes of parabolics are then in bijection with the set of parabolics that contain B, called *the standard parabolics.* From now on, by a parabolic we will mean a standard parabolic, unless explicitly stated otherwise.

For a parabolic P we will denote by $U(P)$ its unipotent radical.

We denote by Γ_G the set of vertices of the Dynkin diagram of G. Parabolics in G are in bijection with subsets of Γ_G. For a parabolic P with Levi quotient M we let $\Gamma_M \subset \Gamma_G$ denote the corresponding subset; it identifies with the set of vertices of the Dynkin diagram of M.

7.1.2. Let Λ_G denote the coweight lattice of G and $\Lambda_G^{\mathbb{Q}} := \mathbb{Q} \underset{\mathbb{Z}}{\otimes} \Lambda_G$. Let $\Lambda_G^+ \subset \Lambda_G$ denote the monoid of dominant coweights and $\Lambda_G^{pos} \subset \Lambda_G$ the monoid generated by positive simple coroots. Let $\Lambda_G^{+,\mathbb{Q}}, \Lambda_G^{pos,\mathbb{Q}} \subset \Lambda_G^{\mathbb{Q}}$ be the corresponding rational cones.

Let $\check{\alpha}_i$, $i \in \Gamma_G$, be the simple roots; we have:

$$\Lambda_G^{+,\mathbb{Q}} = \{\lambda \in \Lambda_G^{\mathbb{Q}} \mid \langle \lambda, \check{\alpha}_i \rangle \geq 0 \ \text{ for } \ i \in \Gamma_G\}.$$

7.1.3. Let P be a parabolic of G and M its Levi quotient. Let $Z_0(M)$ be the neutral connected component of the center of M, then $\Lambda_{Z_0(M)} \subset \Lambda_G$. Set $\Lambda_{G,P}^{\mathbb{Q}} := \Lambda_{Z_0(M)}^{\mathbb{Q}} \subset \Lambda_G^{\mathbb{Q}}$. Explicitly,

$$\Lambda_{G,P}^{\mathbb{Q}} = \{\lambda \in \Lambda_G^{\mathbb{Q}} \mid \langle \lambda, \check{\alpha}_i \rangle = 0 \ \text{ for } \ i \in \Gamma_M\}.$$

Note that

$$\Lambda^{\mathbb{Q}}_{G,G} = \Lambda^{\mathbb{Q}}_{Z_0(G)} \text{ and } \Lambda^{\mathbb{Q}}_{G,B} = \Lambda^{\mathbb{Q}}_{G}.$$

7.1.4. Set $\Lambda^{+,\mathbb{Q}}_{G,P} := \Lambda^{+,\mathbb{Q}}_{G} \cap \Lambda^{\mathbb{Q}}_{G,P}$ and

(7.1)
$$\Lambda^{++,\mathbb{Q}}_{G,P} := \{\lambda \in \Lambda^{\mathbb{Q}}_{G} \mid \langle \lambda, \check{\alpha}_i \rangle = 0 \text{ for } i \in \Gamma_M \text{ and } \langle \lambda, \check{\alpha}_i \rangle > 0 \text{ for } i \notin \Gamma_M\}.$$

In other words, $\Lambda^{++,\mathbb{Q}}_{G,P}$ is the set of those elements of $\Lambda^{+,\mathbb{Q}}_{G,P}$ that are regular (i.e., lie off the walls of $\Lambda^{+,\mathbb{Q}}_{G,P}$). Clearly

$$\Lambda^{+,\mathbb{Q}}_{G} = \bigsqcup_{P} \Lambda^{++,\mathbb{Q}}_{G,P}, \tag{7.2}$$

where the union is taken over the conjugacy classes of parabolics.

7.1.5. Note also that the inclusion $\Lambda^{\mathbb{Q}}_{G,P} \hookrightarrow \Lambda^{\mathbb{Q}}_{G}$ canonically splits as a direct summand: the corresponding projector $\mathrm{pr}_P : \Lambda^{\mathbb{Q}}_{G} \to \Lambda^{\mathbb{Q}}_{G,P}$ is defined so that

$$\ker(\mathrm{pr}_P) = \bigoplus_{i \in \Gamma_M} \mathbb{Q} \cdot \alpha_i.$$

We can also view the map $\Lambda^{\mathbb{Q}}_{G} \to \Lambda^{\mathbb{Q}}_{G,P}$ as follows: it comes from the map

$$\Lambda_G \simeq \Lambda_M \to \Lambda_{M/[M,M]}$$

and the isomorphism

$$\Lambda^{\mathbb{Q}}_{Z_0(M)} \xrightarrow{\sim} \Lambda^{\mathbb{Q}}_{M/[M,M]}$$

induced by the isogeny $Z_0(M) \to M/[M,M]$.

7.1.6. We introduce the partial order on $\Lambda^{\mathbb{Q}}_{G}$ by

$$\lambda_1 \underset{G}{\leq} \lambda_2 \Leftrightarrow \lambda_2 - \lambda_1 \in \Lambda^{pos,\mathbb{Q}}_{G}.$$

The following useful observation is due to S. Schieder:

Lemma 7.1.7. *For a parabolic P, the projection pr_P is order-preserving.*

For a proof, see [Sch, Proposition 3.1.2(a)].

7.2. The degree of a bundle

Fix a connected smooth complete curve X. For any algebraic group H let Bun_H denote the stack of H-bundles on X.

7.2.1. One has a canonical isomorphism $\deg : \pi_0(\mathrm{Bun}_{\mathbb{G}_m}) \xrightarrow{\sim} \mathbb{Z}$. Accordingly, for any torus T one has a canonical isomorphism $\deg_T : \pi_0(\mathrm{Bun}_T) \xrightarrow{\sim} \Lambda_T$.

7.2.2. Let $\widetilde{G}$ be any connected affine algebraic group and let $\widetilde{G}_{\mathrm{tor}}$ be its maximal quotient torus. The composition

$$\pi_0(\mathrm{Bun}_{\widetilde{G}}) \to \pi_0(\mathrm{Bun}_{\widetilde{G}_{\mathrm{tor}}}) \xrightarrow{\deg_{\widetilde{G}_{\mathrm{tor}}}} \Lambda_{\widetilde{G}_{\mathrm{tor}}}$$

will be denoted by $\deg_{\widetilde{G}}$.

If $\widetilde{G} = G$ is reductive then $G_{\mathrm{tor}} = G/[G,G]$, and the map $Z_0(G) \to G_{\mathrm{tor}}$ is an isogeny, so $\Lambda^{\mathbb{Q}}_{G_{\mathrm{tor}}} \simeq \Lambda^{\mathbb{Q}}_{Z_0(G)}$. Therefore one has a locally constant map $\deg_G : \mathrm{Bun}_G \to \Lambda^{\mathbb{Q}}_{Z_0(G)}$. Its fibers are not necessarily connected but have finitely many connected components; this follows from Remark 7.2.4 below.

7.2.3. Let now P be a parabolic subgroup of a reductive group G, and let M be the Levi quotient of P.

Then by Sects. 7.1.3 and 7.2.2, one has the locally constant maps $\deg_M : \mathrm{Bun}_M \to \Lambda^{\mathbb{Q}}_{G,P}$ and therefore $\deg_P : \mathrm{Bun}_P \to \Lambda^{\mathbb{Q}}_{G,P}$.

The preimage of $\lambda \in \Lambda^{\mathbb{Q}}_{G,P}$ in Bun_M (resp. Bun_P) is denoted by Bun^{λ}_M (resp. Bun^{λ}_P).

It is easy to see that Bun^{λ}_M and Bun^{λ}_P are empty unless λ belongs to a certain finitely generated subgroup $A_{G,P} \subset \Lambda^{\mathbb{Q}}_{G,P}$ such that $A_{G,P} \otimes \mathbb{Q} = \Lambda^{\mathbb{Q}}_{G,P}$; namely, $A_{G,P} = \mathrm{pr}_P(\Lambda_G)$, where $\mathrm{pr}_P : \Lambda^{\mathbb{Q}}_G \to \Lambda^{\mathbb{Q}}_{G,P}$ is as in Sect. 7.1.5.

7.2.4. Remark. Let $\widetilde{G}$ be any connected affine algebraic group and $\widetilde{G}_{\mathrm{red}}$ its maximal reductive quotient. Define $\pi_1(\widetilde{G})$ to be the quotient of $\Lambda_{\widetilde{G}_{\mathrm{red}}}$ by the subgroup generated by coroots. It is well known that there is a unique bijection $\pi_0(\mathrm{Bun}_{\widetilde{G}}) \xrightarrow{\sim} \pi_1(\widetilde{G})$ such that the diagram

$$\begin{tikzcd}
\pi_0(\mathrm{Bun}_{\widetilde{B}}) \arrow[r] \arrow[d] & \Lambda_{\widetilde{T}} \simeq \Lambda_{\widetilde{G}_{\mathrm{red}}} \arrow[d] \\
\pi_0(\mathrm{Bun}_{\widetilde{G}}) \arrow[r] & \pi_1(\widetilde{G})
\end{tikzcd}$$

commutes. Here $\widetilde{B}$ is a Borel subgroup of $\widetilde{G}$ and $\widetilde{T}$ is the maximal quotient torus of $\widetilde{B}$.

7.3. Semistability

7.3.1. Let G_{ad} denote the quotient of G by its center and

$$\Upsilon_G : \Lambda_G^{\mathbb{Q}} \to \Lambda_{G_{ad}}^{\mathbb{Q}},$$

the projection.

Let $\mathsf{p}_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ be the natural morphism. Recall that a G-bundle $\mathcal{P}_G \in \mathrm{Bun}_G$ is called *semi-stable* if for every parabolic P such that $\mathcal{P}_G = \mathsf{p}_P(\mathcal{P}_P)$ with $\mathcal{P}_P \in \mathrm{Bun}_P^{\mu}$ we have

$$\Upsilon_G(\mu) \underset{G_{ad}}{\leq} 0.$$

In fact, semi-stability can be tested just using reductions to the Borel:

Lemma 7.3.2. *A G-bundle $\mathcal{P}_G$ is semi-stable if and only if for every reduction $\mathcal{P}_B$ of $\mathcal{P}_G$ to the Borel B with $\mathcal{P}_B \in \mathrm{Bun}_B^{\mu}$, we have $\Upsilon_G(\mu) \underset{G_{ad}}{\leq} 0$.*

Proof. This follows from Lemma 7.1.7 and the fact that every M-bundle admits a reduction to the Borel of M. □

It is known that semi-stable bundles form an open substack $\mathrm{Bun}_G^{ss} \subset \mathrm{Bun}_G$, whose intersection with each connected component of Bun_G is quasi-compact.

7.3.3. More generally, for $\theta \in \Lambda_G^{+,\mathbb{Q}}$ and a G-bundle $\mathcal{P}_G$, we say that $\mathcal{P}_G$ has *Harder-Narasimhan coweight* $\underset{G}{\leq} \theta$ if for every parabolic P such that $\mathcal{P}_G = \mathsf{p}_P(\mathcal{P}_P)$ with $\mathcal{P}_P \in \mathrm{Bun}_P^{\mu}$ we have

$$\mu \underset{G}{\leq} \theta.$$

As in Lemma 7.3.2, it suffices to check this condition for $P = B$.

7.3.4. One shows that G-bundles having Harder-Narasimhan coweight $\underset{G}{\leq} \theta$ form an open substack of Bun_G. The argument repeats the proof of the fact that Bun_G^{ss} is open, given in [Sch, Proposition 6.1.6].

We denote the above open substack by $\mathrm{Bun}_G^{(\leq\theta)}$ and sometimes by $\mathrm{Bun}_G^{(\underset{G}{\leq}\theta)}$. It lies in the (finite) union of connected components of Bun_G corresponding to the image of θ under

$$\Lambda_G^{\mathbb{Q}} \to \Lambda_{G,G}^{\mathbb{Q}} \simeq \Lambda_{Z_0(G)}^{\mathbb{Q}}.$$

Furthermore,

$$\theta_1 \underset{G}{\leq} \theta_2 \Rightarrow \mathrm{Bun}_G^{(\leq\theta_1)} \subset \mathrm{Bun}_G^{(\leq\theta_2)},$$

and

$$\bigcup_{\theta\in\Lambda_G^{+,\mathbb{Q}}} \mathrm{Bun}_G^{(\leq\theta)} = \mathrm{Bun}_G\,.$$

Finally, we have:

Proposition 7.3.5. *The open substack* $\mathrm{Bun}_G^{(\leq\theta)}$ *is quasi-compact.*

We will give two proofs:

Proof 1. With no loss of generality, we can assume that G is of adjoint type. We will use the relative compactification $\overline{\mathsf{p}}_B : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ of the map $\mathsf{p}_B : \mathrm{Bun}_B \to \mathrm{Bun}_G$, see Sect. 7.5.5.

For each connected component $'\mathrm{Bun}_G \subset \mathrm{Bun}_G$ choose a coweight $\lambda \in -\Lambda_G^+$ such that the map $\mathsf{p}_B : \mathrm{Bun}_B^\lambda \to \mathrm{Bun}_G$ lands in $'\mathrm{Bun}_G$ and is smooth (for smoothness, it is enough to take λ so that $\langle\lambda,\check\alpha_i\rangle < -(2g-2)$ for each simple root $\check\alpha_i$). Then the map $\overline{\mathsf{p}}_B : \overline{\mathrm{Bun}}_B^\lambda \to {}'\mathrm{Bun}_G$ is surjective.

It is a basic property of $\overline{\mathrm{Bun}}_B$ (see [Sch, Sect. 6.1.4]) that

$$\overline{\mathsf{p}}_B(\overline{\mathrm{Bun}}_B^\lambda) = \bigcup_{\mu\in\Lambda_G^{pos}} \mathsf{p}_B(\mathrm{Bun}_B^{\lambda+\mu}).$$

Therefore

$$'\mathrm{Bun}_G \cap \mathrm{Bun}_G^{(\leq\theta)} = \bigcup_{\mu\in\Lambda_G^{pos},\, \lambda+\mu\underset{G}{\leq}\theta} \mathsf{p}_B(\mathrm{Bun}_B^{\lambda+\mu}).$$

However, the set

$$\{\mu \in \Lambda_G^{pos} \mid \lambda+\mu \underset{G}{\leq} \theta\}$$

is finite. Hence, $'\mathrm{Bun}_G \cap \mathrm{Bun}_G^{(\leq\theta)}$ is contained in the image of finitely many quasi-compact stacks $\mathrm{Bun}_B^{\lambda+\mu}$, and hence is itself quasi-compact. $\square$

The second proof will be given after Corollary 7.4.5.

7.3.6. By definition, for $\lambda \in \Lambda^{\mathbb{Q}}_{G,G} = \Lambda^{\mathbb{Q}}_{Z_0(G)}$

$$\mathrm{Bun}_G^{ss} \cap \mathrm{Bun}_G^{\lambda} = \mathrm{Bun}_G^{(\leq\lambda)}$$

and

$$\mathrm{Bun}_G^{ss} = \bigcup_{\lambda \in \Lambda^{\mathbb{Q}}_{G,G}} \mathrm{Bun}_G^{(\leq\lambda)}.$$

7.3.7. For each parabolic $P \subset G$ with Levi quotient M we have the corresponding open substack $\mathrm{Bun}_M^{ss} \subset \mathrm{Bun}_M$; let Bun_P^{ss} denote the preimage of Bun_M^{ss} in Bun_P.

For $\lambda \in \Lambda^{\mathbb{Q}}_{G,P}$ we let

$$\mathrm{Bun}_M^{\lambda,ss} := \mathrm{Bun}_M^{ss} \cap \mathrm{Bun}_M^{\lambda} = \mathrm{Bun}_M^{(\underset{M}{\leq}\lambda)}, \ \mathrm{Bun}_P^{\lambda,ss} := \mathrm{Bun}_P^{ss} \cap \mathrm{Bun}_P^{\lambda}.$$

7.4. The Harder-Narasimhan-Shatz stratification of Bun_G

This stratification was defined in [HN, Sh1, Sh2] in the case $G = GL(n)$. For any reductive G it was defined in [R1, R2, R3] and [Beh, Beh1].

7.4.1. We give the following definition:

Definition 7.4.2. *A schematic morphism of algebraic stacks $f : \mathcal{X}_1 \to \mathcal{X}_2$ is an* almost-isomorphism *if f is finite and each geometric fiber of f has a single point.*

The next theorem is a basic result of reduction theory.

Theorem 7.4.3. (1) *Let $\lambda \in \Lambda^{+,\mathbb{Q}}_G$ and let $P \subset G$ be the unique parabolic such that λ belongs to the set $\Lambda^{++,\mathbb{Q}}_{G,P}$ defined by* (7.1). *Then $\mathsf{p}_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ induces an almost-isomorphism between $\mathrm{Bun}_P^{\lambda,ss}$ and a quasi-compact locally closed reduced substack $\mathrm{Bun}_G^{(\lambda)} \subset \mathrm{Bun}_G$.*

($1'$) *If k has characteristic 0 then the morphism $\mathrm{Bun}_P^{\lambda,ss} \to \mathrm{Bun}_G^{(\lambda)}$ is an isomorphism.*

(2) *The substacks $\mathrm{Bun}_G^{(\lambda)}$, $\lambda \in \Lambda^{+,\mathbb{Q}}_G$, are pairwise non-intersecting, and every geometric point of Bun_G belongs to exactly one $\mathrm{Bun}_G^{(\lambda)}$.*

(3) *Let $P' \subset G$ be a parabolic and let λ' be any (not necessarily dominant) element of $\Lambda^{\mathbb{Q}}_{G,P'}$. If $\mathsf{p}_{P'}(\mathrm{Bun}_{P'}^{\lambda'}) \cap \mathrm{Bun}_G^{(\lambda)} \neq \emptyset$ then $\lambda' \underset{G}{\leq} \lambda$.*

Statements (1), ($1'$), (2), and a slightly weaker version of (3) are due to K. Behrend [Beh, Beh1]. A complete proof of the theorem was given by

S. Schieder, see [Sch, Theorem 2.3.3]. In Sect. 7.5 we give a sketch of the proof from [Sch].

7.4.4. We apply Theorem 7.4.3 to obtain the following more explicit description of the open substacks $\mathrm{Bun}_G^{(\leq\theta)}$:

Corollary 7.4.5. *For $\theta \in \Lambda_G^{+,\mathbb{Q}}$ we have:*

$$\mathrm{Bun}_G^{(\leq\theta)} = \bigcup_{\lambda,\, \lambda \underset{G}{\leq} \theta} \mathrm{Bun}_G^{(\lambda)}, \tag{7.3}$$

and the set

$$\{\lambda \in \Lambda_G^{+,\mathbb{Q}} \,|\, \lambda \underset{G}{\leq} \theta \text{ and } \mathrm{Bun}_G^{(\lambda)} \neq \emptyset\} \tag{7.4}$$

is finite.

Proof. The fact that

$$\mathrm{Bun}_G^{(\lambda)} \cap \mathrm{Bun}_G^{(\leq\theta)} \neq \emptyset \Rightarrow \lambda \underset{G}{\leq} \theta$$

follows from the definition of $\mathrm{Bun}_G^{(\leq\theta)}$.

The inclusion

$$\mathrm{Bun}_G^{(\lambda)} \subset \mathrm{Bun}_G^{(\leq\theta)}$$

for $\lambda \underset{G}{\leq} \theta$ follows from Theorem 7.4.3(3).

This proves (7.3) in view of Theorem 7.4.3(2). The finiteness of the set (7.4) follows from the fact that

$$\mathrm{Bun}_G^{(\lambda)} \neq \emptyset \Rightarrow \lambda \in \bigcup_P A_{G,P}\ ,$$

see the end of Sect. 7.2.3. □

As a corollary, we obtain a 2nd proof of Proposition 7.3.5:

Proof 2 (of Proposition 7.3.5). Follows from Corollary 7.4.5 and the fact that each $\mathrm{Bun}_G^{(\lambda)}$ is quasi-compact.[11] □

As another corollary of Corollary 7.4.5 we obtain:

[11] The quasi-compactness of $\mathrm{Bun}_G^{(\lambda)}$ relied on the fact that the open substack $\mathrm{Bun}_M^{\lambda,ss}$ is quasi-compact, which in itself is a particular case of Proposition 7.3.5.

Corollary 7.4.6. *We have:*

$$\mathrm{Bun}_G^{(\theta)} = \mathrm{Bun}_G^{(\leq\theta)} - \bigcup_{\theta', \theta\neq\theta' \underset{G}{\leq} \theta} \mathrm{Bun}_G^{(\leq\theta')}. \tag{7.5}$$

Remark 7.4.7. We could *a priori* define the locally closed substacks $\mathrm{Bun}_G^{(\theta)}$ by formula (7.5). However, without the interpretation of $\mathrm{Bun}_G^{(\theta)}$ via Theorem 7.4.3, it would not be clear that these locally closed substacks are pairwise non-intersecting.

7.4.8. The Harder-Narasimhan map. Let $|\mathrm{Bun}_G(k)|$ denote the set of isomorphism classes of G-bundles on X (or equivalently, of objects of the groupoid $\mathrm{Bun}_G(k)$). We equip $|\mathrm{Bun}_G(k)|$ with the Zariski topology.

By Theorem 7.4.3(2), for every $\mathcal{F} \in \mathrm{Bun}_G(k)$ there exists a unique $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ such that $\mathcal{F} \in \mathrm{Bun}_G^{(\lambda)}(k)$. This λ is called the *Harder-Narasimhan coweight*[12] of $\mathcal{F}$ and denoted by $\mathrm{HN}(\mathcal{F})$. Thus we have a map

$$\mathrm{HN} : |\mathrm{Bun}_G(k)| \to \Lambda_G^{+,\mathbb{Q}}. \tag{7.6}$$

Lemma 7.4.9. *The map* (7.6) *has the following properties.*

(i) *It is upper-semicontinuous, i.e., for each $\lambda_0 \in \Lambda_G^{+,\mathbb{Q}}$ the preimage of the subset*

$$\{\lambda \in \Lambda_G^{+,\mathbb{Q}} \,|\, \lambda \leq \lambda_0\} \tag{7.7}$$

is open.

(ii) *The image of the map* (7.6) *is discrete in the real vector space $\Lambda_G^{\mathbb{R}} := \Lambda_G \otimes \mathbb{R}$.*

(iii) *A subset $S \subset |\mathrm{Bun}_G(k)|$ is quasi-compact if and only if $\mathrm{HN}(S)$ is bounded in $\Lambda_G^{\mathbb{R}}$.*

Proof. Follows from Corollary 7.4.5 and the fact that the substacks $\mathrm{Bun}_G^{(\leq\theta)}$ are open and quasi-compact. ☐

7.4.10. Let us equip the set $\Lambda_G^{+,\mathbb{Q}}$ with the *order topology*, i.e., the one whose base is formed by subsets of the form (7.7). Then statement (i) of Lemma 7.4.9 can be reformulated as follows: the map (7.6) is *continuous*.

[12]By Corollary 7.4.6, this agrees with the usage of the words "Harder-Narasimhan coweight" in Sect. 7.3.4.

Now it is clear that if a subset of $\Lambda_G^{+,\mathbb{Q}}$ is locally closed then so is its preimage in Bun_G. Note that for a subset of $\Lambda_G^{+,\mathbb{Q}}$ it is easy to understand whether it is open, closed, or locally closed, see Lemma A.1.1 from Appendix A. Thus we obtain:

Corollary 7.4.11. *Let $S \subset \Lambda_G^{+,\mathbb{Q}}$ be a subset. Consider the corresponding subset*

$$\mathrm{Bun}_G^{(S)} := \bigcup_{\lambda \in S} \mathrm{Bun}_G^{(\lambda)} \subset \mathrm{Bun}_G .$$

(a) *If S has the property that $\lambda_1 \in S$, $\lambda_1 \underset{G}{\leq} \lambda \Rightarrow \lambda \in S$, then $\mathrm{Bun}_G^{(S)}$ is closed in Bun_G.*
(b) *If S has the property that $\lambda_1 \in S$, $\lambda \underset{G}{\leq} \lambda_1 \Rightarrow \lambda \in S$, then $\mathrm{Bun}_G^{(S)}$ is open in Bun_G.*
(c) *If S has the property that $\lambda_1, \lambda_2 \in S$, $\lambda_1 \underset{G}{\leq} \lambda \underset{G}{\leq} \lambda_2 \Rightarrow \lambda \in S$, then $\mathrm{Bun}_G^{(S)}$ is locally closed in Bun_G.*

In cases (a) and (c) of the lemma we will regard $\mathrm{Bun}_G^{(S)}$ as a substack of Bun_G with the reduced structure.

7.5. On the proof of Theorem 7.4.3

Let us make some remarks regarding the proof of Theorem 7.4.3. For a full proof along these lines see [Sch].

7.5.1. For a G-bundle $\mathcal{P}_G$ let λ be *a*[13] maximal element in $\Lambda_G^{\mathbb{Q}}$, with respect to the $\underset{G}{\leq}$ order relation, such that there exists a parabolic P and $\mathcal{P}_P \in \mathrm{Bun}_P^{\lambda}$ such that $\mathcal{P}_G = \mathsf{p}_P(\mathcal{P}_P)$. One shows using Lemma 7.1.7 that the maximality assumption on λ implies that $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ and that $\mathcal{P}_P \in \mathrm{Bun}_P^{\lambda,ss}$. For details see [Sch, Sect. 6.2].

7.5.2. Using Bruhat decomposition, one shows (see [Sch, Theorem 4.5.1]) that if P' is another parabolic and $\mathcal{P}_{P'} \in \mathrm{Bun}_{P'}^{\lambda'}$ such that $\mathcal{P}_G = \mathsf{p}_{P'}(\mathcal{P}_{P'})$, then $\lambda' \underset{G}{\leq} \lambda$, and the equality takes place if and only if $P' \subset P$ and $\mathcal{P}_P$ is induced from $\mathcal{P}_{P'}$ via the above inclusion.

7.5.3. We obtain that the set of *maximal elements* λ as in Sect. 7.5.1 contains a single element. Moreover, the set of parabolics as in Sect. 7.5.1

[13] The "a" is italicized because we do not yet know that such a maximal element is unique, although we will eventually show that it is.

also contains a unique maximal element P; namely, one for which $\lambda \in \Lambda^{++,\mathbb{Q}}_{G,P}$.

7.5.4. This establishes points (2) and (3) of the theorem, modulo the fact that $\mathrm{Bun}_G^{(\lambda)}$ is locally closed, and not just constructible.

7.5.5. Let λ and P be as in Sect. 7.5.3. To prove point (1), one uses the relative compactification

$$\overline{\mathsf{p}}_P : \overline{\mathrm{Bun}}_P \to \mathrm{Bun}_G$$

of the map $\mathrm{Bun}_P \to \mathrm{Bun}_G$ defined in [BG, Sect. 1.3.2] under the assumption that $[G, G]$ is simply connected and in [Sch, Sect. 7] for an arbitrary reductive G. Since $\overline{\mathsf{p}}_P$ is proper, the images of $\overline{\mathrm{Bun}}_P^{\lambda}$ and $\overline{\mathrm{Bun}}_P^{\lambda} - \mathrm{Bun}_P^{\lambda,ss}$ in Bun_G are both closed. Using Sect. 7.5.2, one shows that the latter does not intersect $\mathrm{Bun}_G^{(\lambda)}$. This implies that p defines a finite map from $\mathrm{Bun}_P^{\lambda,ss}$ to a locally closed substack of Bun_G. It is bijective at the level of k-points by Sect. 7.5.2. See [Sch, Sect. 6.2.2] for details.[14]

7.5.6. Once (1) is proved, statement (1′) is equivalent to the fact that the map $\mathrm{Bun}_P^{\lambda,ss} \to \mathrm{Bun}_G$ is a monomorphism (on S-points for any scheme S). This is proved (see [Sch, Prop. 5.2.1]) using the fact that in characteristic 0, a homomorphism of reductive groups $G_1 \to G_2$ that sends $Z_0(G_1)$ to $Z_0(G_2)$ sends $\mathrm{Bun}^{ss}_{G_1}$ to $\mathrm{Bun}^{ss}_{G_2}$, see [Sch, Prop. 5.2.1] for details. (We will use a similar argument in the proof of Proposition 9.2.2(a) given in Sect. 10).

8. Complements to reduction theory: P-admissible sets

In this section we fix a parabolic $P \subset G$. Let M be the corresponding Levi.

Our goal is to prove Proposition 8.3.3, which allows us to produce locally closed substacks of Bun_G from locally closed substacks of Bun_M.

8.1. Some elementary geometry

Instead of reading the proofs of Lemmas 8.1.2 and 8.1.4 below, the reader may prefer to check the statements in the rank 2 case by drawing the picture, and believe that the statements are true in general.

8.1.1. Recall that according to the definitions from Sect. 7.1.2, we have $\Lambda^{\mathbb{Q}}_G = \Lambda^{\mathbb{Q}}_M$ and $\Lambda^{+,\mathbb{Q}}_G \subset \Lambda^{+,\mathbb{Q}}_M$.

[14] The latter part of the argument will actually be carried out in a slightly more general situation in the proof of Proposition 8.3.3.

Lemma 8.1.2. *Let* $\lambda, \lambda' \in \Lambda^{\mathbb{Q}}_G = \Lambda^{\mathbb{Q}}_M$ *and* $\lambda' \underset{M}{\leq} \lambda$. *Then*

(a) $\langle \lambda' , \check{\alpha}_i \rangle \geq \langle \lambda , \check{\alpha}_i \rangle$ *for* $i \notin \Gamma_M$*;*
(b) *If* $\lambda \in \Lambda^{+,\mathbb{Q}}_G$ *and* $\lambda' \in \Lambda^{+,\mathbb{Q}}_M$ *then* $\lambda' \in \Lambda^{+,\mathbb{Q}}_G$.

Proof. Statement (a) follows from the inequality $\langle \alpha_j , \check{\alpha}_i \rangle \leq 0$ for $i \neq j$.

To prove (b), we have to show that $\langle \lambda' , \check{\alpha}_i \rangle \geq 0$ for all $i \in \Gamma_G$. If $i \in \Gamma_M$ this follows from the assumption that $\lambda' \in \Lambda^{+,\mathbb{Q}}_M$. If $i \notin \Gamma_M$ this follows from (a) and the assumption that $\lambda \in \Lambda^{+,\mathbb{Q}}_G$. □

8.1.3. In Sect. 7.1.3–7.1.5 we defined the subspace $\Lambda^{\mathbb{Q}}_{G,P} \subset \Lambda^{\mathbb{Q}}_G$ and the projector

$$\mathrm{pr}_P : \Lambda^{\mathbb{Q}}_G \to \Lambda^{\mathbb{Q}}_{G,P}.$$

Lemma 8.1.4. *If* $\lambda \in \Lambda^{+,\mathbb{Q}}_G$ *then*

$$\mathrm{pr}_P(\lambda) \underset{M}{\leq} \lambda, \tag{8.1}$$

$$\langle \mathrm{pr}_P(\lambda) , \check{\alpha}_i \rangle \geq \langle \lambda , \check{\alpha}_i \rangle \text{ for } i \notin \Gamma_M\,. \tag{8.2}$$

Proof. On the one hand, for $i \in \Gamma_M$, one has $\langle \lambda - \mathrm{pr}_P(\lambda) , \check{\alpha}_i \rangle = \langle \lambda , \check{\alpha}_i \rangle \geq 0$. On the other hand, $\lambda - \mathrm{pr}_P(\lambda)$ belongs to the subspace generated by the coroots of M. Thus $\lambda - \mathrm{pr}_P(\lambda)$ is in the dominant cone of the root system of M. The latter is contained in $\Lambda^{pos,\mathbb{Q}}_M$, so we get (8.1).

The inequality (8.2) follows from (8.1) by Lemma 8.1.2(a). □

8.2. *P*-admissible subsets of $\Lambda^{+,\mathbb{Q}}_G$

8.2.1. Let S be a subset of $\Lambda^{+,\mathbb{Q}}_G$, and let P be a parabolic.

Definition 8.2.2. *We say that S is P-*admissible *if the following three properties hold:*

$$\text{There exists } \mu \in \Lambda^{\mathbb{Q}}_{G,P} \text{ such that } S \subset \mathrm{pr}_P^{-1}(\mu) \cap \Lambda^{+,\mathbb{Q}}_G. \tag{8.3}$$

$$\text{If } \lambda_1 \in S \text{ and } \lambda_2 \in \Lambda^{+,\mathbb{Q}}_G,\ \lambda_2 \underset{M}{\leq} \lambda_1 \text{ then } \lambda_2 \in S. \tag{8.4}$$

$$\forall \lambda \in S,\ \forall i \in \Gamma_G - \Gamma_M \text{ we have } \langle \lambda , \check{\alpha}_i \rangle > 0. \tag{8.5}$$

Remark 8.2.3. If $S \neq \emptyset$ is P-admissible and $\mathrm{pr}_P(S) = \mu \in \Lambda^{\mathbb{Q}}_{G,P}$ then

$$\mu \in \Lambda^{++,\mathbb{Q}}_{G,P} \subset \Lambda^{+,\mathbb{Q}}_G,$$

where, as before,

$$\Lambda^{++,\mathbb{Q}}_{G,P} := \{\lambda \in \Lambda^{\mathbb{Q}}_G \mid \langle \lambda, \check{\alpha}_i\rangle = 0 \ \text{ for } \ i \in \Gamma_M \ \text{ and } \ \langle \lambda, \check{\alpha}_i\rangle > 0 \ \text{ for } \ i \notin \Gamma_M\}.$$

This follows from (8.2) and (8.5).

8.2.4. Examples.

(i) The subset of $\mathrm{pr}_P^{-1}(\mu) \cap \Lambda^{+,Q}_G$, consisting of elements satisfying (8.5), is P-admissible.

(ii) If $\lambda \in \Lambda^{+,\mathbb{Q}}_G$ is such that $\langle \lambda\,, \check{\alpha}_i\rangle > 0$ for all $i \notin \Gamma_M$ then the set

$$S = \{\lambda' \in \Lambda^{+,\mathbb{Q}}_M \mid \lambda' \underset{M}{\leq} \lambda\}$$

is P-admissible by Lemma 8.1.2.

(iii) If $\mu \in \Lambda^{++,\mathbb{Q}}_{G,P}$ then the one-element set $\{\mu\}$ is P-admissible and moreover, it is the smallest non-empty P-admissible subset of $\mathrm{pr}_P^{-1}(\mu) \cap \Lambda^{+,\mathbb{Q}}_G$. This follows from (8.1).

8.2.5. Let $S \subset \Lambda^{+,\mathbb{Q}}_G$ be P-admissible subset. Note that we can also regard S as a subset of $\Lambda^{+,\mathbb{Q}}_M$.

Lemma 8.2.6. *The subset $S \subset \Lambda^{+,\mathbb{Q}}_M$ is open.*

Proof. Follows from Lemma 8.1.2(b). □

8.3. Reduction theory and P-admissible subsets

8.3.1. Let $S \subset \Lambda^{+,\mathbb{Q}}_G$ be a P-admissible subset.

By Corollary 7.4.11, the subset

$$\mathrm{Bun}_G^{(S)} := \bigcup_{\lambda \in S} \mathrm{Bun}_G^{(\lambda)} \subset \mathrm{Bun}_G \tag{8.6}$$

is locally closed (and thus we can regard it as a substack with the reduced structure).

By Lemma 8.2.6 and Corollary 7.4.11, the subset

$$\mathrm{Bun}_M^{(S)} := \bigcup_{\lambda \in S} \mathrm{Bun}_M^{(\lambda)} \subset \mathrm{Bun}_M$$

is open. Set

$$\mathrm{Bun}_P^{(S)} := \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(S)}.$$

8.3.2. The next proposition is a generalization of Theorem 7.4.3(1); the latter corresponds to the case where the P-admissible subset S has one element, see Example 8.2.4(iii).

Proposition 8.3.3. *Let $S \subset \Lambda_G^{+,\mathbb{Q}}$ be a P-admissible subset. Then the restriction of $\mathsf{p}_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ to $\mathrm{Bun}_P^{(S)}$ defines an almost-isomorphism*

$$\mathrm{Bun}_P^{(S)} \to \mathrm{Bun}_G^{(S)}\,. \tag{8.7}$$

Remark 8.3.4. Later we will show that if k has characteristic 0 then the map (8.7) is, in fact, an *isomorphism* (see Lemma 10.2.1 and Remark 10.2.2 below).

The rest of this section is devoted to the proof of Proposition 8.3.3.

8.3.5. First, let us prove that the map (8.7) maps $\mathrm{Bun}_P^{(S)}$ to $\mathrm{Bun}_G^{(S)}$ and is bijective at the level of k-points. To this end, it suffices to show that if λ is an element of

$$S \subset \Lambda_G^{+,\mathbb{Q}} \subset \Lambda_M^{+,\mathbb{Q}}$$

then the map p_P sends

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\lambda)}$$

bijectively to $\mathrm{Bun}_G^{(\lambda)}$.

Let M' be the Levi of G such that

$$\Gamma_{M'} = \{i \in \Gamma_G \,|\, \langle \lambda, \check{\alpha}_i \rangle = 0\}.$$

By condition (8.5), we have $\Gamma_{M'} \subset \Gamma_M$. Let $P' \subset P$ be the corresponding parabolic. Set $\underline{P}' := P'/U(P)$; this is a parabolic in M whose Levi quotient identifies with M'. We have

$$\lambda \in \Lambda_{G,P'}^{++,\mathbb{Q}} \subset \Lambda_{M,\underline{P}'}^{++,\mathbb{Q}}\,.$$

By the definition of $\mathrm{Bun}_M^{(\lambda)}$, we have a bijection

$$\mathrm{Bun}_{\underline{P}'} \underset{\mathrm{Bun}_{M'}}{\times} \mathrm{Bun}_{M'}^{\lambda,ss} \to \mathrm{Bun}_M^{(\lambda)}\,.$$

Hence, it is enough to show that the map

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{\underline{P}'} \underset{\mathrm{Bun}_{M'}}{\times} \mathrm{Bun}_{M'}^{\lambda,ss} \to \mathrm{Bun}_G$$

defines a bijection onto $\mathrm{Bun}_G^{(\lambda)}$.

However,

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{\underline{P'}} \underset{\mathrm{Bun}_{M'}}{\times} \mathrm{Bun}_{M'}^{\lambda,ss} \simeq \mathrm{Bun}_{P'} \underset{\mathrm{Bun}_{M'}}{\times} \mathrm{Bun}_{M'}^{\lambda,ss},$$

and the required assertion follows from the definition of $\mathrm{Bun}_G^{(\lambda)}$.

8.3.6. To finish the proof of the proposition, we have to show that the map (8.7) is finite. We already know that it is bijective, so it suffices to show the map (8.7) is proper. This will be done by generalizing the argument in Sect. 7.5.5 using the stack $\overline{\mathrm{Bun}}_P$.

Let $\mu \in \Lambda^{\mathbb{Q}}_{G,P}$ be such that $S \subset \mathrm{pr}_P^{-1}(\mu) \cap \Lambda_G^{+,\mathbb{Q}}$. Consider the map

$$\bar{\mathsf{p}}_P : \overline{\mathrm{Bun}}_P^{\mu} \to \mathrm{Bun}_G .$$

This map is proper. So to prove properness of (8.7), it is enough to show that

$$\bar{\mathsf{p}}_P(\overline{\mathrm{Bun}}_P^{\mu} - \mathrm{Bun}_P^{(S)}) \cap \mathrm{Bun}_G^{(S)} = \emptyset. \tag{8.8}$$

We have

$$\overline{\mathrm{Bun}}_P^{\mu} - \mathrm{Bun}_P^{(S)} = (\overline{\mathrm{Bun}}_P^{\mu} - \mathrm{Bun}_P^{\mu}) \cup (\mathrm{Bun}_P^{\mu} - \mathrm{Bun}_P^{(S)}),$$

so to prove (8.8), it suffices to show that

$$\bar{\mathsf{p}}_P(\overline{\mathrm{Bun}}_P^{\mu} - \mathrm{Bun}_P^{\mu}) \cap \mathrm{Bun}_G^{(\lambda)} = \emptyset \quad \text{for all } \lambda \in S \tag{8.9}$$

and

$$\mathsf{p}_P(\mathrm{Bun}_P^{\mu} - \mathrm{Bun}_P^{(S)}) \cap \mathrm{Bun}_G^{(\lambda)} = \emptyset \quad \text{for all } \lambda \in S. \tag{8.10}$$

8.3.7. To prove (8.9), we use the equality

$$\bar{\mathsf{p}}_P(\overline{\mathrm{Bun}}_P^{\mu} - \mathrm{Bun}_P^{\mu}) = \bigcup_{\mu' \in \Lambda^{\mathbb{Q}}_{G,P}\, \mu'-\mu \in \Lambda_G^{\mathrm{pos}} - 0} \mathsf{p}_P(\mathrm{Bun}_P^{\mu'}),$$

which follows from [Sch, Sect. 6.1.4]. This equality shows that if (8.9) were false we would have

$$\mathsf{p}_P(\mathrm{Bun}_P^{\mu'}) \cap \mathrm{Bun}_G^{(\lambda)} \neq \emptyset \tag{8.11}$$

for some $\mu' \in \Lambda^{\mathbb{Q}}_{G,P}$ such that

$$\mu \underset{G}{\leq} \mu' \neq \mu. \tag{8.12}$$

However, (8.11) implies, by Theorem 7.4.3(3), that $\mu' \underset{G}{\leq} \lambda$. So by Lemma 7.1.7,

$$\mu' \underset{G}{\leq} \mathrm{pr}_P(\lambda) = \mu,$$

which contradicts (8.12).

8.3.8. To prove (8.10), we have to show that if $\lambda' \in \Lambda_M^{+,\mathbb{Q}}$ is such that

$$\mathsf{p}_P(\mathrm{Bun}_P^\mu \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\lambda')}) \cap \mathrm{Bun}_G^{(\lambda)} \neq \emptyset \tag{8.13}$$

then $\lambda' \in S$.

If (8.13) holds then $\lambda' \underset{G}{\leq} \lambda$ by Theorem 7.4.3(3). Since $\mathrm{pr}_P(\lambda) = \mu = \mathrm{pr}_P(\lambda')$ this implies that $\lambda' \underset{M}{\leq} \lambda$.

Since $\lambda' \in \Lambda_M^{+,\mathbb{Q}}$ and $\lambda' \underset{M}{\leq} \lambda$ we get $\lambda' \in \Lambda_G^{+,\mathbb{Q}}$ by Lemma 8.1.2(b). Since $\lambda \in S$, $\lambda' \in \Lambda_G^{+,\mathbb{Q}}$, and $\lambda' \underset{M}{\leq} \lambda$ we get $\lambda' \in S$ by the admissibility of S.

9. Proof of the main theorem

9.1. The main result of this section

We wish to prove Theorem 4.1.8 (=Theorem 0.2.5), which says that Bun_G is truncatable.

9.1.1. For each $\theta \in \Lambda_G^{+,\mathbb{Q}}$, we have the quasi-compact open substack $\mathrm{Bun}_G^{(\leq\theta)} \subset \mathrm{Bun}_G$, see Sect. 7.3.4 and formula (7.3). The substacks $\mathrm{Bun}_G^{(\leq\theta)}$ cover Bun_G.

So Theorem 4.1.8 is a consequence of the following fact:

Theorem 9.1.2. *The substack* $\mathrm{Bun}_G^{(\leq\theta)}$ *is co-truncative if for every simple root* $\check{\alpha}_i$ *one has*

$$\langle \theta\, , \check{\alpha}_i \rangle \geq 2g-2, \tag{9.1}$$

where g *is the genus of* X.

In this section we will prove Theorem 9.1.2 modulo a key geometric assertion, Proposition 9.2.2.

Remark 9.1.3. In Theorem 9.1.2 we assume that the ground field k has characteristic 0 (because the notion of truncativeness is defined in terms of D-modules). However, Proposition 9.2.2 (of which Theorem 9.1.2 is an easy consequence) is valid over any k.[15]

Below follow some remarks on the proof of Theorem 9.1.2.

9.1.4. The main difficulty. In Sect. 6 we already proved Theorem 9.1.2 for $G = SL_2$. The proof in the general case is more or less similar.

However, one has to keep in mind the following. If $G = SL_2$ we saw that all but finitely many Harder-Narasimhan-Shatz strata $\mathrm{Bun}_G^{(\lambda)}$ are truncative. This is false already for $G = SL_2 \times SL_2$. Indeed, the stratum of the form $\mathrm{Bun}_{SL_2}^{(n)} \times \mathrm{Bun}_{SL_2}^{(m)}$, with n small relative to the genus of X, is *not* truncative in $\mathrm{Bun}_{SL_2} \times \mathrm{Bun}_{SL_2} = \mathrm{Bun}_{SL_2 \times SL_2}$ because $\mathrm{Bun}_{SL_2}^{(n)}$ is not truncative in Bun_{SL_2}, see Sect. 6.1.3.

For any G, it turns out that $\mathrm{Bun}_G^{(\lambda)}$ is truncative if λ is "deep inside" the interior of some face of the cone $\Lambda_G^{+,\mathbb{Q}}$; the problem arises if λ is close to the boundary of the open face of $\Lambda_G^{+,\mathbb{Q}}$ containing λ.

9.1.5. Resolving the difficulty. We prove that certain *unions* of the strata $\mathrm{Bun}_G^{(\mu)}$ are truncative (see Corollary 9.2.7). In particular, we show that if $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ and

$$S_\lambda := \{\mu \in \Lambda_G^{+,\mathbb{Q}} \mid \lambda - \mu \in \sum_{i \in \Gamma_{G,\lambda}} \mathbb{Q}^{\geq 0} \cdot \alpha_i\}, \tag{9.2}$$

where $\Gamma_{G,\lambda} := \{i \in \Gamma_G \mid \langle \lambda , \check{\alpha}_i \rangle \leq 2g-2\}$ then

$$\bigcup_{\mu \in S_\lambda} \mathrm{Bun}_G^{(\mu)}$$

is a truncative locally closed substack of Bun_G. To finish the proof of Theorem 9.1.2, we show that if θ satisfies (9.1) then the set

$$\{\lambda \in \Lambda_G^{+,\mathbb{Q}} \mid \lambda \underset{G}{\not\leq} \theta\}$$

can be represented as a union of subsets of the form (9.2).

[15] We have in mind future applications to the ℓ-adic derived category on Bun_G, and this category makes sense in any characteristic.

9.1.6. The rank 2 case is representative enough. All the difficulties of the proof of Theorem 9.1.2 appear already if G is a semi-simple group of rank 2. On the other hand, in this case various combinatorial-geometric statements (e.g., the above statement at the end of Sect. 9.1.5) become obvious once you draw a picture.

9.1.7. In Appendix B we give a variant of the proof of Theorem 9.1.2, which has some advantages compared with the one from Sect. 9.3. The relation between the two proofs is explained in Sects. 9.4 and B.4.

9.2. A key proposition

9.2.1. We will deduce Theorem 9.1.2 from the following assertion:

Proposition 9.2.2. *There exists an assignment*

$$i \in \Gamma_G \rightsquigarrow c_i \in \mathbb{Q}^{\geq 0}$$

such that for any parabolic P and any P-admissible subset $S \subset \Lambda_G^{+,\mathbb{Q}}$ satisfying the condition

$$\forall \lambda \in S,\ \forall i \in \Gamma_G - \Gamma_M\ \ \langle \lambda\,, \check{\alpha}_i \rangle > c_i \tag{9.3}$$

(where as usual, M is the Levi quotient of P) the following properties hold:

(a) *The morphism $\mathrm{Bun}_P^{(S)} \to \mathrm{Bun}_G^{(S)}$ induced by $\mathsf{p}_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ is an isomorphism;*

(b) *The locally closed substack $\mathrm{Bun}_G^{(S)} \subset \mathrm{Bun}_G$ is contractive in the sense of Sect. 5.2.1.*

When $\mathrm{char}\, k = 0$ we can take $c_i = \max(0, 2g - 2)$.

9.2.3. Proposition 9.2.2 will be proved in Sects. 10 and 11. Namely, in Sect. 10 we will produce the numbers c_i and prove property (a) for these numbers (and, in fact, for smaller ones). In Sect. 11 we will prove property (b).

Remark 9.2.4. It is only property (b) that will be needed for the proof of Theorem 9.1.2. Property (a) will be used for the proof of property (b).

Remark 9.2.5. Note that the assertion of point (a) of Proposition 9.2.2 differs from that of Proposition 8.3.3 only slightly: the former asserts "isomorphism", while the latter "almost-isomorphism".

9.2.6. We now specialize to the case of char $k = 0$, in which case we have the theory of D-modules and of truncative substacks (see Definition 3.4.1).

We have:

Corollary 9.2.7. *Let $S \subset \Lambda_G^{+,\mathbb{Q}}$ be a P-admissible subset such that*

$$\forall \lambda \in S,\ \forall i \in \Gamma_G - \Gamma_M \text{ we have } \langle \lambda\,, \check{\alpha}_i \rangle > 2g-2. \tag{9.4}$$

Then the locally closed substack $\mathrm{Bun}_G^{(S)} \subset \mathrm{Bun}_G$ is truncative.

Proof. Since S is admissible, the condition $\langle \lambda\,, \check{\alpha}_i \rangle > 2g-2$ from (9.4) is equivalent to the condition $\langle \lambda\,, \check{\alpha}_i \rangle > \max(0, 2g-2)$.

Now apply Proposition 9.2.2 with $c_i = \max(0, 2g-2)$, and the assertion follows from the fact that contractiveness implies truncativeness, see Corollary 5.2.3. □

9.3. Proof of Theorem 9.1.2

9.3.1. We have to show that if

$$\forall i \in \Gamma_G \text{ we have } \langle \theta\,, \check{\alpha}_i \rangle \geq 2g-2 \tag{9.5}$$

and $\theta' \underset{G}{\geq} \theta$ then the substack $\mathrm{Bun}_G^{(\leq \theta')} - \mathrm{Bun}_G^{(\leq \theta)} \subset \mathrm{Bun}_G$ is truncative.

If $g = 0$ then *any* locally closed substack of Bun_G is truncative (see Sect. 3.2.4). So we can and will assume that $g \geq 1$.

9.3.2. By Proposition 3.7.2, it suffices to cover $\mathrm{Bun}_G^{(\leq \theta')} - \mathrm{Bun}_G^{(\leq \theta)}$ by finitely many truncative substacks.

Let $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ be such that $\lambda \underset{G}{\not\leq} \theta$, i.e.,

$$\theta - \lambda \notin \Lambda_G^{pos,\mathbb{Q}} := \sum_{i \in \Gamma_G} \mathbb{Q}^{\geq 0} \cdot \alpha_i\,. \tag{9.6}$$

It suffices to construct for each such λ a subset $S_\lambda \subset \Lambda_G^{+,\mathbb{Q}}$ containing λ such that the substack $\mathrm{Bun}_G^{(S_\lambda)} \subset \mathrm{Bun}_G$ is truncative and

$$\mathrm{Bun}_G^{(\leq \theta)} \cap \mathrm{Bun}_G^{(S_\lambda)} = \emptyset. \tag{9.7}$$

Here is the construction. Let P be the parabolic whose Levi quotient, M, corresponds to the following subset of Γ_G:

$$\Gamma_M = \{ i \in \Gamma_G \,|\, \langle \lambda\,, \check{\alpha}_i \rangle \leq 2g-2 \}. \tag{9.8}$$

Now define

$$S_\lambda := \{\lambda' \in \Lambda_G^{+,\mathbb{Q}} \,|\, \lambda' \underset{M}{\leq} \lambda\}. \tag{9.9}$$

Note that by (9.8), for each $i \in \Gamma_G - \Gamma_M$ we have $\langle \lambda , \check{\alpha}_i \rangle > 2g - 2$, which implies that $\langle \lambda , \check{\alpha}_i \rangle > 0$ (because we are assuming that $g \geq 1$). So by Lemma 8.1.2(a), S_λ is P-admissible and satisfies the condition of Corollary 9.2.7. Hence, the substack $\mathrm{Bun}_G^{(S_\lambda)} \subset \mathrm{Bun}_G$ is truncative.

Therefore, to prove Theorem 9.1.2 it remains to check (9.7).

9.3.3. Proof of equality (9.7). We need the following lemma.

Lemma 9.3.4. *Let $\nu = \sum_{i\in\Gamma_G} a_i \cdot \alpha_i$, $a_i \in \mathbb{Q}$. Assume that $a_i \geq 0$ for $i \notin \Gamma_M$ and $\langle \nu , \check{\alpha}_i \rangle \geq 0$ for $i \in \Gamma_M$. Then $a_i \geq 0$ for all $i \in \Gamma_G$.*

Proof. Set $\nu_M := \sum_{i\in\Gamma_M} a_i \cdot \alpha_i$. We have to show that $\nu_M \in \Lambda_M^{pos,\mathbb{Q}}$. The assumptions of the lemma and the inequality $\langle \alpha_j , \check{\alpha}_i \rangle \leq 0$ for $i \neq j$ imply that $\langle \nu_M , \check{\alpha}_i \rangle \geq 0$ for $i \in \Gamma_M$. Thus ν_M belongs to the dominant cone of the root system of M and therefore to $\Lambda_M^{pos,\mathbb{Q}}$. □

We are now ready to prove (9.7). It suffices to prove the following

Lemma 9.3.5. *There is no $\lambda' \in \Lambda_G^{+,\mathbb{Q}}$ such that $\lambda' \underset{G}{\leq} \theta$ and $\lambda' \underset{M}{\leq} \lambda$.*

Proof. Suppose that such λ' exists. Then $\theta - \lambda$ has the form $\sum_{i\in\Gamma_G} c_i \cdot \alpha_i$, where $c_i \geq 0$ for $i \notin \Gamma_M$. By (9.5) and (9.8), $\langle \theta - \lambda , \check{\alpha}_i \rangle \geq 0$ for $i \in \Gamma_M$. Hence, by Lemma 9.3.4, $\theta - \lambda \in \Lambda_G^{pos,\mathbb{Q}}$, contrary to the assumption (9.6). □

Thus we have proved Theorem 9.1.2.

9.4. A variant of the proof

In the above proof of Theorem 9.1.2 we used substacks of the form $\mathrm{Bun}_G^{(S_\lambda)}$, $\lambda \in \Lambda_G^{+,\mathbb{Q}}$, where S_λ is defined by (9.8)–(9.9). Instead of considering *all* substacks of this form, one could consider only *maximal* ones among them; one can show that they form a *stratification* of Bun_G all of whose strata are truncative. This somewhat "cleaner" picture is explained in Appendix B.

10. The estimates

In this section we produce the numbers c_i mentioned in Proposition 9.2.2 and prove Proposition 9.2.2(a) for these numbers (and, in fact, for smaller ones).

10.1. The vanishing of H^0 and H^1

10.1.1. For what follows, we fix a maximal torus $T \subset B$. This allows us to view the Levi quotient M of a (standard) parabolic P as a subgroup $M \subset P$ (the unique splitting that contains T).

Recall that for a parabolic P we denote by $U(P)$ its unipotent radical. We will use the following notation for Lie algebras: $\mathfrak{g} := \mathrm{Lie}\,(G)$, $\mathfrak{p} := \mathrm{Lie}\,(P)$, $\mathfrak{n}(P) := \mathrm{Lie}\,(U(P))$.

For an algebraic group H, a principal H-bundle $\mathcal{F}_H$ and an H-representation V, we denote by $V_{\mathcal{F}_H}$ the associated vector bundle.

10.1.2. The main result of this section is:

Proposition 10.1.3. *There exists a collection of numbers*

$$c_i', \in \mathbb{Q}, c_i'' \in \mathbb{Q}^{\geq 0}, \quad i \in \Gamma_G$$

such that for any quadruple

$$(P, M, \lambda, \mathcal{F}_M),$$

where P is a parabolic, M the corresponding Levi, $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ and $\mathcal{F}_M \in \mathrm{Bun}_M^{(\lambda)}$, the following statements hold:

(10.1) *if* $\forall i \in \Gamma_G - \Gamma_M$ *we have* $\langle \lambda\,, \check{\alpha}_i \rangle > c_i'$ *then* $H^1(X, \mathfrak{n}(P)_{\mathcal{F}_M}) = 0$;

(10.2) *if* $\forall i \in \Gamma_G - \Gamma_M$ *we have* $\langle \lambda\,, \check{\alpha}_i \rangle > c_i''$ *then* $H^0(X, (\mathfrak{g}/\mathfrak{p})_{\mathcal{F}_M}) = 0$.

If $\mathrm{char}\, k = 0$ *then one can take* $c_i' = 2g - 2$, $c_i'' = 0$.

Proposition 10.1.3 will be proved in Sect. 10.4. For a discussion of the case $\mathrm{char}\, k > 0$, see Sect. 10.5.

10.2. The numbers c_i: proof of Proposition 9.2.2(a)

In this subsection we will assume Proposition 10.1.3.

Let c_i' and c_i'' be as in Proposition 10.1.3. Set

$$c_i := \max(c_i', c_i''). \tag{10.3}$$

Eventually we will show that Proposition 9.2.2 holds for the numbers c_i defined by (10.3). For Proposition 9.2.2(a) this follows from the next lemma (which is slightly sharper than Proposition 9.2.2(a) because the numbers c_i are replaced by $c_i'' \leq c_i$).

Lemma 10.2.1. *Let c_i'' be as in Proposition 10.1.3. Let $S \subset \Lambda_G^{+,\mathbb{Q}}$ be a P-admissible subset such that*

$$\forall \lambda \in S,\ \forall i \in \Gamma_G - \Gamma_M \quad \langle \lambda \,, \check{\alpha}_i \rangle > c_i'' \,. \tag{10.4}$$

Then the morphism $\mathrm{Bun}_P^{(S)} \to \mathrm{Bun}_G^{(S)}$ induced by $\mathsf{p}_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ is an isomorphism.

Remark 10.2.2. The lemma implies that if $\operatorname{char} k = 0$ then the morphism $\mathrm{Bun}_P^{(S)} \to \mathrm{Bun}_G^{(S)}$ is an isomorphism for *any* P-admissible $S \subset \Lambda_G^{+,\mathbb{Q}}$. Indeed, if $\operatorname{char} k = 0$ one can take $c_i'' = 0$ (see the last sentence of Proposition 10.1.3); on the other hand, for $c_i'' = 0$ the inequality (10.4) holds by the definition of P-admissibility, see Definition 8.2.2.

Proof of Lemma 10.2.1. By Proposition 8.3.3, it suffices to show that for any $y \in \mathrm{Bun}_P(k)$, the tangent space at y to the fiber of $\mathsf{p}_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ over $\mathsf{p}_P(y)$ is zero.

The tangent space in question identifies with $H^0(X, (\mathfrak{g}/\mathfrak{p})_{\mathcal{F}_P})$, where $\mathcal{F}_P$ is the P-bundle corresponding to y.

Note that the vector bundle $(\mathfrak{g}/\mathfrak{p})_{\mathcal{F}_M}$ can be identified with the associated graded of $(\mathfrak{g}/\mathfrak{p})_{\mathcal{F}_P}$ with respect to a (canonically defined) filtration on the latter. Hence, (10.2) implies that $H^0(X, (\mathfrak{g}/\mathfrak{p})_{\mathcal{F}_P}) = 0$. ☐

10.3. The notion of strangeness

10.3.1. Let $\widetilde{G}$ be a reductive group and V a finite-dimensional $\widetilde{G}$-module on which $Z_0(\widetilde{G})$ acts by a character $\check{\mu}$.

Lemma 10.3.2. (i) *There exists a number $c \in \mathbb{Q}$ such that for every $\mathcal{F}_{\widetilde{G}} \in \mathrm{Bun}_{\widetilde{G}}^{ss}$ the degree of any line sub-bundle of $V_{\mathcal{F}_{\widetilde{G}}}$ is $\leq \langle \deg_{\widetilde{G}}(\mathcal{F}_{\widetilde{G}}), \check{\mu} \rangle + c$.*

(ii) *If $\operatorname{char} k = 0$ one can take $c = 0$.*

Proof. Statement (i) follows from the fact that the intersection of $\mathrm{Bun}_{\widetilde{G}}^{ss}$ with every connected component of $\mathrm{Bun}_{\widetilde{G}}$ is quasi-compact, and that under the action of $\mathrm{Bun}_{Z_0(\widetilde{G})}$ on $\mathrm{Bun}_{\widetilde{G}}$, the number of orbits of $\pi_0(\mathrm{Bun}_{Z_0(\widetilde{G})})$ on $\pi_0(\mathrm{Bun}_{\widetilde{G}})$ is finite.

Statement (ii) follows from the fact that if $\operatorname{char} k = 0$ then for every $\mathcal{F}_{\widetilde{G}} \in \mathrm{Bun}_{\widetilde{G}}^{ss}$ the vector bundle $V_{\mathcal{F}_{\widetilde{G}}}$ is semistable of slope $\langle \deg_{\widetilde{G}}(\mathcal{F}_{\widetilde{G}}), \check{\mu} \rangle$ (a proof of this fact can be found in [RR, Sect. 3]; for references to other proofs see the introduction to [RR]). ☐

10.3.3. We give the following definition:

Definition 10.3.4. *The* strangeness $\mathrm{strng}_X(\widetilde{G}, V)$ *is the smallest number* $c \in \mathbb{Q}$ *having the property from Lemma 10.3.2(i).*

One always has $\mathrm{strng}_X(\widetilde{G}, V) \geq 0$ (because the trivial $\widetilde{G}$-bundle is semi-stable). If $\mathrm{char}\, k = 0$ then $\mathrm{strng}_X(\widetilde{G}, V) = 0$.

Remark 10.3.5. As before, let $\widetilde{G}$ be a reductive group, V a finite-dimensional $\widetilde{G}$-module on which $Z_0(\widetilde{G})$ acts by a character $\check{\mu}$, and $\mathcal{F} \in \mathrm{Bun}_{\widetilde{G}}^{ss}$. Then

$$H^0(X, V_{\mathcal{F}_{\widetilde{G}}}) = 0 \ \text{ if } \ \langle \deg_{\widetilde{G}}(\mathcal{F}_{\widetilde{G}}), \check{\mu} \rangle < -\,\mathrm{strng}_X(\widetilde{G}, V),$$
$$H^1(X, V_{\mathcal{F}_{\widetilde{G}}}) = 0 \ \text{ if } \ \langle \deg_{\widetilde{G}}(\mathcal{F}_{\widetilde{G}}), \check{\mu} \rangle > 2g - 2 + \mathrm{strng}_X(\widetilde{G}, V^*).$$

In particular, if $\mathrm{char}\, k = 0$ then

$$\begin{aligned} H^0(X, V_{\mathcal{F}}) = 0 \ &\text{ if } \ \langle \deg_{\widetilde{G}}(\mathcal{F}_{\widetilde{G}}), \check{\mu} \rangle < 0, \\ H^1(X, V_{\mathcal{F}}) = 0 \ &\text{ if } \ \langle \deg_{\widetilde{G}}(\mathcal{F}_{\widetilde{G}}), \check{\mu} \rangle > 2g - 2. \end{aligned} \tag{10.5}$$

10.4. Proof of Proposition 10.1.3

10.4.1. Let us introduce some notation. Let $P' \subset G$ be a parabolic and $M' \subset P'$ the corresponding Levi (see our conventions in Sect. 10.1.1); in particular $M' \supset T$.

Given a root $\check{\alpha}$ of G which is not a root of M', define an M'-submodule $V_{M',\check{\alpha}} \subset \mathfrak{g}$ by

$$V_{M',\check{\alpha}} := \bigoplus_{\check{\beta}, \check{\beta} - \check{\alpha} \in R(M')} \mathfrak{g}_{\check{\beta}}, \tag{10.6}$$

where $R(M')$ is the root lattice of M'.

The coefficient of $\check{\alpha}_i$ in a root $\check{\alpha}$ will be denoted by $\mathrm{coeff}_i(\check{\alpha})$.

10.4.2. We are going to formulate a slightly more precise version of Proposition 10.1.3.

Let

$$i \in \Gamma_G \rightsquigarrow c_i', c_i'' \in \mathbb{Q}$$

be numbers satisfying the following inequalities:

For every Levi subgroup $M' \subset G$, every $i \in \Gamma_G - \Gamma_{M'}$, and every root $\check{\alpha}$ of G such that $\mathrm{coeff}_i(\check{\alpha}) > 0$, we have:

$$\mathrm{coeff}_i(\check{\alpha}) \cdot c_i' \geq 2g - 2 + \mathrm{strng}_X(M', (V_{M',\check{\alpha}})^*). \tag{10.7}$$
$$\mathrm{coeff}_i(\check{\alpha}) \cdot c_i'' \geq \mathrm{strng}_X(M', V_{M',-\check{\alpha}}). \tag{10.8}$$

10.4.3. Remark. In the characteristic 0 case we can take $c'_i = 2g-2$ and $c''_i = 0$: indeed, in this case the numbers strng_X from formulas (10.7)–(10.8) are zero.

10.4.4. Here is the promised version of Proposition 10.1.3.

Proposition 10.4.5. *Let c'_i, c''_i be numbers satisfying the conditions from Sect. 10.4.2. Let $P \subset G$ be a parabolic, M be the corresponding Levi, $\lambda \in \Lambda_G^{+,\mathbb{Q}} \subset \Lambda_M^{+,\mathbb{Q}}$ and $\mathcal{F}_M \in \mathrm{Bun}_M^{(\lambda)}$.*

(a) *If $\langle \lambda , \check{\alpha}_i \rangle > c'_i$ for all $i \in \Gamma_G - \Gamma_M$ then $H^1(X, \mathfrak{n}(P)_{\mathcal{F}_M}) = 0$.*
(b) *If $\langle \lambda , \check{\alpha}_i \rangle > c''_i$ for all $i \in \Gamma_G - \Gamma_M$ then $H^0(X, (\mathfrak{g}/\mathfrak{p})_{\mathcal{F}_M}) = 0$.*

Proof. Let P_λ be the parabolic of M corresponding to the subset $\{i \in \Gamma_M \,|\, \langle \lambda , \check{\alpha}_i \rangle = 0\} \subset \Gamma_M$. Let M_λ be the corresponding Levi. The fact that $\mathcal{F}_M \in \mathrm{Bun}_M^{(\lambda)}$ means that $\mathcal{F}_M$ admits a reduction to P_λ such that the corresponding M_λ-bundle $\mathcal{F}_{M_\lambda}$ is semi-stable of degree λ.

Let us prove statement (a). Note that the vector bundle $\mathfrak{n}(P)_{\mathcal{F}_M}$ has a canonical filtration with the associated graded identified with $\mathfrak{n}(P)_{\mathcal{F}_{M_\lambda}}$. Hence, it suffices to show that $H^1(X, \mathfrak{n}(P)_{\mathcal{F}_{M_\lambda}}) = 0$.

By Remark 10.3.5, it suffices to prove that

$$\langle \lambda , \check{\alpha} \rangle > 2g-2 + \mathrm{strng}_X(M_\lambda, (V_{M_\lambda, \check{\alpha}})^*)$$

for any positive root $\check{\alpha}$ of G which is not a root of M.

Let $i \in \Gamma_G - \Gamma_M$ be such that $\mathrm{coeff}_i(\check{\alpha}) > 0$. Since λ is dominant for G and $\langle \lambda , \check{\alpha}_i \rangle > c'_i$ we have $\langle \lambda , \check{\alpha} \rangle \geq \mathrm{coeff}_i(\check{\alpha}) \cdot \langle \lambda , \check{\alpha}_i \rangle > \mathrm{coeff}_i(\check{\alpha}) \cdot c'_i$. Now use (10.7) for $M' = M_\lambda$ (this is possible because $i \notin \Gamma_M$ and therefore $i \notin \Gamma_{M_\lambda}$).

The proof of statement (b) is similar. $\square$

Remark 10.4.6. If k has characteristic $p > 0$ then Proposition 10.4.5 would become really useful if combined with good[16] upper bounds for the strangeness of the relevant representations. It would be interesting to obtain such bounds.

10.5. Remarks on the positive characteristic case

10.5.1. Let V_1 and V_2 be finite-dimensional vector spaces over a field of any characteristic. Let G_i denote the algebraic group $GL(V_i)$. Then the $(G_1 \times$

[16] A good upper bound should have the form $c(p, G, \check{\alpha}) \cdot (g-1)$. The number $c(p, G, \check{\alpha})$ should be independent of X and small enough to explain the phenomenon in Sect. 10.5.1 below.

G_2)-modules $\mathrm{Hom}(V_1, V_2)$ and $V_1 \otimes V_2$ have strangeness 0. This immediately follows from the definition of semi-stability.

Corollary 10.5.2. *Suppose that $G_{ad} \simeq PGL(d_1) \times \ldots \times PGL(d_n)$. Then all inequalities (10.7)–(10.8) hold for $c'_i = 2g-2, c''_i = 0$ (without any assumption on char k).*

10.5.3. Suppose that char $k = 2$. Let V be a 2-dimensional vector space over k. If $g > 1$ then the representation of $GL(V)$ in the symmetric square $\mathrm{Sym}^2(V)$ has strangeness $g-1 > 0$. This follows from Sect. 10.5.1, combined with the exact sequence

$$0 \to V^{(2)} \to \mathrm{Sym}^2(V) \to V \otimes V$$

and the equality $\mathrm{strng}_X(V^{(2)}) = g-1$, which is proved, e.g., in [JRXY, Sect. 4.5].

10.5.4. The assertion in Sect. 10.5.3 implies that if $g > 1$, char $k = 2$ and $G = Sp(2n)$, $n \geq 2$, then some of the inequalities (10.7) *do not* hold for $c'_i = 2g-2$.

10.5.5. The situation for the numbers c''_i is as follows:

J. Heinloth [He] proved that if G is a classical group over a field of odd characteristic then all inequalities (10.8) hold for $c''_i = 0$. He also showed that if char $k = 2$ this is still true if G_{ad} is a product of groups of type A and C.

On the other hand, according to [He, P], some of the inequalities (10.8) do not hold if char $k = 2$ and G_{ad} has one of the following types: G_2, B_n ($n \geq 3$), D_n ($n \geq 4$).

11. Constructing the contraction

The goal of this section is to prove point (b) of Proposition 9.2.2 for the numbers c_i defined in formula (10.3) from Sect. 10.2.

11.1. Morphisms between Bun_P, Bun_{P^-}, Bun_M, and Bun_G

In this subsection and the next we recall some well known facts that will be used in the proof of Proposition 9.2.2(b).

11.1.1. From now on we fix a (standard) parabolic P. Let P^- be the parabolic opposite to P such that $P^- \supset T$. Note that P^- is *not* a standard parabolic! Namely, P^- is the unique parabolic such that $P \cap P^- = M$, when the latter is viewed a subgroup of P, see Sect. 10.1.1.

Lemma 11.1.2. *The morphism* $\mathrm{Bun}_M \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P$ *is an open embedding.*

Proof. An M-bundle on X is the same as a G-bundle $\mathcal{F}_G$ equipped with an M-structure, i.e., a section of $(G/M)_{\mathcal{F}_G}$.

The assertion follows from the fact that the morphism $G/M \to G/P^- \times G/P$ is an open embedding. □

11.1.3. Define open substacks $\mathcal{U}_i \subset \mathrm{Bun}_M$ as follows:

$$\mathcal{U}_0 := \{\mathcal{F}_M \in \mathrm{Bun}_M \mid H^0(X, (\mathfrak{g}/\mathfrak{p})_{\mathcal{F}_M}) = 0\}, \tag{11.1}$$

$$\mathcal{U}_1 := \{\mathcal{F}_M \in \mathrm{Bun}_M \mid H^1(X, \mathfrak{n}(P)_{\mathcal{F}_M}) = 0\}. \tag{11.2}$$

Proposition 11.1.4. *We have:*

(a) *The morphism* $\iota_P : \mathrm{Bun}_M \to \mathrm{Bun}_P$ *induces a smooth surjective morphism*
$$\mathcal{U}_1 \to \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{U}_1.$$
(b) *The morphism* $\mathsf{p}_{P^-} : \mathrm{Bun}_{P^-} \to \mathrm{Bun}_G$ *is smooth when restricted to the open substack*
$$\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathcal{U}_1 \subset \mathrm{Bun}_{P^-}.$$
(c) *The morphism* $\mathsf{q}_{P^-} : \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M$ *is schematic, affine, and smooth over* $\mathcal{U}_0 \subset \mathrm{Bun}_M$.

(d) *The morphism* $\iota_{P^-} : \mathrm{Bun}_M \to \mathrm{Bun}_{P^-}$ *defines a closed embedding*
$$\mathcal{U}_0 \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathcal{U}_0.$$

Proof. Let $\mathcal{F}_M \in \mathcal{U}_1(k)$, i.e., $\mathcal{F}_M$ is an M-torsor on X such that $H^1(X, \mathfrak{n}(P)_{\mathcal{F}_M}) = 0$. Using an appropriate filtration on $U(P)$ one deduces from this that $H^1(X, U(P)_{\mathcal{F}_M}) = 0$, i.e., every $U(P)_{\mathcal{F}_M}$-torsor on X is trivial. This implies the surjectivity part of statement (a).

To prove the smoothness part of (a), it suffices to check that the differential of the morphism $\iota_P : \mathrm{Bun}_M \to \mathrm{Bun}_P$ at any point $\mathcal{F}_M \in \mathcal{U}_1(k) \subset \mathrm{Bun}_M(k)$ is surjective. Its cokernel equals $H^1(X, \mathfrak{n}(P)_{\mathcal{F}_M})$, which is zero by the definition of $\mathcal{U}_1$, see formula (11.2).

Note also that the smoothness part of (a) will follow from (b): to see this, decompose the morphism $\iota_P : \mathrm{Bun}_M \to \mathrm{Bun}_P$ as

$$\mathrm{Bun}_M \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P \to \mathrm{Bun}_P$$

and use Lemma 11.1.2.

To prove (b), we have to show that the differential of $\mathsf{p}_{P^-} : \mathrm{Bun}_{P^-} \to \mathrm{Bun}_G$ at any k-point y of $\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathcal{U}_1$ is surjective. Its cokernel equals $H^1(X, (\mathfrak{g}/\mathfrak{p}^-)_{\mathcal{F}_{P^-}})$, where $\mathcal{F}_{P^-}$ is the P^--bundle corresponding to y. Let $\mathcal{F}_M$ be the corresponding M-bundle. We have $H^1(X, \mathfrak{n}(P)_{\mathcal{F}_M}) = 0$ by the definition of $\mathcal{U}_1$, see formula (11.2). Now, the associated graded of $(\mathfrak{g}/\mathfrak{p}^-)_{\mathcal{F}_{P^-}}$ with respect to a (canonically defined) filtration identifies with $(\mathfrak{g}/\mathfrak{p}^-)_{\mathcal{F}_M}$, and the assertion follows from the fact that the composition

$$\mathfrak{n}(P) \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{p}^-$$

is an isomorphism of M-modules.

To prove (c), consider the filtration

$$U(P^-) = U^1 \supset U^2 \supset \dots,$$

where U^m is the subgroup generated by the root subgroups corresponding to the roots $\check{\alpha}$ of G such that

$$\sum_{i \notin \Gamma_M} \mathrm{coeff}_i(\check{\alpha}) \leq -m.$$

(Here $\mathrm{coeff}_i(\check{\alpha})$ denotes the coefficient of $\check{\alpha}_i$ in $\check{\alpha}$.) Note that each quotient U^m/U^{m+1} is a vector group (i.e., a product of finitely many copies of $\mathbb{G}_a$). To prove (c), it suffices to check that for each m the morphism

$$(\mathrm{Bun}_{P^-/U^m}) \underset{\mathrm{Bun}_M}{\times} \mathcal{U}_0 \to (\mathrm{Bun}_{P^-/U^{m+1}}) \underset{\mathrm{Bun}_M}{\times} \mathcal{U}_0 \tag{11.3}$$

is schematic, affine and smooth. In fact, it is a torsor over a certain vector bundle.

To see this, note that by (11.1), for each $\mathcal{F}_M \in \mathcal{U}_0$ we have

$$H^0(X, (U^m/U^{m+1})_{\mathcal{F}_M}) = 0,$$

so the stack of $(U^m/U^{m+1})_{\mathcal{F}_M}$-torsors on X is a scheme; namely, it is the vector space $H^1(X, (U^m/U^{m+1})_{\mathcal{F}_M})$. As $\mathcal{F}_M$ varies, these vector spaces form a vector bundle on $\mathcal{U}_0$. Let ξ denote its pullback to $(\mathrm{Bun}_{P^-/U^m}) \underset{\mathrm{Bun}_M}{\times} \mathcal{U}_0$, then the morphism (11.3) is a ξ-torsor.

Point (d) follows from point (c) since the map $\mathcal{U}_0 \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathcal{U}_0$ is a section of the map

$$\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathcal{U}_0 \to \mathcal{U}_0,$$

and the latter is schematic and separated. $\square$

11.2. The action of $\mathbb{A}^1$ on Bun_{P^-}

11.2.1. Let $Z(M)$ denote the center of M. Choose a homomorphism $\mu : \mathbb{G}_m \to Z(M)$ such that $\langle \mu , \check{\alpha}_i \rangle > 0$ for $i \notin \Gamma_M$. Then the action of $\mathbb{G}_m$ on P^- defined by

$$\rho_t(x) := \mu(t)^{-1} \cdot x \cdot \mu(t), \quad t \in \mathbb{G}_m , x \in P^- \tag{11.4}$$

extends to an action of the multiplicative monoid $\mathbb{A}^1$ on P^- such that the endomorphism $\rho_0 \in \mathrm{End}(P)$ equals the composition $P^- \twoheadrightarrow M \hookrightarrow P^-$.

11.2.2. The above action of $\mathbb{A}^1$ on P^- induces an $\mathbb{A}^1$-action on Bun_{P^-}. Equip M and Bun_M with the trivial $\mathbb{A}^1$-action. The projection $P^- \to M$ is $\mathbb{A}^1$-equivariant, so *the corresponding morphism* $\mathsf{q}_{P^-} : \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M$ *has a canonical* $\mathbb{A}^1$*-equivariant structure.*

Remark 11.2.3. The above description of ρ_0 implies that the morphism $\mathbf{0}$: $\mathrm{Bun}_{P^-} \to \mathrm{Bun}_{P^-}$ corresponding to $0 \in \mathbb{A}^1$ equals the composition

$$\mathrm{Bun}_{P^-} \xrightarrow{\mathsf{q}_{P^-}} \mathrm{Bun}_M \xrightarrow{\iota_{P^-}} \mathrm{Bun}_{P^-} . \tag{11.5}$$

Remark 11.2.4. The action of $\mathbb{G}_m$ on Bun_{P^-} is trivial: this follows from formula (11.4), which says that the automorphisms $\rho_t \in \mathrm{Aut}(P^-)$, $t \in \mathbb{G}_m$, are inner. Moreover, formula (11.4) provides a *canonical* trivialization of this action.

Remark 11.2.5. Despite the previous remark, it is *not true* that the action of $\mathbb{G}_m$ on each *fiber* of the morphism $\mathrm{Bun}_{P^-} \to \mathrm{Bun}_M$ is trivial. (Note that although $\mathbb{G}_m$ acts on Bun_{P^-} by automorphisms over Bun_M , the trivialization of the $\mathbb{G}_m$-action on Bun_{P^-} provided by (11.4) is *not over* Bun_M .)

Remark 11.2.6. It is not hard to show that the trivialization of the $\mathbb{G}_m$-action on Bun_{P^-} defined in Remark 11.2.4 yields an action of the monoidal stack[17] $\mathbb{A}^1/\mathbb{G}_m$ on Bun_{P^-}. The proof is straightforward; it uses the formula

$$\rho_t(\mu(s)) = \mu(s), \qquad t \in \mathbb{A}^1,\, s \in \mathbb{G}_m\,,$$

which follows from (11.4).

11.3. Proof of Proposition 9.2.2(b)

11.3.1. Let the numbers c_i, $i \in \Gamma_G$, be as in formula (10.3) from Sect. 10.2. Let $S \subset \Lambda_G^{+,\mathbb{Q}}$ be a P-admissible subset, and assume that S satisfies (9.3), i.e.,

$$\forall \lambda \in S,\ \forall i \in \Gamma_G - \Gamma_M \text{ we have } \langle \lambda\,, \check{\alpha}_i \rangle > c_i\,.$$

We have to prove that the locally closed substack $\mathrm{Bun}_G^{(S)} \subset \mathrm{Bun}_G$ is contractive in the sense of Sect. 5.2.1.

11.3.2. Recall that $c_i := \max(c_i', c_i'')$, where c_i' and c_i'' are as in Proposition 10.1.3. So for all $\lambda \in S$ and $i \in \Gamma_G - \Gamma_M$ we have

$$\langle \lambda\,, \check{\alpha}_i \rangle > c_i'\,, \tag{11.6}$$

$$\langle \lambda\,, \check{\alpha}_i \rangle > c_i''\,. \tag{11.7}$$

By (11.6) and the assumption on the numbers c_i' (see Proposition 10.1.3), we have

$$\mathrm{Bun}_M^{(S)} \subset \mathcal{U}_1 := \{\mathcal{F}_M \in \mathrm{Bun}_M \mid H^1(X, (\mathfrak{n}(P))_{\mathcal{F}}) = 0\}. \tag{11.8}$$

Similarly, (11.7) implies that

$$\mathrm{Bun}_M^{(S)} \subset \mathcal{U}_0 := \{\mathcal{F}_M \in \mathrm{Bun}_M \mid H^0(X, (\mathfrak{g}/\mathfrak{p})_{\mathcal{F}_M}) = 0\}. \tag{11.9}$$

11.3.3. Let $\mathrm{Bun}_{P^-}^{(S)} \subset \mathrm{Bun}_{P^-}$ denote the preimage of the open substack $\mathrm{Bun}_M^{(S)} \subset \mathrm{Bun}_M$. The embeddings $M \hookrightarrow P \hookrightarrow G$ and $M \hookrightarrow P^- \hookrightarrow G$ induce a commutative diagram

[17] For any scheme S, the groupoid $(\mathbb{A}^1/\mathbb{G}_m)(S)$ is the groupoid of line bundles over S equipped with a section, so $(\mathbb{A}^1/\mathbb{G}_m)(S)$ is a monoidal category with respect to $\otimes$. In this sense $\mathbb{A}^1/\mathbb{G}_m$ is a monoidal stack.

$$
\begin{tikzcd}
\mathrm{Bun}_M^{(S)} \arrow[r, "\iota_{P^-}^{(S)}"] \arrow[d, "\iota_P^{(S)}"'] & \mathrm{Bun}_{P^-}^{(S)} \arrow[d, "\mathsf{p}_{P^-}^{(S)}"] \\
\mathrm{Bun}_P^{(S)} \arrow[r, "\mathsf{p}_P^{(S)}"] & \mathrm{Bun}_G
\end{tikzcd}
\tag{11.10}
$$

We summarize the properties of the maps in the above diagram in the following lemma:

Lemma 11.3.4. (i) *The morphism* $\mathrm{Bun}_M^{(S)} \to \mathrm{Bun}_P^{(S)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-}^{(S)}$ *defined by diagram* (11.10) *is an open embedding.*

(ii) *The morphism* $\iota_P^{(S)} : \mathrm{Bun}_M^{(S)} \to \mathrm{Bun}_P^{(S)}$ *is surjective and smooth.*

(iii) *The morphism* $\mathsf{p}_{P^-}^{(S)} : \mathrm{Bun}_{P^-}^{(S)} \to \mathrm{Bun}_G$ *is smooth.*

(iv) *The morphism* $\mathsf{p}_P^{(S)}$ *induces an isomorphism* $\mathrm{Bun}_P^{(S)} \to \mathrm{Bun}_G^{(S)}$.

(v) *The morphism* $\iota_{P^-}^{(S)} : \mathrm{Bun}_M^{(S)} \to \mathrm{Bun}_{P^-}^{(S)}$ *is a closed embedding.*

Proof. Statement (i) follows from Lemma 11.1.2.

By (11.8), statements (ii) and (iii) follow from Proposition 11.1.4 points (a) and (b), respectively.

Statement (iv) holds by Proposition 9.2.2(a). By (11.9), statement (v) follows from Proposition 11.1.4(d). □

Remark 11.3.5. One can show, using [Sch, Proposition 4.4.4], that the map in point (i) of Lemma 11.3.4 is an isomorphism for *any* P-admissible set S (i.e., S does not even have to satisfy (9.3).)

11.3.6. Our goal is to prove that the locally closed substack $\mathrm{Bun}_G^{(S)} \subset \mathrm{Bun}_G$ is contractive. By the definition of contractiveness (see Sect. 5.2.1), this follows from Lemma 11.3.4 and the next statement:

Proposition 11.3.7. *Let S be as in Proposition 9.2.2. Then the substack* $\mathrm{Im}(\iota_{P^-}^{(S)}) \subset \mathrm{Bun}_{P^-}^{(S)}$ *from Lemma 11.3.4(v) is contractive.*

Proof. Equip Bun_{P^-} with the $\mathbb{A}^1$-action from Sect. 11.2 corresponding to some $\mu : \mathbb{G}_m \to Z(M)$. The open substack $\mathrm{Bun}_{P^-}^{(S)} \subset \mathrm{Bun}_{P^-}$ is $\mathbb{A}^1$-stable, so we obtain an $\mathbb{A}^1$-action on $\mathrm{Bun}_{P^-}^{(S)}$.

We apply Lemma 5.4.3 to the canonical morphism $\mathsf{q}_{P^-}^{(S)} : \mathrm{Bun}_{P^-}^{(S)} \to \mathrm{Bun}_M^{(S)}$ and the above $\mathbb{A}^1$-action on $\mathrm{Bun}_{P^-}^{(S)}$. We only have to check that the conditions of the lemma hold.

By (11.9) and Proposition 11.1.4(c), the morphism $\mathsf{q}_{P^-}^{(S)} : \mathrm{Bun}_{P^-}^{(S)} \to$

$\mathrm{Bun}_M^{(S)}$ is schematic and affine. Conditions (i)–(ii) from Lemma 5.4.3 hold by Remarks 11.2.3–11.2.4. □

12. Counterexamples

The goal of this section is to show that the property of being truncatable is a purely "stacky" phenomenon, i.e., that it "typically" fails for non quasi-compact schemes.

12.1. Formulation of the theorem

Theorem 12.1.1. *Let Y be an irreducible smooth scheme of dimension n, such that for some (or, equivalently, any) non-empty quasi-compact open $U \subset Y$ the set*

$$\{y \in Y - U \mid \dim_y(Y - U) = \dim Y - 1\} \tag{12.1}$$

is not quasi-compact. Then D-mod(Y) *is not compactly generated.*

The theorem will be proved in Sect. 12.2 below. Here are two examples of schemes Y satisfying the condition of Theorem 12.1.1.

Example 12.1.2. Let I be an infinite set and let Y be the non-separated curve that one obtains from $\mathbb{A}^1 \times I$ by gluing together the open subschemes $(\mathbb{A}^1 - \{0\}) \times \{i\}$, $i \in I$ (in other words, Y is the affine line with the point 0 repeated I times).

Example 12.1.3. Let X_0 be a smooth surface and $x_0 \in X_0$ a point. Set $U_0 = X - \{x_0\}$. Let X_1 be the blow-up of X_0 at x_0. Let $x_1 \in X_1$ be a point on the exceptional divisor. We have an open embedding

$$U_0 = X - \{x_0\} \hookrightarrow X_1 - \{x_1\} = U_1$$

such that $U_1 - U_0$ is a divisor. We can now apply the same process for (X_1, x_1) instead of (X_0, x_0). Thus we obtain a sequence of schemes

$$U_0 \hookrightarrow U_1 \hookrightarrow U_2 \hookrightarrow \ldots$$

Then $Y := \bigcup_i U_i$ satisfies the condition of Theorem 12.1.1. Note that Y is separated if X_0 is.

12.2. Proof of Theorem 12.1.1

We will use facts from Sect. 2.2.10 about the relation between compactness and coherence (in the easier case of smooth schemes).

12.2.1. Let Y be a smooth scheme, $Z \subset Y$ a non-empty divisor, and $Y - Z = U \overset{j}{\hookrightarrow} Y$ be the complementary open embedding.

Lemma 12.2.2. *Suppose that* $\mathcal{N} \in \text{D-mod}(Y)$ *is coherent and* $j_* \circ j^*(\mathcal{N}) = \mathcal{N}$*. Then the singular support* $SS(\mathcal{N}) \subset T^*(Y)$ *is* not *equal to* $T^*(Y)$*.*

Proof. We can assume that Y is affine and Z is smooth. Since j_* is t-exact we can also assume that $\mathcal{N}$ is in $\text{D-mod}(\mathcal{N})^\heartsuit$. Suppose that $SS(\mathcal{N}) = T^*(Y)$. Then there exists an injective map $\mathcal{D}_Y \hookrightarrow \mathcal{N}$, where $\mathcal{D}_Y$ is the D-module of differential operators on Y. We obtain an injective map $j_* \circ j^*(\mathcal{D}_Y) \hookrightarrow j_* \circ j^*(\mathcal{N}) = \mathcal{N}$. But $\mathcal{N}$ is coherent while $j_* \circ j^*(\mathcal{D}_Y)$ is not. $\square$

12.2.3. Let Y be as in Theorem 12.1.1 and $\mathcal{M} \in \text{D-mod}(Y)$ a compact object. Note that by Remark 2.2.13, $\mathcal{M}$ is automatically coherent.

We claim:

Lemma 12.2.4. $SS(\mathcal{M}) \neq T^*(Y)$.

Proof. By Proposition 2.3.7, there exists a quasi-compact open $U \overset{j}{\hookrightarrow} Y$ such that $\mathcal{M} = j_!(j^*(\mathcal{M}))$ or equivalently,

$$\mathbb{D}_Y^{\text{Verdier}}(\mathcal{M}) = j_* \circ j^*(\mathbb{D}_Y^{\text{Verdier}}(\mathcal{M})).$$

We can assume that $U \neq \emptyset$ (otherwise $\mathcal{M} = 0$ and $SS(\mathcal{M}) = \emptyset$). Then the set (12.1) is non-empty, so after shrinking Y we can assume that the set $Z := Y - U$ is a non-empty divisor.

Applying Lemma 12.2.2 to $\mathcal{N} = \mathbb{D}_Y^{\text{Verdier}}(\mathcal{M})$ we get $SS(\mathbb{D}_Y^{\text{Verdier}}(\mathcal{M})) \neq T^*(Y)$. Finally, $SS(\mathcal{M}) = SS(\mathbb{D}_Y^{\text{Verdier}}(\mathcal{M}))$. $\square$

12.2.5. Recall that the full subcategory of compact objects in a DG category $\mathbf{C}$ is denoted by $\mathbf{C}^c$.

Lemma 12.2.6. *Let* $\mathcal{A} \subset \text{D-mod}(Y)$ *be the DG subcategory generated by* $\text{D-mod}(Y)^c$*. If* $\mathcal{M} \in \mathcal{A}$ *is coherent then* $SS(\mathcal{M}) \neq T^*(Y)$*.*

Proof. Let $U \overset{j}{\hookrightarrow} Y$ be a non-empty quasi-compact open subset.

Let $\mathbf{C} \subset \text{D-mod}(U)$ be the full DG subcategory of $\text{D-mod}(U)$ generated by $j^*(\text{D-mod}(Y)^c)$. Since $j^*(\text{D-mod}(Y)^c) \subset \text{D-mod}(U)^c$, we have

$$\mathbf{C}^c = \mathbf{C} \cap \text{D-mod}(U)^c,$$

and by Corollary 1.4.6, the latter is Karoubi-generated by $j^*(\text{D-mod}(Y)^c)$.

This observation, combined with Lemma 12.2.4 and the fact that $T^*(U)$ is dense in $T^*(Y)$, implies that for any $\mathcal{N} \in \mathbf{C}^c$,

$$SS(\mathcal{N}) \neq T^*(U).$$

Now, $j^*(\mathcal{M}) \in \mathbf{C} \cap \text{D-mod}_{\text{coh}}(U)$, and since U is quasi-compact, we have $\text{D-mod}_{\text{coh}}(U) = \text{D-mod}(U)^c$. Hence, $j^*(\mathcal{M}) \in \mathbf{C}^c$, implying the assertion of the lemma. □

Corollary 12.2.7. *The DG category $\mathcal{A}$ from Lemma 12.2.6 does not contain $\mathcal{D}_Y$.*

Theorem 12.1.1 clearly follows from Corollary 12.2.7.

Appendix A. Preordered sets as topological spaces

The material in this section is standard.

A.1. Definition of the topology

Given a preordered set X we equip it with the following topology: a subset $U \subset X$ is said to be *open* if for every $x \in U$ one has $\{y \in X | y \leq x\} \subset U$.

Lemma A.1.1.

(i) *A subset $F \subset X$ is closed if and only if for every $x \in F$ one has $\{y \in X | y \geq x\} \subset F$.*

(ii) *A subset $Z \subset X$ is locally closed if and only if*

$$\forall x_1, x_2 \in Z \quad \{y \in X | x_1 \leq y \leq x_2\} \subset Z. \tag{A.1}$$

(iii) *For every subset $Y \subset X$ the topology on Y corresponding to the induced preordering on Y is induced by the topology on X.*

Proof. We will only prove (ii). Condition (A.1) holds for locally closed subsets because it holds for open and closed ones. Conversely, if Z satisfies (A.1) then Z has the following representation as $F \cap U$ with F closed and U open:

$$F := \bar{Z} = \{x \in X \,|\, \exists z \in Z : \; z \leq x\}, \quad U := \{x \in X \,|\, \exists z \in Z : \; z \geq x\}. \quad \square$$

A.2. Continuous maps

The following is also easy to see:

Lemma A.2.1. *Let X, X' be preordered sets equipped with the above topology. Then a map $f : X \to X'$ is continuous if and only if it is monotone, i.e., $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$.*

Appendix B. The Langlands retraction and coarsenings of the Harder-Narsimhan-Shatz stratification

In Sect. B.1 we recall the definition of the Langlands retraction $\mathfrak{L} : \Lambda_G^{\mathbb{Q}} \to \Lambda_G^{+,\mathbb{Q}}$.

Using this retraction, we define in Sect. B.2 a coarsening of the usual Harder-Narasimhan-Shatz stratification of Bun_G depending on the choice of $\eta \in \Lambda_G^{+,\mathbb{Q}}$ (the usual stratification itself corresponds to $\eta = 0$).

In Sect. B.3 we show that if η is "deep inside" $\Lambda_G^{+,\mathbb{Q}}$ then all the strata of the corresponding stratification are contractive (and therefore truncative if char $k = 0$). Combined with Proposition 9.2.2 this immediately implies Theorem 9.1.2 (see Sect. B.3.5 below).

In Sect. B.4 we explain the relation between the two proofs of Theorem 9.1.2.

B.1. Recollections on the Langlands retraction

Equip $\Lambda_G^{\mathbb{Q}}$ with the $\underset{G}{\leq}$ ordering. The following notion goes back to [La, Sect. 4].

Definition B.1.1. *The* Langlands retraction $\mathfrak{L} : \Lambda_G^{\mathbb{Q}} \to \Lambda_G^{+,\mathbb{Q}}$ *is defined as follows: for $\lambda \in \Lambda_G^{\mathbb{Q}}$, let $\mathfrak{L}(\lambda)$ be the least element of the set $\{\mu \in \Lambda_G^{+,\mathbb{Q}} \mid \mu \underset{G}{\geq} \lambda\}$ in the sense of the $\underset{G}{\leq}$ ordering.*

B.1.2. The existence of the least element is not obvious; it was proved by R.P. Langlands in [La, Sect. 4]. The material from [La, Sect. 4] is known under the name of "Langlands' geometric lemmas". We give a short review of it in [Dr]. In particular, we give there two proofs of the existence of the least element: J. Carmona's "metric" proof (see [Dr, Sections 2-3]) and another one (see Sect. 4 of [Dr], including Example 4.3).

B.1.3. It is clear that the map $\mathfrak{L} : \Lambda_G^{\mathbb{Q}} \to \Lambda_G^{+,\mathbb{Q}}$ is an order-preserving retraction. The following description of the fibers of $\mathfrak{L}$ is given in [La, Sect. 4]; see also [Dr, Cor. 5.3(iii)].

Lemma B.1.4. *For any $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ one has*

$$\mathfrak{L}^{-1}(\lambda) = \lambda + \sum_{i \in I_\lambda} \mathbb{Q}^{\leq 0} \cdot \alpha_i\,, \quad \textit{where} \quad I_\lambda := \{i \in \Gamma_G \,|\, \langle \lambda, \alpha_i \rangle = 0\}.$$

B.2. The η-stratification

B.2.1. The η-shifted Langlands retraction. Let $\eta \in \Lambda_G^{+,\mathbb{Q}}$. The map

$$\mathfrak{L}_\eta^+ : \Lambda_G^{+,\mathbb{Q}} \to (\eta + \Lambda_G^{+,\mathbb{Q}}), \quad \mathfrak{L}_\eta^+(\lambda) := \mathfrak{L}(\lambda - \eta) + \eta \tag{B.1}$$

is an order-preserving retraction (this follows from a similar property of $\mathfrak{L}$). By definition,

$$\forall \lambda' \in \Lambda_G^{+,\mathbb{Q}}, \forall \lambda \in (\eta + \Lambda_G^{+,\mathbb{Q}}) \quad \text{we have} \quad \mathfrak{L}_\eta^+(\lambda') \underset{G}{\leq} \lambda \Leftrightarrow \lambda' \underset{G}{\leq} \lambda. \tag{B.2}$$

B.2.2. The η-stratification of Bun_G. In Sect. 7.4.8 we defined the Harder-Narasimhan map $\mathrm{HN} : |\,\mathrm{Bun}_G(k)| \to \Lambda_G^{+,\mathbb{Q}}$ and formulated three properties of it, see Lemma 7.4.9 (i-iii). Since the map $\mathfrak{L}_\eta^+ : \Lambda_G^{+,\mathbb{Q}} \to (\eta + \Lambda_G^{+,\mathbb{Q}})$ is order-preserving, the map

$$\mathrm{HN}_\eta : |\,\mathrm{Bun}_G(k)| \to (\eta + \Lambda_G^{+,\mathbb{Q}}), \quad \mathrm{HN}_\eta := \mathfrak{L}_\eta^+ \circ \mathrm{HN} \tag{B.3}$$

has the same three properties. So the fibers of the map (B.3) form a stratification of Bun_G with quasi-compact strata. We call it the η-*stratification* of Bun_G; the corresponding strata are

$$\mathrm{Bun}_G^{(\lambda)_\eta} := \bigcup_{\lambda' \in (\mathfrak{L}_\eta^+)^{-1}(\lambda)} \mathrm{Bun}_G^{(\lambda')}, \quad \lambda \in (\eta + \Lambda_G^{+,\mathbb{Q}}). \tag{B.4}$$

It is clear that the η-stratification is coarser than the Harder-Narsimhan-Shatz stratification (the word "coarser" is understood in the non-strict sense).

B.2.3. Open substacks associated to the η-stratification. Recall that for each $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ the open substack $\mathrm{Bun}_G^{(\leq\lambda)} \subset \mathrm{Bun}_G$ is the union of the strata

$$\mathrm{Bun}_G^{(\lambda')}, \quad \lambda' \underset{G}{\leq} \lambda.$$

If one considers similar unions of the strata of the η-stratification then one gets "essentially" the same class of open substacks of Bun_G; more precisely, we claim that for each $\lambda \in (\eta + \Lambda_G^{+,\mathbb{Q}})$ one has

$$\bigcup_{\lambda' \in (\eta + \Lambda_G^{+,\mathbb{Q}}),\, \lambda' \underset{G}{\leq} \lambda} \mathrm{Bun}_G^{(\lambda')_\eta} = \mathrm{Bun}_G^{(\leq\lambda)}.$$

This follows from (B.2) and (B.4).

B.2.4. Changing η. If $\eta' \in (\eta + \Lambda_G^{+,\mathbb{Q}})$ then $\mathfrak{L}^+_{\eta'} \circ \mathfrak{L}^+_\eta = \mathfrak{L}^+_{\eta'}$, so the η'-stratification is coarser than the η-stratification. If η' and η have the same image in $\Lambda^{+,\mathbb{Q}}_{G_{ad}}$ then $\mathfrak{L}^+_{\eta'} = \mathfrak{L}^+_\eta$, so the η'-stratification and the η-stratification are the same.

B.2.5. $(\mathfrak{L}^+_\eta)^{-1}(\lambda)$ as a P-admissible set. Let $\lambda \in (\eta + \Lambda_G^{+,\mathbb{Q}})$. Let P be the parabolic whose Levi quotient, M, corresponds to the following subset of Γ_G:

$$\Gamma_M = \{i \in \Gamma_G \,|\, \langle \lambda - \eta\,, \check{\alpha}_i \rangle = 0\}. \tag{B.5}$$

Equivalently,

$$\Gamma_G - \Gamma_M = \{i \in \Gamma_G \,|\, \langle \lambda\,, \check{\alpha}_i \rangle > \langle \eta\,, \check{\alpha}_i \rangle\}. \tag{B.6}$$

By Lemma B.1.4, the subset $T_\lambda := (\mathfrak{L}^+_\eta)^{-1}(\lambda) \subset \Lambda_G^{+,\mathbb{Q}}$ has the following description in terms of the $\underset{M}{\leq}$ ordering:

$$T_\lambda = \{\lambda' \in \Lambda_G^{+,\mathbb{Q}} \,|\, \lambda' \underset{M}{\leq} \lambda\}. \tag{B.7}$$

So by (B.6) and Lemma 8.1.2(a), the set T_λ is P-admissible in the sense of Definition 8.2.2. Moreover, by (B.4) and (B.7), the stratum $\mathrm{Bun}_G^{(\lambda)_\eta}$ is equal to the locally closed substack $\mathrm{Bun}_G^{(T_\lambda)}$ defined in Sect. 8.3.1 by formula (8.6).

B.3. The case where η is "deep inside" $\Lambda_G^{+,\mathbb{Q}}$

B.3.1. Contractiveness of the strata. Suppose now that

$$\langle \eta\,, \check{\alpha}_i \rangle \geq c_i \quad \text{for all} \quad i \in \Gamma_G\,, \tag{B.8}$$

where the numbers $c_i \in \mathbb{Q}^{\geq 0}$ are as in Proposition 9.2.2.

Proposition B.3.2. *Under these conditions, all strata of the η-stratification are contractive.*

Proof. Let $\lambda \in (\eta + \Lambda_G^{+,\mathbb{Q}})$. By Sect. B.2.5, $\mathrm{Bun}_G^{(\lambda)_\eta} = \mathrm{Bun}_G^{(T_\lambda)}$, where $T_\lambda \subset \Lambda_G^{+,\mathbb{Q}}$ is the P-admissible set defined by (B.7). So by Proposition 9.2.2(b), it suffices to check that for all $\lambda' \in T_\lambda$ and $i \in \Gamma_G - \Gamma_M$ one has $\langle \lambda'\,, \check{\alpha}_i \rangle > c_i$. If $\lambda' = \lambda$ this is clear from (B.6) and (B.8). The general case follows by Lemma 8.1.2(a). □

B.3.3. The characteristic 0 case. Now assume that char $k = 0$. Then by Proposition 9.2.2, one can take $c_i = \max(0, 2g-2)$, where g is the genus of X. In this situation condition (B.8) can be rewritten as

$$\eta \in (\eta_0 + \Lambda_G^{+,\mathbb{Q}}), \quad \text{where} \quad \eta_0 := \max(0, 2g-2)\cdot\rho \tag{B.9}$$

(as usual, ρ denotes the half-sum of positive coroots). E.g., one can take $\eta = \eta_0$.

In characteristic 0 we have the notion of truncativeness and the fact that contractiveness implies truncativeness, see Corollary 5.2.3. Thus we get part (i) of the following

Corollary B.3.4. *Suppose that char $k = 0$ and*

$$\langle \eta\,, \check{\alpha}_i\rangle \geq \max(0, 2g-2) \quad \textit{for all} \quad i \in \Gamma_G\,. \tag{B.10}$$

Then:

(i) *all strata of the η-stratification are truncative;*
(ii) *the open strata of the η-stratification are co-truncative.*

Proof. We have already proved (i). The complement of an open stratum is a union of strata, so statement (ii) follows from Proposition 3.7.2. □

B.3.5. Proof of Theorem 9.1.2. We have to show that the substack $\mathrm{Bun}_G^{(\leq\eta)} \subset \mathrm{Bun}_G$ is co-truncative if $\langle \eta\,, \check{\alpha}_i\rangle \geq 2g-2$ for all $i \in \Gamma_G$. If $g = 0$ then any open substack of Bun_G is co-truncative by Sect. 3.2.4. So we can assume that (B.10) holds. Then the statement follows from Corollary B.3.4(ii) because $\mathrm{Bun}_G^{(\leq\eta)}$ is an open stratum of the η-stratification; namely, it is the stratum corresponding to $\eta \in (\eta + \Lambda_G^{+,\mathbb{Q}})$. □

B.4. Relation between the two proofs of Theorem 9.1.2

Suppose that char $k = 0$ and $g \geq 1$.

In the proof of Theorem 9.1.2 given in Sect. 9.3 we used substacks $\mathrm{Bun}_G^{(S_\lambda)} \subset \mathrm{Bun}_G$, $\lambda \in \Lambda_G^{+,\mathbb{Q}}$, where

$$S_\lambda := \{\lambda' \in \Lambda_G^{+,\mathbb{Q}} \,|\, \lambda' - \lambda \in \sum_{i\in I} \mathbb{Q}^{\leq 0}\cdot \alpha_i\}, \quad I := \{i \in \Gamma_G \,|\, \langle \lambda, \alpha_i\rangle \leq 2g-2\}. \tag{B.11}$$

These substacks are related to the strata of the η_0-stratification, where η_0 is as in (B.9). The relation is as follows. The stratum of the η_0-stratification corresponding to $\lambda \in (\eta_0 + \Lambda_G^{+,\mathbb{Q}})$ equals $\mathrm{Bun}_G^{(S_\lambda)}$. On the other hand, for any $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ the stack $\mathrm{Bun}_G^{(S_\lambda)}$ is a locally closed substack of the stratum of the η_0-stratification corresponding to $\mathfrak{L}^+_{\eta_0}(\lambda)$. (The proof of these facts is left to the reader.)

Appendix C. A stacky contraction principle

The main goal of this appendix is to prove Theorem C.5.3 and Corollaries C.5.4–C.5.5.

Corollary C.5.5 is a "contraction principle", which is slightly more general than Proposition 5.1.2. Theorem C.5.3 and Corollary C.5.4 are generalizations of the classical adjunction from Proposition 5.3.2.

Convention: throughout this appendix algebras are always associative but not necessarily unital; coalgebras are coassociative but not necessarily counital.

C.1. Idempotent algebras in monoidal categories

The notions of algebra and coalgebra make sense in any monoidal category.

Definition C.1.1. *An algebra A in a monoidal category is said to be* idempotent *if the multiplication morphism $A \otimes A \to A$ is an isomorphism.*

Remark C.1.2. The dual notion of idempotent coalgebra is, in fact, equivalent to that of idempotent algebra: an isomorphism $m : A \otimes A \to A$ is an algebra structure if and only if $m^{-1} : A \to A \otimes A$ is a coalgebra structure.

In any monoidal category $\mathcal{M}$ the unit object $\mathbf{1}_{\mathcal{M}}$ has a canonical structure of idempotent algebra.

Here is another example. Any monoid M can be considered as a monoidal category (with M as the set of objects and no morphisms other than the identities). In particular, this applies to $\{0, 1\}$ as a monoid with respect to multiplication. Clearly 0 is an idempotent algebra in the monoidal category $\{0, 1\}$.

Remark C.1.3. The category of idempotent algebras in a monoidal category $\mathcal{M}$ is equivalent to the category of monoidal functors $F : \{0, 1\} \to \mathcal{M}$; namely, the idempotent algebra corresponding to F is $F(0)$.

Let $\mathcal{C}$ be a category. By an *idempotent functor* $\mathcal{C} \to \mathcal{C}$ we mean an idempotent algebra in the monoidal category of functors $\mathcal{C} \to \mathcal{C}$. One can think of idempotent functors in terms of the two mutually inverse constructions below.

Construction 1. Suppose we have categories $\mathcal{A}$ and $\mathcal{C}$, functors $\mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{\pi} \mathcal{A}$, and an isomorphism $f : \pi \circ i \xrightarrow{\sim} \mathrm{Id}_{\mathcal{A}}$. Set $\mathbf{0} := i \circ \pi$. Then $\mathbf{0} : \mathcal{C} \to \mathcal{C}$ is an idempotent functor: the isomorphism $\mathbf{0} \circ \mathbf{0} \xrightarrow{\sim} \mathbf{0}$ is the composition

$$\mathbf{0}\circ\mathbf{0}=(i\circ\pi)\circ(i\circ\pi)=i\circ(\pi\circ i)\circ\pi\xrightarrow{\sim} i\circ\mathrm{Id}_{\mathcal{A}}\circ\pi=i\circ\pi=\mathbf{0}.$$

Construction 2. Let $\mathcal{C}$ be a category equipped with an idempotent functor $\mathbf{0}:\mathcal{C}\to\mathcal{C}$. Equivalently, $\mathcal{C}$ carries an action of the monoid $\{0,1\}$ (see Remark C.1.3). Let $\mathcal{C}^0$ be the category of $\{0,1\}$-equivariant functors $\{0\}\to\mathcal{C}$. Then the $\{0,1\}$-equivariant maps $\{0\}\hookrightarrow\{0,1\}\to\{0\}$ induce functors $\mathcal{C}^0\xleftarrow{\pi}\mathcal{C}\xleftarrow{i}\mathcal{C}^0$ with $\pi\circ i=\mathrm{Id}_{\mathcal{C}^0}$. Equivalently, one can think of $\mathcal{C}^0$ as the category of $\mathbf{0}$-modules[18] c in $\mathcal{C}$ such that the morphism $\mathbf{0}\cdot c\to c$ is an isomorphism; then $i:\mathcal{C}^0\to\mathcal{C}$ is the forgetful functor and $\pi:\mathcal{C}\to\mathcal{C}^0$ is the "free module" functor.

It is easy to check that the above Constructions 1 and 2 are mutually inverse.[19]

Remark C.1.4. In the situation of Construction 1, the algebra $\mathbf{0}$ is unital (or equivalently, is a monad) if and only if (π,i) is an adjoint pair of functors with $f:\pi\circ i\xrightarrow{\sim}\mathrm{Id}_{\mathcal{A}}$ being one of the adjunctions (in this case the unit of $\mathbf{0}$ is the other adjunction). Similarly, $\mathbf{0}$ is a counital coalgebra (or equivalently, a comonad) if and only if (i,π) is an adjoint pair.

C.2. The monodromic subcategory

Let $\mathcal{Y}$ be a QCA stack equipped with a $\mathbb{G}_m$-action. Then one has the quotient stack $\mathcal{Y}/\mathbb{G}_m$ and the canonical morphism $p:\mathcal{Y}\to\mathcal{Y}/\mathbb{G}_m$.

Definition C.2.1. *The* monodromic subcategory $\text{D-mod}(\mathcal{Y})_\mu\subset\text{D-mod}(\mathcal{Y})$ *is the subcategory generated by the essential image of* $p^!:\text{D-mod}(\mathcal{Y}/\mathbb{G}_m)\to\text{D-mod}(\mathcal{Y})$ *(or equivalently, by the essential image of* p^**).*

(A more precise name for $\text{D-mod}(\mathcal{Y})_\mu$ would be "unipotently monodromic subcategory.")

Lemma C.2.2. *If the* $\mathbb{G}_m$*-action on* $\mathcal{Y}$ *is trivial then* $\text{D-mod}(\mathcal{Y})_\mu=\text{D-mod}(\mathcal{Y})$*.*

Proof. A trivialization of the $\mathbb{G}_m$-action on $\mathcal{Y}$ identifies $\mathcal{Y}/\mathbb{G}_m$ with $\mathcal{Y}\times(\mathrm{pt}/\mathbb{G}_m)$ and the morphism $p:\mathcal{Y}\to\mathcal{Y}/\mathbb{G}_m$ with the canonical morphism $\mathcal{Y}=\mathcal{Y}\times\mathrm{pt}\to\mathcal{Y}\times(\mathrm{pt}/\mathbb{G}_m)$. $\square$

[18] This notion makes sense because $\mathbf{0}$ is an algebra in the monoidal category of functors $\mathcal{C}\to\mathcal{C}$.

[19] This is a "baby case" of the theory of retracts and idempotents in ∞-categories from [Lu1, Sect. 4.4.5].

C.3. Recollections on the renormalized direct image

Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of QCA stacks. The *renormalized direct image functor*

$$\pi_\blacktriangle : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2)$$

is defined in [DrGa1, Sect. 9.3] to be the functor dual to $\pi^! : \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1)$ (dual in the sense of Sects. 1.5.2 and 2.2.16 of this article). By definition, $\pi_\blacktriangle$ is continuous. One also has a not necessarily continuous de Rham direct image functor $\pi_{\mathrm{dR},*} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2)$, see [DrGa1, Sect. 7.4]. If $\pi_{\mathrm{dR},*}$ is continuous then one has a canonical isomorphism $\pi_\blacktriangle \xrightarrow{\sim} \pi_{\mathrm{dR},*}$, see [DrGa1, Corollary 9.3.8]. For instance, this happens if the fibers of π are algebraic spaces, see [DrGa1, Corollary 10.2.5].

C.4. Formulation of the theorem

Let $\mathcal{Y}$ be a QCA stack equipped with an action of the multiplicative monoid $\mathbb{A}^1$. Let $\mathbf{0} \in \mathrm{Mor}(\mathcal{Y}, \mathcal{Y})$ denote the endomorphism of $\mathcal{Y}$ corresponding to $0 \in \mathbb{A}^1$. One has continuous functors $\mathbf{0}^!, \mathbf{0}_\blacktriangle : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Y})$ (the functor $\mathbf{0}_{\mathrm{dR},*}$ is continuous only if it equals $\mathbf{0}_\blacktriangle$). By Remark C.1.3, $\mathbf{0}$ is an idempotent algebra in the monoidal category $\mathrm{Mor}(\mathcal{Y}, \mathcal{Y})$. So the functors $\mathbf{0}_\blacktriangle$ and $\mathbf{0}^!$ are idempotent algebras in the monoidal category $\mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y}), \text{D-mod}(\mathcal{Y}))$ and also in the monoidal category $\mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y})_\mu, \text{D-mod}(\mathcal{Y})_\mu)$ (here $\text{D-mod}(\mathcal{Y})_\mu \subset \text{D-mod}(\mathcal{Y})$ is the monodromic subcategory, see Sect. C.2). By Remark C.1.2, one can also consider $\mathbf{0}_\blacktriangle$ and $\mathbf{0}^!$ as idempotent coalgebras.

Theorem C.4.1. *The algebra $\mathbf{0}_\blacktriangle \in \mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y})_\mu, \text{D-mod}(\mathcal{Y})_\mu)$ is unital. The coalgebra $\mathbf{0}^! \in \mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y})_\mu, \text{D-mod}(\mathcal{Y})_\mu)$ is counital.*

A proof will be given in Sect. C.6–C.8. A slightly different proof will be sketched in Sect. C.9.

Corollary C.4.2. *If the $\mathbb{G}_m$-action on $\mathcal{Y}$ is trivial then the algebra*

$$\mathbf{0}_\blacktriangle \in \mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y}), \text{D-mod}(\mathcal{Y}))$$

is unital and the coalgebra

$$\mathbf{0}^! \in \mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y}), \text{D-mod}(\mathcal{Y}))$$

is counital.

Proof. Use Theorem C.4.1 and Lemma C.2.2. $\square$

C.5. Reformulation in terms of adjunctions

Let $\mathcal{Y}$ be as in Sect. C.4. In particular, the submonoid $\{0,1\} \subset \mathbb{A}^1$ acts on $\mathcal{Y}$. Define $\mathcal{Y}^0$ to be the stack of $\{0,1\}$-equivariant maps $\{0\} \to \mathcal{Y}$. Equivalently, for any test scheme S, the groupoid $\mathcal{Y}^0(S)$ is obtained from $\mathcal{Y}(S)$ using Construction 2 from Sect. C.1. It is clear that the stack $\mathcal{Y}^0$ is QCA.

The $\{0,1\}$-equivariant maps $\{0,1\} \to \{0\} \hookrightarrow \{0,1\}$ induce morphisms

$$\mathcal{Y} \xleftarrow{\;i\;} \mathcal{Y}^0 \xleftarrow{\;\pi\;} \mathcal{Y}, \qquad \pi \circ i = \mathrm{Id}_{\mathcal{Y}^0}, \quad i \circ \pi = \mathbf{0}. \tag{C.1}$$

The $\mathbb{A}^1$-action on $\mathcal{Y}$ induces an $\mathbb{A}^1$-action on the diagram (C.1). The $\mathbb{A}^1$-action on $\mathcal{Y}^0$ is canonically trivial (this follows from the identity $\lambda \cdot 0 = 0$ in $\mathbb{A}^1$). So $\text{D-mod}(\mathcal{Y}^0)_\mu = \text{D-mod}(\mathcal{Y}^0)$.

Example C.5.1. Let $\mathcal{Y}$ be the stack Bun_{P^-} equipped with the $\mathbb{A}^1$-action from Sect 11.2. Then $\mathcal{Y}^0 = \mathrm{Bun}_M$ and diagram (C.1) identifies with diagram (11.5).

Remark C.5.2. Since $\pi \circ i = \mathrm{Id}_{\mathcal{Y}^0}$ the morphism i is representable[20] (i.e., its fibers are algebraic spaces rather than stacks). So the renormalized direct image functor $i_\blacktriangle$ equals the "usual" direct image $i_{\mathrm{dR},*}$.

By Remarks C.1.4 and C.5.2, one can reformulate Theorem C.4.1 and Corollary C.4.2 as follows.

Theorem C.5.3. *The functors*

$$\pi_\blacktriangle : \text{D-mod}(\mathcal{Y})_\mu \rightleftarrows \text{D-mod}(\mathcal{Y}^0) : i_{\mathrm{dR},*}\,, \qquad i^! : \text{D-mod}(\mathcal{Y})_\mu \rightleftarrows \text{D-mod}(\mathcal{Y}^0) : \pi^! \tag{C.2}$$

form adjoint pairs with the adjunctions $\pi_\blacktriangle \circ i_{\mathrm{dR},*} \xrightarrow{\sim} \mathrm{Id}_{\text{D-mod}(\mathcal{Y}^0)}$ *and* $\mathrm{Id}_{\text{D-mod}(\mathcal{Y}^0)} \xrightarrow{\sim} i^! \circ \pi^!$ *coming from the isomorphism* $\pi \circ i \xrightarrow{\sim} \mathrm{Id}_{\mathcal{Y}^0}$.

Corollary C.5.4. *If the* $\mathbb{G}_m$*-action on* $\mathcal{Y}$ *is trivial then the functors*

$$\pi_\blacktriangle : \text{D-mod}(\mathcal{Y}) \rightleftarrows \text{D-mod}(\mathcal{Y}^0) : i_{\mathrm{dR},*}\,, \qquad i^! : \text{D-mod}(\mathcal{Y}) \rightleftarrows \text{D-mod}(\mathcal{Y}^0) : \pi^! \tag{C.3}$$

form adjoint pairs with the adjunctions $\pi_\blacktriangle \circ i_{\mathrm{dR},*} \xrightarrow{\sim} \mathrm{Id}_{\text{D-mod}(\mathcal{Y}^0)}$ *and* $\mathrm{Id}_{\text{D-mod}(\mathcal{Y}^0)} \xrightarrow{\sim} i^! \circ \pi^!$ *coming from the isomorphism* $\pi \circ i \xrightarrow{\sim} \mathrm{Id}_{\mathcal{Y}^0}$.

[20] Example C.5.1 shows that i is not necessarily a monomorphism. Simpler example: let G be an affine algebraic group equipped with an $\mathbb{A}^1$-action and set $\mathcal{Y} := \mathrm{pt}/G$, then $\mathcal{Y}^0 = \mathrm{pt}/G^0$, where $G^0 \subset G$ is the subgroup of $\mathbb{A}^1$-fixed points.

Corollary C.5.5. *Suppose that the $\mathbb{G}_m$-action on $\mathcal{Y}$ is trivial and the morphism $i : \mathcal{Y}^0 \to \mathcal{Y}$ is a composition of an almost-isomorphism*[21] $\mathcal{Y}^0 \to \mathcal{Z}$ *and a locally closed embedding $\mathcal{Z} \hookrightarrow \mathcal{Y}$. Then the substack $\mathcal{Z} \subset \mathcal{Y}$ is truncative.*

Remark C.5.6. The assumption of Corollary C.5.5 is equivalent to the following one: $\mathbf{0}(\mathcal{Y}) \subset \mathcal{Z}$ and $\mathbf{0}|_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}$ is an almost-isomorphism.

Remark C.5.7. Corollary C.5.5 holds even if $\mathcal{Y}$ is *locally* QCA (but not necessarily quasi-compact). To show this, we can assume that $\mathcal{Y}^0$ is quasi-compact (otherwise replace $\mathcal{Y}$ by $\mathcal{Y} \times_{\mathcal{Y}^0} S$, where S is any quasi-compact scheme equipped with a smooth morphism to $\mathcal{Y}^0$). Then $\mathcal{Y}^0$ is contained in a quasi-compact open substack $U \subset \mathcal{Y}$. Since the $\mathbb{G}_m$-action on $\mathcal{Y}$ is trivial U is $\mathbb{A}^1$-stable. Applying Corollary C.5.5 to U we see that $\mathcal{Y}^0$ is truncative in U and therefore in $\mathcal{Y}$.

C.6. The key lemma

Similarly to the notion of monoidal groupoid, there is a notion of monoidal stack. Of course, any algebraic group or the multiplicative monoid $\mathbb{A}^1$ are examples of monoidal stacks. In the proof of Theorem C.4.1 we will use the monoidal stack $\mathbb{A}^1/\mathbb{G}_m$. (If you wish, S-points of $\mathbb{A}^1/\mathbb{G}_m$ can be interpreted as line bundles over S equipped with a section; this is a monoidal category with respect to $\otimes$).

Let G be a monoidal QCA stack over k. Then $\text{D-mod}(G)$ is a monoidal category with respect to the convolution

$$M * N := m_\blacktriangle(M \boxtimes N), \quad M, N \in \text{D-mod}(G), \tag{C.4}$$

where $m : G \times G \to G$ is the multiplication map.

For any $g \in G(k)$ define $\underline{g} \in \text{D-mod}(G)$ to be the direct image of $k \in \text{D-mod}(\text{pt})$ under the map $g : \text{pt} \to G$ (this is a kind of "delta-function" at g). The assignment $g \mapsto \underline{g}$ is a monoidal functor $G(k) \to \text{D-mod}(G)$. In particular, $\underline{1} \in \text{D-mod}(G)$ is the unit object.

If $f : G_1 \to G_2$ is a morphism of monoidal stacks then $f_\blacktriangle : \text{D-mod}(G_1) \to \text{D-mod}(G_2)$ is a monoidal functor. If f is only a morphism of semigroups then $f_\blacktriangle$ is a semigroupal[22] functor, so $f_\blacktriangle(\underline{1}) \in \text{D-mod}(G_2)$ is an idempotent algebra.

Applying this to $0 : \text{pt} \to \mathbb{A}^1/\mathbb{G}_m$ we see that $\underline{0} \in \text{D-mod}(\mathbb{A}^1/\mathbb{G}_m)$ is an idempotent algebra.

Lemma C.6.1. *The algebra $\underline{0} \in \text{D-mod}(\mathbb{A}^1/\mathbb{G}_m)$ is unital.*

[21] See Definition 7.4.2.

[22] There exists a precedent of the usage of "semigroupal" in the literature; this word means "monoidal, but without asking that the unit map to the unit."

Proof. Consider the morphisms $\{0\}/\mathbb{G}_m \overset{i}{\hookrightarrow} \mathbb{A}^1/\mathbb{G}_m \xrightarrow{\pi} \{0\}/\mathbb{G}_m$ induced by the morphisms $\{0\} \hookrightarrow \mathbb{A}^1 \to \{0\}$. Set $\mathcal{C} := \text{D-mod}(\{0\}/\mathbb{G}_m)$; this is a monoidal category because $\{0\}/\mathbb{G}_m$ is a monoidal stack. We have a monoidal functor $\pi_{\mathrm{dR},*} : \text{D-mod}(\mathbb{A}^1/\mathbb{G}_m) \to \mathcal{C}$ and a semigroupal functor $i_{\mathrm{dR},*} : \mathcal{C} \to \text{D-mod}(\mathbb{A}^1/\mathbb{G}_m)$ with $\pi_{\mathrm{dR},*} \circ i_{\mathrm{dR},*} = \mathrm{Id}_{\mathcal{C}}$. By definition, $\underline{0} = i_{\mathrm{dR},*}(\mathbf{1}_{\mathcal{C}})$, where $\mathbf{1}_{\mathcal{C}}$ is the unit object of $\mathcal{C}$.

Let us now construct the unit $e : \underline{1} \to \underline{0}$ of the algebra $\underline{0}$. By Sect. 3.3.9, $(\pi_{\mathrm{dR},*}, i_{\mathrm{dR},*})$ is an adjoint pair of functors (this is the "baby case" of Theorem C.5.3). So

$$\mathcal{M}aps(\underline{1}, \underline{0}) = \mathcal{M}aps(\underline{1}, i_{\mathrm{dR},*}(\mathbf{1}_{\mathcal{C}})) = \mathcal{M}aps(\pi_{\mathrm{dR},*}(\underline{1}), \mathbf{1}_{\mathcal{C}}) = \mathcal{M}aps(\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}});$$

more precisely, the map $\pi_{\mathrm{dR},*} : \mathcal{M}aps(\underline{1}, \underline{0}) \to \mathcal{M}aps(\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}})$ is an isomorphism. Define $e : \underline{1} \to \underline{0}$ to be the morphism such that $\pi_{\mathrm{dR},*}(e)$ equals $\mathrm{id} : \mathbf{1}_{\mathcal{C}} \xrightarrow{\sim} \mathbf{1}_{\mathcal{C}}$.

Let us show that e is indeed a unit. Let $f : \underline{0} \to \underline{0}$ denote the composition of the morphism $e * \mathrm{id}_{\underline{0}} : \underline{0} = \underline{1} * \underline{0} \to \underline{0} * \underline{0}$ with the multiplication map $\underline{0} * \underline{0} \to \underline{0}$. We have to prove that $f = \mathrm{id}_{\underline{0}}$. To do this, it suffices to show that $\pi_{\mathrm{dR},*}(f)$ equals the identity. This is clear because $\pi_{\mathrm{dR},*}$ is a monoidal functor and $\pi_{\mathrm{dR},*}(e)$ equals $\mathrm{id} : \mathbf{1}_{\mathcal{C}} \xrightarrow{\sim} \mathbf{1}_{\mathcal{C}}$. $\square$

C.7. Proof of a particular case of Theorem C.4.1

The following statement is a particular case of Theorem C.4.1 and of Corollary C.4.2.

Lemma C.7.1. *Let $\mathcal{Y}$ be a QCA stack equipped with an action of the monoidal stack $\mathbb{A}^1/\mathbb{G}_m$. Then the algebra $\mathbf{0}_{\blacktriangle} \in \mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y}), \text{D-mod}(\mathcal{Y}))$ is unital and the coalgebra $\mathbf{0}^! \in \mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y}), \text{D-mod}(\mathcal{Y}))$ is counital.*

This lemma is an immediate consequence of Lemma C.6.1 and the following general considerations.

Suppose that a monoidal QCA stack G acts on a QCA stack $\mathcal{Y}$. Then the monoidal category $G(k)$ acts on $\text{D-mod}(\mathcal{Y})$ on the left by $g \mapsto g_{\blacktriangle}$, $g \in G(k)$. One also has the right action[23] $g \mapsto g^!$. Each of these two actions extend to an action of $\text{D-mod}(G)$. Namely, the left action is defined by

$$M * N := a_{\blacktriangle}(M \boxtimes N), \qquad M \in \text{D-mod}(G), N \in \text{D-mod}(\mathcal{Y}), \tag{C.5}$$

[23] Usually it does not commute with the left action. E.g., if $g \in G$ is invertible then $g^! = (g^{-1})_{\blacktriangle}$ does not have to commute with $g'_{\blacktriangle}$, $g' \in G$.

where $a : G \times \mathcal{Y} \to \mathcal{Y}$ is the action map. One can get the right action of D-mod(G) on D-mod$(\mathcal{Y})$ from the left one using the equivalence D-mod$(\mathcal{Y})^\vee \simeq$ D-mod$(\mathcal{Y})$ that comes from Verdier duality, see (2.2). One can also define the right action explicitly by

(C.6)
$$N * M := (p_{\mathcal{Y}})_\blacktriangle \left(p_G^!(M) \overset{!}{\otimes} a^!(N) \right), \qquad M \in \text{D-mod}(G), N \in \text{D-mod}(\mathcal{Y}),$$

where $p_G : G \times \mathcal{Y} \to G$ and $p_{\mathcal{Y}} : G \times \mathcal{Y} \to \mathcal{Y}$ are the projections.

Now Lemma C.7.1 is clear. It immediately implies the following statement.

Corollary C.7.2. *Let $\mathcal{Y}$ be a QCA stack equipped with an action of the monoidal stack $\mathbb{A}^1/\mathbb{G}_m$. Then the functors*

(C.7)
$$\pi_\blacktriangle : \text{D-mod}(\mathcal{Y}) \rightleftarrows \text{D-mod}(\mathcal{Y}^0) : i_{\mathrm{dR},*}\,, \qquad i^! : \text{D-mod}(\mathcal{Y}) \rightleftarrows \text{D-mod}(\mathcal{Y}^0) : \pi^!$$

form adjoint pairs with the adjunctions $\pi_\blacktriangle \circ i_{\mathrm{dR},*} \xrightarrow{\sim} \mathrm{Id}_{\text{D-mod}(\mathcal{Y}^0)}$ and $\mathrm{Id}_{\text{D-mod}(\mathcal{Y}^0)} \xrightarrow{\sim} i^! \circ \pi^!$ coming from the isomorphism $\pi \circ i \xrightarrow{\sim} \mathrm{Id}_{\mathcal{Y}^0}$.

C.8. Proof of Theorems C.4.1 and C.5.3

We will deduce them from Corollary C.7.2. First, let us make some general remarks.

If $\mathcal{Z}$ is an algebraic stack equipped with a morphism $\psi : \mathcal{Z} \to B\mathbb{G}_m$ then D-mod$(\mathcal{Z})$ is equipped with the following action of the tensor category $(\text{D-mod}(B\mathbb{G}_m), \overset{!}{\otimes})$:

$$M \otimes \mathcal{F} := \psi^!(M) \overset{!}{\otimes} \mathcal{F}, \qquad M \in \text{D-mod}(B\mathbb{G}_m), \quad \mathcal{F} \in \text{D-mod}(\mathcal{Z}).$$

If $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ is a morphism of QCA stacks over $B\mathbb{G}_m$ then the functors $f_\blacktriangle$ and $f^!$ are compatible with the above action of D-mod$(\mathbb{G}_m)$.

Lemma C.8.1. *Suppose we have a Cartesian diagram of QCA stacks*

$$\begin{array}{ccc} \widetilde{\mathcal{Z}} & \longrightarrow & \mathrm{pt} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathcal{Z} & \xrightarrow{\psi} & B\mathbb{G}_m \end{array}$$

Then one has a canonical isomorphism

(C.8) $$\mathcal{M}aps(p^!(\mathcal{F}_1), p^!(\mathcal{F}_2)) = \mathcal{M}aps(\mathcal{F}_1, A \otimes \mathcal{F}_2), \qquad \mathcal{F}_1, \mathcal{F}_2 \in \text{D-mod}(\mathcal{Z}),$$

where $A := \varphi_{\mathrm{dR},*}(k)[-2]$ *and* $A \otimes \mathcal{F}_2 := \psi^!(A) \overset{!}{\otimes} \mathcal{F}_2$.

Proof. $p^!(\mathcal{F}_1) = p^*_{\mathrm{dR}}(\mathcal{F}_1)[2]$, so

$$\mathcal{M}aps(p^!(\mathcal{F}_1), p^!(\mathcal{F}_2)) = \mathcal{M}aps(\mathcal{F}_1, p_{\mathrm{dR},*} \circ p^!(\mathcal{F}_2)[-2]) = \mathcal{M}aps(\mathcal{F}_1, A \otimes \mathcal{F}_2).$$

□

Now let us prove the assertion of Theorem C.5.3 concerning the pair $(\pi_\blacktriangle, i_{\mathrm{dR},*})$. The proof of the other assertion of Theorem C.5.3 is similar, and Theorem C.4.1 follows from Theorem C.5.3.

We have to show that for any $\widetilde{\mathcal{F}}_1 \in \text{D-mod}(\mathcal{Y})_\mu$ and $\widetilde{\mathcal{F}}_2 \in \text{D-mod}(\mathcal{Y}^0) = \text{D-mod}(\mathcal{Y}^0)_\mu$ the canonical map

(C.9)
$$\mathcal{M}aps(\widetilde{\mathcal{F}}_1, i_{\mathrm{dR},*}(\widetilde{\mathcal{F}}_2)) \to \mathcal{M}aps(\pi_\blacktriangle(\widetilde{\mathcal{F}}_1), \pi_\blacktriangle \circ i_{\mathrm{dR},*}(\widetilde{\mathcal{F}}_2)) = \mathcal{M}aps(\pi_\blacktriangle(\widetilde{\mathcal{F}}_1), \widetilde{\mathcal{F}}_2)$$

is an isomorphism. By the definition of the monodromic subcategory, we can assume that $\widetilde{\mathcal{F}}_1 = p^!(\mathcal{F}_1)$ for some $\mathcal{F}_1 \in \text{D-mod}(\mathcal{Y}/\mathbb{G}_m)$. Since the action of $\mathbb{G}_m$ on $\mathcal{Y}^0$ is trivial, we can also assume that $\widetilde{\mathcal{F}}_2 = (p^0)^!(\mathcal{F}_2)$ (here $p^0 : \mathcal{Y}^0 \to \mathcal{Y}^0/\mathbb{G}_m$). Applying Lemma C.8.1 for $\mathcal{Z} = \mathcal{Y}/\mathbb{G}_m$, $\widetilde{\mathcal{Z}} = \mathcal{Y}$ and for $\mathcal{Z} = \mathcal{Y}^0/\mathbb{G}_m$, $\widetilde{\mathcal{Z}} = \mathcal{Y}^0$ we get

$$\mathcal{M}aps(\widetilde{\mathcal{F}}_1, i_{\mathrm{dR},*}(\widetilde{\mathcal{F}}_2)) = \mathcal{M}aps(\mathcal{F}_1, A \otimes i'_{\mathrm{dR},*}(\mathcal{F}_2)), \quad i' : \mathcal{Y}^0/\mathbb{G}_m \to \mathcal{Y}/\mathbb{G}_m,$$
$$\mathcal{M}aps(\pi_\blacktriangle(\widetilde{\mathcal{F}}_1), \widetilde{\mathcal{F}}_2) = \mathcal{M}aps(\pi'_\blacktriangle(\mathcal{F}_1), A \otimes \mathcal{F}_2), \quad \pi' : \mathcal{Y}/\mathbb{G}_m \to \mathcal{Y}^0/\mathbb{G}_m.$$

The map (C.9) is a particular case of the canonical map

$$\mathcal{M}aps(\mathcal{F}_1, \mathcal{M} \otimes i'_{\mathrm{dR},*}(\mathcal{F}_2)) \to \mathcal{M}aps(\pi'_\blacktriangle(\mathcal{F}_1), M \otimes \mathcal{F}_2) \tag{C.10}$$

which is defined for any $\mathcal{M} \in \text{D-mod}(B\mathbb{G}_m)$. Applying Corollary C.7.2 to the action of $\mathbb{A}^1/\mathbb{G}_m$ on $\mathcal{Y}/\mathbb{G}_m$ we see that the map (C.10) is an isomorphism if $\mathcal{M} = \omega_{B\mathbb{G}_m}$. This implies that (C.10) is an isomorphism for any $\mathcal{M} \in \text{D-mod}_{\mathrm{coh}}(B\mathbb{G}_m)$ (because by connectedness of $\mathbb{G}_m$, $\text{D-mod}_{\mathrm{coh}}(B\mathbb{G}_m)$ is the smallest non-cocomplete triangulated subcategory of $\text{D-mod}(B\mathbb{G}_m)$ containing $\omega_{B\mathbb{G}_m}$). In particular, (C.10) is an isomorphism for $\mathcal{M} = A$, and we are done.

C.9. Sketch of another approach to Theorems C.4.1 and C.5.3

In Sect. C.8 we deduced Theorems C.4.1 and C.5.3 from Corollary C.7.2, which relies on the study of $\text{D-mod}(\mathbb{A}^1/\mathbb{G}_m)$ (see Lemma C.6.1). Here we sketch a slightly different approach, which is based on the study of $\text{D-mod}(\mathbb{A}^1)_\mu$ and does not rely on Corollary C.7.2.

Proposition C.9.1. *The subcategories* $\text{D-mod}(\mathbb{G}_m)_\mu \subset \text{D-mod}(\mathbb{G}_m)$ *and* $\text{D-mod}(\mathbb{A}^1)_\mu \subset \text{D-mod}(\mathbb{A}^1)$ *are closed under convolution. Moreover, they are monoidal categories. The unit object of* $\text{D-mod}(\mathbb{G}_m)_\mu$ *equals* $\underline{1}_\mu$*, where* $\underline{1}$ *is the unit object of* $\text{D-mod}(\mathbb{G}_m)$ *and*

$$\mathcal{M} \mapsto \mathcal{M}_\mu$$

is the "monodromization" functor $\text{D-mod}(\mathbb{G}_m) \to \text{D-mod}(\mathbb{G}_m)_\mu$*, i.e., the functor right adjoint to the embedding* $\text{D-mod}(\mathbb{G}_m)_\mu \hookrightarrow \text{D-mod}(\mathbb{G}_m)$*. The unit object of* $\text{D-mod}(\mathbb{A}^1)_\mu$ *has a similar description and can also be described as* $j_*(\underline{1}_\mu)$*, where* $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ *is the embedding.*

The adjunction mentioned in the proposition defines a canonical morphism $\varepsilon : \underline{1}_\mu \to \underline{1}$.

Remark C.9.2. Let $\Gamma_{\text{dR}}(\mathbb{G}_m, -)$ denote the de Rham cohomology functor $\text{D-mod}(\mathbb{G}_m) \to \text{Vect}$. The pair $(\underline{1}_\mu, \varepsilon)$ is uniquely characterized by the following properties: $\underline{1}_\mu \in \text{D-mod}(\mathbb{G}_m)_\mu$ and the map $\Gamma_{\text{dR}}(\mathbb{G}_m, \underline{1}_\mu) \to \Gamma_{\text{dR}}(\mathbb{G}_m, \underline{1}) = k$ induced by ε is an isomorphism. This implies that $\underline{1}_\mu$ is nothing but the "infinite Jordan block" $I^{-\infty,0}$ from [Be, Sect. 1.3]. In particular, the image of $\underline{1}_\mu$ under the Riemann-Hilbert correspondence is a sheaf (rather than a complex of sheaves).

Similarly to Lemma C.6.1, one has the following statement (which implies Lemma C.6.1).

Lemma C.9.3. *The idempotent algebra* $\underline{0}_\mu \in \text{D-mod}(\mathbb{A}^1)_\mu$ *is unital in* $\text{D-mod}(\mathbb{A}^1)_\mu$*.*

One shows that if $\mathcal{Y}$ is a QCA stack equipped with a $\mathbb{G}_m$-action then $\text{D-mod}(\mathbb{G}_m)_\mu$ acts on $\text{D-mod}(\mathcal{Y})_\mu$ as a monoidal category, i.e., $\underline{1}_\mu$ acts as identity. Similarly, if $\mathcal{Y}$ is equipped with an $\mathbb{A}^1$-action then one has the left and right monoidal action of $\text{D-mod}(\mathbb{A}^1)_\mu$ on $\text{D-mod}(\mathcal{Y})_\mu$. Now Theorem C.4.1 follows from Lemma C.9.3, and Theorem C.5.3 follows from C.4.1.

References

[Be] A. Beilinson, *How to glue perverse sheaves*, in: K-theory, arithmetic and geometry (Moscow, 1984–1986), 42–51, Lecture Notes in Math., **1289**, Springer, Berlin, 1987.

[BeV] A. Beilinson and V. Vologodsky, *A DG guide to Voevodksy's motives*, arXiv:math/0604004.

[Beh] K. Behrend, *Semi-stability of reductive group schemes over curves*, Math. Ann. **301** (1995), no. 2, 281–305.

[Beh1] K. Behrend, *The Lefschetz trace formula for the moduli stack of principal bundles* (Ph.D. thesis), available at http://www.math.ubc.ca/~behrend/preprints.html.

[BoWa] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Mathematical Surveys and Monographs, **67**, AMS, 2000.

[BFN] D. Ben-Zvi, J. Francis and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, arXiv:0805.0157.

[Br] T. Braden, *Hyperbolic localization of Intersection Cohomology*, Transformation Groups **8** (2003), no. 3, 209–216; also: arXiv:math/020225.

[BG] A. Braverman and D. Gaitsgory, *Geometric Eisenstein series*, Invent. Math. **150** (2002), 287–384.

[Dr] V. Drinfeld, *On the Langlands retraction*, arXiv:1301.1940.

[DrGa1] V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*, GAFA **23** (2013), 149–294; also arXiv:1108.5351.

[DrGa2] V. Drinfeld and D. Gaitsgory, *Geometric constant term functor(s)*, arXiv:1311.2071.

[DrGa3] V. Drinfeld and D. Gaitsgory, *Compact generation of the category of D-modules on the stack of G-bundles on a curve*, arXiv:1112.2402, version 4.

[DT] R. Dyckhoff and W. Tholen, *Exponentiable morphisms, partial products, and pullback complements*, J. Pure Appl. Algebra **49** (1987), 103–116.

[Ga1] D. Gaitsgory, *Ind-coherent sheaves*, Moscow Mathematical J. **13** (2013), 399–528; arXiv:1105.4857.

[Ga2] D. Gaitsgory, *A "strange" functional equation for Eisenstein series and Verdier duality on the moduli stack of bundles*, arXiv:1404.6780.

[Ga3] D. Gaitsgory, *Functors given by kernels, adjunctions, and duality*, arXiv:1303.2763.

[GL:DG] Notes on Geometric Langlands, *Generalities on DG categories*, available from http://www.math.harvard.edu/~gaitsgde/GL/.

[GL:Stacks] Notes on Geometric Langlands, *Stacks*, available from http://www.math.harvard.edu/~gaitsgde/GL/.

[GR] D. Gaitsgory and N. Rozenblyum, *Crystals and D-modules*, PAMQ **10**, (2014), no. 1, 57–155; also arXiv:1111.2087.

[He] J. Heinloth, *Bounds for Behrend's conjecture on the canonical reduction*, arXiv:0712.0692.

[HN] G. Harder and M. S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Ann. **212** (1974/75), 215–248.

[JRXY] K. Joshi, S. Ramanan, E. Z. Xia, and J.-K. Yu, *On vector bundles destabilized by Frobenius pull-back*, Compos. Math. **142** (2006), no. 3, 616–630.

[KL] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184.

[KeLa] G. M. Kelly and F. W. Lawvere, *On the complete lattice of essential localizations*, Bulletin de la Société Mathématique de Belgique, Série A, **41** (1989), no 2, 289–319.

[La] R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, in: Representation theory and harmonic analysis on semi-simple Lie groups, edited by P. Sally and D. Vogan, AMS, 1989, 101–170.

[Lu1] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, **170**, Princeton University Press, Princeton, NJ, 2009.

[Lu2] J. Lurie, *Higher Algebra*, available at http://www.math.harvard.edu/~lurie/.

[N] A. Neeman, *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 5, 547–566.

[nLab] Available at http://ncatlab.org/.

[P] C. Pauly, *Orthogonal bundles over curves in characteristic two*, Vector bundles and complex geometry, 131–140, Contemp. Math., **522**, Amer. Math. Soc., Providence, RI, 2010.

[RR] S. Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Math. J. **36** (1984), no. 2, 269–291.

[R1] A. Ramanathan, *Moduli for principal bundles*, in: Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), 527–533, Lecture Notes in Math., **732**, Springer, Berlin, 1979.

[R2] A. Ramanathan, *Moduli for principal bundles over algebraic curves. I*, Proc. Indian Acad. Sci. Math. Sci. **106** (1996), no. 3, 301–328.

[R3] A. Ramanathan, *Moduli for principal bundles over algebraic curves. II*, Proc. Indian Acad. Sci. Math. Sci. **106** (1996), no. 4, 421–449.

[Roz] N. Rozenblyum, *Filtered colimits of categories*, available from http://www.math.harvard.edu/~gaitsgde/GL/.

[Sch] S. Schieder, *On the Harder-Narasimhan stratification of* Bun_G *via Drinfeld's compactifications*, arXiv:1212.6814.

[Sh1] S. Shatz, *Degeneration and specialization in algebraic families of vector bundles*, Bull. Amer. Math. Soc. **82** (1976), no. 4, 560–562.

[Sh2] S. Shatz, *The decomposition and specialization of algebraic families of vector bundles*, Compositio Math. **35** (1977), no. 2, 163–187.

[Sp] T. A. Springer, *A purity result for fixed point varieties in flag manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), no. 2, 271–282.

[Ve] J.-L. Verdier, *Spécialisation de faisceaux et monodromie modérée*, in: Analysis and topology on singular spaces, II, III (Luminy, 1981), 332–364, Astérisque, **101–102**, Soc. Math. France, Paris, 1983.

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