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Algebraic equation of geodesics on the 2D Euclidean space with an exponential density function

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Given two points s and t on the two-dimensional Euclidean space, the straight-line segment connecting s and t gives the shortest path, or geodesic, between them. However, when the 2D plane is equipped with a non-uniform density function, generally one cannot find a closed-form solution to characterize the shortest path, assuming that the endpoints are given. We observe that when the 2D plane is equipped with an exponential-type density function, the algebraic equation of a geodesic path between two points, as well as the corresponding length, can be found explicitly by a variational approach. We systematically give both the theoretical and experimental results in this paper. We also extend the approach to compute the geodesic problem on a polyhedral surface with a pre-defined density function.

1. Introduction

The density-weighted distance, or geodesic distance, is used to measure the distance between two points where the sub-path passing through the high-density area is given more influence. Density based distance is central to many research fields including nonlinear interpolation and clustering [15], applied statistics and data analysis [3, 5, 22], transmission via non-uniform media [9] and feature-preserving adaptive remeshing [2, 26].

Generally speaking, when the domain of interest is equipped with a uniform density function, geodesics usually have a simple form. For example, it's a common sense that the shortest path between two points in $\mathbb{R}^n$ is exactly a straight-line segment. However, if the density function is non-uniform, the distance query problem becomes highly non-trivial. To our best knowledge, no explicit algebraic equation (or parametric equation array) has

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been reported for solving the geodesic problem with a non-uniform density function. Researchers have to use some approximation techniques to solve the density based distance problems. Once the domain has been discreticized, Dijkstra's algorithm [4, 8] and the fast marching method [10] are the two most commonly used techniques.

In this paper, we observe that when the 2D plane is associated with an exponential density function, the algebraic equation of a geodesic path between two points (x_1, y_1) and (x_2, y_2), as well as the corresponding length, can be found by a variational approach. We systematically give a set of theoretical and experimental results assuming the density function to be e^{-y}. Our observations include

1) When $x_1 = x_2$, the geodesic is given by the straight-line between (x_1, y_1) and (x_2, y_2).

2) The geodesic equation is $\cos(x - x_0) = e^{y - y_0}$ if $|x_1 - x_2| < \pi$, where x_0 and y_0 are two parameters determined by the boundary condition. Furthermore, the length of the weighted shortest path is $e^{-y_2} \sin(x_2 - x_0) - e^{-y_1} \sin(x_1 - x_0)$.

3) When $|x_2 - x_1| \geq \pi$, the geodesic path between the two endpoints consists of two rays, one being $(x_1, y_1) \to (x_1, +\infty)$ and the other being $(x_2, y_2) \to (x_2, +\infty)$.

4) Let $\Gamma(x_1, y_1, x_2, y_2)$ be the geodesic curve between (x_1, y_1) and (x_2, y_2). When we move the two endpoints in $(\delta x, \delta y)$ at the same time, the resulting geodesic curve can be obtained by translating $\Gamma(x_1, y_1, x_2, y_2)$ in $(\delta x, \delta y)$.

5) Let y_1, y_2 be fixed. The density-weighted distance is a non-decreasing function with regard to $|x_1 - x_2|$ assuming $|x_1 - x_2| \leq \pi$. When $|x_1 - x_2| \geq \pi$, the density-weighted distance is $e^{-y_1} + e^{-y_2}$.

Finally, when there is a pre-defined density function on a given polyhedral surface, generally one cannot compute exact geodesic paths even if the density function is approximated by a piece-wise linear function. In this paper, we find that it is reasonable to approximate the density function with an exponential function on a sufficiently small triangle. In this way, we can trace geodesic paths accurately even if the input surface is equipped with a non-uniform density function.

2. Formulation

2.1. Problem statement

Suppose that an exponential function $\rho(x, y) = e^{-y}$ is defined on $\mathbb{R}^2$. Given two points $s(x_1, y_1), t(x_2, y_2) \in \mathbb{R}^2$, our task is to find the ρ-weighted shortest path, or geodesic, that can be represented by an algebraic equation formed as

$$g(x, y) = 0 \tag{1}$$

subject to the boundary condition

$$\begin{cases} g(x_1, y_1) = 0 \\ g(x_2, y_2) = 0. \end{cases} \tag{2}$$

Or alternatively, it suffices if we can find the parameterized form $\Gamma(t) = (x(t), y(t)) : [0, 1] \mapsto \mathbb{R}^2$:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \tag{3}$$

subject to

$$\begin{cases} x(0) = x_1, \quad y(0) = y_1 \\ x(1) = x_2, \quad y(1) = y_2. \end{cases} \tag{4}$$

The length of the path $\Gamma(t) : [0, 1] \mapsto \mathbb{R}^2$ is

$$\begin{aligned} L(x_1, y_1, x_2, y_2) &= \int_0^1 \rho(x(t), y(t)) \sqrt{x'^2(t) + y'^2(t)} dt \\ &= \int_0^1 e^{-y(t)} \sqrt{x'^2(t) + y'^2(t)} dt. \end{aligned} \tag{5}$$

2.2. Geodesic equation

In the following, we make an arbitrary assumption that the shortest path be represented by $y = y(x)$, while leaving the discussion of various situations into the next subsection. Our task is to minimize

$$L(y) = \int_0^1 e^{-y}\sqrt{1+y'^2}\mathrm{d}x \triangleq \int_0^1 P(y,y')\mathrm{d}x, \tag{6}$$

where $y_1 = y(x_1), y_2 = y(x_2)$. According to the variational theory, i.e.,

$$P - y'\frac{\partial P}{\partial y'} = C, \tag{7}$$

we have

$$e^{-y}\sqrt{1+y'^2} - \frac{y'^2 e^{-y}}{\sqrt{1+y'^2}} = C \tag{8}$$

or

$$Ce^{y}\sqrt{1+y'^2} = 1 \tag{9}$$

Let $y' = \tan t, t \in [-\pi/2, \pi/2]$, we get

$$y = \ln\cos t - \ln C \triangleq \ln\cos t + y_0. \tag{10}$$

Thus

$$\frac{dx}{dt} = \frac{1}{y'}\cdot\frac{dy}{dt} = -1, \tag{11}$$

or $x = -t + x_0$. Therefore, the geodesic curve can be parameterized into

$$\begin{cases} x = -t + x_0, \\ y = \ln\cos t + y_0. \end{cases} \tag{12}$$

Obviously, it can also be written as the following algebraic equation:

$$y = \ln\cos(x_0 - x) + y_0, \tag{13}$$

or

$$\cos(x - x_0) = e^{y-y_0}. \tag{14}$$

Remark. Note that here x_0, y_0 are two unknown constants and they can be determined by two user-specified endpoints or "one endpoint, one tangent direction".

2.3. Three situations

Without doubt, whether Eq. (14) makes sense depends on the setting of endpoints. In the following, we discuss the boundary condition on three cases: (1) $x_1 = x_2$, (2) $0 < |x_1 - x_2| < \pi$ and (3) $|x_1 - x_2| \geq \pi$.

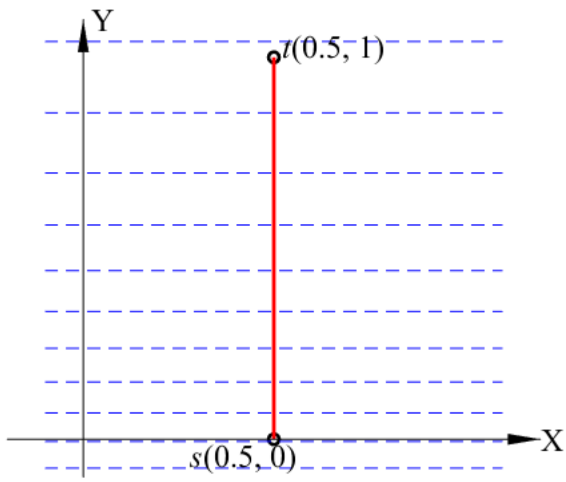


Figure 1: The shortest path when $x_1 = x_2$ on the 2D plane with a density function $\rho(x, y) = e^{-y}$.

2.3.1. $\boldsymbol{x_1 = x_2}$. First, let us consider the case of $x_1 = x_2$. In this case, the parameterized form becomes

$$\begin{cases} x = c \\ y = y(t), \quad 0 \leq t \leq 1, \end{cases} \tag{15}$$

and the length becomes

$$\begin{aligned} L(x_1, y_1, x_2, y_2) &= \int_0^1 e^{-y(t)} \sqrt{x'^2(t) + y'^2(t)} dt \\ &\geq \int_0^1 e^{-y(t)} |y'(t)| dt \\ &\geq |\int_0^1 e^{-y(t)} y'(t) dt| \\ &= |e^{-y_1} - e^{-y_2}|. \end{aligned} \tag{16}$$

At the same time, it is easy to know that when L gets minimized, $x'(t) = 0$ and $y'(t)$ doesn't change its sign for $0 \leq t \leq 1$, which implies that the shortest path is a straight-line segment between (x_1, y_1) and (x_2, y_2), as shown in Figure 1. Furthermore, it is easy to show that the weighted path length is exactly $|e^{-y_1} - e^{-y_2}|$.

Lemma 1. *When $x_1 = x_2$, the shortest path between (x_1, y_1) and (x_2, y_2) is exactly the straight-line segment between (x_1, y_1) and (x_2, y_2), whose length is $|e^{-y_1} - e^{-y_2}|$.*

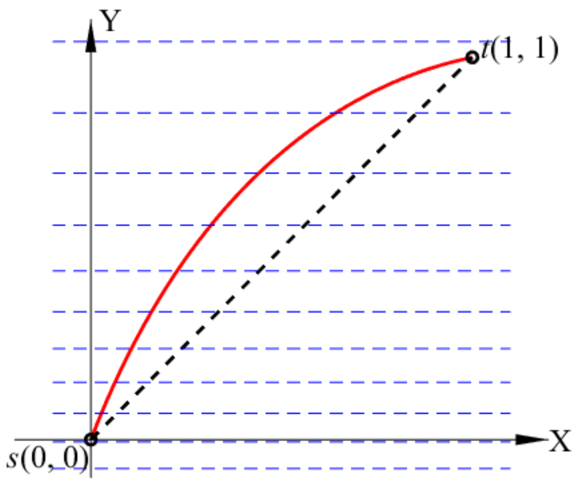


Figure 2: The shortest path when $0 < |x_1 - x_2| < \pi$ on the 2D plane with a density function $\rho(x, y) = e^{-y}$.

2.3.2. $0 < |x_1 - x_2| < \pi$. Without loss of generality, we assume that $0 < x_2 - x_1 < \pi$. Prior to the following discussion, we introduce a lemma first.

Lemma 2. *Suppose $0 < x_2 - x_1 < \pi$ and $\lambda > 0$ is a constant. There exists a unique $x_0^* \in (x_1 - \pi/2, x_2 - \pi/2)$ such that*

$$h(x_0) = \cos(x_2 - x_0) - \lambda \cos(x_1 - x_0) = 0.$$

Furthermore, all the possible values of x_0 can be represented by $x_0 = x_0^* + k\pi$, $k = 0, \pm 1, \pm 2 \cdots$.

Proof. According to

$$h(x_1 - \pi/2) = \sin(x_2 - x_1) > 0$$

and

$$h(x_2 - \pi/2) = -\lambda \sin(x_2 - x_1) < 0,$$

we can verify the existence of the roots between $x_1 - \pi/2$ and $x_2 - \pi/2$.

On the other hand, $h(x_0)$ is a periodic sine function with a least period of 2π, and has two roots in each period. Therefore, the uniqueness of $x_0 \in (x_1 - \pi/2, x_2 - \pi/2)$ is immediate from the fact that the interval length is less than π. □

Theorem 3. *Suppose* $0 < x_2 - x_1 < \pi$. *The shortest path exists and meets the algebraic equation* $\cos(x - x_0) = e^{y-y_0}$ *subject to Eq. (17), and its length is given by*

$$L_{\min}(x_1, y_1, x_2, y_2) = e^{-y_2} \sin(x_2 - x_0) - e^{-y_1} \sin(x_1 - x_0),$$

where x_0 *can be found from Eq. (18). Furthermore, when* $x_2 - x_1 \to \pi-$, $L_{\min}(x_1, y_1, x_2, y_2) \to e^{-y_1} + e^{-y_2}$.

Proof. Consider the boundary condition:

$$\begin{cases} \cos(x_1 - x_0) = e^{y_1 - y_0}, \\ \cos(x_2 - x_0) = e^{y_2 - y_0}. \end{cases} \tag{17}$$

By eliminating y_0, we get

$$\cos(x_2 - x_0) - \cos(x_1 - x_0) \times e^{y_2 - y_1} = 0. \tag{18}$$

Lemma 2 shows that Eq. (18) has one solution $x_0 \in (x_1 - \pi/2, x_2 - \pi/2)$, and thus $y_0 = y_1 - \ln \cos(x_1 - x_0)$ can be found since $-\pi/2 < x_1 - x_0 < \pi/2$. This asserts the existence of the geodesic path meeting the given boundary condition.

According to Eq. (14),

$$\begin{aligned} L_{\min}(x_1, y_1, x_2, y_2) &= \int_{x_1}^{x_2} e^{-y}\sqrt{1+y'^2}\mathrm{d}x \\ &= \int_{x_1}^{x_2} e^{-\ln\cos(x-x_0)-y_0} \times \sec(x-x_0)\mathrm{d}x \\ &= e^{-y_0}\int_{x_1}^{x_2} \sec^2(x-x_0)\mathrm{d}x \\ &= e^{-y_0}[\tan(x_2-x_0) - \tan(x_1-x_0)]. \end{aligned} \tag{19}$$

Considering that Eq. (17) implies $e^{-y_0} = e^{-y_1}\cos(x_1 - x_0) = e^{-y_2}\cos(x_2 - x_0)$, we have

$$\begin{aligned} L_{\min}(x_1, y_1, x_2, y_2) &= e^{-y_0}[\tan(x_2-x_0) - \tan(x_1-x_0)] \\ &= e^{-y_2}\sin(x_2-x_0) - e^{-y_1}\sin(x_1-x_0) \\ &\le e^{-y_1} + e^{-y_2}. \end{aligned} \tag{20}$$

$x_2 - x_1 \to \pi-$ can be replaced by $x_2 = x_1 + \pi - \epsilon$, where $\epsilon \to 0+$. If x_0 is set to $x_1 - \pi/2$, the left side of Eq. (18) becomes $-\sin\epsilon$, which is negative. If x_0 is set to $x_1 - \pi/2 - \epsilon$, the left side of Eq. (18) becomes $\sin\epsilon \times e^{y_2-y_1}$, which is positive. Therefore, it is reasonable to assume $x_0 \to x_1 - \pi/2$. (Strictly speaking, x_0 has infinitely many choices; See Lemma 2. We only consider one of the values close to $x_1 - \pi/2$.) It is easy to show $\sin(x_2 - x_0) \to 1$ and $\sin(x_1 - x_0) \to -1$ when $\epsilon \to 0$. Taking them into Eq. (20), we finish the proof. □

Figure 2 shows a geodesic path between $(0, 0)$ and $(1, 1)$ on the 2D plane with a density setting $\rho(x, y) = e^{-y}$. From Eq. (17), we can easily observe that the geodesic path is unchanged up to a translation.

Corollary 1. *Suppose* $0 < x_2 - x_1 < \pi$. *Let* $\Gamma(x_1, y_1, x_2, y_2)$ *denote the geodesic path between* (x_1, y_1) *and* (x_2, y_2). *Then the geodesic path* $\Gamma(x_1 + \delta x, y_1 + \delta y, x_2 + \delta x, y_2 + \delta y)$ *can be obtained by a translation of* $(\delta x, \delta y)$ *on* $\Gamma(x_1, y_1, x_2, y_2)$.

Besides, we observe the length of the geodesic path will increase, or at least remains unchanged, if we fix y_1, y_2 while moving x_1 and x_2 away from each other.

Theorem 4. *Suppose* $0 \le x_2 - x_1 < \pi$. $L_{\min}(x_1, y_1, x_2, y_2)$ *is non-decreasing with regard to* $x_2 - x_1$.

Proof. Suppose that $0 < x_2^{(1)} - x_1 < x_2^{(2)} - x_1 < \pi$. Let

$$a = \frac{x_2^{(1)} - x_1}{x_2^{(2)} - x_1}.$$

Then the mapping $(x, y) \longrightarrow (a(x - x_1) + x_1, y)$ is able to transform the geodesic $\Gamma(x_1, y_1, x_2^{(2)}, y_1)$ to a path (may not be the shortest) between (x_1, y_1) and $(x_2^{(1)}, y_1)$. Let $L_{\text{scaled}}(x_1, y_1, x_2^{(1)}, y_1)$ be the length of the resulting curve. We have

$$\begin{aligned} L_{\min}(x_1, y_1, x_2^{(1)}, y_1) &\le L_{\text{scaled}}(x_1, y_1, x_2^{(1)}, y_1) \\ &= \int_{x_1}^{x_2^{(1)}} e^{-y}\sqrt{1 + y'^2}\mathrm{d}x \\ &= \int_{x_1}^{x_2^{(2)}} ae^{-y}\sqrt{1 + \frac{y'^2}{a^2}}\mathrm{d}x \\ &= \int_{x_1}^{x_2^{(2)}} e^{-y}\sqrt{a^2 + y'^2}\mathrm{d}x \\ &\le \int_{x_1}^{x_2^{(2)}} e^{-y}\sqrt{1 + y'^2}\mathrm{d}x \\ &= L_{\min}(x_1, y_1, x_2^{(2)}, y_1). \end{aligned}$$

□

2.3.3. $|x_1 - x_2| \ge \pi$. Without loss of generality, we assume $x_2 - x_1 \ge \pi$. It's impossible that for all $x \in [x_1, x_2]$, $\cos(x - x_0)$ is always positive. Therefore, Eq. (14) cannot give the shortest path for the case of $|x_1 - x_2| \ge \pi$.

As Theorem 3 states, when $x_2 - x_1 \to \pi-$, the path length approaches $e^{-y_1} + e^{-y_2}$. In fact, when $x_2 - x_1 \ge \pi$, the geodesic path between the two endpoints can be understood in this way:

$$(x_1, y_1) \xrightarrow{e^{-y_1}} (x_1, +\infty) \xrightarrow{0} (x_2, +\infty) \xrightarrow{e^{-y_2}} (x_2, y_2).$$

The first segment is e^{-y_1} in length and the last segment is e^{-y_2} in length, which is immediate from Lemma 1. See an example in Figure 3.

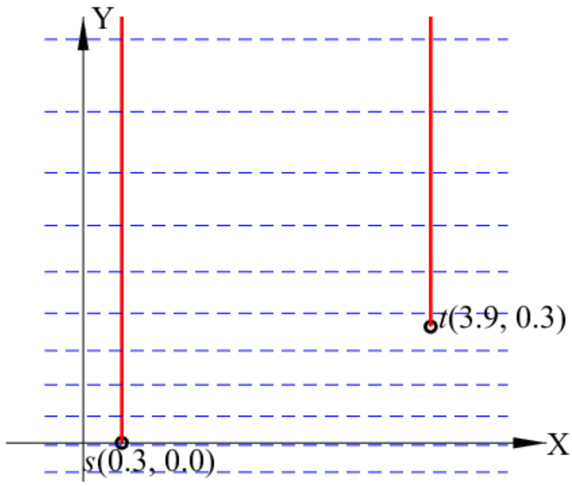
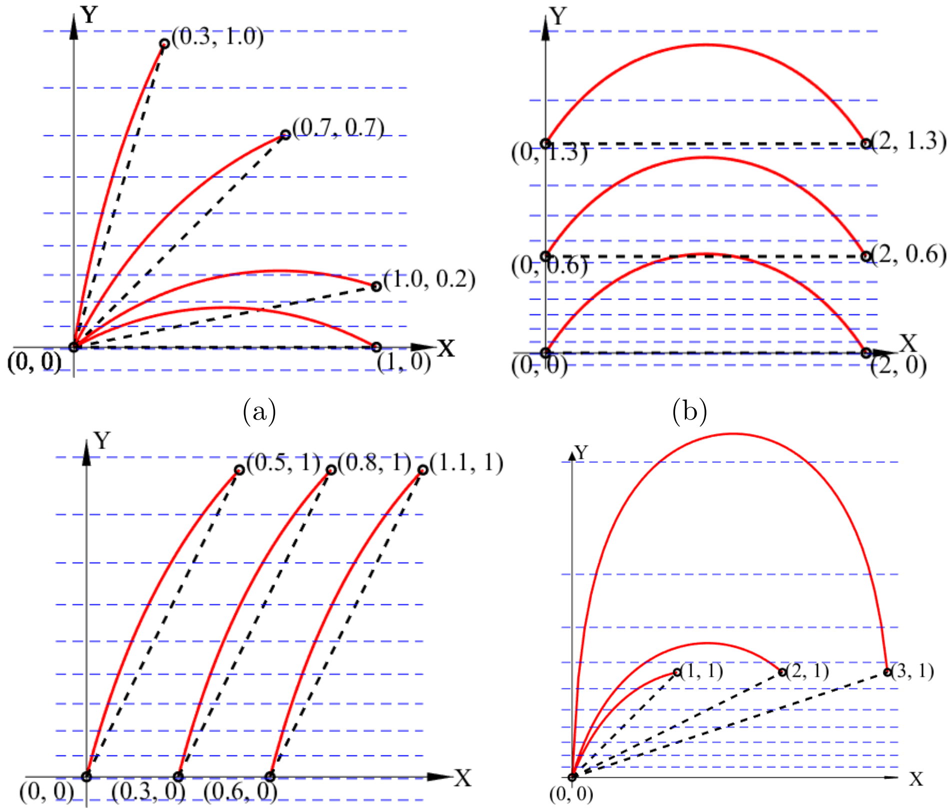


Figure 3: The shortest path when $|x_1 - x_2| \geq \pi$ on the 2D plane with a density function $\rho(x, y) = e^{-y}$.

2.4. Examples

We give more examples to help readers understand the geodesic equation on the 2D plane with an exponential density function. Figure 4(a) shows 4 geodesics rooted at the origin point. It can be seen that when we fix one endpoint while moving the other endpoint (assuming $|x_2 - x_1| < \pi$), the resulting geodesics don't intersect each other except at the starting point. Figure 4(b-c) shows that the geodesic shape doesn't have any change if we move the endpoints with the same quantity, whether vertically or horizontally. Figure 4(d) shows that when we increase the gap between x_1 and x_2, the geodesic tends to have an arched shape. When $|x_2 - x_1| = \pi$, the geodesic degenerates into two rays, one being $x = x_1, y \geq y_1$, and the other being $x = x_2, y \geq y_2$.



(a) (b) (c) (d)

Figure 4: More examples.

3. Extension to polygonal surfaces with non-uniform density setting

Computing geodesic distances and paths has broad applications in computer graphics, such as sampling [28], non-rigid registration [6], surface classification [7], geodesic Voronoi diagram [12, 24], intrinsic Delaunay triangulation [11], shape segmentation [21], texture mapping [19], parameterization [17, 29], image segmentation [13], and many others. To our best knowledge, existing exact geodesic algorithms [1, 14, 16, 18, 20, 23, 25, 27] fail to deal with the case that the input surface is equipped with a non-uniform density function. Although some approximate algorithms [10] support a non-uniform density setting, they cannot achieve a high accuracy requirement.

Rather than viewing the density function as a piece-wise linear function, we approximate the density function in each triangle with an exponential function. Suppose the density function at v_1, v_2, v_3, the vertices of a triangle face, is respectively $\rho_{v_1}(x_1, y_1), \rho_{v_2}(x_2, y_2), \rho_{v_3}(x_3, y_3)$. Considering that there may be a rotation and a translation with regard to the coordinate system, we have

$$\begin{cases} e^{-c(-\sin\theta\cdot x_1+\cos\theta\cdot y_1+y_0)} = \rho_{v_1} \\ e^{-c(-\sin\theta\cdot x_2+\cos\theta\cdot y_2+y_0)} = \rho_{v_2} \\ e^{-c(-\sin\theta\cdot x_3+\cos\theta\cdot y_3+y_0)} = \rho_{v_3}, \end{cases} \tag{21}$$

from which we can solve the three unknown constants c, y_0 and θ. We can further trace a geodesic path according to Eq. (14); See Figure 5 for illustration.

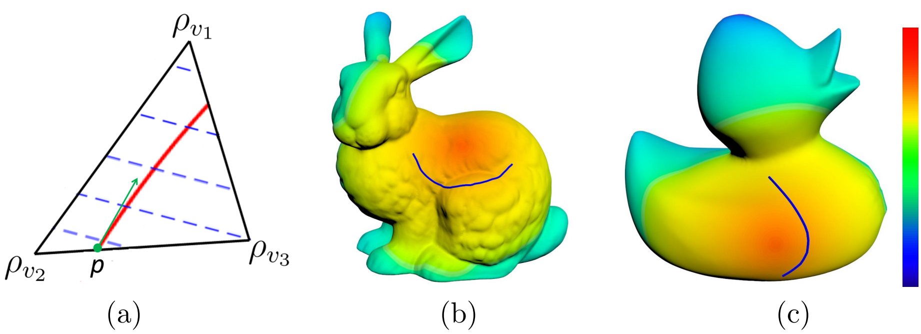


Figure 5: Tracing geodesics on surfaces with a pre-defined density function, where the warm color means high density value. (a) Approximating the density function with an exponential function inside a triangle to facilitate tracing geodesic paths. (b-c) Solving the initial value problem of the discrete geodesics assuming a non-uniform density function.

4. Conclusions and future works

In this paper, we found that the algebraic equation of geodesic exists on the 2D plane associated with an exponential density function. Based on the variational approach, we systematically give a set of theoretical and experimental results. It's easy to extend the conclusions in this paper to the case where the density function is set to $\rho(x, y) = e^{ax+by}$ in 2D or $\rho(x, y) = e^{ax+by+cz}$ in 3D. We also use the result to solve the initial value problem of the discrete geodesics on surfaces with a pre-defined density function.

In the future, we will investigate whether our algorithm can be extended to surfaces with general anisotropic metrics.

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