Communications in Information and Systems
Volume 15, Number 4, 489–519, 2015

Properties for a class of multi-type mean-field models

Tuan Anh Hoang and George Yin

Dedicated to Peter Caines on the occasion of his 70th birthday

This work develops properties of a class of multi-type mean-field models represented by solutions of stochastic differential equations with random switching. Using stochastic calculus, we prove the existence and uniqueness of the global solution and its positivity. In addition to deriving bounds on the moments of the solutions, we derive upper and lower bounds of the growth, and decay rates of the solutions.

Keywords and Phrases: mean-field model, moment bound, recurrence.

1. Introduction

Mean-field models are originated from statistical mechanics and physics (for instance, in the derivation of Boltzmann or Vlasov equations in kinetic gas theory). They are concerned with many particle systems having weak interactions. To overcome the complexity of interactions due to a large number of particles (or many body problems), all interactions with each particle are replaced by a single average interaction. Studying the limits of mean-field models has been a long-standing problem and presents many technical difficulties. Some of questions were concerned with characterization of the limit of the empirical probability distribution of the systems when the size of the systems tend to infinity, the fluctuations and large deviations of the systems around the limit. The first breakthroughs were due to Henry McKean; see, e.g., [25, 26]. The problems were then subsequently investigated in various contexts by a host of authors such as Braun and Hepp [4], Dawson [9, 12],

This research was supported in part by the Air Force Office of Scientific Research under FA9550-15-1-0131.

Tanaka and collaborators [31–34], etc. A very nice and systematic introduction to the topic and many related problems can be found in Sznitman [30].

There has been interest in mean-field models in the past decades. Initiated independently by Huang, Malhamé, and Caines [14, 15], and Lasry and Lions [21–23], mean-field differential games have drawn much attentions and became a very active area of research; see Huang et al. [16], Nguyen and Huang [27], Nourian and Caines [29], Bensoussan et al. [2], Carmona et al. [6], among others. Along with the renewed interest in the classical models, the studies for some other type of mean-field models were also carried out; see for example, the regime-switching models [35] and models with two-time scales [13]. Another type of mean-field models being investigated recently is the class of models with multi-types. In these models, the particles come from finitely many different populations or types, which appear in social sciences [8], statistical mechanics [7], neurosciences [1], as well as finance [3]. In particular, in [1], the authors established a result on law of large numbers and propagation of chaos for a class of such models. In [5], the authors study fluctuations about the law of large numbers of a class of multi-type mean-field models where there is one more driving Brownian motion used by all particles in the system. In [3], the authors considered a sector-wise allocation in a portfolio consisting of a very large number of stock modeled by a multi-type mean-field model. The authors established the existence and uniqueness and the McKean-Vlasov limit of the model. However, in the aforementioned study, the regularity and asymptotic properties with a fixed and finite number of particles have not been considered yet. This work aims to provide a better understanding about the system in this direction.

The rest of the paper is organized as follows. Section 2 provides the detailed problem formulation. Section 3 proceeds with the main results. We prove the existence and uniqueness of the strong solution for the model and establish certain bounds on the moments. Section 4 concentrates of asymptotic bounds. In addition to positive recurrent, we derive estimates on the upper and lower bounds of the growth, and decay rates of the solutions. We conclude the paper with some further remarks in Section 5.

2. Problem formulation

This section provides the set up of our problem and gives the assumptions and notations used throughout the paper. Let $(\Omega, \{\mathcal{F}_t\}_{\{t\geq 0\}}, \mathcal{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e. the filtration is increasing, right continuous and complete. For $N \geq 1$, denote $\mathbb{R}^N_+ :=$

$\{(x_1, \ldots, x_N) : x_i > 0, i = 1, \ldots, N\}$; and for each $x \in \mathbb{R}^N$, denote $S(x) = \sum_{i=1}^{N} x_i$, and denote by $|x|$ the Euclidean norm of x. We will use C to denote a generic constant whose value may change from appearance to appearance in the paper. Let $x_1^N, \ldots, x_N^N$ be $\mathbb{R}$-valued stochastic processes that represent trajectories of N particles, each belongs to one of K types (populations) with membership map denoted by $\mathbf{p} : \{1, \ldots, N\} \to \{1, \ldots, K\} := \mathbf{K}$. So the i-th particle is of type α if $\mathbf{p}(i) = \alpha$. For $\alpha \in \mathbf{K}$, let $\mathbf{N}_\alpha = \{i \in \mathbf{N} : \mathbf{p}(i) = \alpha\}$ and we use N_α to denote the number of particles belonging to the α-th population, namely $N_\alpha = |\mathbf{N}_\alpha|$, the cardinal of $\mathbf{N}_\alpha$. The dynamics of the system are given by a collection of stochastic differential equations. The N stochastic processes interact with each other through the coefficients of the SDEs which, for the i-th process, with $\mathbf{p}(i) = \alpha$, depend on not only the i-th state process and the α-type but also the empirical measures defined as follows: For any Borel set $A \subset \mathbb{R}$,

$$\begin{aligned}\mu^{\gamma,N}(t, A) &= \frac{1}{N_\gamma} \sum_{j:\mathbf{p}(j)=\alpha} \delta_{x_j^N(t)}(A) \\ &= \frac{1}{N_\gamma}\{\#\text{ of } \ j\text{'s} : j \in \{1, \ldots, N\}, \\ &\qquad\qquad \mathbf{p}(j) = \gamma, x_j(t) \in A\}, \quad \gamma \in \mathbf{K}.\end{aligned} \tag{2.1}$$

To simplify the notation, we shall suppress the A in what follows and write it simply as $\mu^{\gamma,N}(t)$ instead. To be more precise, for $i \in \mathbf{N}_\alpha, \alpha \in \mathbf{K}$

$$\begin{aligned} dx_i^N(t) &= x_i^N(t) \sum_{\gamma=1}^{K} \langle b_{\alpha\gamma}(x_i^N(t), \cdot), \mu^{\gamma,N}(t)\rangle dt \\ &\qquad + x_i^N(t) \sum_{\gamma=1}^{K} \langle \sigma_{\alpha\gamma}(x_i^N(t), \cdot), \mu^{\gamma,N}(t)\rangle dw_\gamma(t) \\ x_i^N(0) &= x_{i,0}^N. \end{aligned} \tag{2.2}$$

where $b_{\alpha\gamma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\sigma_{\alpha\gamma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are suitable functions where the related conditions will be specified later. Here $w = (w_1(\cdot), \ldots, w_K(\cdot))$ is an $\mathbb{R}^K$-valued standard Brownian motion. We also use the notation $\langle f, \mu\rangle$ to denote the integral $\int f d\mu$ of a function f with respect to a probability measure μ. Since we concentrate on the asymptotic properties of the fixed and finite size system, we will, from now on, suppress the superscript N in the solution x_i^N as well as the induced empirical measure $\mu^{\gamma,N}$.

Throughout the paper, we use $\mathcal{L}$ to denote the infinitesimal generator associated with the system (2.2). For any sufficient smooth real-valued function $V : \mathbb{R}_+^N \to \mathbb{R}$ (e.g., $V \in C_c^2(\mathbb{R}^N)$), $\mathcal{L}V$ is defined by

$$\begin{aligned}\mathcal{L}V(x) &= \sum_{i=1}^{N} \frac{\partial V}{\partial x_i}(x) x_i \sum_{\gamma=1}^{K} \langle b_{\mathbf{p}(i)\gamma}(x_i, \cdot), \mu^\gamma \rangle \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2 V}{\partial x_i x_j} x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j),\end{aligned} \tag{2.3}$$

where

$$\tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j) = \sum_{\gamma=1}^{K} \langle \sigma_{\mathbf{p}(i)\gamma}(x_i, \cdot), \mu^\gamma \rangle \langle \sigma_{\mathbf{p}(j)\gamma}(x_j, \cdot), \mu^\gamma \rangle$$

and μ^γ is the empirical measure induced by components of x that belong to the population γ as was defined in (2.1). We assume the following assumptions on the drift and diffusion part of the system (2.2) throughout the paper.

A1. For all $\alpha, \gamma \in \mathbf{K}$, the function $b_{\alpha\gamma}$ are locally Lipschitz, continuous and there exists $c_2 \in \mathbb{R}$ and $d_2 \in \mathbb{R}_+$ such that

$$b_{\alpha\gamma}(x, y) \le c_2 - d_2 y, \quad \text{for all } x, y \in \mathbb{R}_+,$$

where $\mathbb{R}_+$ is the set of positive real numbers.

The above condition indicates that the growth of $b_{\alpha,\gamma}$ can be controlled by a linear function. Along with the above condition, we will discuss the results corresponding to the two following set of conditions regarding the diffusion coefficients.

A2. For all $\alpha, \gamma \in \mathbf{K}$, the functions $\sigma_{\alpha,\gamma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous. Moreover, for all $\alpha, \gamma \in \mathbf{K}$, for all $x \in \mathbb{R}_+^N$, we have $\tilde{a}_{\alpha,\gamma}(\cdot, \cdot) \ge 0$, $\tilde{a}_{\alpha,\alpha}(\cdot, \cdot) > 0$ and there exists $\Lambda_1, \Lambda_2 > 0$ such that

$$\begin{aligned}\frac{|x|^2}{\Lambda_1} &\le \sum_{i,j=1}^{N} \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j) \le \Lambda_1 |x|^2, \\ \frac{|x|^4}{\Lambda_2} &\le \sum_{i,j=1}^{N} x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j) \le \Lambda_2 |x|^4.\end{aligned}$$

A2'. For all $\alpha, \gamma \in \mathbf{K}$, the functions $\sigma_{\alpha,\gamma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous. Moreover, for all $\alpha, \gamma \in \mathbf{K}$, for all $x \in \mathbb{R}_+^N$, we have $\tilde{a}_{\alpha,\gamma}(\cdot, \cdot) \ge 0$,

$\tilde{a}_{\alpha,\alpha}(\cdot,\cdot) > 0$ and there exists $\Lambda_1, \Lambda_2 > 0$ such that

$$\frac{1}{\Lambda_1} \le \sum_{i,j=1}^{N} \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j) \le \Lambda_1, \quad \frac{|x|^2}{\Lambda_2} \le \sum_{i,j=1}^{N} x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j) \le \Lambda_2 |x|^2.$$

Remark 2.1. We note the following points regarding the assumptions (A1), (A2), (A2') and the driving Brownian motion.

(i) By localization procedure and using some simple estimates in the proofs, we can indeed weaken the assumption (A1) as follows: There exists $c_\alpha \in \mathbb{R}, d_{\alpha,\gamma} \in \mathbb{R}_+$ such that for all $\alpha, \gamma \in \mathbf{K}$, the function $b_{\alpha\gamma}$ satisfy $b_{\alpha\gamma}(x,y) \le c_\alpha - d_{\alpha,\gamma} y$, for $x, y \in \mathbb{R}_+$ large enough. To maintain the clarity of the exposition and avoid cumbersome computations, we will however content with assumption (A1).

(ii) We can easily see that if all $\sigma_{\alpha,\gamma}$ are of the form σx and $\sigma > 0$, the assumption (A2) holds; if $\sigma_{\alpha,\gamma}$ are positive constants, the assumption (A2') trivially holds.

(iii) In (2.2), if we replace dw_γ by $dw_{i,\gamma}$ and $dw_{i,\gamma}$ are independent for all $i = 1, \dots, N, \gamma \in \mathbf{K}$, i.e., the random fluctuations now depend both on the state of the particle and the population that the particle interacts with, we can still obtain the similar results by using the same method with some minors changes in arguments and computations.

3. Existence and uniqueness of solution

In this section, we prove the existence and uniqueness of the solution to the system (2.2). First, we derive here some estimates regarding the coefficients of the system to be used frequently in the paper. For $x \in \mathbb{R}_+^N$, using (A1) and the definition of the empirical measure μ^γ we have

$$\begin{aligned}
\sum_{\gamma=1}^{K} \langle b_{\mathbf{p}(i)\gamma}(x_i(s), \cdot), \mu^{\gamma,N}(s) \rangle &= \sum_{\gamma=1}^{K} \frac{1}{N_\gamma} \sum_{j:\mathbf{p}(j)=\gamma} b_{\mathbf{p}(i)\gamma}(x_i(s), x_j(s)) \\
&\le \sum_{\gamma=1}^{K} \left(c_2 - \frac{d_2}{N_\gamma} \sum_{\mathbf{p}(i)=\gamma} x_i(s) \right) \\
&\le K c_2 - \frac{d_2}{\min_{\gamma \in \mathbf{K}} N_\gamma} S(x).
\end{aligned} \tag{3.1}$$

For $p \geq 1$, an application of Schwartz inequality gives us $\frac{S^p(x)}{n^{p/2}} \leq |x|^p \leq S^p(x)$. Using this inequality, under assumption (A2), we have

$$\frac{S^2(x)}{N\Lambda_1} \leq \frac{|x|^2}{\Lambda_1} \leq \sum_{i,j=1}^{N} \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j) \leq \Lambda_1 |x|^2 \leq \Lambda_1 S^2(x) \tag{3.2}$$

$$\frac{S^4(x)}{N^2\Lambda_2} \leq \frac{|x|^4}{\Lambda_2} \leq \sum_{i,j=1}^{N} x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j) \leq \Lambda_2 |x|^4 \leq \Lambda_2 S^4(x), \tag{3.3}$$

for all $x \in \mathbb{R}_+^N$. Similarly, using (A2'), we derive

$$\frac{S^2(x)}{N\Lambda_2} \leq \frac{|x|^2}{\Lambda_2} \leq \sum_{i,j=1}^{N} x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j) \leq \Lambda_2 |x|^2 \leq \Lambda_2 S^2(x). \tag{3.4}$$

We first prove the existence and uniqueness of the solution to (2.2) and establish some bounds on the moments of the solution. Note that in [3], the authors used a comparison principle coupled with Khasminskii's criterion to prove the global existence of the solution. Here we use the Lyapunov function to obtain the same conclusion. Moreover, using this technique, we are also able to prove that the solution is indeed a.s positive at every $t \geq 0$. This is an important property that was only mentioned briefly in a simple case there.

Theorem 3.1. *Assume* (A1) *and* (A2). *Then for any initial conditions* $x(0) = x_0 \in \mathbb{R}_+^N$, *there is a unique solution* $x(t)$ *to* (2.2) *on* $t \geq 0$, *and the solution remains in* $\mathbb{R}_+^N$ *a.s.* (*almost surely*), *i.e.*, $x(t) \in \mathbb{R}_+^N$ *a.s. for any* $t \geq 0$.

Proof. Since the coefficients of (2.2) are locally Lipschitz, there is a unique local solution $x(t)$ on $t \in [0, \zeta)$, where ζ is the explosion time (please see [19]). Let $k_0 \in \mathbb{N}$ be sufficiently large such that every components of the initial value x_0 are contained in the open interval $(\frac{1}{k_0}, k_0)$. For each $k \geq k_0$, we define the corresponding stopping time

$$\tau_k := \inf \left\{ t \in [0, \zeta) : x_i(t) \notin (\frac{1}{k}, k) \text{ for some } i = 1, 2, \ldots, N \right\} \tag{3.5}$$

It is straight forward to see that the sequence τ_k is non-decreasing. Let $\tau_\infty := \lim_{k\to\infty} \tau_k$, then $\tau_\infty \leq \zeta$. We prove $\tau_\infty = \infty$ and thus $\zeta = \infty$ and the solution is indeed the global one. Assume that $\tau_\infty < \infty$, then there would exist $T > 0$ and $\varepsilon > 0$ such that $\mathbb{P}\{\tau_\infty \le T\} > \varepsilon$. By the definition of τ_k, there exists $k_1 > k_0$ such that

$$\mathbb{P}\{\tau_k \le T\} > \varepsilon, \quad \text{for all } k \ge k_1. \tag{3.6}$$

We now consider the following Lyapunov function

$$V(x) = \sum_{i=1}^{N} \left(x_i^{1/2} - 1 - \frac{1}{2} \ln x_i \right) \quad x \in \mathbb{R}_+^N$$

It is clear that $V(x) \ge 0$ for every $x \in \mathbb{R}_+^N$ and for $i, j = 1, 2, \ldots, N$,

$$\frac{\partial V}{\partial x_i}(x) = \frac{1}{2}(x_i^{-1/2} - x_i^{-1}); \quad \frac{\partial^2 V}{\partial x_i^2}(x) = \frac{1}{4}(2x_i^{-2} - x_i^{-3/2})$$
$$\frac{\partial^2 V}{\partial x_i x_j}(x) = 0, \quad \text{for } i \ne j.$$

Using (A1), (A2), (3.1), and (3.2), we compute

$$\begin{aligned}
\mathcal{L}V(x) &= \frac{1}{2}\sum_{i=1}^{N}(x_i^{1/2} - 1)\sum_{\gamma=1}^{K}\langle b_{\mathbf{p}(i)\gamma}(x_i, \cdot), \mu^\gamma \rangle \\
&\quad + \frac{1}{8}\sum_{i=1}^{N}(2 - x_i^{1/2})\tilde{a}_{\mathbf{p}(i)\mathbf{p}(i)}(x_i, x_i) \\
&\le \sum_{i=1}^{N}(x_i^{1/2} - 1)\left(c_2 - \frac{d_2}{\min_{\gamma\in\mathbf{K}} N_\gamma} S(x) \right) \\
&\quad + \frac{N\Lambda_1}{4}|x|^2 - \frac{1}{8\Lambda_1}\sum_{i=1}^{N} x_i^{1/2}|x|^2 \\
&\le C\sum_{i=1}^{N}(x_i^{1/2} - 1) - \frac{d_2}{\min_{\gamma\in\mathbf{K}} N_\gamma}\sum_{i=1}^{N} x_i^{3/2} + CS(x) \\
&\quad + \frac{N\Lambda_1}{4}|x|^2 - \frac{1}{8\Lambda_1}\sum_{i=1}^{N} x_i^{1/2}|x|^2 \le C,
\end{aligned} \tag{3.7}$$

for some positive constant C. This fact is obtained by noting that the coefficient of the leading term in the last expression is negative. It then follows from Itô's formula that for any $k \geq k_1$,

$$\begin{aligned}&V(x(\tau_k \wedge T)) - V(x(0))\\&= \int_0^{\tau_k\wedge T} \mathcal{L}V(x(s))ds + \sum_{i=1}^N \int_0^{\tau_k\wedge T} (x_i - 1) \sum_{\gamma=1}^K \langle \sigma_{\alpha\gamma}(x_i(s),\cdot), \mu^\gamma(s)\rangle dw_\gamma.\end{aligned}$$

Taking expectation on both sides, using (3.7) and noting the last term being a martingale, we get

$$\mathbb{E}V(x(\tau_k \wedge T)) - V(x(0)) = \mathbb{E}\int_0^{\tau_k\wedge T} \mathcal{L}V(x(s))ds \leq CT$$

and it follows that,

$$CT + V(x(0)) \geq \mathbb{E}V(x(\tau_k \wedge T)) \geq \mathbb{E}[V(x(\tau_k \wedge T)\mathbb{1}_{\{\tau_k\leq T\}})]. \tag{3.8}$$

Note that on the set $\{\tau_k \leq T\}$, there is some component i of x such that $x_i(\tau_k) \geq k$ or $x_i(\tau_k) \leq \frac{1}{k}$. Hence, by properties of V, one deduces that on $\{\tau_k \leq T\}$

$$V(x(\tau_k)) \geq \left(k^{1/2} - 1 - \frac{1}{2}\ln k\right) \wedge \left(\frac{1}{k^{1/2}} - 1 - \frac{1}{2}\ln\frac{1}{k}\right) \tag{3.9}$$

Therefore, in view of (3.6), (3.9), and (3.8), we get

$$CT + V(x(0)) \geq \varepsilon\left[\left(k^{1/2} - 1 - \frac{1}{2}\ln k\right) \wedge \left(\frac{1}{k^{1/2}} - 1 - \frac{1}{2}\ln\frac{1}{k}\right)\right].$$

Sending $k \to \infty$ we get a contradiction and thus $\tau_\infty = \infty$ a.s. □

Using (A1) and (A2'), we can obtain the following lemma regarding the existence of global solution to the system (2.2).

Lemma 3.2. *Assume* (A1) *and* (A2'). *Then for any initial conditions* $x(0) = x_0 \in \mathbb{R}^N_+$, *there is a unique solution* $x(t)$ *to* (2.2) *on* $t \geq 0$, *and the solution remains in* $\mathbb{R}^N_+$ *almost surely.*

Proof. The proof can be carried out by considering the Lyapunov function, $V(x) = \sum_{i=1}^N (x_i - 1 - \ln x_i), x \in \mathbb{R}^N_+$ and repeating the arguments of Theorem 3.1. □

Theorem 3.3. *Assume* (A1) *and* (A2') *hold. Then the following assertions hold.*

(i) *For any* $p > 0$, $\sup_{t\geq 0} \mathbb{E}S^p(x(t)) < \infty$,

(ii) *The process* $x(t)$ *is stochastically bounded. That is, for any* $\varepsilon > 0$, *there exists a constant* $H = H(\varepsilon)$ *such that* $\limsup_{t\to\infty} \mathbb{P}\{|x(t)| \leq H\} \geq 1 - \varepsilon$ *for any* $x_0 \in \mathbb{R}^N_+$.

Proof. To prove (i), let $V(x) = S^p(x), x \in \mathbb{R}^N_+$. A straightforward calculation gives us

$$\frac{\partial V}{\partial x_i}(x) = pS^{p-1}(x), \quad \text{and} \quad \frac{\partial^2 V}{\partial x_i x_j}(x) = p(p-1)S^{p-2}(x),$$

and thus, by (2.3),

$$\begin{aligned}
\mathcal{L}V(x) &= pS^{p-1}(x)\sum_{i=1}^{N} x_i \sum_{\gamma=1}^{K}\langle b_{\mathbf{p}(i)\gamma}(x_i,\cdot),\mu^\gamma\rangle \\
&\quad + \frac{1}{2}p(p-1)S^{p-2}\sum_{i,j=1}^{N} x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i,x_j) \\
&\leq pS^{p-1}(x)\sum_{i=1}^{N} x_i\left(Kc_2 - \frac{d_2}{\min_{\gamma\in\mathbf{K}} N_\gamma}S(x)\right) \\
&\quad + \frac{1}{2}p(p-1)S^{p-2}\Lambda_2 S^2(x) \\
&\leq \left(Kpc_2 + \frac{1}{2}p(p-1)\Lambda_2\right)S^p(x) - \frac{pd_2}{\min_{\gamma\in\mathbf{K}} N_\gamma}S^{p+1}(x) \\
&\leq C,
\end{aligned} \tag{3.10}$$

for some constant C. Applying the Itô lemma to $e^t V(x(t))$, we have

$$\begin{aligned}
&\mathbb{E}e^{t\wedge\tau_k}V(x(t\wedge\tau_k)) \\
&= V(x(0)) + \int_0^{t\wedge\tau_k} e^s(V+\mathcal{L}V)(x(s))ds \\
&\quad + p\sum_{i=1}^{N}\int_0^{t\wedge\tau_k} e^s x_i^p(s)\sum_{\gamma=1}^{K}\langle\sigma_{\mathbf{p}(i)\gamma}(x_i(s),\cdot),\mu^\gamma(s)\rangle dw_\gamma(s).
\end{aligned}$$

where τ_k is the stopping time defined in (3.5). Thus taking expectation both sides, one yields

$$\mathbb{E}e^{t\wedge\tau_k}V(x(t\wedge\tau_k)) \leq V(x(0)) + \mathbb{E}\int_0^{t\wedge\tau_k} Ce^s ds \leq V(x(0)) + C(e^t - 1).$$

Recall that $\lim_{k\to\infty}\tau_k = \infty$, as in the proof of Theorem 3.1, letting $k \to \infty$ and applying Fatou's lemma yields, $\mathbb{E}e^t S^p(x(t)) \le S^p(x(0)) + C(e^t - 1)$, or $\mathbb{E}S^p(x(t)) \le e^{-t}S^p(x(0)) + C(1 - e^{-t}) < \infty$. We then obtain the desired claim by taking sup with respect to $t \ge 0$. An application of Chebysev inequality gives us (ii). □

Lemma 3.4. *Let assumption* (A1) *and* (A2') *hold and assume further that for all* $\alpha, \gamma \in \mathbf{K}$ *and* $x, y \in \mathbb{R}_+$, *there exists* $c_1 > 0, d_1 > 0$ *such that*

$$c_1 - d_1 y \le b_{\alpha\gamma}(x, y).$$

and $\frac{\Lambda_2}{2} - Kc_1 < 0$. *Then for any given initial value* $x(0) \in \mathbb{R}^N_+$, *there exists a constant* $\theta > 0$ *such that* $\limsup_{t\to\infty} \mathbb{E}(\frac{1}{S^\theta(x(t))}) \le C$, *where* C *is a constant.*

Proof. By similar calculation as in (3.1), with the additional assumption on $b_{\alpha,\gamma}$ we can derive

$$\sum_{\gamma=1}^{K} \langle b_{\mathbf{p}(i)\gamma}(x_i(s), \cdot), \mu^{\gamma,N}(s)\rangle \ge Kc_1 - \frac{d_1}{N} S(x(s)) \tag{3.11}$$

We now choose $\theta, \rho > 0$ and small enough and such that $\frac{\Lambda_2}{2}\theta(\theta+1) + \rho - \theta Kc_1 < 0$. This can be done thanks to our assumptions. For such θ and ρ, let $U : \mathbb{R}^N_+ \to \mathbb{R}$ be defined by $U(x) = \frac{1}{S(x)}$ and let $V(x) = (1 + U(x))^\theta$. It is straightforward to compute that

$$\begin{aligned}
\frac{\partial V}{\partial x_i} &= -\theta U^2(x)(1 + U(x))^{\theta-1}\\
\frac{\partial V}{\partial x_i \partial x_j} &= \theta(\theta-1)U^4(x)(1+U(x))^{\theta-2} + 2\theta U^3(x)(1+U(x))^{\theta-1}.
\end{aligned}$$

Using (3.1), (3.4) and (3.11), we derive the following estimate

$$\begin{aligned}
\mathcal{L}V(x) = {}& -\theta U^2(x)(1+U(x))^{\theta-1} \sum_{i=1}^{N} x_i \sum_{\gamma=1}^{K} \langle b_{\mathbf{p}(i),\gamma}(x_i, \cdot), \mu^\gamma\rangle\\
&+ \frac{1}{2}\big[\theta(\theta-1)U^4(x)(1+U(x))^{\theta-2}\\
&\quad + 2\theta U^3(x)(1+U(x))^{\theta-1}\big]\\
&\times \sum_{i,j=1}^{N} x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j)
\end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&\leq -\theta U^2(x)(1+U(x))^{\theta-1}\left(Kc_1 S(x) - \frac{d_1}{\min_{\gamma\in\mathbf{K}} N_\gamma} S^2(x)\right)\\
&\quad + \frac{1}{2}\big[\theta(\theta-1)U^4(x)(1+U(x))^{\theta-2}\\
&\qquad + 2\theta U^3(x)(1+U(x))^{\theta-1}\big]\Lambda_2 S^2(x)\\
&= \theta(1+U(x))^{\theta-2}\Bigg[-(1+U(x))U^2(x)(Kc_1S(x)\\
&\qquad - \frac{d_1}{\min_{\gamma\in\mathbf{K}} N_\gamma} S^2(x))\\
&\qquad + (1+U(x))U^3(x)\Lambda_2 S^2(x)\\
&\qquad + \frac{1}{2}(\theta-1)U^4(x)\Lambda_2 S^2(x)\Bigg]\\
&\leq \theta(1+U(x))^{\theta-2}\Bigg[\left(\frac{\Lambda_2}{2}(\theta+1) - Kc_1\right)U^2(x)\\
&\qquad + \left(\Lambda_2 - Kc_1 + \frac{d_1}{\min_{\gamma\in\mathbf{K}} N_\gamma}\right)U(x)\\
&\qquad + \frac{d_1}{\min_{\gamma\in\mathbf{K}} N_\gamma}\Bigg].
\end{aligned}
$$

By virtue of Itô's lemma, we have

$$
\begin{aligned}
de^{\rho t}V(x) &= (\rho e^{\rho t}V(x) + e^{\rho t}\mathcal{L}V(x))dt\\
&\quad + \sum_{i=1}^{N} e^{\rho t}\frac{\partial V(x)}{\partial x_i} x_i \sum_{\gamma=1}^{K}\langle \sigma_{\mathbf{p}(i)\gamma}(x_i,\cdot),\mu^\gamma\rangle dw_\gamma.
\end{aligned}
$$

Using the estimate (3.12),

$$
\begin{aligned}
&\rho e^{\rho t}V(x) + e^{\rho t}\mathcal{L}V(x) \qquad (3.13)\\
&\leq e^{\rho t}(1+U(x))^{\theta-2}\Bigg[\rho(1+U(x))^2 + \theta(\frac{\Lambda_2}{2}(\theta+1) - Kc_1)U^2(x)\\
&\quad + \theta(\Lambda_2 - Kc_1 + \frac{d_1}{\min_{\gamma\in\mathbf{K}} N_\gamma})U(x) + \frac{\theta d_1}{\min_{\gamma\in\mathbf{K}} N_\gamma}\Bigg]\\
&= e^{\rho t}(1+U(x))^{\theta-2}[(\frac{\Lambda_2}{2}\theta(\theta+1) + \rho - \theta Kc_1)U^2(x)\\
&\quad + (\Lambda_2\theta - Kc_1\theta + \frac{\theta d_1}{\min_{\gamma\in\mathbf{K}} N_\gamma} + 2\rho)U(x)\\
&\quad + (\rho + \frac{\theta d_1}{\min_{\gamma\in\mathbf{K}} N_\gamma})].
\end{aligned}
$$

Thanks to the way we chose θ and ρ, we deduce that

$$\begin{aligned}\sup_{x\in\mathbb{R}^N_+}(1+U(x))^{\theta-2}\Bigg[&\left(\frac{\Lambda_2}{2}\theta(\theta+1)+\rho-\theta Kc_1\right)U^2(x)\\ &+\left(\Lambda_2\theta-Kc_1\theta+\frac{\theta d_1}{\min_{\gamma\in\mathbf{K}}N_\gamma}+2\rho\right)U(x)\\ &+\left(\rho+\frac{\theta d_1}{\min_{\gamma\in\mathbf{K}}N_\gamma}\right)\Bigg]\le C.\end{aligned}\tag{3.14}$$

Therefore,

$$\mathbb{E}e^{\rho t}V(x(t\wedge\tau_k))\le V(x(0))+C\int_0^{t\wedge\tau_k}e^{\rho s}ds,$$

where τ_k is the stopping time defined in (3.5). Combining this fact and repeating the arguments used at the end of proof of Theorem 3.3 we have

$$\begin{aligned}\limsup_{t\to\infty}\mathbb{E}\left(\frac{1}{S^\theta(x(t))}\right)&\le\limsup_{t\to\infty}\mathbb{E}(V(x(t)))^\theta\\ &\le e^{-\rho t}(1+V(x(0)))+\frac{C}{\rho}\le C.\end{aligned}\tag{3.15}$$

which is the desired result. □

Theorem 3.5. *Assume* (A1) *and* (A2) (or (A2')) *hold. For any initial condition* $x(0)\in\mathbb{R}^N_+$*, the solution* $x(t)$ *of* (2.2) *is positive recurrent with respect to the domain* $G_\rho:=\{x\in\mathbb{R}^N_+:0<x_i<\rho,i=1,2,\dots,N\}$*, where* ρ *is a positive number to be specified later in the proof.*

Proof. Thanks to Theorem 3.9 and Corollary 3.2 in [19], to show the positive recurrence of $x(t)$, it suffices to prove that it is regular and there exist a nonnegative twice continuously differentiable function V defined on $\mathbb{R}^N_+$ such that $\mathcal{L}V(s)\le -1$ on $\mathbb{R}^N_+-G_\rho$. By Theorem 3.1, $x(t)$ is regular. Consider $V:\mathbb{R}^N\to\mathbb{R}$ defined by $V(x)=S(x)$. It is easy to see that

$$\begin{aligned}\mathcal{L}V(x)&=\sum_{i=1}^N x_i\sum_{\gamma=1}^K\langle b_{\mathbf{p}(i)\gamma}(x_i,\cdot),\mu^\gamma\rangle\\ &\le\sum_{i=1}^N x_i\left(c_2-\frac{d_2}{\min_{\gamma\in\mathbf{K}}N_\gamma}S(x)\right)\\ &\le c_2S(x)-\frac{d_2}{\min_{\gamma\in\mathbf{K}}N_\gamma}S^2(x).\end{aligned}$$

Since $d_2 > 0$, $\mathcal{L}V$ is bounded above on $\mathbb{R}^N_+$ and $\lim_{|x|\to\infty} \mathcal{L}V(x) = -\infty$, hence there exists $\rho > 0$ such that $\mathcal{L}V(x) \le -1$ on $\mathbb{R}^N_+ - G_\rho$. The desired claim then follows. □

4. Asymptotic bounds

In this section, we prove several bounds on the growth rate of the system (2.2). The main thrust of the proofs, beside using suitable Lyapunov functions, is to utilize the exponential martingale inequality and the Borel - Cantelli lemma. These techniques are very popular and used extensively to investigate the longtime behavior of population models (please see, for example, [24, 28, 36] and references there in).

4.1. Upper bounds

We first prove some upper bounds on the growth rate of the solution to the system (2.2) under the assumption (A2) on the diffusion part. We remark that if we replace (A2) by (A2') we still achieve the same results as those in Theorem 4.1 and Theorem 4.2 below by using the same Lyapunov functions and repeating the same arguments (with some minor changes) as those used in the Theorems.

Theorem 4.1. *Assume* (A1) *and* (A2) *hold. Then for any initial condition* $x(0) \in \mathbb{R}^N_+$, *the solution* $x(\cdot)$ *of* (2.2) *satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\left[\ln S(x(t)) + \frac{1}{4\kappa}\int_0^t S^2(x(s))ds\right] \le C \quad a.s.,$$

where κ *and* C *are constants* (*to be specified in the proof*).

Proof. Define $V(x) = \ln S(x), x \in \mathbb{R}^N_+$. Then we can readily verify that

$$\frac{\partial V}{\partial x_i}(x) = \frac{1}{S(x)}, \quad \text{and} \quad \frac{\partial^2 V}{\partial x_i x_j}(x) = -\frac{1}{S^2(x)}.$$

Applying Itô's lemma to $V(\cdot)$ yields,

$$
\begin{aligned}
\text{(4.1)} \qquad & \ln S(x(t)) - \ln S(x(0)) \\
&= \int_0^t \Bigg[\frac{1}{S(x)} \sum_{i=1}^{N} x_i \sum_{\gamma=1}^{K} \langle b_{\mathbf{p}(i)\gamma}(x_i(s), \cdot), \mu^\gamma(s) \rangle \\
&\quad - \frac{1}{2S^2(x(s))} \sum_{i,j=1}^{N} x_i(s) x_j(s) \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i(s), x_j(s)) \Bigg] ds + M(t),
\end{aligned}
$$

where

$$
M(t) = \int_0^t \sum_{i=1}^{N} \frac{x_i(s)}{S(x(s))} \sum_{\gamma=1}^{K} \langle \sigma_{p(i)\gamma}(x_i(s), \cdot), \mu^\gamma(s) \rangle dw_\gamma(s), \quad i = 1, 2, \ldots, N,
$$

is a real-valued continuous local martingale vanishing at $t = 0$ with quadratic variation

$$
\langle M(t) \rangle = \int_0^t \sum_{i,j=1}^{N} \frac{x_i(s) x_j(s)}{S^2((s))} \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i(s), x_j(s)) ds.
$$

For an arbitrary but fixed $0 < \varepsilon < 1$, for any $k \geq 1$, using the exponential martingale inequality (see [11]) gives us

$$
\mathbb{P} \left\{ \sup_{0 \leq s \leq k} [M_i(s) - \frac{\varepsilon}{4} \langle M_i(s) \rangle] > \frac{4 \ln k}{\varepsilon} \right\} \leq \frac{1}{k^2}.
$$

By virtue of the Borel - Cantelli lemma, we can find a set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and for any $\omega \in \Omega_0$, there exists $k_0(\omega)$ such that, for all $k \geq k_0(\omega)$,

$$
\sup_{0 \leq s \leq k} \left[M(s) - \frac{\varepsilon}{4} \langle M(s) \rangle \right] \leq \frac{4 \ln k}{\varepsilon}.
$$

Thus for any $\omega \in \Omega_0$ and for all $k \geq k_0(\omega)$.

$$
\text{(4.2)} \qquad M(s) < \frac{\varepsilon}{4} \langle M(s) \rangle + \frac{4 \ln k}{\varepsilon}, \quad \text{for all } 0 \leq s \leq k.
$$

Substituting (4.2) into (4.1) we get

$$\begin{aligned}
(4.3)\qquad & \ln S(x(t)) + \int_0^t \frac{1}{4S^2(x(s))} \sum_{i,j=1}^N x_i(s)x_j(s)\tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i(s), x_j(s)) \\
& \le \ln S(x(0)) + \int_0^t \left[\frac{1}{S(x)} \sum_{i=1}^N x_i \sum_{\gamma=1}^K \langle b_{\mathbf{p}(i)\gamma}(x_i(s), \cdot), \mu^\gamma(s) \rangle \right. \\
& \quad \left. - \frac{1-\varepsilon}{4S^2(x(s))} \sum_{i,j=1}^N x_i(s)x_j(s)\tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i(s), x_j(s)) \right] ds + \frac{4\ln k}{\varepsilon},
\end{aligned}$$

Using (3.2) and letting $\kappa = N^2\Lambda_2$, we have

$$(4.4)\qquad \frac{S^4(x)}{\kappa} \le \sum_{i,j=1}^N x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j) \le \kappa S^4(x), \quad \forall x \in \mathbb{R}_+^N.$$

This implies

$$\begin{aligned}
& \ln S(x(t)) + \int_0^t \frac{1}{4S^2(x(s))} \sum_{i,j=1}^N x_i(s)x_j(s)\tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i(s), x_j(s)) \\
& \ge \ln S(x(t)) + \frac{1}{4\kappa} \int_0^t S^2(x(s))ds.
\end{aligned}$$

Using (3.1), we deduce

$$\begin{aligned}
\frac{1}{S(x)} \sum_{i=1}^N x_i \sum_{\gamma=1}^K \langle b_{\mathbf{p}(i)\gamma}(x_i, \cdot), \mu^\gamma \rangle & \le \frac{1}{S(x)} \sum_{i=1}^N x_i \left(Kc_2 - \frac{d_2}{\min_{\gamma\in\mathbf{K}} N_\gamma} S(x) \right) \\
& \le Kc_2 - \frac{d_2}{\min_{\gamma\in\mathbf{K}} N_\gamma} S(x).
\end{aligned}$$

Combining this inequality and (4.4), we get

$$\begin{aligned}
& \frac{1}{S(x)} \sum_{i=1}^N x_i \sum_{\gamma=1}^K \langle b_{\mathbf{p}(i)\gamma}(x_i, \cdot), \mu^\gamma \rangle - \frac{1-\varepsilon}{4S^2(x)} \sum_{i,j=1}^N x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j) \\
& \le Kc_2 - \frac{d_2}{\min_{\gamma\in\mathbf{K}} N_\gamma} S(x) - \frac{(1-\varepsilon)}{4\kappa} S^2(x) \le C,
\end{aligned}$$

for some constant C. We therefore arrive at

$$\begin{aligned}\ln S(x(t)) + \frac{1}{4\kappa}\int_0^t S^2(x(s))ds &\le \ln(S(x(0))) + \int_0^t C ds + \frac{4\ln k}{\varepsilon}\\ &\le \ln(S(x(0))) + Ct + \frac{4\ln k}{\varepsilon},\end{aligned}$$

for any $\omega \in \Omega_0, k \ge k_0(\omega)$ and $0 \le t \le k$. Hence, if $k-1 \le t \le k$ with $k \ge k_0(\omega)$ then

$$\begin{aligned}\frac{1}{t}\left[\ln S(x(t)) + \frac{1}{4\kappa}\int_0^t S^2(x(s))ds\right] &\le C + \frac{1}{t}\left[\ln(S(x(0))) + \frac{4\ln k}{\varepsilon}\right]\\ &\le C + \frac{1}{t}\left[\ln(S(x(0))) + \frac{4\ln(t+1)}{\varepsilon}\right],\end{aligned}$$

so

$$\limsup_{t\to 0}\frac{1}{t}\left[\ln S(x(t)) + \frac{1}{4\kappa}\int_0^t S^2(x(s))ds\right] \le C$$

as desired. □

Theorem 4.2. *Assume* (A1) *and* (A2) *hold. Then for any initial condition* $x(0) \in \mathbb{R}^N_+$*, the solution* $x(t)$ *of* (2.2) *satisfies*

$$\limsup_{t\to\infty}\frac{\ln S(x(t))}{\ln t} \le 1 \quad a.s. \tag{4.5}$$

Proof. Define $V(t,x) = e^t \ln S(x)$ for $(t,x) \in [0,\infty)\times\mathbb{R}^N_+$. Applying Itô's lemma to $V(\cdot,\cdot)$, one obtains

$$\begin{aligned}&e^t\ln S(x(t)) - \ln S(x(0))\\ &= \int_0^t \frac{e^s}{S(x(s))}\sum_{i=1}^N x_i(s)\sum_{\gamma=1}^K \langle b_{\mathbf{p}(i)\gamma}(x_i(s),\cdot),\mu^\gamma(s)\rangle ds\\ &\quad - \int_0^t \frac{e^s}{2S^2(x(s))}\sum_{i,j=1}^N x_i(s)x_j(s)\tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i(s),x_j(s))ds\\ &\quad + \int_0^t e^s \ln S(x(s))ds\\ &\quad + \int_0^t e^s\sum_{i=1}^N \frac{x_i(s)}{S(x(s))}\sum_{\gamma=1}^K\langle\sigma_{p(i)\gamma}(x_i(s),\cdot),\mu^\gamma(s)\rangle dw_\gamma(s).\end{aligned} \tag{4.6}$$

Let

$$M(t) = \int_0^t e^s \sum_{i=1}^N \frac{x_i(s)}{S(x(s))} \sum_{\gamma=1}^K \langle \sigma_{p(i)\gamma}(x_i(s), \cdot), \mu^\gamma(s)\rangle dw_\gamma(s).$$

Then $M(t)$ is a real-valued continuous local martingale whose quadratic variation is

$$\langle M(t)\rangle = \int_0^t e^{2s} \sum_{i,j=1}^N \frac{x_i(s)x_j(s)}{S^2((s))} \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i(s), x_j(s))ds.$$

By virtue of the exponential martingale inequality, and the Borel-Cantelli lemma, for $0 < \varepsilon < 1, \theta > 1$ and $\lambda > 0$, there exists $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) = 1$ and for each $\omega \in \Omega_1$, there exists $k_1(\omega)$ such that for every $k \geq k_1(\omega)$,

$$M(t) \leq \frac{\varepsilon e^{-k\lambda}}{2}\langle M(t)\rangle + \frac{\theta e^{k\lambda} \ln k}{\varepsilon}, \quad 0 \leq t \leq \lambda k. \tag{4.7}$$

Combining (4.6) and (4.7), we have that

(4.8)

$$\begin{aligned}
&e^t \ln S(x(t)) - \ln S(x(0)) \\
&= \int_0^t \frac{e^s}{S(x(s))} \sum_{i=1}^N x_i(s) \sum_{\gamma=1}^K \langle b_{\mathbf{p}(i)\gamma}(x_i(s), \cdot), \mu^\gamma(s)\rangle ds \\
&\quad - \int_0^t \frac{e^s}{2S^2(x(s))} \sum_{i,j=1}^N x_i(s)x_j(s)\tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i(s), x_j(s))ds \\
&\quad + \int_0^t \frac{\varepsilon e^{-k\lambda}}{2} \frac{e^{2s}}{S^2(x(s))} \sum_{i,j=1}^N x_i(s)x_j(s)\tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i(s), x_j(s))ds + \frac{\theta e^{k\lambda} \ln k}{\varepsilon}.
\end{aligned}$$

Using similar arguments as that in Theorem 4.1, we can prove that for all $0 \leq s \leq k\lambda$ and $x \in \mathbb{R}_+^N$,

$$\begin{aligned}
&\frac{e^s}{S(x(s))} \sum_{i=1}^N x_i(s) \sum_{\gamma=1}^K \langle b_{\mathbf{p}(i)\gamma}(x_i(s), \cdot), \mu^\gamma(s)\rangle \\
&\quad - \frac{e^s(1 - \varepsilon e^{-k\lambda+s})}{2S^2(x(s))} \sum_{i,j=1}^N x_i(s)x_j(s)\tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i(s), x_j(s)) \\
&\leq e^s K c_2 - \frac{e^s d_2}{\min_{\gamma \in \mathbf{K}} N_\gamma} S(x(s)) - \frac{e^s(1 - \varepsilon e^{-k\lambda+s})}{2\kappa} S^2(x(s)) \leq C.
\end{aligned} \tag{4.9}$$

From (4.8) and (4.9), it follows that for all $0 \le t \le k\lambda$ with $k \ge k_1(\omega)$,

$$
\begin{aligned}
e^t \ln S(x(t)) &\le \ln S(x(0)) + \int_0^t Ce^s ds + \frac{\theta e^{k\lambda} \ln k}{\varepsilon} \\
&= \ln S(x(0)) + C(e^t - 1) + \frac{\theta e^{k\lambda} \ln k}{\varepsilon}
\end{aligned}
$$

and thus

$$
\ln S(x(t)) \le e^{-t} \ln S(x(0)) + C(1 - e^{-t}) + e^{-t} \frac{\theta e^{k\lambda} \ln k}{\varepsilon}.
$$

Hence for $(k-1)\lambda \le t \le k\lambda$ and $k \ge k_1(\omega)$, we have

$$
\frac{\ln S(x(t))}{\ln t} \le \frac{e^{-(k-1)\lambda}}{\ln(k-1)\lambda} (\ln S(x(0)) - C) + \frac{C}{\ln(k-1)\lambda} + \frac{\theta e^{\lambda} \ln k}{\ln(k-1)\lambda}.
$$

Now letting $k \to \infty$ (and so $t \to \infty$), we then obtain $\limsup_{t\to\infty} \frac{\ln S(x(t))}{\ln t} \le \frac{\theta e^{\lambda}}{\varepsilon}$ a.s. Finally, the above inequality holds for every $\lambda > 0, \varepsilon < 1$ and $\theta > 1$, by sending $\lambda \downarrow 0, \varepsilon \uparrow 1$, and $\theta \downarrow 1$, we have $\limsup_{t\to\infty} \frac{\ln S(x(t))}{\ln t} \le 1$ a.s. The proof is complete. □

4.2. Lower bounds

We are now in position to prove a lower bound on the growth rate of the solution to the system (2.2) under the assumption (A2').

Theorem 4.3. *Suppose the assumptions of Lemma 3.4 hold. Then, with probability 1, the solution $x(t)$ of (2.2) with any initial value $x(0) \in \mathbb{R}_+^N$ satisfies*

$$
\liminf_{t\to\infty} \frac{\ln S(x(t))}{\ln t} \ge -\frac{1}{\theta}. \tag{4.10}
$$

Proof. Similar to the proof of Lemma 3.4, let $U : \mathbb{R}_+^N \to \mathbb{R}$ be defined by $U(x) = \frac{1}{S(x)}$ and let $V(x) = (1 + U(x))^{\theta}$. Thanks to (3.12), we have

$$
\begin{aligned}
\mathcal{L}V(x) \le \theta(1 + U(x))^{\theta-2} \Bigg[&\left(\frac{\Lambda_2}{2}(\theta + 1) - Kc_1 \right) U^2(x) \\
&+ \left(\Lambda_2 - Kc_1 + \frac{d_1}{\min_{\gamma\in\mathbf{K}} N_\gamma} \right) U(x) + \frac{d_1}{\min_{\gamma\in\mathbf{K}} N_\gamma} \Bigg].
\end{aligned}
$$

Hence

$$
\begin{aligned}
(4.11)\qquad & d[(1+U(x))^{\theta}] \\
&\le \theta(1+U(x))^{\theta-2}[C_1U^2(x)+C_2U(x)+C_3] \\
&\quad - \theta(1+U(x))^{\theta-1}U^2(x)\sum_{i=1}^{N} x_i \sum_{\gamma=1}^{K}\langle \sigma_{\mathbf{p}(i),\gamma}(x_i,\cdot),\mu^{\gamma}\rangle dw_{\gamma}.
\end{aligned}
$$

where C_1, C_2, and C_3 are some constants. As a by product of Lemma 3.4, $\mathbb{E}[(1+U(x(t))^{\theta}] \le C$, for all $t \ge 0$, which together with (4.11) implies that for $k = 1, 2, \ldots$ and for any given $\delta > 0$,

$$
\begin{aligned}
(4.12)\qquad & \mathbb{E}\left[\sup_{(k-1)\delta<t<k\delta}(1+U(x(t)))^{\theta}\right] \\
&\le \mathbb{E}\left[1+U(x((k-1)\delta)))^{\theta}\right] \\
&\quad + \mathbb{E}\Bigg[\sup_{(k-1)\delta<t<k\delta}\bigg|\int_{(k-1)\delta}^{t}\theta(1+U(x(s)))^{\theta-2} \\
&\qquad \times (C_1U^2(x(s))+C_2U(x(s))+C_3)ds\bigg|\Bigg] \\
&\quad + \mathbb{E}\Bigg[\sup_{(k-1)\delta<t<k\delta}\bigg|\int_{(k-1)\delta}^{t}\theta(1+U(x))^{\theta-1}U^2(x) \\
&\qquad \times \sum_{i=1}^{N} x_i \sum_{\gamma=1}^{K}\langle \sigma_{\mathbf{p}(i),\gamma}(x_i,\cdot),\mu^{\gamma}\rangle dw_{\gamma}\bigg|\Bigg].
\end{aligned}
$$

We now compute

$$
\begin{aligned}
(4.13)\qquad & \mathbb{E}\Bigg[\sup_{(k-1)\delta<t<k\delta}\bigg|\int_{(k-1)\delta}^{t}\theta(1+U(x(s)))^{\theta-2} \\
&\qquad\qquad (C_1U^2(x(s))+C_2U(x(s))+C_3)ds\bigg|\Bigg] \\
&\le \mathbb{E}\Bigg[\int_{(k-1)\delta}^{k\delta}\big|\theta(1+U(x(s)))^{\theta-2} \\
&\qquad\qquad (C_1U^2(x(s))+C_2U(x(s))+C_3)\big|ds\Bigg] \\
&\le \delta\theta C\mathbb{E}\left[\sup_{(k-1)\delta<t<k\delta}(1+U(x(s)))^{\theta}\right]
\end{aligned}
$$

By the well-known Burkholder-Davis-Gundy inequality, and using the assumption (A2'), we have the following estimate for the stochastic integral

$$
\begin{aligned}
(4.14)\qquad & \mathbb{E}\Bigg[\sup_{(k-1)\delta<t<k\delta}\Bigg|\int_{(k-1)\delta}^{t}\theta(1+U(x))^{\theta-1}U^2(x) \\
&\qquad\qquad \sum_{i=1}^{N}x_i\sum_{\gamma=1}^{K}\langle\sigma_{\mathbf{p}(i),\gamma}(x_i,\cdot),\mu^{\gamma}\rangle dw_{\gamma}\Bigg|\Bigg] \\
&\le 3\theta C\mathbb{E}\Bigg[\int_{(k-1)\delta}^{k\delta}(1+U(x))^{2(\theta-1)}U^2(x)\frac{1}{|x(s)|^2} \\
&\qquad \times\sum_{i,j=1}^{N}x_i(s)x_j(s)\tilde{a}_{\mathbf{p}(i),\mathbf{p}(j)}(x_i(s),x_j(s))ds\Bigg]^{\frac{1}{2}} \\
&\le 3\theta C\mathbb{E}\Bigg[\int_{(k-1)\delta}^{k\delta}(1+U(x))^{2\theta}ds\Bigg]^{\frac{1}{2}} \\
&\le 3\delta^{\frac{1}{2}}\theta C\mathbb{E}\Bigg[\sup_{(k-1)\delta<t<k\delta}(1+U(x))^{\theta}\Bigg]
\end{aligned}
$$

Substituting (4.13) and (4.14) into (4.12) yields

$$
\begin{aligned}
(4.15)\qquad & \mathbb{E}\left[\sup_{(k-1)\delta<t<k\delta}(1+U(x(t)))^{\theta}\right] \\
&\le \mathbb{E}\left[1+U(x((k-1)\delta)))^{\theta}\right] \\
&\quad + C\theta(\delta+\delta^{\frac{1}{2}})\mathbb{E}\left[\sup_{(k-1)\delta<t<k\delta}(1+U(x))^{\theta}\right].
\end{aligned}
$$

Choosing δ small enough so that $C\theta(\delta+\delta^{\frac{1}{2}})<\frac{1}{2}$ and using the fact that $\mathbb{E}[1+U(x((k-1)\delta)))^{\theta}]$ is bounded we obtain that

$$
\mathbb{E}\left[\sup_{(k-1)\delta<t<k\delta}(1+U(x(t)))^{\theta}\right]\le C.
$$

Let $\varepsilon>0$ be arbitrary. Then, by the Chebyshev inequality, we have

$$
\mathbb{P}\left\{\omega:\sup_{(k-1)\delta<t<k\delta}(1+U(x(t)))^{\theta}>(k\delta)^{1+\varepsilon}\right\}\le\frac{C}{(k\delta)^{1+\varepsilon}},\quad k=1,2,\ldots.
$$

By virtue of Borel-Cantelli lemma again, there exist Ω_2 such that $\mathbb{P}(\Omega_2) = 1$ and for all $\omega \in \Omega_2$, there exists $k_2(\omega) > \frac{1}{\delta} + 2$ such that

$$\sup_{(k-1)\delta<t<k\delta} (1 + U(x(t)))^\theta \le (k\delta)^{1+\varepsilon}$$

holds whenever $k > k_2(\omega)$. Consequently, for almost all $\omega \in \Omega$, if $k > k_2(\omega)$ and $(k-1)\delta < t < k\delta$,

$$\frac{\ln(1 + U(x(t)))^\theta}{\ln t} \le \frac{(1+\varepsilon)\ln(k\delta)}{\ln((k-1)\delta)} \to 1 + \varepsilon.$$

Hence, we have

$$\limsup_{t\to\infty} \frac{\ln(1 + U(x(t)))^\theta}{\ln t} \le 1 + \varepsilon \quad \text{a.s.}$$

Letting $\varepsilon \to 0$, we have

$$\limsup_{t\to\infty} \frac{\ln(1 + U(x(t)))^\theta}{\ln t} \le 1 \quad \text{a.s.}$$

By definition of U, from the previous estimates we obtain

$$\limsup_{t\to\infty} \frac{\ln(\frac{1}{S(x(t))})^\theta}{\ln t} \le 1 \quad \text{a.s.}$$

which further implies

$$\liminf_{t\to\infty} \frac{\ln(S(x(t)))}{\ln t} \ge -\frac{1}{\theta} \quad \text{a.s.}$$

This concludes the proof. □

4.3. Bounds on decay rate

In this section we study the decay rate to zero of the system (2.2) under the assumption (A2) on the diffusion part. The idea is to use the time change method and comparison principle (please see [17, Section 4, Chapter IV and Section 4, Chapter VI respectively]) to study a one-dimensional stochastic differential equation that is closely related to $S^{-1}(x(t))$. This investigation is inspired by the work [28] and the motivation for it comes from the following observation. Let us consider the following one-dimensional stochastic differential equation

$$dy(t) = y(t)(c + dy(t))dt + \sigma(x(t))dw(t),$$

where the coefficients satisfy $c > 0, d < 0$ and $\frac{C_1}{x^2} < \sigma^2(x) < \frac{C_2}{x^2}$ for some constants $C_1, C_2 > 0$. Then by Theorem 3.1 in Chapter VI of [17], we have that

$$\mathbb{P}(\liminf_{t\to\infty} y(t) = 0) = \mathbb{P}(\limsup_{t\to\infty} y(t) = \infty) = 1.$$

Since the above equation serves as the one-dimensional prototype for the system (2.2), we expect that (2.2) behaves somehow the same and this leads to the question about the decay rate to zero of system (2.2).

Theorem 4.4. *Suppose that* (A1) *and* (A2) *hold and that for all* $\alpha, \gamma \in \mathbf{K}$ *and* $x, y \in \mathbb{R}_+$, *there exists* $c_1 > 0, d_1 > 0$ *such that*

$$c_1 - d_1 y \le b_{\alpha\gamma}(x, y).$$

Then, with probability 1, there exists two positive constants Υ_1, Υ_2 *such that*

$$\limsup_{t\to\infty} \frac{1}{\sqrt{\ln t}\sum_{i=1}^N x_i(t)} \ge \Upsilon_1,$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{1+\varepsilon}\sum_{i=1}^N x_i(t)} \le \Upsilon_2.$$

Proof. For convenience, for each $i \in \{1, \cdots, N\}$, let

$$\bar{b}_i(x(t)) = x_i^N(t) \sum_{\gamma=1}^K \langle b_{\mathbf{p}(i)\gamma}(x_i^N(t), \cdot), \mu^\gamma(t)\rangle,$$

and

$$\bar{\sigma}_{i\gamma}(x(t)) = x_i^N(t)\langle \sigma_{\mathbf{p}(i)\gamma}(x_i^N(s), \cdot), \mu^\gamma(t)\rangle, \quad \gamma = 1, 2, \ldots, K.$$

Then (2.2) becomes

$$dx_i(t) = \bar{b}_i(x(t))dt + \sum_{\gamma=1}^K \bar{\sigma}_{i\gamma}(x(t))dw_\gamma(t). \tag{4.16}$$

Let $p(x) = \frac{1}{\sum_{i=1}^N x_i} = \frac{1}{S(x)}$. It is easy to see that,

$$\begin{aligned}\mathcal{L}p(x) = &- \frac{1}{S^2(x)} \sum_{i=1}^N x_i \sum_{\gamma=1}^K \langle b_{\alpha\gamma}(x_i, \cdot), \mu^\gamma(s)\rangle \\ &+ \frac{1}{S^3(x)} \sum_{i,j=1}^N x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j)\end{aligned}$$

We further define

$$\begin{aligned}a(x) &:= \sum_{i,j=1}^N \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x) \frac{\partial p}{\partial x_i} \frac{\partial p}{\partial x_j} \\ &= \frac{1}{S^4(x)} \sum_{i,j=1}^N x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x) > 0, \quad \forall x \in \mathbb{R}_+^N,\end{aligned} \tag{4.17}$$

$$b(x) := \frac{\mathcal{L}p(x)}{a(x)} = S(x) - S^2(x) \frac{\sum_{i=1}^N x_i \sum_{\gamma=1}^K \langle b_{\mathbf{p}(i)\gamma}(x_i, \cdot), \mu^\gamma\rangle}{\sum_{i,j=1}^N x_i x_j \tilde{a}_{\mathbf{p}(i)\mathbf{p}(j)}(x_i, x_j)}. \tag{4.18}$$

As noticed in (4.4), there exists $\kappa > 0$ such that

$$\frac{1}{\kappa} \le a(x) \le \kappa, \quad \forall x \in \mathbb{R}_+^N. \tag{4.19}$$

Let

$$\begin{cases} a^+(\xi) & = \sup_{x \in D(\xi,p)} a(x), \\ b^+(\xi) & = \sup_{x \in D(\xi,p)} b(x), \end{cases} \qquad \begin{cases} a^-(\xi) & = \inf_{x \in D(\xi,p)} a(x), \\ b^-(\xi) & = \inf_{x \in D(\xi,p)} b(x), \end{cases}$$

where $D(\xi, p) = \{x \in \mathbb{R}_+^N : p(x) = \xi\}$ for every $\xi > 0$. From the definition of a and since $\sigma_{\alpha,\gamma}$ are locally Lipschitzian, we can check that $a^\pm(\xi)$ and $b^\pm(\xi)$ are local Lipschitz positive continuous functions. Let Φ^+ and Φ^- are two functions defined by

$$\Phi^+(t) = \int_0^t \frac{a(x(s))}{a^+(p(x(s))}ds, \quad \Phi^-(t) = \int_0^t \frac{a(x(s))}{a^-(p(x(s)))}ds,$$

It can be easily seen that

$$\Phi^+(t) \le t \le \Phi^-(t), \quad \forall t \ge 0. \tag{4.20}$$

Let Ψ^+ and Ψ^- be the inverse functions of Φ^+ and Φ^- respectively. From (4.20), it is also easy to check that

$$\Psi^-(t) \le t \le \Psi^+(t), \quad \forall t \ge 0. \tag{4.21}$$

Let $x^+(t) = x(\Psi^+(t))$ and $x^-(t) = x(\Psi^-(t))$. As in Chapter IV, Section 4 of [17], we can conclude that x^+ and x^- satisfy the following stochastic differential equations

$$\begin{cases} dx_i^+(t) = \left[\frac{a^+(p(x^+(t)))}{a(x^+(t))}\right] \bar{b}_i(x^+(t))dt \\ \qquad\qquad + \left[\frac{a^+(p(x^+(t)))}{a(x^+(t))}\right]^{1/2} \sum_{\gamma=1}^K \bar{\sigma}_{i\gamma}(x^+(t))dB_\gamma^+(t), \\ x_0^+ = x_0, \quad i = 1, 2, \ldots, N, \end{cases}$$

and

$$\begin{cases} dx_i^-(t) = \left[\frac{a^-(p(x^-(t)))}{a(x^-(t))}\right] \bar{b}_i(x^-(t))dt \\ \qquad\qquad + \left[\frac{a^-(p(x^-(t)))}{a(x^-(t))}\right]^{1/2} \sum_{\gamma=1}^K \bar{\sigma}_{i\gamma}(x^-(t))dB_\gamma^-(t), \\ x_0^- = x_0, \quad i = 1, 2, \ldots, N, \end{cases}$$

where

$$B^+(t) = (B_1^+(t), \ldots, B_K^+(t)) \quad B^-(t) = (B_1^-(t), \ldots, B_K^-(t))$$

are two K-dimensional Brownian motions defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Using the Itô formula, we have

$$\begin{aligned} dp(x^+(t)) &= \left[\frac{a^+(p(x^+(t)))}{a(x^+(t))}\right] (\mathcal{L}p(x^+(t)))dt \\ &\quad + \left[\frac{a^+(p(x^+(t)))}{a(x^+(t))}\right]^{1/2} \sum_i^N \frac{\partial p}{\partial x_i}(x^+(t)) \sum_{\gamma=1}^K \bar{\sigma}_{i,\gamma}(x^+(t))dB_\gamma^+(t). \end{aligned}$$

By (4.17) and Lévy's celebrated martingale characterization of Brownian motion (see [18, Theorem 3.16]), if we set

$$\tilde{B}^\pm = \frac{1}{a^{1/2}(x^\pm(t))} \int_0^t \sum_i^N \frac{\partial p}{\partial x_i}(x^\pm(t)) \sum_{\gamma=1}^K \bar{\sigma}_{i,\gamma}(x^\pm(t))dB_\gamma^\pm(t)$$

then $\tilde{B}^+$ and $\tilde{B}^-$ are 1-dimensional Brownian motions and we have

$$\begin{aligned}dp(x^+(t)) &= a^+(p(x^+(t)))b(x^+(t))dt \\ &\quad + [a^+(p(x^+(t)))]^{1/2}d\tilde{B}^+(t),\end{aligned} \tag{4.22}$$

and similarly,

$$\begin{aligned}&dp(x^-(t)) \\ &= \left[\frac{a^-(p(x^-(t)))}{a(x^-(t))}\right](\mathcal{L}p(x^-(t)))dt \\ &\quad + \left[\frac{a^-(p(x^-(t)))}{a(x^-(t))}\right]^{1/2}\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}(x^-(t))\sum_{\gamma=1}^{K}\sigma_{i\gamma}(x^-(t))d\tilde{B}_\gamma^-(t) \\ &= a^-(p(x^-(t)))b(x^-(t))dt + [a^-(p(x^-(t)))]^{1/2}d\tilde{B}^-(t).\end{aligned} \tag{4.23}$$

From (A1), (4.17), (4.18), and (4.19), it is readily seen that there exists constants $\bar{\eta} > 0$, $\bar{\lambda} > 0$, and $\bar{\nu} > 0$ such that

$$a^+(p(x))b(x) \le -\bar{\eta}p(x) + \frac{\bar{\lambda}}{p(x)} + \bar{\nu}.$$

For any fix $\varepsilon > 0$, by (4.5) we can find $T > 0$ such that

$$\frac{1}{p(x^+(t))} = S(x^+(t)) \le t^{1+\varepsilon}, \forall t \ge T.$$

Therefore (4.22) implies

$$\begin{aligned}e^{\bar{\eta}t}p(x^+(t)) &= p(x^+(T)) + \int_T^t e^{\bar{\eta}s}[a^+(p(x^+(s)))]^{1/2}d\tilde{B}^+(s) \\ &\quad + \int_T^t e^{\bar{\eta}s}[a^+(p(x^+(s))) + \bar{\eta}p(x^+(s))]ds \\ &\le p(x^+(T)) + \int_T^t e^{\bar{\eta}s}[a^+(p(x^+(s)))]^{1/2}d\tilde{B}^+(s) \\ &\quad + \int_T^t e^{\bar{\eta}s}\left[\frac{\bar{\lambda}}{p(x)} + \bar{\nu}\right]ds \\ &\le p(x^+(T)) + \int_T^t e^{\bar{\eta}s}[a^+(p(x^+(s)))]^{1/2}d\tilde{B}^+(s) \\ &\quad + \int_T^t e^{\bar{\eta}s}[\bar{\lambda}s^{1+\varepsilon} + \bar{\nu}]ds\end{aligned}$$

Because a^+ is bounded we thus have

$$\limsup_{t\to\infty} \frac{\int_T^t e^{\bar\eta s}[a^+(p(x^+(s)))]^{1/2} d\tilde B^+(s)}{e^{\bar\eta t} t^{1+\varepsilon}} = 0.$$

Therefore,

$$\limsup_{t\to\infty} \frac{p(x^+(t))}{t^{1+\varepsilon}} \le \frac{\bar\lambda}{\bar\eta}.$$

By virtue of (4.21), $t \le \Psi^+(t)$ for every $t > 0$ it follows that

$$\begin{aligned}\limsup_{t\to\infty} \frac{p(x(t))}{t^{1+\varepsilon}} &= \limsup_{t\to\infty} \frac{p(x(\Psi^+(t)))}{[\Psi^+(t)]^{1+\varepsilon}} = \limsup_{t\to\infty} \frac{p(x(\Psi^+(t)))}{t^{1+\varepsilon}} \frac{t^{1+\varepsilon}}{[\Psi^+(t)]^{1+\varepsilon}} \\ &\le \limsup_{t\to\infty} \frac{p(x^+(t))}{t^{1+\varepsilon}} \le \frac{\bar\lambda}{\bar\eta}.\end{aligned}$$

On the other hand, by (A1), (4.17), (4.18) and (4.19), we can verify that there exist constants $\underline{\eta} > 0$ and $\underline{\nu} > 0$ such that

$$a^-(p(x))b(x) \ge -\underline{\eta} p(x) + \underline{\nu}.$$

Therefore (4.23) implies

$$\begin{aligned}
&e^{\underline{\eta} t} p(x^-(t)) \\
&= p(x^-(0)) + \int_0^t e^{\underline{\eta} s}[a^-(p(x^-(s)))]^{1/2} d\tilde B^-(s) \\
&\quad + \int_0^t e^{\underline{\eta} s}[a^-(p(x^-(s))) + \bar\eta p(x^-(s))] ds \\
&\ge p(x^-(0)) + \int_0^t e^{\underline{\eta} s}[a^-(p(x^-(s)))]^{1/2} d\tilde B^-(s) + \int_0^t e^{\underline{\eta} s} \bar\nu ds.
\end{aligned} \tag{4.24}$$

Let $M(t)$ be

$$\int_0^t e^{\underline{\eta} s}[a^-(p(x^-(s)))]^{1/2} d\tilde B^-(s)$$

then M is a real-valued continuous martingale with quadratic variation

$$\langle M(t)\rangle = \int_0^t e^{2\underline{\eta} s}[a^-(p(x^-(s)))] ds$$

and, by virtue of law of iterated logarithm, we also have

$$\limsup_{t\to\infty} \frac{M(t)}{2\langle M(t)\rangle \ln\ln\langle M(t)\rangle} = 1 \quad \text{a.s.} \tag{4.25}$$

In addition, by (4.4), it is easy to deduce that

$$\frac{1}{\kappa}\frac{e^{2\underline{\eta}t}-1}{2\underline{\eta}} \leq \langle M(t)\rangle \leq \kappa\frac{e^{2\underline{\eta}t}-1}{2\underline{\eta}}.$$

Hence

$$\begin{aligned}\sqrt{2\langle M(t)\rangle \ln\ln\langle M(t)\rangle} &\geq \sqrt{2\frac{1}{\kappa}\frac{e^{2\underline{\eta}t}-1}{2\underline{\eta}}\ln\ln(\frac{1}{\kappa}\frac{e^{2\underline{\eta}t}-1}{2\underline{\eta}})}\\ &\sim \sqrt{\frac{1}{\kappa\underline{\eta}}}e^{\underline{\eta}t}\sqrt{\ln t}\end{aligned}$$

and thus

$$\liminf_{t\to\infty}\frac{\sqrt{2\langle M(t)\rangle \ln\ln\langle M(t)\rangle}}{e^{\underline{\eta}t}\sqrt{\ln t}} \geq \sqrt{\frac{1}{\kappa\underline{\eta}}}. \tag{4.26}$$

Combining (4.25) and (4.26), we get

$$\begin{aligned}\limsup_{t\to\infty}\frac{M(t)}{e^{\underline{\eta}t}\sqrt{\ln t}} &\geq \limsup_{t\to\infty}\frac{M(t)}{2\langle M(t)\rangle \ln\ln\langle M(t)\rangle}\liminf_{t\to\infty}\frac{\sqrt{2\langle M(t)\rangle \ln\ln\langle M(t)\rangle}}{e^{\underline{\eta}t}\sqrt{\ln t}}\\ &\geq \sqrt{\frac{1}{\kappa\underline{\eta}}}.\end{aligned}$$

Using this inequality in (4.24), it then follows that

$$\limsup_{t\to\infty}\frac{p(x^-(t))}{\sqrt{\ln t}} \geq \sqrt{\frac{1}{\kappa\underline{\eta}}}.$$

By virtue of (4.21) we have $t \leq \Psi^+(t)$ for every $t > 0$ and hence

$$\begin{aligned}\limsup_{t\to\infty}\frac{p(x(t))}{\sqrt{\ln t}} &= \limsup_{t\to\infty}\frac{p(x(\Psi^-(t)))}{\sqrt{\ln\Psi^-(t)}} = \limsup_{t\to\infty}\frac{p(x(\Psi^-(t)))}{\sqrt{\ln t}}\frac{\sqrt{\ln t}}{\sqrt{\ln\Psi^-(t)}}\\ &\geq \limsup_{t\to\infty}\frac{p(x(\Psi^-(t)))}{\sqrt{\ln t}} = \limsup_{t\to\infty}\frac{p(x^-(t))}{\sqrt{\ln t}} \geq \sqrt{\frac{1}{\kappa\underline{\eta}}},\end{aligned}$$

or

$$\limsup_{t\to\infty}\frac{1}{\sqrt{\ln t}S(x(t))} \geq \sqrt{\frac{1}{\kappa\underline{\eta}}}$$

and the proof is complete. □

5. Concluding remarks

This paper focuses on multi-type mean-field models. The problems under consideration enrich the mean-field models and enlarge their applicability. The equations under consideration are highly nonlinear with an additional empirical measure. The use of multiple types further complicates the matter. First, sufficient conditions are presented, under which not only are the existence and uniqueness of solutions obtained, but also the solutions are shown to remain in the positive (first) quadrant of $\mathbb{R}^N$. Moreover, starting with a point outside a bounded open set with compact closure, we derive conditions to ensure that the expect return time to the bounded set is finite. Such positive recurrence is the key for the study of ergodicity. Furthermore, we derive upper and lower bounds on the moments of the solutions. These results will be essential for subsequent study on control, optimization, and related issues.

References

[1] J. Baladron, D. Fasoli, O. Faugeras and J. Touboul, *Mean-Field description and propagation of chaos in networks of Hodgkin-Huxley and Fitz Hugh-Nagumo neurons*. Journal of Mathematical Neuroscience, **2**(10): 1–50, 2012.

[2] A. Bensoussan, J. Frehse and P. Yam, *Mean-field games and mean field type control theory*. Springer Briefs in Mathematics, Springer, New York, 2013.

[3] V. S. Borkar and K. S. Kumar, *McKean-Vlasov Limit in Portfolio Optimization*. Stochastic Analysis and Applications, **28**: 884–906, 2010.

[4] W. Braun and K. Hepp, *The Vlasov dynamics and its fluctuations in the 1/n limit of interacting classical particles*. Commun. Math. Phys., **56**(2): 101–113, 1977.

[5] A. Budhiraja and W. Ruoyu, *Some Fluctuation Results for Weakly Interacting Multi-type Particle Systems*. `arXiv:1505.00856`.

[6] R. Carmona, F. Delarue and D. Lacker, *Mean field games with common noise*. Preprint, `arXiv:1407.6181`.

[7] F. Collet, *Macroscopic limit of a bipartite Cuire-Weiss model: A dynamical approach*. `arXiv:1406.4044v1`, 2014.

[8] P. Contucci, I. Gallo and G. Menconi, *Phase transitions in social sciences: Two-population mean field theory*. International Journal of Modern Physics B, **22**: 2199–2212, 2008.

[9] D. A. Dawson, *Critical dynamics and fluctuations for a mean-field model of cooperative behavior*. J. Statist. Phys., **31**(1): 29–85, 1983.

[10] D. A. Dawson and J. Gartner, *Large deviations from the McKean-Vlasov limit for weakly interacting diffusions*. Stochastics, **20**(4): 247–308, 1987.

[11] A. Friedman, *Stochastic Differential Equations and Applications*, vols. I and II. Academic Press, New York, 1971.

[12] J. Gartner, *On the McKean-Vlasov limit for interacting diffusions*. Math. Nachr., **137**: 197–248, 1988.

[13] T. A. Hoang, S. L. Nguyen and G. Yin, *Nearly optimal and nearly equilibrium controls with wide band-width noise for hybrid systems*. To appear in: Dynamics of Continuous Discrete and Impulsive Systems, Series A, Mathematical Analysis.

[14] M. Huang, R. P. Malhamé and P. E. Caines, *Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle*. Commun. Inf., Syst., **6**(3): 221–251, 2006.

[15] M. Huang, P. E. Caines and R. P. Malhamé, *An invariance principle in large population stochastic dynamic games*. J. Syst. Sci. Complex, **20**(2): 162–172, 2007.

[16] M. Huang, P. E. Caines and R. P. Malhamé, *Social optima in mean field LQG control: centralized and decentralized strategies*. IEEE Trans. Automat. Control, **57**(7): 1736–1751, 2012.

[17] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*. North-Holland, Amsterdam, 1981.

[18] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*. (2nd ed.), Springer, New York, 1991.

[19] R. Z. Khasminskii, *Stochastic Stability of Differential Equations*. Springer, Heidelberg, 2012.

[20] V. N. Kolokoltsov, J. Li, and W. Yang, *Mean field games and nonlinear Markov processes*. Preprint, `arXiv:1112.3744`.

[21] J. M. Lasry and P. L. Lions, *Jeux champ moyen. I. Le cas stationnaire.* (French) [Mean field games. I. The stationary case] C. R. Math. Acad. Sci. Paris, **343**(9): 619–625, 2006.

[22] J. M. Lasry and P. L. Lions, *Jeux à champ moyen. II. Horizon fini et contrôle optimal* (French) [Mean field games. II. Finite horizon and optimal control], C. R. Math. Acad. Sci. Paris, **343**(10):679–684, 2006.

[23] J. M. Lasry and P. L. Lions, *Mean field games.* Jpn. J. Math, **2**(1): 229–260, 2007.

[24] X. Mao, S. Sabais, and E. Renshaw, *Asymptotic Behavior of stochastic Lotka-Volterra model.* J. Math. Anal. Appl., **287**: 141–156, 2003.

[25] H. P. McKean, *A class of Markov processes associated with nonlinear parabolic equations.* Proc. Nat. Acad. Sci. U.S.A., **56**: 1907–1911, 1966.

[26] H. P. McKean, *Propagation of chaos for a class of non-linear parabolic equations.* In: Stochastic Differential Equations, Arlington: Air Force Office Sci. Res: 41–57, 1967. [Lecture Series in Differential Equations, vol. 7.]

[27] S. L. Nguyen and M. Huang, *Linear-quadratic-Gaussian mixed games with continuum-parametrized minor players.* SIAM J. Control Optim., **50**(5): 2907–2937, 2012.

[28] D. H. Nguyen and S. H. Vu, *Dynamics of a stochastic Lotka-Volterra model perturbed by white noise.* J. Math. Anal. Appl., **324**: 82–97, 2006.

[29] M. Nourian and P. E. Caines, *ϵ-Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents.* SIAM J. Control Optim, **51**(4): 3302–3331, 2013.

[30] A. S. Sznitman, *Topics in propagation of chaos.* Ecole d'été de Probabilités de Saint-Flour XIX–1989, page 165–251, Lecture Notes in Math., **1464**, Springer, Berlin, 1991.

[31] H. Tanaka, *Probabilistic treatment of the Boltzmann equation of Maxwellian molecules.* Probab Theory Relat Fields, **46**: 67–105, 1978.

[32] H. Tanaka and M. Hitsuda, *Central limit theorem for a simple diffusion model of interacting particles.* Hiroshima Math J., **11**(2): 415–423, 1981.

[33] H. Tanaka, *Some probabilistic problems in the spatially homogeneous Boltzmann equation.* In: Theory and Application of Random Fields,

Lecture Notes in Control and Information Sciences. Edited by Kallianpur G. Springer, Berlin, page 258–267, 1983.

[34] H. Tanaka, *Limit theorems for certain diffusion processes with interaction.* Stochastic Analysis. Amsterdam, North-Holland, page 469–488, 1984. [North-Holland Mathematical Library, vol. 32.]

[35] G. Yin, G. Zhao, and F. Xi, *Mean-Field models involving continuous-state-dependent random switching: Nonnegativity constraints, moment bounds, and two-time-scale limits.* Taiwanese Journal of Mathematics, **15**:1783–1805, 2011.

[36] C. Zhu and G. Yin, *On competitive Lotka-Volterra model in random environments,* J. Math. Anal. Appl., **357**: 154–170, 2009.

Department of Mathematics, Wayne State University
Detroit, Michigan 48202 USA
E-mail address: `tuan.hoang@wayne.edu`

Department of Mathematics, Wayne State University
Detroit, Michigan 48202
E-mail address: `gyin@math.wayne.edu`