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Some solvable stochastic differential games in $SU(3)$

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Dedicated to Professor Peter Caines on the occasion of his seventieth birthday

Some stochastic differential games are formulated and explicitly solved in the special unitary group $SU(3)$. This Lie group is probably the most elementary simple Lie group of rank two, that is, the Cartan subalgebra has dimension two. The game is to influence the roots of $SU(3)$ which are directly related to the eigenvalues of an element in $SU(3)$. The group $SU(3)$ has particular interest in physics because the Gell-Mann matrices are generators for $SU(3)$ that mediate Quantum Chromodynamics (QCD) which is also known as the Strong Force. Since the radial parts of Laplacians for higher rank Lie groups have a similar structure as the radial part of the Laplacian for $SU(3)$, stochastic differential games for $SU(3)$ are important to understand for extensions of possible solvable stochastic differential games for other Lie groups.

1. Introduction

Two general methods are available for solving stochastic differential games. The first method uses the Hamilton-Jacobi-Isaacs (HJI) equation which can be considered as the game analogue of the Hamilton-Jacobi-Bellman equation of stochastic control and the Hamilton-Jacobi equation of deterministic control. A number of results (e.g. Fleming and Hernandez-Hernandez [14], Fleming and Souganidis [15]) have been obtained that provide conditions for the existence of solutions of HJI equations for stochastic differential games. The second method is backward stochastic differential equations that generalized the use of backward stochastic equations from stochastic control. This approach is more recent than the HJI equation approach and fewer results about these backward equations are available (e.g. Buckdahn and Li [5], Hamadene et al. [17]). The approach used here for the solutions of

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stochastic differential games is more direct than the two aforementioned methods. One family of stochastic differential games that has explicit solutions is the collection of games described by linear stochastic differential equations with Brownian motions and quadratic payoffs (LQ games) (e.g. Basar and Bernhard [3]). Jacobson [19] showed that the solution of a linear quadratic stochastic differential game uses the same Riccati equation as the solution of a suitable linear exponential quadratic Gaussian control problem. A more direct method to solve the linear exponential quadratic Gaussian control problem and thereby its relation to LQ games is given in [6] and a stochastic differential game with an exponential quadratic payoff is explicitly solved in [9]. This family of explicit solutions for LQ stochastic differential games has been generalized by allowing the linear stochastic differential equations to be driven by an arbitrary square integrable process with continuous sample paths [7]. Some infinite dimensional LQ games with fractional Brownian motions are solved in [8].

To obtain explicit optimal control strategies for nonlinear stochastic differential games it is fairly clear that the nonlinear systems should possess some special structure that can be exploited for the optimization solutions. In this paper some stochastic differential games are formulated and solved in $SU(3)$, the special unitary group of degree three. This group is realized as 3×3 unitary matrices with determinant one. The payoff functionals for these problems have some special symmetries that reduce the analysis to a two dimensional subspace that is obtained from the radial part of the Laplacian for $SU(3)$. The only other work on stochastic differential games in symmetric spaces seems to be the results for spheres in [10, 11] and some projective spaces [12] which are all rank one symmetric spaces so the analysis is reduced to a one dimensional subspace.

The stochastic differential game problem that is formulated and solved in this paper is to control the roots of a process in the Lie algebra $su(3)$. Since $SU(3)$ is simply connected, this game problem can be viewed in the Lie algebra, $su(3)$. The group $SU(3)$ has particular interest in physics because the Gell-Mann matrices are generators for $SU(3)$ that mediate Quantum Chromodynamics (QCD) [16] which is also known as the Strong Force. In theoretical physics QCD is the theory of strong interactions that is a fundamental force describing the interactions between quarks and gluons which comprise hadrons such as the proton, neutron and pion. This theory is an important part of the Standard Model of particle physics.

2. Some properties of $SU(3)$

The simply connected Lie group $SU(3)$ is the family of 3×3 unitary matrices with determinant one, that is, $g \in SU(3)$ if $gg^* = I$, $det(g) = 1$. This Lie group has dimension eight as a real manifold. It is a simple Lie group. This Lie group has rank two, that is, the dimension of the Cartan subalgebra is two. A Weyl chamber is a special convex cone in $\mathbb{R}^2$. The maximal torus is all diagonal matrices with determinant one and the Weyl group is S_3 that is the family of signed permutation matrices. The Lie algebra, $su(3)$, is the family of zero trace skew Hermitian 3×3 complex matrices. $SU(3)$ contains $SO(3)$ as a double cover. The two (simple) positive roots are not orthogonal and $\mathbb{R}^2$ can be decomposed into six disjoint open convex cones (Weyl chambers) and their boundaries. Some symmetries for the stochastic differential games are considered so the strategies for the game are determined in the positive Weyl chamber. The positive Weyl chamber is one of the six Weyl chambers and this chamber can be arbitrarily chosen among the Weyl chambers. The payoff functional is obtained from an eigenfunction of the radial part of the Laplacian so that the game can be determined by its evolution in the positive Weyl chamber.

Initially the radial part of the Laplacian is explicitly described to verify some eigenfunctions and to determine optimal strategies for the two players. Let the positive Weyl chamber be denoted a_+ so that $a_+ = \{t_1 M_{e_1} + t_2 M_{e_2} : t_1 < t_2\}$ where e_1, e_2 is an orthonormal basis and define the functions φ and ψ as $\varphi(t_1, t_2) = e^{t_1} e_1 + e^{t_2} e_2$ and $\psi(t_1, t_2) = \sinh^2(\frac{t_1}{2}) e_1 + \sinh^2(\frac{t_2}{2}) e_2$.

Now the radial part of the Laplacian, $R(\Delta)$, is described. Consider the Laplacian in $SU(3)$ projected to the positive Weyl chamber a_+. While this is known e.g. [13], it is considered in some detail here to understand a particular eigenfunction.

$$
\begin{aligned}
(1) \qquad R(\Delta) = \Delta_K = {} & \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + c\left(\coth(t_1)\frac{\partial}{\partial t_1} + \coth(t_2)\frac{\partial}{\partial t_2}\right) \\
& + b\left(\coth\frac{t_1}{2}\frac{\partial}{\partial t_1} + \coth\frac{t_2}{2}\frac{\partial}{\partial t_2}\right) \\
& + a\left(\frac{\sinh t_1}{\cosh t_2 - \cosh t_1}\frac{\partial}{\partial t_1} + \frac{\sinh t_2}{\cosh t_1 - \cosh t_2}\frac{\partial}{\partial t_2}\right)
\end{aligned}
$$

The simplest eigenfunction, φ_1, is considered, that is,

$$
(2) \qquad \varphi_1(t_1, t_2) = k + \sinh^2\frac{t_1}{2} + \sinh^2\frac{t_2}{2}
$$

where $k = 1 + 2b + c$. Using the following elementary properties of the hyperbolic functions

$$\sinh(x+y) = \cosh(x)\sinh(y) + \sinh(x)\cosh(y) \tag{3}$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \tag{4}$$

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)} \tag{5}$$

it is verified that $\varphi_1(t_1, t_2)$ is an eigenfunction of Δ_K. This eigenfunction is largely the same as for the product Lie group $SU(2) \times SU(2)$, that is the cartesian product of two rank one symmetric spaces.

Some computations are exhibited now to verify that φ_1 is an eigenfunction and to determine the eigenvalue for φ_1.

$$\frac{\partial^2}{\partial t^2}\left(\sinh^2 \frac{t}{2}\right) = \frac{1}{2} + \sinh^2 \frac{t}{2} \tag{6}$$

$$b \coth \frac{t}{2} \frac{\partial}{\partial t}\left(\sinh^2 \frac{t}{2}\right) = b\left(1 + \sinh^2 \frac{t}{2}\right) \tag{7}$$

$$\coth(t) = \frac{1}{2}\coth \frac{t}{2} + \frac{1}{2}\frac{1}{\coth \frac{t}{2}} \tag{8}$$

$$c \coth(t) \frac{\partial}{\partial t}\left(\sinh^2 \frac{t}{2}\right) = c\left(\frac{1}{2} + \sinh^2 \frac{t}{2}\right) \tag{9}$$

$$\begin{aligned} &a \frac{\sinh t_1}{(\cosh t_2 - \cosh t_1)} \sinh \frac{t_1}{2} \cosh \frac{t_1}{2} \\ &+a \frac{\sinh t_2}{(\cosh t_1 - \cosh t_2)} \sinh \frac{t_2}{2} \cosh \frac{t_2}{2} \\ &= a\left(\sinh^2 \frac{t_1}{2} + \sinh^2 \frac{t_2}{2}\right) \end{aligned} \tag{10}$$

From the above equalities it follows that the eigenvalue λ_1 for φ_1 is

$$\lambda_1 = a + b + c \tag{11}$$

For $SU(3)$ it can be verified that $a = b = c = 1$. However these parameters are continued to indicate the changes that are made for other Lie groups. A coordinate transformation can be made in the Weyl chamber e.g. $x_i = e_i^t$ so that the eigenfunctions can be expressed as monomial symmetric functions. Some processes are constructed from monomial symmetric functions that are called Macdonald processes e.g. [4]. The monomial symmetric function that is used for the above eigenfunction is $x_1 + x_2$ so it is clearly the most elementary case.

3. Stochastic differential game formulation and solution

The stochastic differential game is described by a stochastic differential equation that has terms from the strategies of the two players and terms from the radial part of the Laplacian.

$$dX_1(t) = \frac{1}{2}\left(c \coth X_1(t) + b \coth \frac{X_1(t)}{2} + a \frac{\sinh X_1(t)}{\cosh X_2(t) - \cosh X_1(t)}\right) dt + \alpha U_1(t)dt + \beta V_1(t)dt + dB_1(t) \tag{12}$$

$$dX_2(t) = \frac{1}{2}\left(c \coth X_2(t) + b \coth \frac{X_2(t)}{2} + a \frac{\sinh X_2(t)}{\cosh X_1(t) - \cosh X_2(t)}\right) dt + \alpha U_2(t)dt + \beta V_2(t)dt + dB_2(t) \tag{13}$$

$$X_1(0) = x_{10} \tag{14}$$

$$X_2(0) = x_{20} \tag{15}$$

The process $((B_1(t), B_2(t)), t \in [0,T])$ is an $\mathbb{R}^2$-valued standard Brownian motion that is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0,T])$ is the filtration for the Brownian motion (B_1, B_2), x_{10} and x_{20} are constants and α, β are strictly positive constants. Player I has the control pair (U_1, U_2) and player II has the control pair (V_1, V_2). It is assumed that the positive real numbers α, β satisfy $\alpha^2 - \beta^2 > 0$. The symmetry of the two scalar equations for X_1 and X_2 is inherited from the coordinate symmetry for the radial part of the Laplacian.

The payoff functional, $J(U,V)$, is

$$J^0(U,V) = \int_0^T \left(\sinh^2 \frac{X_1(t)}{2} + \sinh^2 \frac{X_2(t)}{2} + (U_1^2(t) - V_1^2(t)) \cosh^2 \frac{X_1(t)}{2} + (U_2^2(t) - V_2^2(t)) \cosh^2 \frac{X_2(t)}{2} \right) dt \tag{16}$$

$$J(U,V) = \mathbb{E} J^0(U,V) \tag{17}$$

The families of admissible strategies for (U_1, U_2) is $\mathcal{U}$ and for (V_1, V_2) is $\mathcal{V}$ and are defined as follows

$$\mathcal{U} = \Big\{ U = (U_1, U_2) : U \text{ is progressively measurable with respect to } (\mathcal{F}(t), t \in [0,T]) \text{ and } \int_0^T |U|^2 dt < \infty \text{ a.s.} \Big\}$$

$$\mathcal{V} = \Big\{ V = (V_1, V_2) : V \text{ is progressively measurable with respect to } (\mathcal{F}(t), t \in [0,T]) \text{ and } \int_0^T |V|^2 dt < \infty \text{ a.s.} \Big\}$$

To determine optimal control strategies for the two players, the positive solution of the the following scalar Riccati equation and the solution of the following linear differential equation are used. The occurrence of a Riccati equation is natural from the symplectic geometry property of the optimization, e.g. [1, 2].

$$\frac{dg}{dt} = -\frac{1}{2}\lambda_1 g + \frac{1}{4} g^2(\alpha^2 - \beta^2) - 1 \tag{18}$$

$$g(T) = 0 \tag{19}$$

$$\frac{dh}{dt} = -\frac{k}{2} g(t) \tag{20}$$

$$h(T) = 0 \tag{21}$$

The following theorem provides optimal strategies for the two players. These strategies form a Nash equilibrium [20], that is, the optimal strategy of one player is not influenced by the strategy of the other player.

Theorem 3.1. *The stochastic differential game given by* (12), (13), *and* (16) *has the following optimal strategies,* (U^*, V^*), *that form a Nash equilibrium*

$$U_1^*(t) = -\frac{1}{2}\alpha g(t) \tanh \frac{X_1(t)}{2} \tag{22}$$

$$U_2^*(t) = -\frac{1}{2}\alpha g(t) \tanh \frac{X_2(t)}{2} \tag{23}$$

$$V_1^*(t) = \frac{1}{2}\beta g(t) \tanh \frac{X_1(t)}{2} \tag{24}$$

$$V_2^*(t) = \frac{1}{2}\beta g(t) \tanh \frac{X_2(t)}{2} \tag{25}$$

The optimal payoff is

$$J(U^*, V^*) = g(0)\left(\sinh^2 \frac{x_{10}}{2} + \sinh^2 \frac{x_{20}}{2}\right) + h(0) \tag{26}$$

Proof. Let $f(t, x_1, x_2) = g(t)(\sinh^2 \frac{x_1}{2} + \sinh^2 \frac{x_2}{2}) + h(t)$ and define $Y(t) = f(t, X_1(t), X_2(t))$. Apply the Ito differential rule to the process $(Y(t), t \in [0, T])$ and integrate it to obtain the following equality.

$$\begin{aligned}
Y(T) - Y(0) = \int_0^T \Bigg(& \frac{1}{2}\left(g\lambda_1 \sinh^2 \frac{X_1}{2} + g\lambda_1 \sinh^2 \frac{X_2}{2}\right) \\
& + \Bigg(g \sinh \frac{X_1}{2} \cosh \frac{X_1}{2}(\alpha U_1 + \beta V_1) \\
& \quad + g \sinh \frac{X_2}{2} \cosh \frac{X_2}{2}\Bigg)(\alpha U_2 + \beta V_2) \\
& + \frac{dg}{dt} \sinh^2 \frac{X_1}{2} + \frac{dg}{dt} \sinh^2 \frac{X_2}{2}\Bigg) dt \\
+ \int_0^T \Bigg(& g \sinh \frac{X_1}{2} \cosh \frac{X_1}{2} dB_1(t) \\
& + g \sinh \frac{X_2}{2} \cosh \frac{X_2}{2} dB_2(t)\Bigg) \\
= \int_0^T \Bigg(& \frac{1}{2}\left(g\lambda_1 \sinh^2 \frac{X_1}{2} + g\lambda_1 \sinh^2 \frac{X_2}{2}\right) \\
& + g \sinh \frac{X_1}{2} \cosh \frac{X_1}{2}(\alpha U_1 + \beta V_1) \\
& + g \sinh \frac{X_2}{2} \cosh \frac{X_2}{2}(\alpha U_2 + \beta V_2) \\
& + \sinh^2 \frac{X_1}{2}\left(-\frac{1}{2}\lambda_1 g + \frac{1}{4} g^2(\alpha^2 - \beta^2) - 1\right) \\
& + \sinh^2 \frac{X_2}{2}\left(-\frac{1}{2}\lambda_1 g + \frac{1}{4} g^2(\alpha^2 - \beta^2) - 1\right)\Bigg) dt \\
+ \int_0^T \Bigg(& g \sinh \frac{X_1}{2} \cosh \frac{X_1}{2} dB_1(t) \\
& + g \sinh \frac{X_2(t)}{2} \cosh \frac{X_2}{2} dB_2(t)\Bigg)
\end{aligned} \tag{27}$$

This equality used the fact that $k + \sinh^2 \frac{x_1}{2} + \sinh^2 \frac{x_2}{2} = \varphi_1(x_1, x_2)$ is an eigenfunction of $R(\Delta)$ with eigenvalue λ_1.

Using the definition of J^0 the following equality is satisfied, that is, add the terms from J^0 that contain the strategies for the two players to both sides of the above equation and transpose the terms from the state in J^0 from the RHS to the LHS. The following equality is obtained.

$$
\begin{aligned}
(28)\qquad & J^0(U,V) - g(0)\Big(\sinh^2\frac{x_{10}}{2} + \sinh^2\frac{x_{20}}{2}\Big) - h(0)\\
&= \int_0^T \Big(U_1^2\cosh^2\frac{X_1}{2} + g\alpha U_1 \sinh\frac{X_1}{2}\cosh\frac{X_1}{2}\Big)\\
&\quad - \Big(V_1^2\cosh^2\frac{X_1}{2} - g\beta V_1\sinh\frac{X_1}{2}\cosh\frac{X_2}{2}\Big)\\
&\quad + \frac{1}{4}g^2(\alpha^2-\beta^2)\sinh^2\frac{X_1}{2} + g\sinh\frac{X_1}{2}\cosh\frac{X_1}{2}dB_1(t)\\
&\quad + \Big(U_2^2\cosh^2\frac{X_2}{2} + g\alpha U_2\sinh\frac{X_2}{2}\cosh\frac{X_2}{2}\Big)\\
&\quad - \Big(V_2^2\cosh^2\frac{X_2}{2} - g\beta V_2\sinh\frac{X_2}{2}\cosh\frac{X_2}{2}\Big)dt\\
&\quad + \frac{1}{4}g^2(\alpha^2-\beta^2)\sinh^2\frac{X_2}{2} + g\sinh\frac{X_2}{2}\cosh\frac{X_2}{2}dB_2(t)\\
&= \int_0^T \Bigg(\Big(U_1^2\cosh^2\frac{X_1}{2} + g\alpha U_1\sinh\frac{X_1}{2}\cosh\frac{X_1}{2}\\
&\qquad\quad + \frac{1}{4}\alpha^2 g^2\sinh^2\frac{X_1}{2}\Big) - \Big(V_1^2\cosh^2\frac{X_1}{2}\\
&\qquad\quad - g\beta V_1\sinh\frac{X_1}{2}\cosh\frac{X_1}{2}dt + \frac{1}{4}\beta^2 g^2\sinh^2\frac{X_1}{2}\Big)\Bigg)dt\\
&\quad + \int_0^T \Bigg(\Big(U_2^2\cosh^2\frac{X_2}{2} + g\alpha U_2\sinh\frac{X_2}{2}\cosh\frac{X_2}{2}\\
&\qquad\quad + \frac{1}{4}\alpha^2 g^2\sinh^2\frac{X_2}{2}\Big) - \Big(V_2^2\cosh^2\frac{X_2}{2}\\
&\qquad\quad - g\beta V_2\sinh\frac{X_2}{2}\cosh\frac{X_2}{2}dt + \frac{1}{4}\beta^2 g^2\sinh^2\frac{X_2}{2}\Big)\Bigg)dt\\
&\quad + \int_0^T \Big(g\sinh\frac{X_1}{2}\cosh\frac{X_1}{2}dB_1(t) + g\sinh\frac{X_2}{2}\cosh\frac{X_2}{2}dB_2(t)\Big)
\end{aligned}
$$

The two stochastic integrals are martingales by comparison of their integrands with integrands formed by replacing X_1 and X_2 by Brownian motions. Thus

$$J(U,V) = g(0)\left(\sinh^2\frac{x_{10}}{2} + \sinh^2\frac{x_{20}}{2}\right) + h(0) \tag{29}$$
$$+ \mathbb{E}\int_0^T \left(\cosh^2\frac{X_1}{2}\left(U_1 + \frac{1}{2}\alpha g \tanh\frac{X_1}{2}\right)^2\right.$$
$$- \cosh^2\frac{X_1}{2}\left(V_1 - \frac{1}{2}\beta g \tanh\frac{X_1}{2}\right)^2$$
$$+ \cosh^2\frac{X_2}{2}\left(U_2 + \frac{1}{2}\alpha g \tanh\frac{X_2}{2}\right)^2$$
$$\left. - \cosh^2\frac{X_2}{2}\left(V_2 - \frac{1}{2}\beta g \tanh\frac{X_2}{2}\right)^2\right) dt$$

It follows from this equality that the optimal strategies for the two players are

$$\begin{aligned} U_1^*(t) &= -\frac{1}{2}\alpha g(t)\tanh\frac{X_1(t)}{2} \\ U_2^*(t) &= -\frac{1}{2}\alpha g(t)\tanh\frac{X_2(t)}{2} \\ V_1^*(t) &= \frac{1}{2}\beta g(t)\tanh\frac{X_1(t)}{2} \\ V_2^*(t) &= \frac{1}{2}\beta g(t)\tanh\frac{X_2(t)}{2} \end{aligned} \tag{30}$$

It is clear that the optimal strategies form a Nash equilibrium. It is elementary to verify that the solution of the optimal system does not hit the walls of the Weyl chamber almost surely by comparison with a scalar Bessel process. Thus the optimal payoff is

$$J(U^*,V^*) = g(0)\left(\sinh^2\frac{x_{10}}{2} + \sinh^2\frac{x_{20}}{2}\right) + h(0) \tag{31}$$

□

4. Concluding remarks

This stochastic differential game is a special case of games in Lie groups and symmetric spaces. The group $SU(3)$ has some well known applications especially to physics and it provides a basis to consider more general stochastic differential games because it has rank greater than one.

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