Communications in Information and Systems
Volume 15, Number 1, 47–55, 2015

Stochastic volatility models with volatility driven by fractional Brownian motions

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In this paper the price of a risky asset that has a stochastic volatility being a function of a fractional Brownian motion is considered. Such models can provide a long range dependence for the volatility. The probability density function for the price at a given time is given explicitly under some natural, verifiable conditions. An option pricing model is also considered with some explicit results.

AMS Subject Classifications: 91B25, 91G20, 91G80, 60H30.

Keywords and Phrases: stochastic volatility, fractional Brownian motion, financial models.

1. Introduction

Some research indicates the presence of long range dependence in financial data and in volatility dynamics (e.g. Anderson and Bollerslew [1], Casas and Gao [2], Comte and Renault [4], [5] and Comte et al [3], Ding and Granger [6], Fukasawa [7]). Based on these investigations a market with volatility determined by fractional Brownian motion (FBM) is considered. Specifically it is assumed that the asset price process X satisfies the following stochastic equation

$$dX(t) = f(W^H(t))g(t)X(t)\, dW(t), \tag{1.1}$$

where $X(0)$ is a positive constant, the process W is a standard Brownian motion, W^H is a standard fractional Brownian motion with Hurst parameter $H \in (0,1)$, $f : \mathbb{R} \to \mathbb{R}^+$ is Borel measurable and $g : \mathbb{R} \to \mathbb{R}^+$ is Borel measurable and bounded such that the equation (1.1) has a unique strong solution. This approach is motivated by [9]. Following Jakubowski and Wiśniewolski

Research supported by NSF grants DMS 1108884 and DMS 1411412, AFOSR grant FA9550-12-1-0384, ARO grant W911NF-14-10390, Polish MNiSW grant N N201 547838.

[9] this model (1.1) is called a linear stochastic volatility model with volatility driven by a fractional Brownian motion.

Under some natural assumptions it is verified that the distribution of the asset price X has a probability density function that admits a probabilistic representation (Theorem 2.1).

Subsequently examples of such models are given where the probabilistic representations of the asset price density function is important. The first example is for volatility being a function of a fractional Brownian motion. The second example that is given is the case where the volatility is a geometric fractional Brownian motion.

In Section 3 the probabilistic representations for European call and put option prices in some linear stochastic volatility models are given.

In this paper a similar approach as in Jakubowski and Wiśniewolski [9] is used where the probabilistic representations for the density and European call and put option prices with a linear stochastic volatility model have been given.

2. The density function of the asset price in a volatility model with volatility given by fractional Brownian motions

Consider a market defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $T < \infty$, satisfying the usual conditions and $\mathcal{F} = \mathcal{F}_T$. Without loss of generality it is assumed that the savings account is constant and identically equal to one. Moreover, it is assumed that the price X of the underlying asset has a stochastic volatility given by a function of a standard fractional Brownian motion, so the dynamics of X is given by

$$dX(t) = f(W^H(t))g(t)X(t)\; dW(t), \tag{2.1}$$

where $X(0)$ is a positive constant, the process W is a standard Brownian motion, W^H is a standard fractional Brownian motion with the Hurst parameter $H \in (0,1)$, $f : \mathbb{R} \to \mathbb{R}^+$ is Borel measurable and $g : [0,T] \to \mathbb{R}^+$ is Borel measurable and bounded. It is well known (e.g. Nualart [10]) that W^H has a representation

$$W^H(t) = \int_0^t K_H(t,s)d\widehat{W}(s), \tag{2.2}$$

where for $H < 1/2$

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \tag{2.3}$$

where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}\right]^{\frac{1}{2}}$, $t > s$, and for $H > 1/2$

$$K_H(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right], \tag{2.4}$$

where $c_H = \left[\frac{2H}{(1-2H)\beta(1-2H,H+\frac{1}{2})}\right]^{\frac{1}{2}}$, $t > s$, and $\widehat{W}$ is a standard Brownian motion. Assume that the stochastic basis is rich enough so that W and $\widehat{W}$ are well defined on it. Let

$$\int_0^T f^2(W^H(u))du < \infty, \quad \mathbb{P} - a.s. \tag{2.5}$$

If f is a continuous function, then clearly (2.5) is always satisfied. Since g is bounded, then

$$\int_0^T f^2(W^H(u))g^2(u)du < \infty, \quad \mathbb{P} - a.s. \tag{2.6}$$

So, under the assumption (2.5) there exists a unique strong solution of (2.1) and the process X can be expressed as

$$X(t) = X_0 \exp\left(\int_0^t f(W^H(u))g(u)dW(u) - \frac{1}{2} \int_0^t f^2(W^H(u))g^2(u)du \right) \tag{2.7}$$

(see, e.g., Revuz and Yor [11]). The process X is a local martingale, so there is no arbitrage in the market so defined.

Following [9] this model is called a *linear stochastic volatility model*, because the stochastic differential equation (2.1) governing the asset price is linear with respect to the asset itself with the coefficient being the stochastic volatility driven by a fractional Brownian motion.

Now, for a fixed t, the probability density function for a stochastic volatility model with volatility given by a function of FBM, which can be correlated with the Brownian motion driving the price equation (2.1) is described.

Theorem 2.1. *Let $t \in [0, T]$, X be given by (2.1), W^H by (2.2) and W, $\widehat{W}$ be correlated Brownian motions, $d\langle W, \widehat{W}\rangle_t = \rho(t)dt$ with a measurable, deterministic function $\rho : [0, T] \to (-1, 1)$. Under (2.5) the random variable X_t has a probability density function h_{X_t} satisfying*

$$h_{X(t)}(s) = \mathbb{E}\Big[\frac{1}{s\sigma_H}\varphi\Big(\frac{\ln \frac{s}{X_0} - \int_0^t f(W^H(u))g(u)\rho(u)d\widehat{W}(u)}{\sigma_H} + \frac{\frac{1}{2}\int_0^t f^2(W^H(u))g^2(u)du}{\sigma_H}\Big)\Big], \tag{2.8}$$

where $s > 0$, φ is the probability density of a standard Gaussian random variable $N(0, 1)$, and

$$\sigma_H^2 = \int_0^t f^2(W^H(u))g^2(u)(1 - \rho^2(u))du. \tag{2.9}$$

Proof. Let B be a standard Brownian motion independent of $\widehat{W}$ such that

$$W(t) = \rho(t)\widehat{W}(t) + \sqrt{1 - \rho^2(t)}B(t). \tag{2.10}$$

Let V and Z be defined as follows

$$Z = \int_0^t f(W^H(u))g(u)\sqrt{(1 - \rho^2(u))}dB(u) - \frac{1}{2}\int_0^t f^2(W^H(u))g^2(u)(1 - \rho^2(u))du, \tag{2.11}$$

$$V = \int_0^t f(W^H(u))g(u)\rho(u)d\widehat{W}(u) - \frac{1}{2}\int_0^t f^2(W^H(u))g^2(u)\rho^2(u)du, \tag{2.12}$$

Then, by (2.7), $Y := \ln X_t = \ln X_0 + V + Z$. By the independence of B and $\widehat{W}$ it follows that Z, conditioned on $\mathcal{F}_t^{\widehat{W}}$, has the Gaussian distribution $N\big(-\frac{\sigma_H^2}{2}, \sigma_H^2\big)$. So to verify (2.8) it is sufficient to note that the cumulative distribution function for $X(t)$ is

$$\begin{aligned}\mathbb{P}(X(t) \le s) &= \mathbb{E}\mathbb{E}\Big[1_{\{Z \le \ln \frac{s}{X_0} - V\}} \Big| \mathcal{F}_t^{\widehat{W}}\Big] \\ &= \mathbb{E}\Big[\Phi\Big(\frac{\ln \frac{s}{X_0} - \int_0^t f(W^H(u))g(u)\rho(u)d\widehat{W}(u)}{\sigma_H} \\ &\qquad + \frac{\frac{1}{2}\int_0^t f^2(W^H(u))g^2(u)du}{\sigma_H}\Big)\Big],\end{aligned}$$

where Φ denotes the cumulative distribution function of a standard Gaussian random variable. Hence, by the Fubini theorem for nonnegative functions the equality (2.8) follows because

$$\begin{aligned}&\frac{\partial}{\partial s}\Phi\Big(\frac{\ln \frac{s}{X_0} - \int_0^t f(W^H(u))g(u)\rho(u)d\widehat{W}(u) + \frac{1}{2}\int_0^t f^2(W^H(u))g^2(u)du}{\sigma_H}\Big) \\ &= \frac{1}{s\sigma_H}\phi\Big(\frac{\ln \frac{s}{X_0} - \int_0^t f(W^H(u))g(u)\rho(u)d\widehat{W}(u)}{\sigma_H} \\ &\qquad + \frac{\frac{1}{2}\int_0^t f^2(W^H(u))g^2(u)du}{\sigma_H}\Big).\end{aligned}$$

□

Remark 2.2. If W and $\widehat{W}$ are independent Brownian motions (so W and W^H are independent Gaussian processes), then

$$\sigma_H^2 = \int_0^t f^2(W^H(u))g^2(u)du \tag{2.13}$$

and the equality (2.8) takes a simpler form

$$h_{X(t)}(u) = \mathbb{E}\Big[\frac{1}{u\sigma_H}\varphi\Big(\frac{\ln \frac{u}{X_0} + \sigma_H^2/2}{\sigma_H}\Big)\Big], \tag{2.14}$$

So to determine the probability density of $X(t)$ it is sufficient to determine the distribution of $\sigma_H^2 = \int_0^t f^2(W^H(u))g^2(u)du$, provided that (2.5) is satisfied. Since W^H is a Gaussian process there are estimation methods to estimate the distribution of σ_H^2.

Remark 2.3. It follows directly that for $H \neq \frac{1}{2}$

$$corr(W(t), W^H(t)) = \int_0^t K_H(t,s)ds,$$

where K_H is given by (2.3) for $H < 1/2$, and by (2.4) for $H > 1/2$.

Theorem 2.1 allows to include two important cases of volatility, the volatility being a function of a fractional Brownian motion and the volatility being a function of a geometric fractional Brownian motion.

Remark 2.4. a) Taking $g \equiv 1$ the problem for volatility being a function of a fractional Brownian motion is solved, that is, $f(W^H(t))$, so

$$dX(t) = f(W^H(t))X(t)\, dW(t),$$

provided (2.5) is satisfied.

b) Let the volatility Y be a geometric fractional Brownian motion, that is, Y satisfies the stochastic equation

$$dY(t) = Y(t)(adt + bdW^H(t)), \tag{2.15}$$

and X is defined by

$$dX(t) = Y(t)X(t)\, dW(t).$$

Using the form of the unique solution of (2.15) and defining

$$f(x) = \exp(bx), \quad g(t) = \exp\left(at - \frac{1}{2}b^2t^{2H}\right),$$

by Theorem 2.1, the probability density of $X(t)$ is obtained. Indeed, assumption (2.5) is satisfied by continuity of the sample paths.

Example 2.5. If $g \equiv 1$, $f(x) = |x|$, then (2.5) is satisfied, and in fact the following expectation is easily computed

$$\mathbb{E}\left[\int_0^t (W_u^H)^2 du\right]. \tag{2.16}$$

3. Pricing in the model with stochastic volatility driven by a fractional Brownian motion

Using the probabilistic representation of the density, a closed form of the probability density function is determined in some cases. Initially, recall that X given by (2.1) is a local martingale. If the integrability condition

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T f^2(W^H(u))g^2(u)du\right)\right) < \infty. \tag{3.1}$$

is satisfied then X is a martingale (e.g. [11]).

Note that having the probability density, a formula for the price can be determined for many financial derivatives, indeed from Theorem 2.1 it follows immediately

Corollary 3.1. *Let X be the price of an asset with dynamics given by* (2.1)*, and Y be an attainable European contingent claim of the form $Y = F(X_T)$ with maturity at time T. If $Y \in L^2(P)$ then its price at time* 0 *is equal to*

$$\int_0^\infty F(u)h_{X_T}(u)du,$$

where h_{X_T} is the density of X_T.

Therefore for X the closed form solution of prices can be determined and they can be explicitly calculated or numerical methods can be used.

In the next proposition a representation of a vanilla option price is determined. These formulae generalize the famous Black-Scholes formulae as well as a result of Hull and White for a stochastic volatility model with uncorrelated noises [8].

Proposition 3.2. *Let $K > 0$ be fixed and the price X be given by* (2.1)*. Then*

$$\mathbb{E}[X(t) - K]^+ = X_0\mathbb{E}\big[e^V\Phi(d_1)\big] - K\mathbb{E}\Phi(d_2), \tag{3.2}$$

$$\mathbb{E}[K - X(t)]^+ = K\mathbb{E}\Phi(-d_2) - X_0\mathbb{E}\left[e^V\Phi(-d_1)\right], \tag{3.3}$$

where

$$d_1 = \frac{\ln\frac{X_0}{K} + V + \frac{\sigma_H^2}{2}}{\sigma_H}, \quad d_2 = d_1 - \sigma_H,$$

and σ_H and V are given by (2.9) *and* (2.12)*, respectively.*

Proof. Recall that $X(t) = X_0 \exp(V + Z)$, by (2.7), where Z is given by (2.11). It follows that V is $\mathcal{F}_t^{\widehat{W}}$-measurable, so

$$\mathbb{E}(K - X_t)^+ = \mathbb{E}\left[X_0 e^V \mathbb{E}\left(\left(\frac{K}{X_0 e^V} - e^Z\right)^+ \Big| \mathcal{F}_t^{\widehat{W}}\right)\right] := I.$$

Since Z, conditioned on $\mathcal{F}_t^{\widehat{W}}$, has the Gaussian distribution $N(-\frac{\sigma_H^2}{2}, \sigma_H^2)$ (by independence of B and $\widehat{W}$) using some classical results it follows that

$$\begin{aligned} I &= \mathbb{E}\left[X_0 e^V \frac{K}{X_0 e^V} \Phi\left(\frac{-\ln\frac{X_0}{K} - V + \frac{\sigma_H^2}{2}}{\sigma_H}\right) - X_0 e^V \Phi\left(\frac{-\ln\frac{X_0}{K} - V - \frac{\sigma_H^2}{2}}{\sigma_H}\right)\right] \\ &= K\mathbb{E}\Phi(-d_2) - X_0 \mathbb{E}\left[e^V \Phi(-d_1)\right]. \end{aligned}$$

By the same arguments it follows that

$$\begin{aligned} &\mathbb{E}(X_t - K)^+ \\ &= \mathbb{E}\left[X_0 e^V \mathbb{E}\left(\left(e^Z - \frac{K}{X_0 e^V}\right)^+ \Big| \mathcal{F}_t^{\widehat{W}}\right)\right] \\ &= \mathbb{E}\left[X_0 e^V \Phi\left(\frac{\ln\frac{X_0}{K} + V + \frac{\sigma_H^2}{2}}{\sigma_H}\right) - X_0 e^V \frac{K}{X_0 e^V} \Phi\left(\frac{\ln\frac{X_0}{K} + V - \frac{\sigma_H^2}{2}}{\sigma_H}\right)\right] \\ &= X_0 \mathbb{E}\left[e^V \Phi(d_1)\right] - K\mathbb{E}\Phi(d_2). \end{aligned}$$

□

References

[1] T. G. Anderson and T. Bollerslew, *Heterogeneous information arrivals and return volatility dynamics: Uncovering the long-run in high frequency returns.* J. Finance, **52** (1997), 975–1005.

[2] I. Casas and J. Gao, *Econometric estimation in long-range dependent volatility models: Theory and practice.* J. Econometrics, **147** (2008), 72–83.

[3] F. Comte, L. Coutin and E. Renault, *Affine fractional stochastic volatility models.* Ann. Finance, **8** (2012), 337–378.

[4] F. Comte and E. Renault, *Long memory continuous-time models.* J. Econometrics, **73** (1996), 101–149.

[5] F. Comte and E. Renault, *Long memory in continuous-time stochastic volatility models.* Mathematical Finance, **8** (1998), 291–323.

[6] Z. Ding and C. W. J. Granger, *Modelling persistence of speculative returns: A new approach.* J. Econometrics, **73** (1996), 185–215.

[7] M. Fukasawa, *Asymptotic analysis for stochastic volatility: martingale expansion.* Finance and Stochastics, **15** (2011), 635–654.

[8] J. Hull and A. White, *The pricing of options on assets with stochastic volatilities.* J. Finance, **42** (1987), 281–300.

[9] J. Jakubowski and M. Wiśniewolski, *Linear stochastic volatility models.* Preprint, arXiv:0909.4765, 2009.

[10] D. Nualart, *The Malliavin Calculus and Related Topics* (2nd edition). Springer-Verlag, 2006.

[11] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion* (3rd edition). Springer-Verlag, 2005.

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Received August 17, 2015
Accepted August 27, 2015