Communications in Information and Systems
Volume 15, Number 1, 1–14, 2015

# On the design of quantized control using randomization

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*Dedicated to Wing-Shing Wong*

In this paper we describe an approach to the design of quantized feedback control making use of randomization. The methodology described here provides a basis for designing uniform or nonuniform quantization suitable in situations where fine quantization is not possible. We show that a suitable inhomogeneous finite state Markov chain whose transition rates depend on measurements of the error, can be used to generate low information rate regulators capable of achieving basic control objectives. One attractive aspect of this approach is the possibility of using classic feedback control design methods to determine the properties of the quantizer.

## 1. Introduction

In [1, 2] Wong and the present author pursued a line of inquiry whose goal was to describe the limitations that information content places on the quality of the control one can achieve. Earlier work by Delchamps [3] focused on the design of quantized control for stabilization and subsequent work has produced a number of further results in that direction, such as those of Baillieul [4] and Nair [5]. Because the full specification of even a single real number requires an infinite number of bits, and because in most work on control theory both the domains over which processes are defined, and the ranges of the variables to be controlled are described in terms of real numbers, information theoretic ideas have proven to be difficult to fit in. In traditional "computer control" this is circumvented by uniformly discretizing time and uniformly quantizing the signals that the compensator transmits. This works well if fine quantization is possible. Here we describe an approach in which the values of the feedback signals come from a finite set of values

This work was supported in part by the US Army Research Office under grant DAAG 55 97 1 0114.

and are sent at random times with the average rate of transmission being specified as part of the design.

If the transition rates of a finite state Markov process are allowed to depend on exogenous variables then clearly the process conveys information about theses variables. This is a popular model, for example, in neuroscience where spiking rates are known to carry significant information about the environment. In that setting the stochastic calculus provides tools for determining how effective such schemes are in specific situations. See, for example, [6]. We will show that if the counting rates properly code information on the state of the system to be controlled then feedback based on such signals can create a containment region in the state space of the system to be controlled, even though the controller is finite state and the feedback control takes on only a finite set of values. The size of the containment region and the size of the steady state error can be estimated from the properties of an associated variance equation. If the system to be controlled is linear, and if the quantized signals are generated in accordance with suitable rules, the equations for the mean and variance are linear. In this way the model provides guidance for selecting the various parameters specifying the quantizer.

## 2. Sample path descriptions of Markov chains

The usual starting point in studying continuous time, finite state, Markov chains is the infinitesimal generator $F$ which describes the evolution of the vector of probabilities, as in $\dot{p} = Fp$. With the exception of work on stochastic control, much of the literature on Markov chains is unconcerned with sample path descriptions. By way of contrast, in this paper we will be concerned with both sample path descriptions and the corresponding statistical properties of solutions.

One general method for realizing a sample path description of inhomogeneous finite state Markov chain, is based on variable rate Poisson counters. A Poisson counter is a random processes taking on values in the set of non-negative integers, monotone increasing, and having a counting rate $\lambda$ such that for $t > \tau$

$$\mathcal{E}N(t) - \mathcal{E}N(\tau) = \int_{\tau}^{t} \lambda(\sigma) d\sigma.$$

A suitable collection of such counters can be used to generate sample path realizations of Markov chains. Given a set of $d$ points $\mathcal{S} = \{s_1, s_2, \ldots, s_d\}$ in some vector space $\mathcal{V}$, consider the stochastic equation

$$dz = \sum_{i=1}^{m} g_{ij}(z)dN_{ij}, \quad z(t) \in S = \{s_1, s_2, \ldots, s_d\}$$

with the $g_{ij}$ being such that for each $i$ and $j$, $g_{ij}(s_j) = s_i - s_j$ and $g_{ij}(s_k) = 0$ for $k \neq j$. If $z(0)$ belongs to $\mathcal{S}$ then $z(t)$ belongs to $\mathcal{S}$ for all time. Given an infinitesimal generator $F$, it is possible to use Poisson counters to realize a process such that $\dot{p} = Fp$ for $p_i$ being the probability that $z(t) = s_i$.

This approach is used in [7] where $\mathcal{V}$ is taken to be $\mathbb{R}^d$ and $\mathcal{S}$ the set of standard basis elements in $\mathbb{R}^d$; i.e., $\mathcal{S} = \{e_1, e_2, \ldots, e_d\}$. Given this representation of the states we can take $g_{ij}(z)$ to be of the form $G_{ij}z$ where the $G_{ij} = e_i e_j^T - e_j e_j^T$. There are a total of $n(n-1)$ matrices of this form. The sample path realization takes the form

$$dz = \sum G_{ij} z dN_{ij}.$$

If $\lambda_{ij}$ is the counting rate for $N_{ij}$ then the expected value of $z$ satisfies the linear differential equation

$$\frac{d}{dt}\mathcal{E}z = \sum G_{ij}\lambda_{ij}\mathcal{E}z.$$

Note that if $p_i$ is the probability that $z = e_i$ then $p = \mathcal{E}z$. If we let $F$ be a generator such that for $i \neq j$ and $f_{ij} = \lambda_{ij}$ then $\dot{p} = Fp$. In general, it will require $d(d-1)$ independent counters to realize a Markov process with $d$ states.

Within this framework there is complete freedom with respect to the choice of the vector space $\mathcal{V}$ and $\mathcal{S} \subset \mathcal{V}$ but circumstances may suggest particular choices. A second possibility which also finds use here is to let $\mathcal{V} = \mathbb{R}$ and to let $\mathcal{S} = \{z_1, z_2, \ldots, z_d\}$ where now the $z_i$ are distinct real numbers. Define polynomials $\phi_{ij}(z)$ as follows. Let $\phi(z) = (z - z_1)(z - z_2)\cdots(z - z_d)$ and let $\phi_i(z) = \phi(z)/(z - z_i)$. If $z_k$ is in $\mathcal{S}$ then exactly one of the $\phi_i(z_k)$ is nonzero. The scalar equation

$$dz = \sum (z_j - z_i)\frac{\phi_i(z)}{\phi_i(z_i)} dN_{ij}$$

then defines a realization of $\dot{p} = Fp$ in which $z$ takes on values in $\{z_1, z_2, \ldots, z_d\}$. The $\phi_i$ are of degree $d-1$ and if we arrange their coefficients in a $d$-dimensional vector as suggested by $v_i = [p_{d-1}, p_{d-2}, \ldots, p_0]^T$ then the $v_i$

take on values in a $d$-element set and satisfy an equation of the form

$$dv = \sum PG_{ij}P^{-1}vdN_{ij}$$

where the $G_{ij}$ and $N_{ij}$ are as above and $P = [v_1, v_2, \ldots, v_n]$.

**Example.** Let $z$ be a stochastic process taking on values in $\{1,\ 0,\ -1\}$. Let $\phi_1(z) = z(z-1)/2,\ \phi_2(z) = z(z+1)/2,\ \phi_3(z) = z^2 - 1$. For $i = 1$ to 6 let $N_i$ be Poisson counters with rates $\lambda_i$. Suppose that $z$ satisfies the Itô equation

$$dz = \phi_1(dN_1 + 2dN_2) - \phi_2(dN_3 + 2dN_4) + \phi_3(dN_5 - dN_6)$$

then if $p_1,\ p_2,\ p_3$ are the probabilities that $z$ is 1, 0, -1, respectively, then

$$\frac{d}{dt}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -\lambda_3 - \lambda_4 & \lambda_6 & \lambda_2 \\ \lambda_3 & -\lambda_5 - \lambda_6 & \lambda_1 \\ \lambda_4 & \lambda_5 & -\lambda_1 - \lambda_2 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

Now let $z$ be a stochastic process taking on values $e_1, e_2, e_3 \in \mathbb{R}^3$ and let $p_i$ be the probability that $z_i = e_i$, as above. Suppose that the Markov chain description is

$$\frac{d}{dt}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

Define $\lambda_i$ so that $F$ relates to $\lambda_i$ as above. Let $G_{ij} = -e_ie_j^T - e_je_j^T$. If the evolution of $z$ is described by

$$dz = \sum G_{ij}zdN_{ij}$$

and if we let $h^T = [1,\ 0,\ -1]$ then the values assumed by the scalar process $h^Tz$ have the same probabilistic description as the scalar process $z$ defined above.

## 3. Sample path description of controlled Markov chains

We are interested in situations in which it is possible to exercise some control over the transition rates of a Markov process, leading to the consideration of $\dot{p} = F(u)p$. In our work on optimal control [7] the dependence of $F$ on $u$

is linear and $\dot{p} = (F + \sum F_i u_i)p$. In such models the $u_i$ must be restricted to a certain convex set if $F + \sum u_i F_i$ is to have nonnegative off-diagonal elements. Here we are concerned with a special case in which the effect of $u$ is to influence the probability of the next state in a way that is independent of the present state. Suppose $F$ is an infinitesimal generator and that $F_i = g_i e^T$ with $g_i$ being a vector whose components sum to zero and $e^T = [1, 1, \dots, 1]$. Using the fact that $e^T p = 1$ for any probability vector $p$, we see that in this case the evolution equation for $p$ can be written as a linear system

$$\dot{p} = Fp + \sum u_i g_i.$$

We can construct a sample path description of a process whose probabilities evolve in accordance with $\dot{p} = Fp + \sum u_i g_i$ by slightly extending the methods just described. Suppose that $z \in \mathbb{R}^d$ takes on values in the set $\{e_1, e_2, \dots, e_d\}$ and satisfies

$$dz = \sum G_{ij} z dN_{ij}.$$

Choose the $\lambda_{ij}$, the $u$-dependent counting rates for $N_{ij}$, to be $\lambda_{ij} = f_{ij} + g_i u$. Again, the values of $u$ must be restricted to assure that $f_{ij} + g_i u \geq 0$. If all off-diagonal elements of $F$ are strictly positive this requirement will be satisfied for all sufficiently small $u$.

**Theorem 1.** *Suppose that $\mu$ and $\nu$ are related by*

$$dz = \sum G_{ij} z dN_{ij}, \quad \nu = h^T z$$

*with the counting rate for $N_{ij}$ being $f_{ij} + g_i \mu$. Suppose that $f_{ij} > 0$ for all $i \neq j$. Then if*

$$\|\mu\| = \sup_{t \geq 0} |\mu(t)|$$

*is sufficiently small the expected value of $\nu$ is related to $\mu$ via the linear system*

$$\frac{d}{dt}\mathcal{E}z = F\mathcal{E}z + \mu g, \quad \mathcal{E}\nu = h^T \mathcal{E}z.$$

*Moreover, if $\mu$ is differentiable then variance of $\nu$ for scaled system obtained by replacing $f_{ij} + g_i \mu$ by $\alpha(f_{ij} + g_i \mu)$, satisfies*

$$\lim_{\alpha \to \infty} \mathcal{E}(\nu(t) - \mathcal{E}\nu(t))^2 \to 0$$

*provided that $||\mu|| + ||\dot{\mu}||$ is sufficiently small.*

The first statement is just a short calculation. The second statement asserts that in the limit of high counting rates the relationship becomes effectively deterministic provided that $u$ is not too irregular. This fact will not be used here and we will not prove it.

## 4. Stochastic quantization in a feedback loop

The application of these ideas to the control of linear systems gives useful insight as to their power and limitations. In most studies of how the performance of a feedback system depends on quantized data there is a divide between systems which are open loop stable, those that are open loop neutrally stable and those that are open loop unstable. In the last case much of the literature focuses on the basic requirement of achieving asymptotic stability. Whereas in the first case the literature is more diverse with various possible choices of a performance measure having been explored. Using a dynamically scaled quantizer, it was shown in [8] that one can stabilize an unstable system with low resolution quantized feedback but here we do not discuss dynamic scaling.

Looking at quantization of real valued signals as a form of lossy coding, what is being proposed here lacks the sophistication of the block coding techniques that lie at the heart of digital communication theory but has the advantage of being causal and not requiring time delay. In much of communication theory these attributes are not considered to be of importance in assessing the performance of source coders and decoders, but for feedback control delays can be very significant because they often have a destabilizing effect. This has limited the use of coding theory in real time feedback control applications. The causal coders described here incorporate a degree of smoothing but have no delay. For a specified range of input values, they transform real valued signals into functions which take on only a finite set of values and have the significant property of producing an output whose expected value relates to the input in a simple way. We begin with a description of such a quantizer having input $u$ and output $y$. We suppose that the internal state $z$ takes on one of $d$ possible values.

Consider a scalar input, scalar output, (generalization is clearly possible) linear constant coefficient system coupled to a continuous time finite state Markov process as suggested by Figure 1. The sample path description is taken to be

$$\dot{x} = Ax + bh^T z, \quad y = c^T x, \quad dz = \sum_{i,j=1}^{d} G_{ij} z dN_{ij}$$

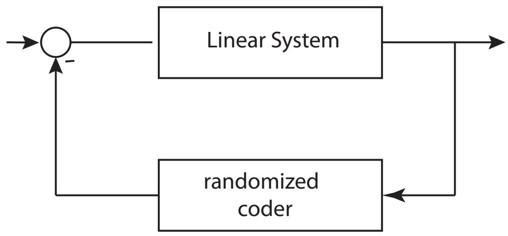


Figure 1: The type of feedback loop of considered here.

The counting rates of the $N_i$ are assumed to be affine functions of $c^T x$ given by

$$\lambda_{ij} = f_{ij} + g_i e_j c^T x \ ; \ i \neq j$$

subject to the constraint that the sum of the components of $g$ is zero. Using the fact that $e^T z$ is identically one, the differential equation for the expectation is linear and takes the form

$$\frac{d}{dt}\mathcal{E}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & bh^T \\ gc^T & F \end{bmatrix}\mathcal{E}\begin{bmatrix} x \\ z \end{bmatrix}.$$

In a design problem it is the triple $(F, g, h)$ which needs to be chosen subject to the constraints that $F$ is an infinitesimal generator, $h$ determines the levels of the quantizer and $g$ shapes the extent to which a nonzero value of $u$ favors the transition to one state over another. Recall that $g$ is a vector whose entries sum to zero. The allowable range for the input to the quantizer is determined by smallest value of $f_{ij}/g_j$.

**Theorem 2.** *Suppose that $x \in \mathbb{R}^n$ and $z \in \{e_1, e_2, \ldots, e_d\}$ satisfy*

$$\dot{x} = Ax + bh^T z, \quad y = c^T x, \quad dz = \sum_{i \neq j}(E_{ij} - E_{ii})z dN_{ij}$$

*and for $i \neq j$, $N_{ij}$ has the counting rate $f_{ij} + g_i c^T x$. Let*

$$r = \min_{i \neq j} \frac{f_{ij}}{|g_i|}.$$

*Suppose that for* $c^T x = 0$ *the expected value of* $z$ *satisfies* $(d/dt)\mathcal{E}z = F\mathcal{E}z$ *with* $F$ *irreducible and that* $p_\infty$ *is the unique probability vector such that* $Fp = 0$. *Assume that* $h^T p_\infty = 0$ *and let* $k_\infty$ *denote the* $L_\infty$ *gain of the system*

$$\dot{x} = Ax + bu, \quad y = c^T x.$$

*If for all* $i$ *we have* $h_i k_\infty < r$ *and if*

$$M = \begin{bmatrix} A & bh^T \\ gc^T & F \end{bmatrix}$$

*has* $n + d - 1$ *eigenvalues with negative real parts, then there exists* $\epsilon > 0$ *such that for all* $x(0), z(0)$ *with* $||x(0)|| \leq \epsilon$ $x$ *remains bounded on* $[0, \infty)$, *the expectations satisfy*

$$\frac{d}{dt}\mathcal{E}x = A\mathcal{E}x + bh^T\mathcal{E}z,$$
$$\frac{d}{dt}\mathcal{E}z = gc^T\mathcal{E}x + F\mathcal{E}z,$$

*and*

$$\lim_{t\to\infty} \mathcal{E}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ p_\infty \end{bmatrix}.$$

*Proof.* Because $h^T p_\infty = 0$ the vector $[0,\ p_\infty]^T$ is in the null space of $M$ so that under the hypothesis $M$ has exactly one zero eigenvalue. From the assumptions on the products $h_i k_\infty$ it follows that $|c^T x|$ will be less than $r$ provided that $x(0)$ is sufficiently small. If there are $n + d - 1$ eigenvalues with negative real parts the expected value of $x$ will converge to zero, as indicated. □

**Example.** Many of the basic ideas are present in the following example, which involves controlling a neutrally stable harmonic oscillator with a three level quantizer. Not all of the assumptions of Theorem 2 are fulfilled in this case but a partial analysis can be made. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \quad y = x_1.$$

We constrain the probabilistic description of the quantizer so that there is symmetry under changing $x$ to $-x$. With some further choices, we postulate that

$$\dot{p} = Fp = \begin{bmatrix} -1 & .5 & .3 \\ .7 & -1 & .7 \\ .3 & .5 & -1 \end{bmatrix} p + \begin{bmatrix} g \\ 0 \\ -g \end{bmatrix} \mathcal{E}y, \quad h^T = [h,\ 0,\ -h].$$

As seen above, $p$ together with the expected value of $x$ satisfy a 5-dimensional linear system,

$$\frac{d}{dt}\begin{bmatrix} \mathcal{E}x_1 \\ \mathcal{E}x_2 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & h & 0 & -h \\ g & 0 & -1 & .5 & .3 \\ 0 & 0 & .7 & .-1 & .7 \\ -g & 0 & .3 & .5 & -1 \end{bmatrix}\begin{bmatrix} \mathcal{E}x_1 \\ \mathcal{E}x_2 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

If there is to be a steady state value for the expectations with the steady state value of $x$ and $\dot{x}$ being zero then it will necessarily result in $p_1 = 5/17,\ p_2 = 10/17,\ p_3 = 5/17$. For this value of $p$ we have $h^T p = 0$. Because of the special form of this 5-by-5 matrix we see that if we multiply $h$ by $\alpha$ and at the same time divide $g$ by $\alpha$ the characteristic equation does not change. The larger $\alpha$ is, the smaller $g/\alpha$ is and, from the point of view of $c^T x$, the larger the admissible range of $c^T x$. At the same time, larger $\alpha$ means that the quantization levels have a larger absolute value so that the control can be expected to be less precise. Of course the vector $p$, being a probability vector, can not go to zero. However we can ask if $\mathcal{E}x$ and $p - p_\infty$ go to zero as $t$ goes to infinity. As we will see in the next section, with this controller solutions will have the desired stability property if $-23.27 < gh < 0$.

## 5. Transfer functions and design

Focusing attention now on the equations for the expectations,

$$\frac{d}{dt}\begin{bmatrix} \mathcal{E}x \\ \mathcal{E}z \end{bmatrix} = \begin{bmatrix} A & bh^T \\ gc^T & F \end{bmatrix}\begin{bmatrix} \mathcal{E}x \\ \mathcal{E}z \end{bmatrix}$$

we see that it is possible to interpret these as arising from a feedback system with $m(s) = c^T(Is - A)^{-1}b$ in the forward part of the loop and $n(s) = h^T(Is - F)^{-1}g$ in the feedback path. The components of $h$ are the quantization levels and hence, together with $F$ and $g$, they determine the transfer function appearing in the feedback path. Given $m(s)$ the selection of a feedback compensator $n(s)$ to achieve suitable dynamics is a standard design problem in feedback control theory. What is remarkable here is that the quantization levels enter into the description of $n(s)$ in such a direct way.

Recall the following restrictions on $F, g, h$.

1) $F$ must be the infinitesimal generator of a Markov process. This means the columns sum to zero and the off diagonal elements are nonnegative. However, more is needed because $f_{ij}$ has a role in determining the dynamic range for the error, $c^T x$.
2) $g$ is a vector whose components sum to zero and the larger the ratio $f_{ij}/|g_i|$, the larger the admissible range of $c^T x$.
3) $h$ must be chosen so that $h^T p_\infty = 0$ if the expected value of $x$ is to go to zero. In addition, the values of $h_i$ determine the location of the zeroes of $n(s)$ and hence they play a role in determining the closed loop dynamics.

**Remark.** Given any set of negative real numbers together with 0 there exists an infinitesimal generator with these eigenvalues. To construct such a generator simply put the given numbers on the diagonal with 0 in the lower right and fill in below the diagonal with nonnegative numbers so as to make the columns sum to zero. Moreover, If the generator is $d$-by-$d$ then there is a $d$-dependent bound on the ratio of the imaginary part of an eigenvalue to its real part. In the situations described here it may be necessary that the smallest off diagonal element be greater than or equal to some preassigned number $\alpha > 0$; in that case there are further limitations on the eigenvalues.

**Continuing the example.** Putting the example of the previous section into this form, we have $m(s) = 1/(s^2+1)$ and for the given $F$

$$
\begin{aligned}
&(Is - F)^{-1} \\
&= \frac{1}{s^3 + 3s^2 + 2.21s} \begin{bmatrix} s^2 + 2s + .65 & .5s + .35 & .3s + .65 \\ .7s + .91 & s^2 + 2s + .91 & .7s + .91 \\ .3s + .65 & .5s + .35 & s^2 + 2s + .65 \end{bmatrix}.
\end{aligned}
$$

The symmetry conditions demand that $h$ and $g$ take the form $h^T = [\gamma, 0, -\gamma]$ and $g^T = [\beta,\ 0,\ -\beta]$ so that

$$
n(s) = \frac{2\beta\gamma(s + 1.7)}{s^2 + 3s + 2.21}.
$$

The symmetries combine to force a cancelation of the pole at zero. The characteristic equation defining the eigenvalues of the equation for the expected values is

$$p(s) = (s^2 + 1)(s^2 + 3s + 2.21) - \beta\gamma(4s + 3).$$

If the equation for the expected values is to be asymptotically stable then the Routh Hurwitz conditions imply that it is necessary that

$$9.63(3 + \beta\gamma) - 9(2.21 + 3\beta\gamma) - (3 + \beta\gamma)^2 = (9.63 - 27 - 6)\beta\gamma - (\beta\gamma)^2 > 0.$$

Thus stability considerations limit $\beta\gamma$ to values between -23.27 an 0. If the off diagonal entries of the infinitesimal generator are to be nonnegative it is necessary to restrict $c^T x$ to the interval $[-3/\beta,\ 3/\beta]$.

## 6. The second moments

The behavior of the second moments is of obvious interest, not only as a measure of error variance but also because of the need to assure that that $c^T x$ does not exceed the allowed bound. The equations for the second moments can be derived as follows. Starting from the closed loop stochastic equations

$$dx = (Ax + bh^T z)dt, \quad dz = \sum G_{ij} z dN_{ij}, \quad \lambda_{ij} = f_{ij} + g_i c^T x$$

and using the Itô rule to evaluate derivatives, the equations governing $xx^T$ and $xz^T$ are

$$\begin{aligned} dxx^T &= \left(Axx^T + bh^T zx^T + xx^T A^T + xz^T hb^T\right) dt, \\ dxz^T &= \left(Axz^T + bh^T zz^T\right) dt + x \left(\sum G_{ij} z dN_{ij}\right)^T . \end{aligned}$$

Notice that $zz^T$ is diagonal because $z$ has only one nonzero entry at any moment in time. In fact, $\mathcal{E}zz^T =$ diag $\mathcal{E}$z, and the equation for $\mathcal{E}z$ was determined in the previous section.

**Example.** Consider a simpler version of the above example in which a harmonic oscillator is to be controlled with a two state quantizer. In this case the equilibrium, if any, will take the form of an invariant probability distribution. Here we use a representation of the finite state system in which the states are the real numbers $\pm 1$, not unit vectors.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} z,$$

$$dz = -2z dN, \quad z(0) \in \{\pm 1\}.$$

It is convenient to relabel $z$ as $x_3$ and to describe the system and controller using a single vector equation expressed in Itô notation

$$\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} dt + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} dN.$$

Assuming a counting rate of $\lambda = f + cx_2$, and taking expectations, it follows that

$$\frac{d}{dt}\mathcal{E}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & b \\ 2c & 0 & -2f \end{bmatrix} \mathcal{E}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The relevant characteristic equation is $s^3 + 2fs^2 + s + 2(f - bc) = 0$ so that if $0 < bc < f$ the eigenvalues have negative real parts. The corresponding equation for the matrix of second moments

$$\Sigma = \mathcal{E}xx^T = \mathcal{E}\begin{bmatrix} x_1^2 & x_1x_2 & x_1z \\ x_1x_2 & x_2^2 & x_2z \\ x_1z & x_2z & z^2 \end{bmatrix}$$

is given by

$$\dot{\Sigma} = H\Sigma + \Sigma H^T + R$$

with

$$H = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & b \\ 2c & 0 & -2f \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & b & 0 \end{bmatrix}.$$

Thus we have

$$\frac{d}{dt}\mathcal{E}x_1^2 = 2\mathcal{E}x_1x_2, \quad \frac{d}{dt}\mathcal{E}x_1x_2 = 2c\mathcal{E}(x_2^2 - -x_1^2 + bx_1x_3),$$
$$\frac{d}{dt}\mathcal{E}x_1x_3 = \mathcal{E}(x_2x_3 + 2cx_1^2 - 2fx_1x_3),$$
$$\frac{d}{dt}\mathcal{E}x_2^2 = \mathcal{E}(-2x_1x_2) + 2b,$$
$$\frac{d}{dt}\mathcal{E}x_2x_3 = 2c\mathcal{E}(x_1x_2 - x_1x_3 - 2x_2x_3) + 2b.$$

Solving for the steady state values for these variances we have

$$\mathcal{E}x_1^2 = \frac{fb}{f + bc}, \quad \mathcal{E}x_1x_3 = \frac{bc}{f + bc}, \quad \mathcal{E}x_2^2 = b, \quad \mathcal{E}x_3^2 = 1$$

and the remaining values are 0.

As noted, this analysis will only be meaningful when trajectories lie in the region of $(x_1, x_2)$-space having the property that $f + cx_1 \geq 0$.

## 7. A deterministic translation

There will be situations in which randomness is unacceptable. Here we describe a heuristic procedure for converting a randomized quantizer of the form discussed above into a deterministic one. Suppose that we are given the $F, g, h$ corresponding to a quantizer with $d$ levels and wish to find an "equivalent" deterministic quantizer. Without loss of generality we assume that the levels are ordered, $h_1 \geq h_2 \geq \cdots \geq h_d$ and that $g_1 \geq g_2 \geq \cdots \geq g_d$. Associated with the invariant distribution $p_\infty$ we define a division of the interval $[-.5,\ .5]$ into $d$ subintervals labeled $I_i$

$$I_1 = [.5,\ .5 - p_1), \quad I_2 = [.5 - p_1\ .5 - p_1 - p_2) \cdots [-.5 + p_n,\ -.5).$$

Define $q : \mathbb{R} \to \{h_1, h_2, \ldots, h_d\}$ such that on the $i^{th}$ interval $q$ takes on the value $h_i$. The suggested deterministic version then transforms $c^T x$ into $u$ according to

$$\dot{z} = Fz + gc^T x, \quad u = \alpha^{-1} q(\alpha h^T z)$$

with $\alpha = h_1 - h_d$. Suppose that the entries of $g$ and $h$ are symmetric in the sense that $g_i = -g_{d-i}$ and $h_i = -h_{d-i}$. In this case, for $\alpha h^T z$ in the interval $-.5 \leq c^T x \leq .5$ a harmonic balance analysis, i.e., a describing function approach, suggests a qualitative similarity between the stochastic quantizer and this deterministic one.

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Received September 30, 2015
Accepted October 12, 2015