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Characterizing heterogeneity in cooperative networks from a resource distribution view-point

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A network of agents in which agents with a diverse set of resources or capabilities interact and coordinate with each other to accomplish various tasks constitutes a heterogeneous cooperative network. In this paper, we investigate heterogeneity in terms of resources allocated to agents within the network. The objective is to distribute resources in such a way that every agent in the network should be able to utilize all these resources through local interactions. In particular, we formulate a graph coloring problem in which each node is assigned a subset of labels from a labeling set, and a graph is considered to be completely heterogeneous whenever all the labels in the labeling set are available in the closed neighborhood of every node. The total number of different resources that can be accommodated within a system under this setting depends on the underlying graph structure of the network. This paper provides an analysis of the assignment of multiple resources to nodes and the effect of these assignments on the overall heterogeneity of the network.

1. Introduction

Heterogeneity has emerged as an important aspect of multiagent and cooperative networks in which agents with assorted functionalities and capabilities interact with each other to accomplish complex tasks. Agents with various capabilities and properties when integrated together in the framework of heterogeneous networks, become specialized to achieve sub-goals which may not be accomplished by a team of homogeneous agents efficiently. Several applications of such heterogeneous systems have been studied in the literature. For instance, it is shown in [1] that the reliability and lifetime of a sensor network can be increased by introducing nodes in the network that are different in terms of power consumption and communication capabilities. In [2], it is shown that heterogeneity can be exploited to reduce the number of sensors required in a sensor network without compromising on the coverage and the broadcast reach of the network. Heterogeneity among agents is explored to make a decentralized system more stable and efficient in [3]. Several other applications are also reported in other domains including multirobot systems (e.g., [4, 5, 6, 8]), and wireless sensor networks (e.g., [9, 10]) to name a few.

One of the challenges in heterogeneous networks is to optimally distribute agents having different capabilities and resources within a network. A part of the challenge is to articulate the notion of *heterogeneity* in order to guide the analysis and design of heterogeneous networks in a systematic way. Heterogeneity in cooperative networks can be understood along a number of dimensions. We can broadly classify the studies in this area into two categories; one that quantify heterogeneity from *agents' perspectives*, including functional or behavioral dissimilarities among individual agents (e.g., [11, 12, 13, 14]), and the relative number of non-homogeneous agents (e.g., [15, 16, 17]); second that measures heterogeneity in terms of the *underlying graph structure* of the network with an aim to quantify degree distribution among the nodes while treating all the nodes similar[1] (e.g., [18, 19, 20]). Despite their merits, all these studies address one of the two attributes of a cooperative network at a time, either agents' classification or the topological properties of the underlying graph structure. However, to obtain a holistic overview of heterogeneity, a unified framework incorporating agents' classification as well as network topology is needed.

In this paper, we extend our initial work in [7, 8], and investigate a resource assignment problem over a graph in the context of heterogeneous cooperative networks. The objective is to provide a framework for a network topology based characterization of heterogeneity in cooperative networks while distinguishing among agents. We investigate interactions among various types of agents, which are distributed throughout a heterogeneous network, by formulating a *graph coloring* problem. A network is modeled by a graph, in which vertices represent agents and edges abstract interactions among agents. Vertices corresponding to agents of a similar type are assigned a distinct color. An agent interacts locally with other types of agents to perform some task. The goal is to maximally exploit heterogeneous resources available within the network through these local interactions. Using the graph coloring formulation, we examine an inter-relationship between network topology and distribution of agents with various capabilities in heterogeneous networks. This framework also provides a way to characterize various network topologies in terms of their capabilities to incorporate heterogeneous entities under constraints on interactions among various types

[1] I.e., there is no distinction among the nodes to account for their functional or behavioral differences.

of agents. The role and significance of individual agents and interactions on the overall heterogeneity of the network is also explored.

The paper is organized as follows: In Section 2, a graph coloring based model of heterogeneous networks is given. Using this model, distribution of resources among nodes within a network is analyzed in Section 3. In Section 4, the issue of maximum number of resources that can be incorporated within a network under the constraints of the model is addressed. A resource assignment problem in R-disk proximity graphs, which is a widely used inter-connection model in multiagent, multirobot, and sensor networks is investigated in Section 5. The paper is concluded in Section 6 along with some future directions of the work.

2. Graph coloring based model of heterogeneous networks

Graph theoretic tools are frequently applied to model and analyze various cooperative networks including sensor networks, multiagent, and multirobot networks. In this paper, a network is modeled by a graph $G(V, E)$ in which the vertex set V represents agents and the edges in the edge set E correspond to interactions among agents. In the case of heterogeneous networks in which agents may be different from each other in terms of their resources or capabilities (for instance, sensing, actuation, dynamics, capabilities, resources, hardware, or software etc.), heterogeneity is modeled by associating a unique *color (or label)* with each resource type available in the network. Moreover, all of the vertices in an underlying graph of the network are assigned colors (or labels) in accordance with the resources contained by the corresponding agents. A vertex may have multiple labels if the corresponding agent has multiple types of resources. In heterogeneous cooperative networks, agents interact and utilize each others' resources to accomplish tasks such as surveillance, coverage, data analysis, and computation to name a few. The availability of resources of different types in the local neighborhood of an agent determines the agent's overall capability to perform various tasks. Correlation between network topology and distribution of agents with various types of resources can be studied by casting a graph coloring formulation of the problem in which vertices are assigned labels in accordance with the resources contained by the agents.

Throughout this paper, a *graph* $G(V, E)$ with a vertex set V and an edge set E, is a simple undirected graph. An edge between nodes v_i and v_j is denoted by $v_i \sim v_j$. The *open neighborhood* of a vertex $v \in V(G)$, denoted by $\mathcal{N}(v)$, is the set of vertices adjacent to v. Its *closed neighborhood*, denoted by $\mathcal{N}[v]$, is $\mathcal{N}(v) \cup \{v\}$. The degree of a vertex v, $deg(v)$, is the cardinality of $\mathcal{N}(v)$. The minimum degree of a graph, $\delta(G)$, is $min\{deg(v) \mid v \in V\}$ and the maximum degree of a graph, $\Delta(G)$, is $max\{deg(v) \mid v \in V\}$. The terms color and label are used interchangeably.

Let $\mathbf{r} = \{1, 2, \cdots, r\}$ be a set of labels representing r different types of resources (or capabilities) available within a heterogeneous network. Furthermore, vertices are assigned labels according to the map,

$$f : \quad V \longrightarrow 2^{\mathbf{r}}$$

Here, $2^{\mathbf{r}}$ is the set of all subsets of $\mathbf{r}$. $f(v)$ is a subset of resources contained by agent v, which interacts and utilizes the resources of its neighbors to perform some task. Thus, the *heterogeneity* of an agent v within the network depends on the resources (or capabilities) contained by v and its neighbors, and is defined as

$$\mathcal{H}(v) = \bigcup_{u \in \mathcal{N}[v]} f(u) \tag{1}$$

Moreover, an agent is *maximally* heterogeneous within the network whenever $\mathcal{H}(v) = \mathbf{r}$, as v can exploit all different functionalities and resources available within the network by interacting with its neighbors. Thus, from the the network topological view-point, a *completely heterogeneous* graph is defined as

Definition 2.1. *A graph $G(V, E)$ in which every $v \in V$ is assigned a subset of labels $f(v)$ from the set $\mathbf{r} = \{1, 2, \cdots, r\}$, is completely heterogeneous with r labels if*

$$\mathcal{H}(v) = \mathbf{r}, \quad \forall v \in V$$

Note that in a *completely heterogeneous* network as defined above, every agent is capable of exploiting a complete set of resources and functionality available within the network to perform various tasks by working in conjunction with its neighbors.

2.1. Examples

Consider an industrial location in which some manufacturing process depends on environmental conditions, including temperature (t), light (ℓ), humidity (h), and air pressure (p). A specific environmental condition that depends on all of the above parameters is needed to be maintained to get a desired yield. Let this condition be denoted by $\omega(t, \ell, h, p)$. Sensors for each of the above parameters t, l, h and p are mounted at various data collection points, which are inter-connected with each other and exchange data.

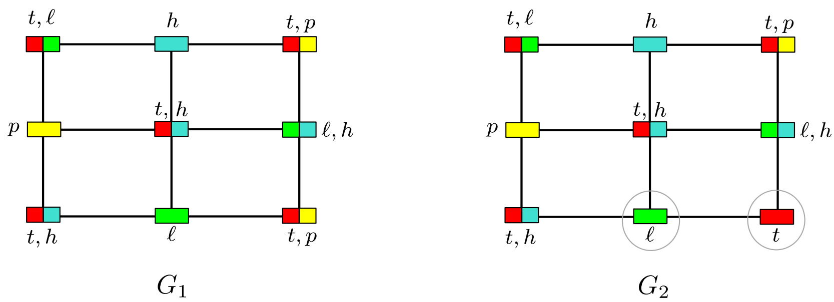

Figure 1: G_1 is a completely heterogeneous graph with four labels $\{t, h, l, p\}$. In G_2, label p is missing from the closed neighborhoods of the circled nodes. Thus, G_2 is not completely heterogeneous.

The environmental condition $\omega(t, \ell, h, p)$ is computed at every such data collection point. The distribution of sensors with assorted sensing capabilities constitutes a heterogeneous network. It is further assumed that owing to some constraints (e.g., hardware, power, economical etc.), only a subset of sensors can be mounted at each data collection point. Since all four parameters are needed for the computation of $\omega(t, \ell, h, p)$, sensors need to be distributed in such a way that all of the four types of sensors are available in the closed neighborhood of every data collection point. In other words, underlying graph of the network needs to be completely heterogeneous with the set of labels $\{t, l, h, p\}$ as shown in Fig. 1.

As another example, consider a society of some 'species' in which each member of the society has been assigned a specific role. Some members are food providers, some are shelter providers, while others hold the task of providing security to the members they interact with. In such a society, every member depends on other members to ensure the availability of all the facilities. For instance, a food providing member must interact with a shelter provider and a security provider for shelter and security respectively. This kind of cooperation constitutes a heterogeneous network in which availability of all the resources to each member of the society is possible if the underlying graph of the network is completely heterogeneous with three distinct labels.

2.2. Major issues related to the notion of completely heterogeneous graph

In the context of heterogeneous cooperative networks, the notion of completely heterogeneous graph has three major aspects. First, given a colored

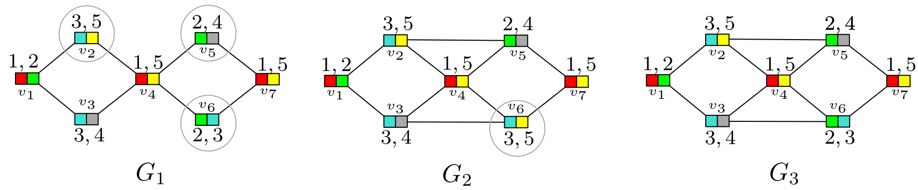

Figure 2: G_1 can not be made completely heterogeneous with five labels by assigning at most two distinct labels to each node. In the case of G_2, although it is possible to assign two colors to each vertex and obtain a completely heterogeneous graph with five colors, under the given labeling v_6 is missing label 2 in its closed neighborhood. In G_3, each vertex has a complete set of five labels in its closed neighborhood.

graph, how can we analyze the distribution of colors to nodes, i.e., determine in a systematic way which nodes are missing colors in their closed neighborhoods? what is the most deficient color in the network? which edges or nodes are relatively more 'crucial' or significant? This analysis will provide a way to transform a given coloring of a graph to a completely heterogeneous one through simple graph operations, such as adding or removing certain edges.

Second, given a coloring set $\mathbf{r}$ and a constraint on the maximum number of colors each node can have, is it possible to color the nodes to get a completely heterogeneous graph? For instance, in Fig. 2, it is impossible to color G_1 to get a completely heterogeneous graph with five colors if each node is allowed to have at most two colors. This issue is significant in characterizing network topology in terms of its overall capability to incorporate heterogeneous resources.

Third, once we know that a labeling scheme exists to make a graph completely heterogeneous with a certain number of labels and constraint on the number of labels each node can have, how can we assign colors to nodes to achieve such a labeling scheme? An example is illustrated in Fig. 2.

3. Analyis of the distribution of heterogeneous agents within a network

In this section, distribution of resources or capabilities in heterogeneous cooperative networks is analyzed using the model introduced in Section 2. Various tasks performed by individual agents in such networks depend on the resources available locally to agents. Thus, information regarding missing resources in the closed neighborhoods of agents along with the interactions needed to make these resources available is crucial. In this section, these issues are addressed.

Given a graph with n nodes, in which every node v_i is assigned a subset of colors (labels) $f(v_i)$ from the set of colors $\mathbf{r} = \{1, 2, \cdots, r\}$. We define a *color matrix*, denoted by C, as a binary matrix with dimensions $n \times r$ as follows:

$$C_{ij} = \begin{cases} 1 & \text{if } j \in f(v_i), \quad \text{where } f(v_i) \subseteq \mathbf{r} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In (2), $f(v_i)$ indicates the colors assigned to the vertex v_i. The column index of C indicates the color, thus $C_{ij} = 1$ means that color j has been assigned to the vertex v_i.

Using the color matrix C, and the adjacency matrix A of the graph, another integer matrix of dimensions $n \times r$ is defined and named as the *color distribution matrix* as follows:

$$\Phi = \mathcal{A}C, \tag{3}$$

where $\mathcal{A} = (A + I)$. Here, I is the identity matrix of dimensions $n \times n$.

The color distribution matrix gives information regarding the distribution of various colors within the network. In fact, it describes the exact number of different colors available in the closed neighborhood of any node.

Lemma 3.1. *Φ_{ij} is the number of nodes with the color j in the closed neighborhood of node v_i.*

Proof. The entries in the i^{th} row matrix $\mathcal{A}$, denoted by $\mathcal{A}_i$, are 1 only for the vertices in $\mathcal{N}[v_i]$ and 0 otherwise. The entries in the j^{th} column of C, denoted by C_j, are 1 only for the vertices with the color j and 0 otherwise. Thus, $\mathcal{A}_i C_j$ is the number of vertices that have color j in the closed neighborhood of vertex v_i. □

The color distribution matrix turns out to be a useful object in characterizing the distribution of colors to vertices in a graph. For instance, it allows to determine the extra edges required to transform a given labeling of a graph into a completely heterogeneous one. In fact, $\Phi_{ij} = 0$ means that v_i is missing color j in its closed neighborhood. Thus, an extra edge is needed to connect v_i to some v_u with a color j. Upper and lower bounds on the number of extra edges required to get a completely heterogeneous graph with r labels from a given coloring of G are presented in the following result.

Theorem 3.2. *Let s be the maximum number of labels assigned to any vertex in a graph G. The number of extra edges $\mathcal{E}$, needed to get a completely heterogeneous graph with r labels from a given coloring of G is*

$$\left\lceil \frac{z(\Phi)}{2s} \right\rceil \leq \mathcal{E} \leq \; z(\Phi), \tag{4}$$

where $z(\Phi)$ is the number of 0's in the color distribution matrix Φ.

Proof. Let $v_i \sim v_j$ be an extra edge connecting vertex v_i with colors $\kappa_1, \cdots, \kappa_s$, to vertex v_j with colors $\tau_1, \tau_2, \cdots, \tau_s$. Since every vertex can have at most s distinct colors, $v_i \sim v_j$ can add at most s missing colors in $\mathcal{N}[v_i]$ and also at most s missing colors in $\mathcal{N}[v_j]$. This is possible whenever v_i is missing colors $\tau_1, \tau_2, \cdots, \tau_s$ in $\mathcal{N}[v_i]$ given by $\Phi_{i\tau} = 0, \; \forall \tau \in \{\tau_1, \cdots, \tau_s\}$, and v_j is missing $\kappa_1, \kappa_2, \cdots, \kappa_s$ in $\mathcal{N}[v_j]$, given by $\Phi_{j\kappa}, \; \forall \kappa \in \{\kappa_1, \cdots, \kappa_s\}$. In this case, $v_i \sim v_j$ edge will change $2s$ zero entries in the Φ matrix to ones. In any other case, i.e., v_i has at least one of the $\tau_1, \tau_2, \cdots, \tau_s$ colors in its closed neighborhood or v_j has at least one of the $\kappa_1, \kappa_2, \cdots, \kappa_s$ colors in $\mathcal{N}[v_j]$, the number of zeros in Φ that will be converted to 1 will be less than $2s$. Thus, $\lceil \frac{z(\Phi)}{2s} \rceil \leq \mathcal{E}$.

The upper bound is straight forward as $\Phi_{i\tau} = 0$ means that v_i is missing a color τ in $\mathcal{N}[v_i]$, and the color τ can always be made available in $\mathcal{N}[v_i]$ by the addition of a single edge $v_i \sim v_j$, where v_j is any vertex with the color τ. $\square$

As an illustration, consider G shown in Fig. 3. Every node has at most two labels from the set of five labels given by $\{1, 2, 3, 4, 5\}$. The corresponding C and Φ matrices are,

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Since $\Phi_{43} = \Phi_{51} = 0$, v_4 is missing label 3 in $\mathcal{N}[v_4]$ and v_5 is missing label 1 in its closed neighborhood. By adding $\mathcal{E}$ number of edges, where $1 \leq \mathcal{E} \leq 2$ (by Theorem 3.2), G can be transformed into a completely heterogeneous graph. Note that by adding a single edge, $v_4 \sim v_5$, a completely heterogeneous graph with five labels is obtained.

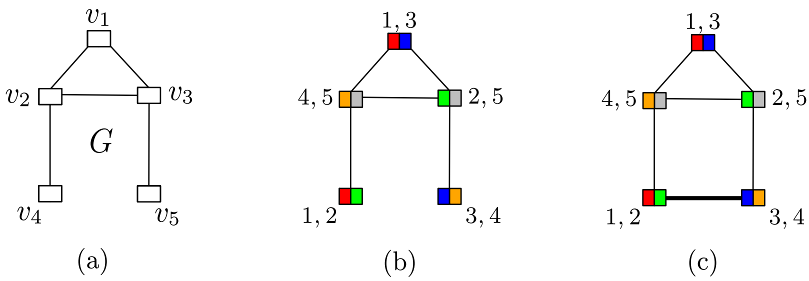

Figure 3: (a) A graph G has five vertices. (b) Two distinct labels are assigned to each vertex of G from the labeling set, $\{1, 2, \cdots, 5\}$. (c) As label 3 is missing from $\mathcal{N}[v_4]$, and label 1 is missing from $\mathcal{N}[v_5]$, $v_4 \sim v_5$ edge is needed to make G completely heterogeneous with five labels.

3.1. Redundant edges

In dynamic networks, some edges may be lost. These edge deletions may take away the availability of certain resources in the neighborhood of an agent. Thus, we need to characterize edges whose deletion is not critical in the sense that their removal will preserve the number of resources available in the neighborhood of any agent. Let us define the *deficiency of a node* v as the number of colors from the coloring set $\{1, 2, \cdots, r\}$ that are missing in $\mathcal{N}[v_i]$. Similarly, *deficiency of a network* is the sum of all the node deficiencies. Based on this notion, we define a *redundant edge* to be the one whose deletion does not increase the deficiency of the network.

$\Phi_{ij} = 1$ means that v_i has only one neighbor with the color j. Thus, an edge between v_i and the j colored node is *not* redundant. Similarly, $\Phi_{ij} > 1$ implies that v_i has multiple nodes with the color j in $\mathcal{N}[v_i]$. As a result, there may be a redundant edge between v_i and some of its neighbors.

Theorem 3.3. *Let v_i be a node with colors $\kappa_1, \kappa_2, \cdots, \kappa_s$, and v_j be its neighbor with colors $\tau_1, \tau_2, \cdots, \tau_s$. An edge $v_i \sim v_j$ is redundant if and only if $\Phi_{i\tau_1}, \Phi_{i\tau_2}, \cdots, \Phi_{i\tau_s}$, and $\Phi_{j\kappa_1}, \Phi_{j\kappa_2}, \cdots, \Phi_{j\kappa_s}$ are all greater than one simultaneously.*

Proof. Let $v_i \sim v_j$ be a redundant edge, then by definition v_i has at least two neighbors for each of the colors $\tau_1, \tau_2, \cdots, \tau_s$ in $\mathcal{N}[v_i]$. In other words, $\Phi_{i\tau_1}, \Phi_{i\tau_2}, \cdots, \Phi_{i\tau_s}$ are all greater than 1. Similarly, for v_j, the redundancy of a $v_i \sim v_j$ edge implies that for each of the colors, $\kappa_1, \kappa_2, \cdots, \kappa_s$, vertex v_j has at least two neighbors in $\mathcal{N}[v_j]$, implying that $\Phi_{j\kappa_1}, \Phi_{j\kappa_2}, \cdots, \Phi_{j\kappa_s}$ are all greater than 1.

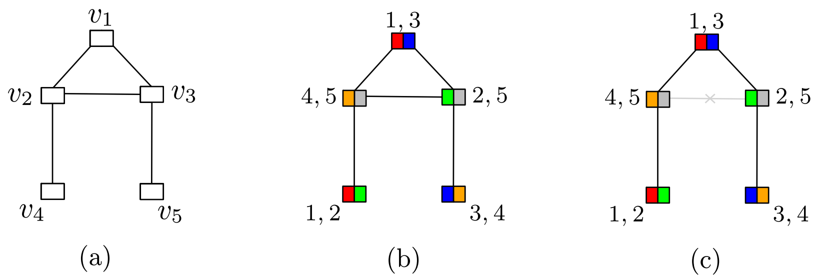

Figure 4: Every vertex is assigned two distinct labels from the set $\{1, 2, \cdots, 5\}$. $v_2 \sim v_3$ edge is redundant and removing this edge will not increase the deficiency of any node.

If $v_i \sim v_j$ is not redundant, then at least one of the following is true. (a) there exists a $\tau \in \{\tau_1, \tau_2, \cdots, \tau_s\}$, such that v_i has only v_j as a τ colored vertex in $\mathcal{N}[v_i]$, i.e., $\Phi_{i\tau} = 1$ for some $\tau \in \{\tau_1, \tau_2, \cdots, \tau_s\}$. (b) there exists a $\kappa \in \{\kappa_1, \kappa_2, \cdots, \kappa_s\}$, such that v_j has only v_i as a κ colored vertex in $\mathcal{N}[v_j]$, i.e., $\Phi_{j\kappa} = 1$ for some $\kappa \in \{\kappa_1, \kappa_2, \cdots, \kappa_s\}$.

In both cases, $\Phi_{i\tau_1}, \Phi_{i\tau_2}, \cdots, \Phi_{i\tau_s}$ and $\Phi_{j\kappa_1}, \Phi_{j\kappa_2}, \cdots, \Phi_{j\kappa_s}$, are not all greater than 1 simultaneously, proving the required result. $\square$

As an example, consider the graph in Fig. 4. Note that v_2 has labels 4 and 5, while v_3 has labels 2 and 5. In the color distribution matrix, $\Phi_{22}, \Phi_{25}, \Phi_{34}$, and Φ_{35} are all greater than 1. By Lemma 3.3, $v_2 \sim v_3$ edge is redundant and its deletion is not increasing the deficiency of any node in the network.

3.2. Most deficient color and the effect of node deletion

The *most deficient color* in the network is the one that is missing from the closed neighborhood of the maximum number of vertices in G. The j^{th} column of Φ tells about the availability of the color j in the closed neighborhood of all the vertices in G. By Lemma 3.1, $\Phi_{ij} = 0$ means v_i does not have a color j in $\mathcal{N}[v_i]$. Thus, the column index of Φ with the *maximum number of zeros* will be the most deficient color in the given labeling of G.

The deletion of a vertex from a graph may increase the deficiency of the remaining vertices. If vertex v_i is the only vertex with the color κ in the closed neighborhood of vertex v_j, deleting v_i will make v_j deficient in κ. However, if v_j has multiple vertices with the color κ in $\mathcal{N}[v_j]$, which is indicated by $\Phi_{j\kappa} \geq 2$, removal of v_i will not increase the deficiency of v_j for the color κ. Using this observation, we can write a matrix $\mathcal{U}$ in which $\mathcal{U}_{i\kappa}$

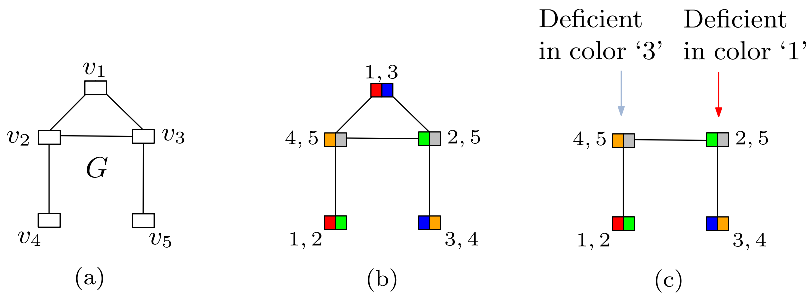

Figure 5: A graph G along with the labeling of its vertices from the set $\{1, 2, \cdots, 5\}$. Removal of v_1 makes v_2 deficient in color 3 and v_3 deficient in color 1. Thus, deletion of v_1 will increase the deficiency of the network by two, which is also indicated by the row sum of the first row of $\mathcal{U}$.

is the number of vertices that will become deficient in the color κ upon the deletion of v_i from the graph G. If C is the color matrix, and Φ be the color distribution matrix, then

$$\mathcal{U}_{i\kappa} = \begin{cases} |\{v_j : \quad v_j \in \mathcal{N}(v_i), \text{ and } \Phi_{j\kappa} = 1\}|, & \text{if } C_{i\kappa} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{U}$ is an integer matrix with dimensions $n \times r$, where n is the total number of vertices in G, and r is the total number of colors in the labeling of G. The i^{th} row sum of $\mathcal{U}$ indicates the increase in the deficiency of the network as a result of the deletion of v_i. If we define a *critical node* as the one whose removal from the network maximizes the increase in the deficiency of the remaining network, then the row index of $\mathcal{U}$ corresponding to the maximum row sum indicates the *most critical* vertex.

As an example, consider G shown in Fig. 5. The $\mathcal{U}$ matrix for G is

$$\mathcal{U} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\mathcal{U}_{13} = 1$, indicates that a single vertex will become deficient in color '3' upon the removal of v_1 from G. Note that both v_2 and v_3 are critical vertices here as both second and third rows have the maximum row sum.

4. Heterogeneity in terms of the maximum number of resources available within the network

In this section, we address the second issue highlighted in Section 2.2, which is related to the maximum number of resources' types that can be incorporated within the system under the constraint that every node v can find every resource type in $\mathcal{N}[v]$. If r different types of resources are available within the network and each agent is equipped with at most s of these resources, then the maximum value of r, denoted by r^*, such that the the underlying graph of the network can be made completely heterogeneous with r^* labels is a crucial attribute of heterogeneous cooperative networks. In fact, in a completely heterogeneous network with r^* different types of resources, every agent can find all r^* types of resources in its closed neighborhood to accomplish various tasks. Thus, a higher value of r^* implies that more types of resources can be made available to agents in a completely heterogeneous network. As a result, agents can perform tasks of higher complexity. It is to be noted that, for a given graph G and a bound on the number of resources an agent can have, i.e., $\mid f(v) \mid \leq s$, if $r > r^*$, then it is impossible to distribute resources among the nodes to get a completely heterogeneous graph with r labels. We utilize the notion of *domination in graphs* to address this issue. A *dominating set* is a fundamental object in the field of domination in graphs (see [21] for details).

Definition 4.1. *(Dominating Set) A set D is a dominating set if for each $v \in V$, either $v \in D$, or v is adjacent to some $u \in D$.*

Note that if D is a dominating set, then $\bigcup_{u \in D} \mathcal{N}[u] = V$. The cardinality of the smallest dominating set in a graph is known as the *domination number* of the graph. A graph can have multiple disjoint[2] dominating sets. A related and important concept is that of the *domatic number*.

Definition 4.2. *(Domatic Number) The maximum number of disjoint dominating sets in a graph is the domatic number of the graph.*

If D_i is a dominating set of a graph G and all the vertices in D_i are assigned a particular label i, then i is available in the closed neighborhood of every vertex in G. In other words, if $f(u) = i$, $\forall u \in D_i$, then $i \in \mathcal{H}(v)$, $\forall v \in V$. Furthermore, if the domatic number of a graph G is γ and every vertex is assigned a single label, then by the definition of domatic number, G can be completely heterogeneous with at most γ labels, i.e., $r^* = \gamma$. The notion of so-called (r, s)-configuration [22] defined below is also helpful in this context.

[2]The intersection of distinct dominating sets is empty.

Definition 4.3. *((r,s)-Configuration of a Graph) Let $\mathbf{r} = \{1, 2, \cdots, r\}$ be a set of labels (colors). A function*

$$f : V \longrightarrow [\mathbf{r}]_s,$$

where $[\mathbf{r}]_s$ is a collection of all s-subsets of $\mathbf{r}$, is called an (r,s)-configuration of a graph G, whenever $\bigcup_{u \in \mathcal{N}[v]} f(u) = \mathbf{r}$, $\forall v \in V$.

Thus, the maximum value of r in an $(r,1)$-configuration of a graph is the domatic number of the graph. Moreover, if γ is the domatic number of a graph, it is obvious that for $s > 1$, there always exists an (r,s)-configuration for $r = s\gamma$. In other words, each vertex in the graph can always be labeled with at most s colors such that the overall graph is completely heterogeneous with $s\gamma$ labels. However, there are graphs for which (r,s)-configurations exist for $r > s\gamma$. For example, cycle graphs C_n in which n is not a multiple of 3, have a domatic number of 2, but $(5,2)$-configurations of such graphs exist [22]. Here, we present a sufficient condition for a graph to have an (r,s)-configuration with $r = s\gamma + \lfloor \frac{s}{2} \rfloor$. A labeling scheme to obtain such a configuration can also be derived using this condition.

We begin by defining some terms that will be used to prove Theorem 4.1, which is the main result of this section.

Definition 4.4. *(Minimal Partition of G) Let G be a graph with domatic number γ, and vertex set V. A minimal partition of G, denoted by Π, is a partitioning of V into $\gamma + 1$ disjoint sets such that,*

$$\Pi = D_1 \cup D_2 \cup \cdots D_\gamma \cup V_\Pi, \tag{5}$$

where D_i is a minimal dominating set, $\forall i \in \{1, 2, \cdots, \gamma\}$, and $V_\Pi = V - (\cup_{i=1}^{\gamma} D_i)$ is the set of vertices that are not included in any minimal dominating set D_i. □

We term V_Π in (5) as the set of *non-critical vertices* with respect to the minimal partition Π. Note that $V_\Pi \cap (\cup_{i=1}^{\gamma} D_i) = \emptyset$.

Consider a minimal partition of G, denoted by Π. Let $D_{\gamma+1}$ be a dominating set such that $V_\Pi \subseteq D_{\gamma+1}$. Since $dom(G) = \gamma$, and V_Π is not a dominating set, we have

$$D_{\gamma+1} = V_\Pi \cup I_\Pi,$$

where $I_\Pi \subset (\cup_{i=1}^{\gamma} D_i)$. We call a set I_Π with the smallest cardinality, a set of *common vertices* with respect to the minimal partition Π.

The notions of minimal partition Π, set of non-critical vertices with respect to Π, and set of common vertices with respect to Π are shown in Fig. 6.

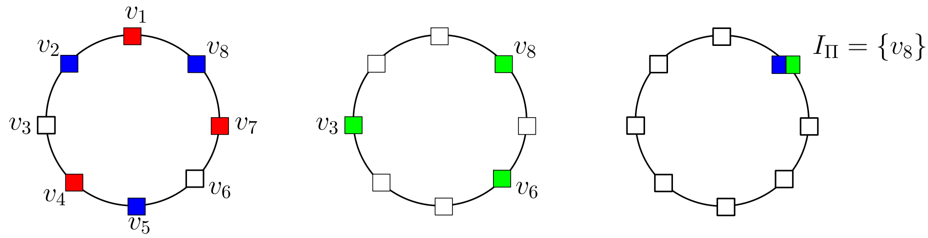

Figure 6: A cycle graph, C_8 having a domatic number $\gamma = 2$. A minimal partition $\Pi = D_1 \cup D_2 \cup V_\Pi$, where $D_1 = \{v_1, v_4, v_7\}$ and $D_2 = \{v_2, v_5, v_8\}$ are minimal dominating sets, and $V_\Pi = \{v_3, v_6\}$ is the set of non critical vertices with respect to Π. We can take another dominating set D_3 as $D_3 = V_\Pi \cup I_\Pi$, where $I_\Pi = \{v_8\}$ is a set of common vertices with respect to the minimal partition Π.

Theorem 4.1. *Let G be a graph with domatic number γ. Let Π be a minimal partition of G and I_Π be a set of common vertices with respect to Π. If there exists another minimal partition of G, say $\tilde{\Pi} \neq \Pi$, such that $I_\Pi \subseteq V_{\tilde{\Pi}}$, then G has an (r, s)-configuration with $r = s\gamma + \lfloor \frac{s}{2} \rfloor$. Here, $V_{\tilde{\Pi}}$ is a set of non-critical vertices with respect to $\tilde{\Pi}$,*

Proof. Let $\Pi = \bigcup_{i=1}^{\gamma} D_i \cup V_\Pi$ in which V_Π is the set of non-critical vertices with respect to the minimal partition Π. Let $D_{\gamma+1}$ be a dominating set with $D_{\gamma+1} = V_\Pi \cup I_\Pi$, in which I_Π is a set of common vertices with respect to Π. Assign $\lfloor \frac{s}{2} \rfloor$ distinct labels to all the vertices in a dominating set D_i, for every $i \in \{1, 2, \cdots, \gamma + 1\}$[3]. Under this labelling scheme, vertices in I_Π will have $(2\lfloor \frac{s}{2} \rfloor)$ distinct labels as they are included in two different dominating sets, including $D_{\gamma+1}$ and some other D_i for $i \in \{1, 2, \cdots, \gamma\}$. Note that the vertices in I_Π are the only ones with $(2\lfloor \frac{s}{2} \rfloor)$ labels. Moreover, every $v \in V$ has a $\lfloor \frac{s}{2} \rfloor(\gamma + 1)$ distinct labels in its closed neighborhood.

Consider another minimal partition of G, $\tilde{\Pi} = \bigcup_{i=1}^{\gamma} S_i \cup V_{\tilde{\Pi}}$, with $V_{\tilde{\Pi}}$ being the set of non-critical vertices with respect to $\tilde{\Pi}$, and each S_i being a minimal dominating set. Let $\tilde{\Pi}$ be such that $I_\Pi \subseteq V_{\tilde{\Pi}}$. It means that every vertex in $V - V_{\tilde{\Pi}}$ has $\lfloor \frac{s}{2} \rfloor$ labels. Since $S_i \subseteq (V - V_{\tilde{\Pi}})$ for any $i \in \{1, 2, \cdots, \gamma\}$, every vertex $v \in S_i$ has $\lfloor \frac{s}{2} \rfloor$ labels. Furthermore, assign $\lceil \frac{s}{2} \rceil$ unique labels to each vertex in S_i, $\forall i$. Since each S_i is a dominating set, every $v \in V$ has a set of $\lceil \frac{s}{2} \rceil\gamma$ unique labels in $\mathcal{N}[v]$. Moreover, since $\lfloor \frac{s}{2} \rfloor(\gamma + 1)$ unique labels are already available in the closed neighborhood of every vertex, all

[3] $\lfloor \frac{s}{2} \rfloor$ labels assigned to the vertices of D_i are different from the ones assigned to the vertices in D_j where $i \neq j$.

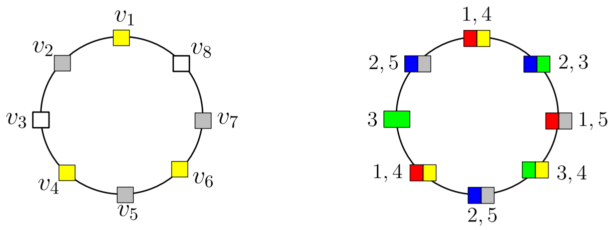

Figure 7: (a) $\tilde{\Pi} = S_1 \cup S_2 \cup V_{\tilde{\Pi}}$, in which $S_1 = \{v_1, v_4, v_6\}$ and $S_2 = \{v_2, v_5, v_7\}$ are disjoint minimal dominating sets, and $V_{\tilde{\Pi}} = \{v_3, v_8\}$ is the set of non-critical vertices with respect to $\tilde{\Pi}$. (b) A $(5, 2)$-configuration of C_8.

vertices in V have $\lfloor \frac{s}{2} \rfloor (\gamma + 1) + \lceil \frac{s}{2} \rceil \gamma = s\gamma + \lfloor \frac{s}{2} \rfloor$ distinct labels in their closed neighborhoods. Note that each vertex is assigned at most s distinct labels. Thus, an (r, s)-configuration of G with $r = s\gamma + \lfloor \frac{s}{2} \rfloor$ is obtained. $\square$

As an example, consider a $(5, 2)$-configuration of C_8. Domatic number of C_8 is 2, i.e., $\gamma = 2$. We consider two minimal partitions of C_8, denoted by Π and $\tilde{\Pi}$, where Π is shown in Fig. 6. For $\tilde{\Pi}$, we take $\tilde{\Pi} = S_1 \cup S_2 \cup V_{\tilde{\Pi}}$, as shown in Fig. 7. Since $I_\Pi \subseteq V_{\tilde{\Pi}}$, $(5, 2)$-configuration exists for C_8.

5. Assignment of multiple resources in *R*-disk proximity graphs

In this section, a resource assignment problem based on the graph coloring formulation in Section 2 is investigated for the R-disk proximity graphs, which are frequently employed to model inter-connections among nodes in multiagent networks. In such a model, a disk of radius R, which represents interaction range of a node, is associated with every node v that lies at the center of the disk. A node forms an edge with other nodes if and only if they exist within the R radius disk of the node. Applications of this model include ad hoc communication networks, wireless sensor networks (e.g., see [25]), multiagent and multirobot systems (see e.g., [26]), and other broadcast networks with limited range transmitters and receivers.

Analysis of (r, s)-configurations of R-disk proximity graphs is of significance, particularly in the context of heterogeneous multiagent systems. Here, we show that under some mild conditions, R-disk graphs always have an (r, s)-configuration for $r = \lfloor \frac{5s}{2} \rfloor$, where s is any positive integer. It is assumed that agents, which are equipped with multiple capabilities or resources, are lying in a plane, and interactions among them are modeled by the R-disk proximity graph model.

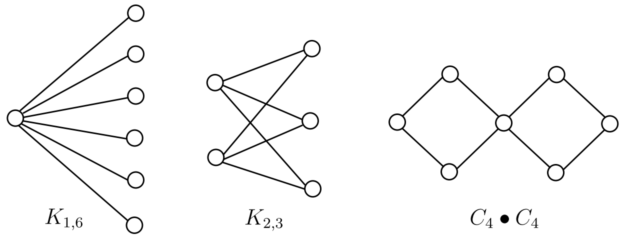

Figure 8: $K_{1,6}$ and $K_{2,3}$, which are examples of complete bi-partite graphs, are shown along with the double cycle graph $C_4 \bullet C_4$, which is obtained by identifying a vertex of C_4 with a vertex of another C_4.

We start by translating the geometric property of such graphs into a graph-theoretic one by first defining the following special graphs. A graph G is a *complete bi-partite graph* if there exists a partition of its vertex set, $V = X \cup Y$, such that an edge $u \sim v$ exists whenever $u \in X$ and $v \in Y$. If $|X| = x$ and $|Y| = y$, then we denote a complete bi-partite graph by $K_{x,y}$. Examples of complete bi-partite graphs are shown in Fig. 8. We also define a *double cycle graph*, denoted by $C_4 \bullet C_4$, as the one obtained by identifying a vertex of C_4 with a vertex of another C_4, as shown in Fig. 8. Furthermore, a graph G is said to be an *H-free graph*, if H is not an induced subgraph of G.

It is shown in [23] that $K_{2,3}$ cannot be an R-disk graph. In the following Lemma, it is shown that R-disk graphs are always $K_{1,6}$-free.

Lemma 5.1. *An R-disk proximity graph is $K_{1,6}$ free.*

Proof. Let $G(V,E)$ be an R-disk proximity graph. Let $v \in V$, such that $\mathcal{N}(v) = \{v_1, v_2, \cdots, v_p\}$, where $p \geq 6$. We define $\theta_{(v_i v v_j)}$ to be the angle v makes with v_i and v_j. If $\|v_i, v_j\|$ is the euclidean distance between v_i and v_j, then it is easy to see that $\|v_i, v_j\| > R$, whenever $\theta_{(v_i v v_j)} > 60°$. Thus, $v_i, v_j \in \mathcal{N}(v)$ are non-adjacent if and only if $\theta_{(v_i v v_j)} > 60°$. To have $K_{1,6}$ as an induced subgraph of G, there must be a subset of $\mathcal{N}(v)$ with six nodes, say $\{x_1, x_2, \cdots, x_6\} \subseteq \mathcal{N}(v)$, such that $\theta_{(x_i v x_j)} > 60°$, $\forall x_i, x_j$. This will give $\sum_{i=1}^{5} \theta_{x_i v x_{(i+1)}} + \theta_{x_6 v x_1} > 360°$, which is not possible. Thus, an R-disk graph is $K_{1,6}$-free. □

Since every R-disk graph is $K_{1,6}$-free, we can focus on the results regarding (r,s)-configurations of $K_{1,6}$-free graphs to study the resource assignment problem in R-disk graphs. A useful result regarding (r,s)-configurations of $K_{1,6}$-free graphs is stated in [24].

Theorem 5.2. *[24] For any positive integer s, a connected $K_{1,6}$-free graph G with a minimum degree of at least two has a $(5,2)$-configuration whenever G is not isomorphic to $C_4, C_7, K_{2,3}$, or $C_4 \bullet C_4$.*

The above result can be generalized for any positive integer s.

Corollary 5.3. *For any positive integer s, a connected $K_{1,6}$-free graph G with a minimum degree of at least two has an (r,s)-configuration with $r = \lfloor \frac{5s}{2} \rfloor$ whenever G is not isomorphic to $C_4, C_7, K_{2,3}$, or $C_4 \bullet C_4$.*

Proof. If s is an even number, let $s = 2s'$. Obtain a $(5,2)$-configuration of G. Repeat this process s' number of times assigning distinct labels to vertices in each step. At the end of s' steps, at most $2s'$ labels are assigned to every v, i.e., $|f(v)| \leq 2s' = s$, and $5s' = \frac{5s}{2}$ distinct labels are available in the closed neighborhood of every v. Thus, a $(\frac{5s}{2}, s)$-configuration of G is obtained.

If s is an odd number, let $s - 1 = 2s'$, then $(\frac{5(s-1)}{2}, s-1)$-configuration can be obtained as above. Furthermore, using the fact that every connected graph has a domination number of at least 2, it is possible to assign a single label to each vertex such that every vertex has at least two distinct labels in its closed neighborhood. Thus, for a given positive odd integer s, an (r,s)-configuration is possible with $r = \frac{5(s-1)}{2} + 2 = \frac{5s}{2} - \frac{1}{2} = \lfloor \frac{5s}{2} \rfloor$. □

Now, using Theorem 5.2, Lemma 5.1, and the fact that an R-disk graph can never have a component isomorphic to $K_{2,3}$ graph, we state the following result regarding (r,s)-configurations of R-disk graphs.

Theorem 5.4. *For any positive integer s, an R-disk proximity graph G with a minimum degree of at least 2 has an (r,s)-configuration with $r = \lfloor \frac{5s}{2} \rfloor$ whenever G has no component isomorphic to C_4, C_7, or $C_4 \bullet C_4$.*

5.1. Example

Consider a group of robots deployed in a planar region $\mathcal{D}$ for the purpose of environment modeling. These robots interact and exchange information with each other, and this interaction is modeled by an R-disk proximity graph model. For environment modeling, five different parameters are considered, including temperature, relative humidity, air pressure, light, and soil texture. Every robot performs some spatio-temporal data processing using the data of the environment parameters, which is obtained from the sensors mounted on robots. If each sensor observes a specific environment parameter, five different types of sensors are needed.

One possible course of action is to install all five sensors on each robot so that robots can collect information of all parameters for further processing. However, this might not be a feasible approach for one or more of the following reasons: first a large amount of power supplies will be needed to keep all five sensors operational on every robot, and power is always a limiting factor for a continuous operation of such networks over an extended period of time; second a large number of sensors of each type will be required which may not be cost effective; third mounting all sensors on a single robot may not be feasible from a hardware view-point in certain cases. Another approach is to install a subset of sensors on each robot and utilize the fact that robots exchange information with each other. However, this approach requires sensors to be distributed among robots in such a way that each robot can obtain the data of the missing parameters from its neighbor robots. In other words, sensors missing on a robot are installed on its neighbor robots. This set-up requires a much smaller number of sensors of each type for the overall operation of the system. If each robot is allowed to have at most two of the five sensors' types, then sensors need to be installed in such a way that each robot can find a complete set of five distinct sensors in its closed neighborhood. However, it is possible if and only if the underlying R-disk graph of the network has a $(5, 2)$-configuration. It is shown in Theorem 5.4 that every R-disk graph has a $(5, 2)$-configuration under some conditions. Thus, it is possible to make a robot network completely heterogeneous with five labels under the restriction that each robot can have at most two labels. An example is illustrated in Fig. 9.

6. Discussion and conclusions

The notion of completely heterogeneous graph can be extended and generalized in different ways. In Section 2, a completely heterogeneous graph with a set of r unique labels is defined as the one in which every node finds a complete set of r labels in its closed neighborhood. There might be situations in which a complete set of resources might not be required in the closed neighborhood of all the nodes. This motivates to further extend the notion of a completely heterogeneous graph for more general scenarios. For instance, the notion of neighborhood can be extended to the *k-neighborhood*. If the distance between nodes v and u, denoted by $d(u, v)$, is the length of the shortest path in a graph G, then the *open k-neighborhood* of node v is the set $\mathcal{N}_k(v) = \{u \in V : \ d(u, v) \leq k\}$. Likewise, the *closed k-neighborhood* of v is $\mathcal{N}_k[v] = \mathcal{N}_k(v) \cup v$. We can define a *completely k-heterogeneous graph* with r labels as the one in which every vertex finds a complete set of r labels in its

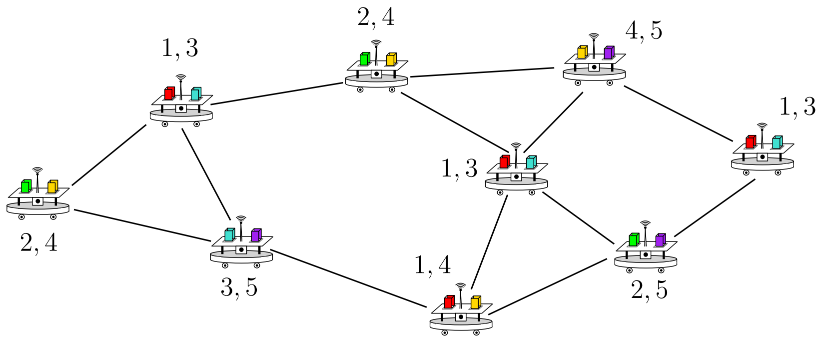

Figure 9: A group of robots connected via R-disk proximity graph model. Each robot is assigned two labels from the set $\{1, 2, 3, 4, 5\}$ in such a way that a complete set of five labels is available in the closed neighborhood of every node.

closed k-neighborhood. The concept of completely k-heterogeneous graph with r labels is particularly useful in situations in which r is quite large, i.e., a large number of different types of resources exist within the network, and it might not be possible to ensure the availability of all these resources in the closed neighborhood of every node. In such situations, one can aim to distribute resources among nodes to get a completely k-heterogeneous graph with r labels for a small value of k to ensure that each node finds all r resources within a small distance k from it.

Analogous to the color distribution matrix introduced in Section 3 for the analysis purpose, we can define *k-color distribution matrix* for the case of completely k-heterogeneous graphs as

$$\Phi(k) = \mathcal{A}_k C.$$

Here, $\mathcal{A}_k = A_k + I$ and C is the color matrix. A_k is an $n \times n$ matrix whose ij^{th} entry is 1 whenever $d(v_i, v_j) \leq k$ and $i \neq j$.

It is to be observed that ij^{th} entry of the $\Phi(k)$ matrix is the number of vertices with the color j in the closed k-neighborhood of v_i. Thus, we can use the same approach as in Section 3 to analyze the distribution of labels when using the notion of k-neighborhood for $k > 1$.

In conclusion, we investigated heterogeneity in cooperative and multiagent systems from a network topology view-point in this paper. The notion of (r, s)-configuration of a graph was used to characterize a distribution of agents with multiple capabilities (or resources). In such a distribution, every agent could find all types of resources available in the network in its closed neighborhood. The role of individual agents and interactions among them in attaining (r, s)-configurations was also examined. The study not only analyzed the role of network topology in the context of heterogeneous multiagent systems, but also provided ways to design network structures in which agents equipped with various resources coordinate and compliment each others capabilities to accomplish complex tasks.

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