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Asymptotic behavior of BV solutions to the equations of nonlinear viscoelasticity

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For Marshall Slemrod on the occasion of his 70th birthday

It is shown that the total variation of admissible BV solutions to the Cauchy problem for a system modeling the equations of motion of one-dimensional nonlinear viscoelastic media with fading memory decays to zero as $t \to \infty$.

1. Introduction

The equations of motion of one-dimensional viscoelastic media with fading memory are modeled by simple systems in the form

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x f(u) - \int_{-\infty}^{t} a'(t-\tau)\partial_x u d\tau = 0, \end{cases} \tag{1.1}$$

where u denotes the strain and v stands for velocity. The function f, encoding the instantaneous elastic response of the medium, is strictly increasing, normalized by $f'(0) = 1$, and nonlinear.

Under suitable assumptions on the relaxation kernel a, (1.1) exhibits the typical behavior of hyperbolic systems of balance laws with relaxation: Smooth and small initial data generate globally defined smooth solutions to the Cauchy problem [6]. However, when the gradient of (even smooth) initial data gets large, waves break and shocks develop [1]. In that case one may hope, at best, for the existence of weak solutions, in the large.

As shown in [3], when the initial data have small total variation and decay rapidly to zero as $|x| \to \infty$, the Cauchy problem for (1.1) possesses a unique admissible BV solution, defined on the entire upper half-plane. The aim here is to demonstrate that the total variation of this solution decays to zero, as $t \to \infty$.

At the outset, in order to simplify the analysis, we make the assumption that the relaxation kernel is in the form

$$a(t) = \frac{1}{\mu} e^{-\mu t}. \tag{1.2}$$

The parameter μ is fixed bigger than 1, in order to secure that the relaxed response of the medium is elastic.

For the above choice of relaxation kernel, and upon introducing the internal variable w,

$$w(x,t) = \int_{-\infty}^{t} e^{-\mu(t-\tau)} u(x,\tau) d\tau, \tag{1.3}$$

(1.1) reduces to the strictly hyperbolic system

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x f(u) + \partial_x w = 0 \\ \partial_t w - u + \mu w = 0. \end{cases} \tag{1.4}$$

We prescribe initial data

$$u(x,0) = u_0(x), \;\; v(x,0) = v_0(x), \;\; w(x,0) = w_0(x), \;\; -\infty < x < \infty, \tag{1.5}$$

where (u_0, v_0, w_0) have bounded variation,

$$\mathrm{TV}u_0(\cdot) + \mathrm{TV}v_0(\cdot) + \mathrm{TV}w_0(\cdot) = \delta\,, \tag{1.6}$$

and decay to zero, as $|x| \to \infty$, so that

$$\int_{-\infty}^{\infty} (1+|x|)^{2s} [u_0^2(x) + v_0^2(x) + w_0^2(x)] dx = \sigma^2, \tag{1.7}$$

for some $s > 1$. Furthermore, in order to eliminate trivial rigid motions,

$$\int_{-\infty}^{\infty} u_0(x) dx = \int_{-\infty}^{\infty} v_0(x) dx = 0. \tag{1.8}$$

The principal result is

Theorem 1.1 *There are positive numbers δ_0 and σ_0 such that when the initial data (u_0, v_0, w_0) satisfy* (1.6), *for $\delta < \delta_0$,* (1.7), *for $\sigma < \sigma_0$, and* (1.8), *then the Cauchy problem* (1.4), (1.5) *possesses a unique admissible* BV *solution (u, v, w), defined on $(-\infty, \infty) \times [0, \infty)$. Furthermore,*

$$\mathrm{TV}u(\cdot,t) + \mathrm{TV}v(\cdot,t) + \mathrm{TV}w(\cdot,t) \to 0, \quad \text{as } t \to \infty. \tag{1.9}$$

The assertion on the existence of the solution has already been established in [3] and [4], so it will be assumed known. Furthermore, the total variation of the solution over $(-\infty, \infty)$ remains bounded and small, uniformly in time on $[0, \infty)$:

$$\text{TV}u(\cdot, t) + \text{TV}v(\cdot, t) + \text{TV}w(\cdot, t) \leq a\delta + b\sigma\,, \quad 0 \leq t < \infty. \tag{1.10}$$

The goal here is to verify the statement (1.9). For that purpose, we shall draw freely from [2], [4] and [5].

2. Energy estimates and decay of the L^1 norm

Let (u, v, w) be the admissible BV solution to the Cauchy problem (1.4), (1.5), with initial data (u_0, v_0, w_0) satisfying (1.6), (1.7) and (1.8), which is defined on $(-\infty, \infty) \times [0, \infty)$ and takes values in a small neighborhood of the origin. To avoid proliferation of symbols, we shall employ throughout c as a generic positive constant that may depend on μ and on bounds of f and its derivatives on a small neighborhood of the origin, but is independent of δ and σ.

The system (1.4) is endowed with an entropy-entropy flux pair

$$\eta = g(u) + \frac{1}{2}v^2 + \frac{1}{2}\mu w^2 - uw\,, \qquad q = -vf(u) + vw, \tag{2.1}$$

where

$$g(u) = \int_0^u f(\omega)d\omega. \tag{2.2}$$

Since $f'(0) = 1$ and $\mu > 1$, η is convex on the range of the solution, which is admissible, and thus

$$\partial_t \eta(u, v, w) + \partial_x q(u, v, w) + r(u, w) \leq 0, \tag{2.3}$$

in the sense of distributions, where r is the entropy production,

$$r(u, w) = (u - \mu w)^2. \tag{2.4}$$

Lemma 2.1 *We have*

$$\int_0^\infty \int_{-\infty}^\infty [u^2(x, t) + v^2(x, t) + w^2(x, t)]dxdt \leq c\sigma^2. \tag{2.5}$$

Proof. It is taken from [4]; see also [7]. We introduce the "potential" functions

$$\phi(x,t)=\int_{-\infty}^{x}u(y,t)dy\,,\quad \psi(x,t)=\int_{-\infty}^{x}v(y,t)dy. \tag{2.6}$$

Note that, by virtue of (1.7) and (1.8),

$$\int_{-\infty}^{\infty}[\phi^2(x,0)+\psi^2(x,0)]dx\le\frac{\sigma^2}{2(s-1)(2s-1)}. \tag{2.7}$$

We now define the functions

$$H=\phi^2+\frac{1}{\mu}w^2+\frac{1}{\mu(\mu-1)}[\mu\psi-w]^2-\frac{1}{2\mu^2}\phi v, \tag{2.8}$$

$$Q=-2\phi\psi+\frac{1}{2\mu^2}\phi f(u)-\frac{1}{2\mu^2}\phi w, \tag{2.9}$$

$$R=2w^2+\frac{1}{2\mu^2}v^2-\frac{1}{2\mu^2}u^2+\frac{1}{2\mu^2}uw+\Theta, \tag{2.10}$$

where

$$\Theta=[-\frac{1}{2\mu^2}u+\frac{2}{\mu-1}w-\frac{2\mu}{\mu-1}\psi][f(u)-u]. \tag{2.11}$$

After a laborious but straightforward calculation,

$$\partial_t H(v,w,\phi,\psi)+\partial_x Q(u,w,\phi,\psi)+R(u,v,w,\psi)=0. \tag{2.12}$$

Furthermore, it is easy to see that

$$\eta(u,v,w)+H(v,w,\phi,\psi)\ge 0, \tag{2.13}$$

$$r(v,w)+R(u,v,w,\psi)\ge c(|u|^2+|v|^2+|w|^2). \tag{2.14}$$

Therefore, integrating the sum of (2.3) and (2.12) over $(-\infty,\infty)\times[0,\infty)$, we arrive at (2.5). This completes the proof.

Lemma 2.2 *We have*

$$\int_{-\infty}^{\infty}[u^2(x,t)+v^2(x,t)+w^2(x,t)]dx\le\frac{c\sigma^2}{t},\quad 0<t<\infty, \tag{2.15}$$

$$t\int_{-\infty}^{\infty}[u^2(x,t)+v^2(x,t)+w^2(x,t)]dx\to 0,\quad \text{as } t\to\infty. \tag{2.16}$$

Proof. By virtue of (2.5),

$$\int_0^\infty \int_{-\infty}^\infty \eta(u(x,t),v(x,t),w(x,t))dxdt \le c\sigma^2. \tag{2.17}$$

Moreover, by (2.3), $t \mapsto \int_{-\infty}^\infty \eta(u(x,t),v(x,t),w(x,t))dx$ is nonincreasing and hence

$$\int_{-\infty}^\infty \eta(u(x,t),v(x,t),w(x,t))dx \le \frac{c\sigma^2}{t}\,, \qquad 0 < t < \infty, \tag{2.18}$$

whence (2.15) follows.

Fix any $\varepsilon > 0$. On account of (2.17), there is $\tau > 0$ such that

$$\tau \int_{-\infty}^\infty \eta(u(x,\tau),v(x,\tau),w(x,\tau))dx < \frac{\varepsilon}{2}, \tag{2.19}$$

$$\int_\tau^\infty \int_{-\infty}^\infty \eta(u(x,t),v(x,t),w(x,t))dxdt < \frac{\varepsilon}{2}. \tag{2.20}$$

By (2.3) and (2.4),

$$\partial_t[t\eta(u,v,w)] + \partial_x[tq(u,v,w)] \le \eta(u,v,w). \tag{2.21}$$

For any $t > \tau$, integrating (2.21) over $(-\infty,\infty) \times (\tau,t)$ and using (2.19), (2.20) we deduce

$$t\int_{-\infty}^\infty \eta(u(x,t),v(x,t),w(x,t))dx < \varepsilon, \qquad \tau < t < \infty, \tag{2.22}$$

which implies (2.16). The proof is complete.

Lemma 2.3 *Let*

$$I(t) = \int_{-\infty}^\infty [|u(x,t)| + |v(x,t)| + |w(x,t)|]dx, \quad 0 \le t < \infty. \tag{2.23}$$

Then

$$I(t) \le c\sigma, \qquad 0 < t < \infty, \tag{2.24}$$

$$I(t) \to 0, \qquad \text{as} \quad t \to \infty. \tag{2.25}$$

Proof. We fix λ large enough for

$$|q(u,v,w)| \le \lambda\eta(u,v,w), \tag{2.26}$$

for all (u,v,w) in the range of the solution. We also fix any $t>0$ and for $m=0,1,2,\dots$ we integrate (2.3) over the set $\{(x,\tau): 0\le\tau\le t, \lambda(t+\tau)+m\le x<\infty\}$. Using (2.4) and (2.26),

$$\int_{2\lambda t+m}^{\infty}\eta(u(x,t),v(x,t),w(x,t))dx\le\int_{\lambda t+m}^{\infty}\eta(u_0(x),v_0(x),w_0(x))dx. \tag{2.27}$$

On account of (1.7) this yields

$$\int_{2\lambda t+m}^{\infty}[u^2(x,t)+v^2(x,t)+w^2(x,t)]dx\le c(1+\lambda t+m)^{-2s}\sigma^2. \tag{2.28}$$

Therefore, by Schwarz's inequality,

$$\int_{2\lambda t+m}^{2\lambda t+m+1}[|u(x,t)|+|v(x,t)|+|w(x,t)|]dx\le c\sigma(1+\lambda t+m)^{-s}. \tag{2.29}$$

Since

$$\sum_{m=0}^{\infty}(1+\lambda t+m)^{-s}\le(1+\lambda t)^{-s}+\int_0^{\infty}(1+\lambda t+\xi)^{-s}d\xi\le\frac{s}{s-1}(1+\lambda t)^{1-s}, \tag{2.30}$$

(2.29) implies

$$\int_{2\lambda t}^{\infty}[|u(x,t)|+|v(x,t)|+|w(x,t)|]dx\le c\sigma(1+\lambda t)^{1-s}. \tag{2.31}$$

A similar argument yields

$$\int_{-\infty}^{-2\lambda t}[|u(x,t)|+|v(x,t)|+|w(x,t)|]dx\le c\sigma(1+\lambda t)^{1-s}. \tag{2.32}$$

At the same time, by Schwarz's inequality,

$$\begin{aligned}\int_{-2\lambda t}^{2\lambda t}&[|u(x,t)|+|v(x,t)|+|w(x,t)|]dx\\ &\le\{12\lambda t\int_{-\infty}^{\infty}[u^2(x,t)+v^2(x,t)+w^2(x,t)]dx\}^{\frac12}.\end{aligned} \tag{2.33}$$

Combining (2.31), (2.32), (2.33) and (2.15), we arrive at (2.24). Furthermore, (2.31), (2.32), (2.33) and (2.16), yield (2.25). The proof is complete.

3. Redistribution of damping

Our system, in its present form (1.4), does not lend itself to establishing the decay of the total variation of solutions, because the first two equations appear damping-free while the source seems to induce damping only to the third equation. As explained in [2], one may escape this predicament by reformulating the system through a state vector transformation in such a manner that the damping is redistributed more equitably among the equations. For the case of (1.4), it is natural to pass from (u, v, w) to the state vector (u, z, ω), where

$$z = v + \frac{1}{2}\phi, \quad \omega = w - \psi, \tag{3.1}$$

with ϕ and ψ defined by (2.6). In terms of the new state vector, (1.4) takes the form

$$\begin{cases} \partial_t u - \partial_x z + \frac{1}{2}u = 0 \\ \partial_t z - \partial_x f(u) + \partial_x \omega + \frac{1}{2}z = \frac{1}{4}\phi \\ \partial_t \omega + f(u) - u + (\mu - 1)\omega = -(\mu - 1)\psi. \end{cases} \tag{3.2}$$

The initial conditions read

(3.3)

$$u(x,0) = u_0(x), z(x,0) = z_0(x) = v_0(x) + \tfrac{1}{2}\phi(x,0), \omega(x,0) = w_0(x) - \psi(x,0).$$

There is now a fair distribution of damping among the equations of (3.2) and thus, following [2], one may estimate the variation of the solution by combining the random choice method with operator splitting. In the process, ϕ and ψ on the right-hand side of (3.2) are treated as known functions, whose contributions to the variation of (u, z, ω) is estimated through their own variation. By virtue of (2.6) and (2.23),

$$\mathrm{TV}\phi(\cdot, t) + \mathrm{TV}\psi(\cdot, t) \le I(t), \qquad 0 \le t < \infty. \tag{3.4}$$

A detailed exposition of the estimation is found in [2,5], so it will suffice to record here the conclusion:

$$\begin{aligned} &\mathrm{TV}u(\cdot, t) + \mathrm{TV}z(\cdot, t) + \mathrm{TV}\omega(\cdot, t) \\ &\quad \le ce^{-\nu t}[\mathrm{TV}u_0(\cdot) + \mathrm{TV}z_0(\cdot) + \mathrm{TV}\omega_0(\cdot)] + c\int_0^t e^{-\nu(t-\xi)} I(\xi) d\xi, \end{aligned} \tag{3.5}$$

for some $\nu > 0$. In terms of the original state vector:

$$
\begin{aligned}
&\mathrm{TV}u(\cdot,t) + \mathrm{TV}v(\cdot,t) + \mathrm{TV}w(\cdot,t) \\
&\quad \leq ce^{-\nu t}[\mathrm{TV}u_0(\cdot) + \mathrm{TV}v_0(\cdot) + \mathrm{TV}w_0(\cdot)] \\
&\qquad + cI(t) + c\int_0^t e^{-\nu(t-\xi)} I(\xi)d\xi.
\end{aligned} \tag{3.6}
$$

Thus, upon using (1.6) and (2.24),

$$
\mathrm{TV}u(\cdot,t) + \mathrm{TV}v(\cdot,t) + \mathrm{TV}w(\cdot,t) \leq c\delta e^{-\nu t} + c\sigma. \tag{3.7}
$$

We finally write the analog of (3.6) for a pair of times $0 \leq \tau < t < \infty$:

$$
\begin{aligned}
&\mathrm{TV}u(\cdot,t) + \mathrm{TV}v(\cdot,t) + \mathrm{TV}w(\cdot,t) \\
&\quad \leq ce^{-\nu(t-\tau)}[\mathrm{TV}u(\cdot,\tau) + \mathrm{TV}v(\cdot,\tau) + \mathrm{TV}w(\cdot,\tau)] \\
&\qquad + cI(t) + c\int_\tau^t e^{-\nu(t-\xi)} I(\xi)d\xi.
\end{aligned} \tag{3.8}
$$

On account of (3.7) and (2.25), given any $\varepsilon > 0$, one can find $0 \leq \tau < \tilde{t} < \infty$, depending on δ, σ and ν, such that the right-hand side of (3.8) becomes less than ε, for all $t \geq \tilde{t}$. We thus arrive at (1.9), and the proof of Theorem 1.1 is now complete.

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