Communications in Information and Systems
Volume 13, Number 2, 123–149, 2013

Two-species flocking particles immersed in a fluid

Young-Pil Choi and Bongsuk Kwon

We present a new particle-fluid model describing the motion of two-species flocking particles immersed in an incompressible viscous fluid. The flocking particles are directly affected by the incompressible fluid through a drag force, and they are also coupled with each other via the fluid. On the other hand, the two-species particles are coupled with each other via the viscous fluid. For this proposed model, we show the global existence of a unique strong solution when the initial data is sufficiently small, and we also investigate the large-time behavior of the solutions under suitable conditions.

1. Introduction

In this paper, we present a new model for the two-species flocking particles interacting with an incompressible fluid. The model consists of the Vlasov-type equations with flocking force terms for the particles and the incompressible Navier-Stokes equations for the fluid. The flocking particles are directly affected by the Navier-Stokes equations through a drag force, and they are also coupled with each other via the fluid. For the proposed model, we first show the existence, uniqueness and regularity of the strong solutions when the initial data is sufficiently small, and investigate the large-time behavior of the solutions using *a priori* estimates.

More specifically, let $f_i = f_i(x, \xi, t)$, $i = 1, 2$ be the one-particle distribution function of the flocking particles at the phase-space position $(x, \xi) \in \Omega \times \mathbb{R}^3$ at time t, and $u = u(x, t)$ be the bulk velocity of the incompressible fluid. Here Ω is either a periodic space $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$ or a whole space $\mathbb{R}^3$. Then the particles and fluid are governed by the following equations:

(1.1)
$$\begin{aligned}
&\partial_t f_1 + \xi \cdot \nabla_x f_1 + \nabla_\xi \cdot \left(\left(F_a^1[f_1] + F_d[u] \right) f_1 \right) = 0, \quad (x, \xi) \in \Omega \times \mathbb{R}^3, \quad t > 0,\\
&\partial_t f_2 + \xi \cdot \nabla_x f_2 + \nabla_\xi \cdot \left(\left(F_a^2[f_2] + F_d[u] \right) f_2 \right) = 0, \quad (x, \xi) \in \Omega \times \mathbb{R}^3, \quad t > 0,\\
&\partial_t u + u \cdot \nabla_x u + \nabla_x p - \mu \Delta_x u\\
&\quad = -\int_{\mathbb{R}^3} F_d[u](f_1 + f_2)\, d\xi, \quad \nabla_x \cdot u = 0, \quad x \in \Omega,\ t > 0,
\end{aligned}$$

subject to initial data:

$$(f_1, f_2, u)|_{t=0} = (f_{10}, f_{20}, u_0), \quad \nabla_x \cdot u_0 = 0 \quad \text{in } \Omega \times \mathbb{R}^3, \tag{1.2}$$

and, in the case of $\Omega = \mathbb{R}^3$, the end state condition for u is imposed:

$$u(x) \to u_\infty \quad \text{as} \quad |x| \to \infty. \tag{1.3}$$

Here F_a^i, $i = 1, 2$, and F_d are the flocking alignment forces and the drag force per unit mass, respectively:

$$F_a^i[f_i](x, \xi, t) := \int_{\Omega \times \mathbb{R}^3} \psi_i(x, y)(\xi_* - \xi) f_i(y, \xi_*, t) d\xi_* dy, \quad i = 1, 2,$$
$$F_d[u](x, \xi, t) := u(x, t) - \xi,$$

where the communication weight function $\psi_i : \Omega \times \Omega \to \mathbb{R}_+$ is a $\mathcal{C}^1$-function satisfying the symmetric and nonnegative conditions:

$$\psi_i(x, y) = \psi_i(y, x), \quad \psi_i \geq 0, \quad i = 1, 2. \tag{1.4}$$

For the communication weight functions, there are various possibilities to adopt. For example, in the case of $\Omega = \mathbb{R}^3$, a regular kernel as in the Cucker-Smale models can be chosen:

$$\psi_i(x, y) = \frac{\alpha_i}{(1 + |x - y|^2)^{\beta_i/2}}, \quad \alpha_i > 0, \quad \beta_i \geq 0, \quad i = 1, 2.$$

Throughout the paper, we assume $\mu = 1$ and $u_\infty = 0$. In fact, a more general condition on the viscosity coefficient $\mu > 0$ does not yield any difficulties for our analysis, and the assumption on the far-field u_∞ is reasonable due to the Galilean invariance for the fluid equations.

The collective behavior of the interaction particle systems such as flocking, aggregation, and synchronization has received a bulk of attention from various research fields arising in physics, biology, robotics, control theory and other disciplines [1, 5, 10, 14, 15, 16, 24, 27, 28, 30]. The interaction between the flocking particles and fluids is first studied in [2]. They considered the kinetic equation for the Cucker-Smale flocking particles coupled with the incompressible Navier-Stokes equations, and showed the global existence of weak solutions. We refer readers to [2] and references therein for a detailed description of the modeling and the related literature. Later in [3] they showed the global strong solutions for sufficiently small and regular initial data, and the large-time behavior of the classical solutions in three space dimensions was obtained under suitable assumptions. Concerning two dimensions case, Choi and Lee [11] established the global existence of weak and strong solutions. Unlike the three dimensions case, the smallness assumption on the initial data has been removed to show the global existence of strong solutions. More recently, the interaction between the Cucker-Smale type flocking particle and the compressible viscous fluid is studied in [4]. For the other related particle interacting with fluid, we refer to [6, 7, 8, 9, 13, 17, 22, 26].

We now extend the previous result for the one-species problems to the case for two particle species. We would like to describe the situation in which more than one type of flocking particle interacting with each other in the fluid. Two-species models have many applications such as pedestrian flows [29], opinion formation between two groups with different leanings [18, 19], and so on. A mathematical study of existence, stability, finite-time blow up, and the large-time behavior for two competitive populations of biological species which are attracted by random diffusion and chemotaxis is another recent active research area [12, 20, 23, 31]. We also refer to [21, 25] for nonlocal interaction PDEs with two-species.

Our first result is concerned with the global existence of a unique strong solution to the system (1.1)-(1.4) when the initial data is sufficiently small and regular. For this purpose, we do not restrict the communication weight function to the specific ones. Rather various alignment forces can be adopted for various physical situations. In particular, as we mentioned before, the regular alignment force as in the Cucker-Smale model can be considered.

Theorem 1.1. *Suppose that the initial data f_{10} and f_{20} have a compact support in position and velocity. For $T \in (0, \infty)$, there exists a positive constant ϵ_0 such that if $\|f_{10}\|_{W^{1,\infty}(\Omega\times\mathbb{R}^3)} + \|f_{20}\|_{W^{1,\infty}(\Omega\times\mathbb{R}^3)} + \|u_0\|_{H^2(\Omega)} < \epsilon_0$, the system (1.1)-(1.4) has a unique strong solution (f_1, f_2, u) satisfying*

$$
\begin{aligned}
&(i) \quad f_1,\, f_2 \in W^{1,\infty}(\Omega \times \mathbb{R}^3 \times (0,T)),\\
&(ii) \quad u \in \mathcal{C}([0,T]; H^2(\Omega)) \cap L^2(0,T; H^3(\Omega)) \quad \textit{and}\\
&\qquad u_t \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)),\\
&(iii)\ p \in \mathcal{C}([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)).
\end{aligned}
$$

Our second result concerns the large-time behavior of the strong solutions to the system (1.1)-(1.4) for the periodic spatial domain, i.e., $\Omega = \mathbb{T}^3$. To this end, we first introduce a total energy-variance function $\mathcal{E}(t)$:

$$
\mathcal{E}(t) := \frac{1}{2}\bigg(\int_{\mathbb{T}^3\times\mathbb{R}^3} |\xi - \xi_c^1|^2 f_1 + |\xi - \xi_c^2|^2 f_2 dx d\xi
$$

$$+\int_{\mathbb{T}^3}|u-u_c|^2dx+\frac{1}{2}|u_c-(\xi_c^1+\xi_c^2)|^2\bigg),$$

where

$$\xi_c^i(t):=\frac{\int_{\mathbb{T}^3\times\mathbb{R}^3}\xi f_i dxd\xi}{\int_{\mathbb{T}^3\times\mathbb{R}^3}f_i dxd\xi}\quad \text{and} \quad u_c(t):=\int_{\mathbb{T}^3}udx.$$

Note that $\xi_c^i(t)$ and $u_c(t)$ are the mean velocities for the i-th particles and the fluid, respectively. For later use, let ρ_{f_i} denote a local particle density:

$$\rho_{f_i}(x,t):=\int_{\mathbb{R}^3}f_i(x,\xi,t)d\xi,\quad i=1,2.$$

In the following theorem, we find out that the system exhibits the exponential alignment between the flocking particles and the fluid.

Theorem 1.2. *Let (f_1,f_2,u) be classical solutions to the system (1.1)-(1.4) satisfying*

(*i*) $\|f_{10}\|_{L^1(\Omega\times\mathbb{R}^3)}=\|f_{20}\|_{L^1(\Omega\times\mathbb{R}^3)}=1,$

(*ii*) $\lim_{|\xi|\to\infty}|\xi|^2\big(f_1(x,\xi,t)+f_2(x,\xi,t)\big)=0,\quad (x,t)\in\mathbb{T}^3\times[0,T).$

Suppose that $\mathcal{E}(0)<\infty$, and $\|\rho_{f_i}\|_{L^{\frac{3}{2}}}$, $i=1,2$ are sufficiently small. Then $\mathcal{E}(t)$ verifies the decay estimate:

$$\mathcal{E}(t)\le\mathcal{E}(0)e^{-ct},\quad t\in[0,T]\quad \textit{for some } c>0.$$

Remark 1.1. *Theorem 1.2 shows that the system exhibits the exponential alignment between the particles and fluid. More precisely, since the total momentum of the system is conserved:*

$$\frac{d}{dt}\left(\int_{\mathbb{T}^3\times\mathbb{R}^3}\xi(f_1+f_2)dxd\xi+\int_{\mathbb{T}^3}udx\right)=0,$$

we obtain

$$\begin{aligned}\big|u_c(t)-(\xi_c^1(t)+\xi_c^2(t))\big|&=2\left|u_c(t)-\frac{1}{2}(\xi_c^1(0)+\xi_c^2(0)+u_c(0))\right|\\&=2\left|\xi_c^1(t)+\xi_c^2(t)-\frac{1}{2}(\xi_c^1(0)+\xi_c^2(0)+u_c(0))\right|.\end{aligned}$$

This implies that if all conditions in Theorem 1.2 hold for $T = \infty$, then the sum of particle-velocity and the fluid-velocity both converge to the half of the initial total momentum of the particles and fluid as time tends to infinity.

The paper is organized as follows. In Section 2, we prove the global existence of the unique strong solution to the nonlinear Cauchy problem of the quasi-linearized problem (1.1)-(1.4). In Section 3, we construct the approximation solutions to the original system and provide the existence of invariant sets for the approximations by employing the results in Section 2. We then derive the convergence of the approximation solutions, and this yields global existence of the unique strong solution. In Section 4, we provide the large-time behavior of classical solutions to the system (1.1)-(1.4). This result implies the system exhibits the exponential alignment between the flocking particles and fluid. Finally, Section 5 is devoted to summarize the main results and give our future work in this direction.

Notations. For a function $f(x,\xi)$, we denote by $\|f\|_{L^p}$ the usual $L^p(\Omega\times\mathbb{R}^3)$-norm, and if g is a function of x only, $\|g\|_{L^p}$ is the usual $L^p(\Omega)$-norm, otherwise specified. For simplicity, we drop x-dependence of differential operators ∂_{x_i} $(i = 1,2,3)$, ∇_x, and Δ_x, i.e.,

$$\partial_i f := \partial_{x_i} f, \quad \nabla f := \nabla_x f, \quad \text{and} \quad \Delta f := \Delta_x f.$$

2. Global existence for the linearized system

In this section, we first linearize the system (1.1) with respect to the fluid velocity u in the drag forces, and show the existence for the linearized system. We also provide uniform boundness of the unique solution.

Consider the linearized system:

(2.1)
$$\begin{aligned}
&\partial_t f_1 + \xi\cdot\nabla f_1 + \nabla_\xi\cdot\left(\left(F_a^1[f_1] + F_d[v]\right)f_1\right) = 0, \quad (x,\xi)\in\Omega\times\mathbb{R}^3, \quad t>0,\\
&\partial_t f_2 + \xi\cdot\nabla f_2 + \nabla_\xi\cdot\left(\left(F_a^2[f_2] + F_d[v]\right)f_2\right) = 0, \quad (x,\xi)\in\Omega\times\mathbb{R}^3, \quad t>0,\\
&\partial_t u + v\cdot\nabla u + \nabla p - \Delta u\\
&\quad = -\int_{\mathbb{R}^3} F_d[v](f_1+f_2)d\xi, \quad \nabla\cdot u = 0, \quad x\in\Omega,\ t>0,
\end{aligned}$$

where v is a known vector field. For this system, we present the existence, smallness, and regularity results as follows.

Theorem 2.1. *Let $T \in (0, \infty)$. Suppose that the initial data (f_{10}, f_{20}, u) satisfy the smallness, regularity and compactly supported conditions:*

$$(2.2) \qquad \begin{aligned} &(i)\ \|f_{10}\|_{W^{1,\infty}} + \|f_{20}\|_{W^{1,\infty}} + \|u_0\|_{H^2} < \varepsilon, \\ &(ii)\ f_{10} \text{ and } f_{20} \text{ have a compact support in position and velocity.} \end{aligned}$$

Furthermore, v satisfies the smallness and regularity conditions:

(2.3)

$$\|v\|_{\mathcal{C}([0,T];H^2)} + \|v\|_{L^2(0,T;H^3)} \le \varepsilon^{\alpha} \quad \text{and} \quad \|v_t\|_{\mathcal{C}([0,T];L^2)} + \|v_t\|_{L^2(0,T;H^1)} \le \varepsilon^{\alpha^-},$$

where $\alpha^- := \alpha - \varepsilon_1$ for sufficiently small $\varepsilon_1 > 0$. Then there exists a unique strong solution (f_1, f_2, u) to the Cauchy problem (2.1), (1.2)-(1.4) such that

$$(2.4) \qquad \begin{aligned} &(i)\ \|f_1\|_{L^\infty(0,T;W^{1,\infty})},\ \|f_2\|_{L^\infty(0,T;W^{1,\infty})} \le \varepsilon^{\beta}, \\ &(ii)\ \|u\|_{\mathcal{C}([0,T];H^2)} + \|u\|_{L^2(0,T;H^3)} \le \varepsilon^{\alpha} \quad \text{and} \\ &\qquad \|u_t\|_{\mathcal{C}([0,T];L^2)} + \|u_t\|_{L^2(0,T;H^1)} \le \varepsilon^{\alpha^-}. \end{aligned}$$

Here α and β are positive numbers with $1 > \beta > \alpha > 0$, and $\varepsilon > 0$ is a sufficiently small constant such that $\varepsilon \approx e^{-\mathcal{O}(1)T}$.

We first solve the Vlasov-type equations in (2.1). We notice that the theory of local existence and regularity of a unique strong solution to the system $(2.1)_1$ and $(2.1)_2$ have been well-known when $v \in \mathcal{C}([0,T];H^2)$. For the estimate of uniform bound of f_i in (2.4), we need to control the propagation of support of f_i in velocity. For this, we introduce new notations here. Let $\Sigma_x^i(t)$ and $\Sigma_\xi^i(t)$ be the x, ξ-projections of $\mathrm{supp} f_i(\cdot,\cdot,t)$, respectively, $i = 1, 2$:

$$\begin{aligned} \Sigma_x^i(t) &:= \left\{ x \in \Omega \ : \ \exists\, (x,\xi) \in \Omega \times \mathbb{R}^3 \quad \text{such that} \quad f_i(x,\xi,t) \neq 0 \right\}, \\ \Sigma_\xi^i(t) &:= \left\{ \xi \in \mathbb{R}^3 \ : \ \exists\, (x,\xi) \in \Omega \times \mathbb{R}^3 \quad \text{such that} \quad f_i(x,\xi,t) \neq 0 \right\}. \end{aligned}$$

Then we define $R_x^i(t)$ and $R_\xi^i(t)$ by

$$R_x^i(t) := \max_{x \in \Sigma_x^i(t)} |x|, \quad R_\xi^i(t) := \max_{\xi \in \Sigma_\xi^i(t)} |\xi|.$$

We now present the estimates for the support of f_i in position and velocity.

Lemma 2.1. *For $i = 1, 2$, let $(X_i(s), V_i(s)) := (X_i(s; 0, x, \xi), V_i(s; 0, x, \xi))$*

be the forward particle trajectories solving the following ODEs:

$$\begin{aligned}\frac{dX_i(s)}{ds} &= V_i(s),\\ \frac{dV_i(s)}{ds} &= \int_{\Omega\times\mathbb{R}^3} \psi_i\left(X_i(s),y\right)(\xi_* - V_i(s)) f_i(y,\xi_*,s)dyd\xi_*\\ &\quad + v(X_i(s),s) - V_i(s)\end{aligned}$$

with initial data $(X_i(0),V_i(0)) = (x,\xi)$. *Then we have*

$$\begin{aligned}|X_i(t)| &\le |R_x^i(0)| + \left(|R_\xi^i(0)| + \|v\|_{L^1(0,T;L^\infty)}\right)T,\\ |V_i(t)| &\le |R_\xi^i(0)| + \|v\|_{L^1(0,T;L^\infty)}, \qquad i=1,2.\end{aligned}$$

Proof. From the regularity results for f_i and v, we find that $R_\xi^i(t)$ is a Lipschitz function and differentiable with respect to time t almost everywhere. This enable us to choose $V_i(t)$ so that $\frac{d}{dt}R_\xi^i(t)$ exists and $R_\xi^i(t) = |V_i(t)|$. Then we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(R_\xi^i(t)\right)^2 = \frac{1}{2}\frac{d}{dt}|V_i(t)|^2 = V_i(t)\cdot\frac{dV_i(t)}{dt}\\ &\quad = \int_{\Omega\times\mathbb{R}^3}\psi_i\left(X_i(t),y\right)(\xi_* - V_i(t))\cdot V_i(t) f_i(y,\xi_*,t)dyd\xi_*\\ &\quad + v\left(X_i(t),t\right)\cdot V_i(t) - |V_i(t)|^2\\ &\quad \le |v(t)|_{L^\infty}|V_i(t)| - |V_i(t)|^2,\end{aligned}$$

where we used

$$(\xi_* - V_i(t))\cdot V_i(t) \le 0 \quad \text{for} \quad \xi_* \in \Sigma_\xi^i(t).$$

This implies

$$|R_\xi^i(t)| \le |R_\xi^i(0)| + \|v\|_{L^1(0,T;L^\infty)},$$

and

$$|R_x^i(t)| \le |R_x^i(0)| + |R_\xi^i(0)|T + \|v\|_{L^1(0,T;L^\infty)}T. \qquad \square$$

For notational simplicity, we set

$$R_x^{i,\infty} := \sup_{0\le t\le T} R_x^i(t), \quad R_\xi^{i,\infty} := \sup_{0\le t\le T} R_\xi^i(t), \quad \pi(R_x^{i,\infty}) := \mathrm{vol}(B_{R_x^{i,\infty}}),$$

and

$$\pi(R_\xi^{i,\infty}) := \mathrm{vol}(B_{R_\xi^{i,\infty}}), \quad \text{for} \quad i=1,2.$$

We now establish the estimate of uniform bound of f_i in $W^{1,\infty}$-norm. For this, we first present the following simple calculations without proof.

Lemma 2.2. *For $i = 1, 2$, let f_i be classical solutions to the system (2.1), (1.4) with compactly supported initial data f_{i0} in velocity. Suppose that v satisfies (2.3). Then the following estimates hold.*

$$\begin{aligned}
(i)\quad & -\nabla_\xi \cdot \left(F_a^i[f_i] + F_d[v]\right) = 3\int_{\Omega\times\mathbb{R}^3} \psi_i(x,y) f_i(y,\xi_*)\,dyd\xi_* - 3 \\
&\qquad \leq 3\|\psi_i\|_{L^\infty} M_{i0}, \\
(ii)\quad & -\partial_j \left(F_a^i[f_i] + F_d[v]\right) = -\int_{\Omega\times\mathbb{R}^3} \partial_j\psi_i(x,y)(\xi_* - \xi) f_i(y,\xi_*)\,dyd\xi_* - \partial_j v \\
&\qquad \leq 2R_\xi^{i,\infty}\|\partial_j\psi_i\|_{L^\infty} M_{i0} + \|\partial_j v\|_{L^\infty}, \\
(iii)\quad & -\nabla_\xi \cdot \partial_j \left(F_a^i[f_i] + F_d[v]\right) = 3\int_{\Omega\times\mathbb{R}^3} \partial_j\psi_i(x,y) f_i(y,\xi_*)\,dyd\xi_* \\
&\qquad \leq 3\|\partial_j\psi_i\|_{L^\infty} M_{i0},
\end{aligned}$$

where $M_{i0} := \int_{\Omega\times\mathbb{R}^3} f_{i0}(x,\xi)\,dxd\xi < \infty$.

Lemma 2.3. *For $i = 1, 2$, let f_i be classical solutions to the system (2.1), (1.4) with the initial data f_{i0} satisfying (2.2). If v satisfies the smallness conditions (2.3), then we have*

$$\|f_i\|_{L^\infty(0,T;W^{1,\infty})} < \varepsilon^\beta, \quad i = 1, 2.$$

Proof. Similarly as the arguments in [2], we introduce a nonlinear operator $\mathcal{N}_i := \partial_t + \xi\cdot\nabla + (F_a^i[f_i] + F_d[v])\cdot\nabla_\xi$ which is associated with f_i. Then we employ the estimates in Lemma 2.2 to obtain

$$\begin{aligned}
\mathcal{N}_i(f_i) &= -\nabla_\xi\cdot\left(F_a^i[f_i] + F_d[v]\right) f_i \leq 3\|\psi_i\|_{L^\infty} M_{i0}, \\
\mathcal{N}_i(\partial_j f_i) &= -\partial_j\left(F_a^i[f_i] + F_d[v]\right)\cdot\nabla_\xi f_i - \left(\nabla_\xi\cdot\partial_j\left(F_a^i[f_i] + F_d[v]\right)\right) f_i \\
&\quad - \left(\nabla\cdot\left(F_a^i[f_i] + F_d[v]\right)\right)\partial_j f_i \\
&\leq \left(2R_\xi^i(t)\|\psi_i\|_{L^\infty} M_{i0} + \|\partial_j v\|_{L^\infty}\right)|\nabla_\xi f_i| + 3\|\partial_j\psi_i\|_{L^\infty} M_{i0}|f_i| \\
&\quad + 3\left(\|\psi_i\|_{L^\infty} M_{i0} + 1\right)|\partial_j f_i|, \\
\mathcal{N}_i(\partial_{\xi_j}(f_i)) &= -\partial_j f_i - \partial_{\xi_j}\left(F_a^i[f_i] + F_d[v]\right)\cdot\nabla_\xi f_i \\
&\quad - \left(\nabla_\xi\cdot\left(F_a^i[f_i] + F_d[v]\right)\right)\partial_{\xi_j} f_i \\
&\leq |\partial_j f_i| + \left(\|\psi_i\|_{L^\infty} M_{i0} + 1\right)|\nabla_\xi f_i| + 3\left(\|\psi_i\|_{L^\infty} M_{i0} + 1\right)|\nabla_{\xi_j} f_i|,
\end{aligned}$$

for $i = 1, 2$. We set $\mathcal{F}_i(t)$ measuring f in $W^{1,\infty}$-norm:

$$\mathcal{F}_i(t) := \sum_{0 \le |\alpha| + |\beta| \le 1} \left\| \nabla^\alpha \nabla_\xi^\beta f_i(t) \right\|_{L^\infty}.$$

Then using the previous estimates, we obtain

$$\frac{d\mathcal{F}_i(t)}{dt} \le C \left(1 + \|\nabla v\|_{L^\infty}\right) \mathcal{F}_i(t), \quad t \in (0, T).$$

This yields

$$\mathcal{F}_i(t) \le \mathcal{F}_i(0) \exp\left(C\left(T + \sqrt{T}\|v\|_{L^2(0,T;H^3)}\right)\right), \quad t \in (0, T).$$

Since $1 > \beta > \alpha$, for the sufficiently small $\varepsilon \ll 1$, we obtain

$$\|f_i\|_{W^{1,\infty}} \le \varepsilon \exp\left(C\left(T + \sqrt{T}\varepsilon^\alpha\right)\right) < \varepsilon^\beta.$$

This yields that for $i = 1, 2$,

$$\sup_{0 \le t \le T} \|f_i\|_{W^{1,\infty}} \le \varepsilon^\beta. \qquad \square$$

Remark 2.1. *From the structure of Vlasov-type equations, one can easily check that*

$$f_i \in W^{1,\infty}(\Omega \times \mathbb{R}^3 \times (0, T)), \quad i = 1, 2.$$

Proof of Theorem 2.1. We first notice that $f_1, f_2 \in W^{1,\infty}(\Omega \times \mathbb{R}^3 \times (0, T))$, $v \in \mathcal{C}([0, T]; H^2) \cap L^2(0, T; H^3)$, and $v_t \in \mathcal{C}([0, T]; L^2) \cap L^2(0, T; H^1)$. Thus the existence and regularity of the unique solution u can be proved by a standard method. Then we obtain the estimates of uniform bounds in (2.4). This proof is a rather lengthy, so we divide it into five steps.

- Step A.- Estimate of $\|u\|_{L^\infty(0,T;L^2)} + \|\nabla u\|_{L^2(0,T;L^2)}$: It follows from $(2.1)_3$ that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 &= -\int_\Omega (v \cdot \nabla u) \cdot u dx \\ &\quad - \int_{\Omega \times \mathbb{R}^3} (v \cdot u - u \cdot \xi)(f_1 + f_2) dx d\xi \\ &=: I_1 + I_2, \end{aligned} \tag{2.5}$$

where $I_j, j = 1, 2$ are estimated as follows.

$$I_1 = \frac{1}{2}\int_\Omega (\nabla \cdot v)\,|u|^2 dx \le \frac{1}{2}\|\nabla v\|_{L^3}\|u\|_{L^6}\|u\|_{L^2} \le \frac{1}{2}\|\nabla v\|_{H^1}\|\nabla u\|_{L^2}\|u\|_{L^2}$$
$$\le \varepsilon^\alpha \|\nabla u\|_{L^2}\|u\|_{L^2}$$

and

(2.6)

$$\begin{aligned} I_2 &\le \|v\|_{L^2}\|u\|_{L^2}\left\|\int_{\mathbb{R}^3} f_1 + f_2\, d\xi\right\|_{L^\infty} + \|u\|_{L^2}\left\|\int_{\mathbb{R}^3} \xi(f_1+f_2) d\xi\right\|_{L^2} \\ &\le \left(\sum_{j=1}^2 \|f_i\|_{L^\infty}\pi(R_\xi^{i,\infty})\right)\|v\|_{L^2}\|u\|_{L^2} \\ &\quad + \left(\sum_{j=1}^2 \|f_i\|_{L^\infty} R_\xi^{i,\infty}\pi(R_\xi^{i,\infty})\pi(R_x^{i,\infty})\right)\|u\|_{L^2}. \end{aligned}$$

Here we used

$$\left\|\int_{\mathbb{R}^3} f_i\, d\xi\right\|_{L^\infty} \le \|f_i\|_{L^\infty}\pi(R_\xi^{i,\infty}) \quad \text{and}$$
$$\left\|\int_{\mathbb{R}^3} \xi f_i\, d\xi\right\|_{L^2} \le \|f_i\|_{L^\infty} R_\xi^{i,\infty}\pi(R_\xi^{i,\infty})\pi(R_x^{i,\infty}),$$

for $i = 1, 2$. Combining (2.5) and (2.6), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 &\le C\|u\|_{L^2}\left(\varepsilon^\alpha\|\nabla u\|_{L^2} + \varepsilon^{\alpha+\beta} + \varepsilon^\beta\right) \\ &\le \frac{1}{2}\|u\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2 + C\varepsilon^{2\beta}. \end{aligned}$$

Applying the Gronwall's inequality for $\|u\|_{L^2}^2$ and integrating over $[0, T]$, we obtain

(2.7)

$$\sup_{0\le t\le T}\|u\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2 dt \le C\Big(\|u_0\|_{L^2}^2 + \varepsilon^{2\beta}\Big)e^T \le C\Big(\varepsilon^2 + \varepsilon^{2\beta}\Big)e^T \le \varepsilon^{2\alpha}.$$

- Step B.- Estimate of $\|\nabla u\|_{L^\infty(0,T;L^2)} + \|\nabla^2 u\|_{L^2(0,T;L^2)}$: We differentiate

$(2.1)_3$ with respect to x and integrate over Ω to find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \\ &= -\int_\Omega \nabla(v\cdot\nabla u)\cdot\nabla u dx - \int_{\Omega\times\mathbb{R}^3}\nabla\left(F_d[v](f_1+f_2)\right)\cdot\nabla u dx d\xi \\ &=: J_1 + J_2.\end{aligned}$$

We estimate $J_j, j=1,2$, as follows:

$$\begin{aligned}J_1 &\le \int_\Omega |\nabla v||\nabla u|^2 dx + \int_\Omega |v||\nabla^2 u||\nabla u| dx \\ &\le \|\nabla v\|_{L^3}\|\nabla u\|_{L^6}\|\nabla u\|_{L^2} + \|v\|_{L^\infty}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2} \\ &\le C\varepsilon^\alpha \|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}J_2 &\le \int_{\Omega\times\mathbb{R}^3}|\nabla v||\nabla u||f_1+f_2| dx d\xi + \int_{\Omega\times\mathbb{R}^3}|v-\xi||\nabla u||\nabla f_1 + \nabla f_2| dx d\xi \\ &\le C\left(\sum_{i=1}^2 \|f_i\|_{L^\infty}\pi(R_\xi^{i,\infty})\right)\|\nabla v\|_{L^2}\|\nabla u\|_{L^2} \\ &\quad + C\left(\sum_{i=1}^2 \|\nabla f_i\|_{L^\infty}\pi(R_\xi^{i,\infty})\right)\|v\|_{L^2}\|\nabla u\|_{L^2} \\ &\quad + C\left(\sum_{i=1}^2 \|\nabla f_i\|_{L^\infty}R_\xi^{i,\infty}\pi(R_\xi^{i,\infty})\pi(R_x^{i,\infty})^{\frac{1}{2}}\right)\|\nabla u\|_{L^2} \\ &\le C\varepsilon^\beta\|\nabla u\|_{L^2}.\end{aligned}$$

Then we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 &\le C\varepsilon^\alpha\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2} + C\varepsilon^\beta\|\nabla u\|_{L^2} \\ &\le \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|\nabla^2 u\|_{L^2}^2 + C\varepsilon^{2\beta},\end{aligned}$$

and this implies that

$$\begin{aligned}\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2 + \int_0^T \|\nabla^2 u\|_{L^2}^2 dt &\le C\left(\|\nabla u_0\|_{L^2}^2 + \varepsilon^{2\beta}\right)e^T \\ &\le C\left(\varepsilon^2 + \varepsilon^{2\beta}\right)e^T \le \varepsilon^{2\alpha}.\end{aligned} \tag{2.8}$$

• Step C.- Estimate of $\|\nabla^2 u\|_{L^\infty(0,T;L^2)} + \|\nabla^3 u\|_{L^2(0,T;L^2)}$: We take the spatial differential operator ∇^2 to the system $(2.1)_3$ and integrate over Ω to the following equality.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 \\
&\quad = -\int_\Omega \nabla^2(v\cdot\nabla u) : \nabla^2 u \, dx - \int_{\Omega\times\mathbb{R}^3} \nabla^2 \left(F_d[v](f_1+f_2)\right) : \nabla^2 u \, dxd\xi \\
&\quad =: K_1 + K_2,
\end{aligned}$$

where $K_j, j = 1,2$ are estimated as

$$\begin{aligned}
K_1 &\le C\int_\Omega |\nabla^2 v||\nabla u||\nabla^2 u| dx + C\int_\Omega |\nabla v||\nabla^2 u||\nabla^2 u| dx \\
&\quad + C\int_\Omega |v||\nabla^3 u||\nabla^2 u| dx \\
&\le C\|\nabla^2 v\|_{L^2}\|\nabla u\|_{L^3}\|\nabla^2 u\|_{L^6} + C\|\nabla v\|_{L^3}\|\nabla^2 u\|_{L^6}\|\nabla^2 u\|_{L^2} \\
&\quad + C\|v\|_{L^\infty}\|\nabla^3 u\|_{L^2}\|\nabla^2 u\|_{L^2} \\
&\le C\varepsilon^\alpha \left(\|\nabla u\|_{L^2}^{\frac{1}{2}} + \|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\right)\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2} \\
&\le \frac{1}{4}\|\nabla^2 u\|_{L^2}^2 + \frac{1}{4}\|\nabla^3 u\|_{L^2}^2 + C\varepsilon^{6\alpha},
\end{aligned}$$

and

$$\begin{aligned}
K_2 &\le C\int_{\Omega\times\mathbb{R}^3} |\nabla^2 v||f_1+f_2||\nabla^2 u| dxd\xi + \int_{\Omega\times\mathbb{R}^3} |\nabla v||\nabla f_1 + \nabla f_2||\nabla^2 u| dxd\xi \\
&\quad + \int_{\Omega\times\mathbb{R}^3} |v-\xi||\nabla f_1 + \nabla f_2||\nabla^3 u| dxd\xi \\
&\le C\left(\sum_{i=1}^2 \|f_i\|_{L^\infty}\pi(R_\xi^{i,\infty})\right)\|\nabla^2 v\|_{L^2}\|\nabla^2 u\|_{L^2} \\
&\quad + C\left(\sum_{i=1}^2 \|\nabla f_i\|_{L^\infty}\pi(R_\xi^{i,\infty})\right)\|\nabla v\|_{L^2}\|\nabla^2 u\|_{L^2} \\
&\quad + C\left(\sum_{i=1}^2 \|\nabla f_i\|_{L^\infty}\pi(R_\xi^{i,\infty})\left(\|v\|_{L^2} + R_\xi^{i,\infty}\left(\pi(R_x^{i,\infty})\right)^{\frac{1}{2}}\right)\right)\|\nabla^3 u\|_{L^2} \\
&\le C\varepsilon^{\alpha+\beta}\|\nabla^2 u\|_{L^2} + C\varepsilon^\beta(1+\varepsilon^\alpha)\|\nabla^3 u\|_{L^2} \\
&\le \frac{1}{4}\|\nabla^2 u\|_{L^2}^2 + \frac{1}{4}\|\nabla^3 u\|_{L^2}^2 + C\varepsilon^{2\beta}.
\end{aligned}$$

This yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 \le \frac{1}{2}\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + C\varepsilon^{6\alpha} + C\varepsilon^{2\beta},$$

and

$$\begin{aligned} \sup_{0\le t\le T}\|\nabla^2 u\|_{L^2}^2 + \int_0^T \|\nabla^3 u\|_{L^2}^2 dt &\le C\left(\|\nabla^2 u_0\|_{L^2}^2 + \varepsilon^{2\beta} + \varepsilon^{4\alpha}\right)e^T \\ &\le C\left(\varepsilon^2 + \varepsilon^{2\beta} + \varepsilon^{4\alpha}\right)e^T \\ &\le \varepsilon^{2\alpha}. \end{aligned} \tag{2.9}$$

Combining the estimates (2.7), (2.8), and (2.9), we arrive at

$$\|u\|_{L^\infty(0,T;H^2)} + \|\nabla u\|_{L^2(0,T;H^2)} \le \varepsilon^\alpha. \tag{2.10}$$

• Step D.- Estimate of $\|u_t\|_{L^\infty(0,T;L^2)} + \|u_t\|_{L^2(0,T;H^1)}$: We first multiply $(2.1)_3$ by $\partial_t u$ and integrate over Ω to find

$$\begin{aligned} \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{d}{dt}\int_\Omega |\nabla u|^2 dx &\le \int_\Omega |v|^2|\nabla u|^2 dx \\ &\quad + \int_\Omega \left(\int_{\mathbb{R}^3} |v-\xi|(f_1+f_2)d\xi\right)^2 dx \\ &\le C\left(\varepsilon^{4\alpha} + \varepsilon^{2\beta}(1+\varepsilon^{2\alpha})\right). \end{aligned} \tag{2.11}$$

Then we get

$$\int_0^T \int_\Omega |u_t|^2 dxdt \le \int_\Omega |\nabla u_0|^2 dx + C\left(\varepsilon^{4\alpha} + \varepsilon^{2\beta}\right)T \le \varepsilon^{2\alpha}.$$

In order to derive the higher regularity estimates, we next differentiate $(2.1)_3$ with respect to t to obtain

(2.12)

$$u_{tt} + v\cdot\nabla u_t + \nabla p_t - \Delta u_t = -v_t\cdot\nabla u - \int_{\mathbb{R}^3} v_t(f_1+f_2)\,d\xi - \int_{\mathbb{R}^3}(v-\xi)(f_1+f_2)_t\,d\xi.$$

Multiplying (2.12) by u_t and integrating over Ω, one can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2dx + \int_\Omega |\nabla u_t|^2 dx \\
&\quad \le \|u_t\|_{L^2}\|v_t\|_{L^2}\|\nabla u\|_{L^\infty} + \|u_t\|_{L^2}\|v\|_{L^\infty}\|\nabla u_t\|_{L^2} \\
&\qquad + \|u_t\|_{L^2}\|v_t\|_{L^2}\left(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty}\right) \\
&\qquad + C\|\nabla u_t\|_{L^2}\left(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty}\right)\left(\|v\|_{L^2}+1\right) \\
&\qquad + C\|u_t\|_{L^2}\|\nabla v\|_{L^2}\left(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty}\right) \\
&\qquad + C\|u_t\|_{L^2}\left(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty}\right) \\
&\qquad + C\|u_t\|_{L^2}\left(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty}\right)\left(\|v\|_{L^2}+1\right) \\
&\quad \le C\|u_t\|_{L^2}^2 + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + C\varepsilon^{2\alpha}\|\nabla u\|_{L^\infty}^2 + C\varepsilon^{2\beta}.
\end{aligned}$$

This implies

$$\int_\Omega |u_t|^2(t)dx + \int_s^t\int_\Omega |\nabla u_t|^2 dx d\tau \le \int_\Omega |u_t|^2(s)dx e^{CT} + C\left(\varepsilon^{4\alpha}+\varepsilon^{2\beta}\right)e^{CT}. \tag{2.13}$$

On the other hand, similarly as in (2.11), one can have that

$$\begin{aligned}
\int_\Omega |u_t|^2 dx &= \int_\Omega \left(\int_{\mathbb{R}^3}(\xi - v)(f_1+f_2)d\xi - v\cdot\nabla u + \Delta u\right)\cdot u_t dx \\
&\le C\int_\Omega\left(\int_{\mathbb{R}^3}(\xi - v)(f_1+f_2)\right)^2 dx + C\int_\Omega |v|^2|\nabla u|^2 dx \\
&\quad + C\int_\Omega |\nabla^2 u|^2 dx \\
&\le C\left(\varepsilon^{2\alpha}+\varepsilon^{2\beta}\right) \le C\varepsilon^{2\alpha}.
\end{aligned} \tag{2.14}$$

Hence we combine (2.13) and (2.14) to have

$$\begin{aligned}
\sup_{0\le t\le T}\int_\Omega |u_t|^2 dx + \int_0^T\int_\Omega |\nabla u_t|^2 dxdt &\le \limsup_{s\to 0}\int_\Omega |u_t|^2(s)dx e^{CT} \\
&\quad + C\left(\varepsilon^{4\alpha}+\varepsilon^{2\beta}\right) \\
&\le C\varepsilon^{2\alpha} \le \varepsilon^{2\alpha^-},
\end{aligned}$$

and this concludes

$$\|u_t\|_{L^\infty(0,T;L^2)} + \|u_t\|_{L^2(0,T;H^1)} \le \varepsilon^{2\alpha^-}.$$

• Step E.- $(u, u_t) \in \mathcal{C}([0,T]; H^2(\Omega)) \times \mathcal{C}([0,T]; L^2(\Omega))$: From Step C and D, we find that

$$u \in L^2(0,T; H^3(\Omega)) \quad \text{and} \quad u_t \in L^2(0,T; H^1(\Omega)).$$

Thus, by using a standard Sobolev embedding, we obtain $u \in \mathcal{C}([0,T]; H^2(\Omega))$. Then by this continuity of u in $H^2(\Omega)$, we deduce from the momentum equations that $u_t \in \mathcal{C}([0,T]; L^2(\Omega))$. This completes the proof. □

3. Proof of Theorem 1.1

In this section, we give the proof of our first main result. For this, we consider the following approximated sequences:

(3.1)
$$\begin{aligned}
&\partial_t f_1^{n+1} + \xi \cdot \nabla f_1^{n+1} + \nabla_\xi \cdot \big((F_a^1[f_1^{n+1}] + F_d[u^n]) f_1^{n+1} \big) = 0,\\
&\quad (x,\xi) \in \Omega \times \mathbb{R}^3, \quad t > 0,\\
&\partial_t f_2^{n+1} + \xi \cdot \nabla f_2^{n+1} + \nabla_\xi \cdot \big((F_a^2[f_2^{n+1}] + F_d[u^n]) f_2^{n+1} \big) = 0,\\
&\quad (x,\xi) \in \Omega \times \mathbb{R}^3, \quad t > 0,\\
&\partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} + \nabla p^{n+1} - \Delta u^{n+1} = -\int_{\mathbb{R}^3} F_d[u^n](f_1^{n+1} + f_2^{n+1})\, d\xi,\\
&\quad x \in \Omega,\ t > 0, \quad \nabla \cdot u^{n+1} = 0, \quad x \in \Omega,\ t > 0,
\end{aligned}$$

with initial data and first iterate:

$$\begin{aligned}
(f_{10}^n(x,\xi), f_{20}^n(x,\xi), u_0^n(x)) &= (f_{10}(x,\xi), f_{20}(x,\xi), u_0(x)),\\
&\quad (x,\xi) \in \Omega \times \mathbb{R}^3, \quad n \geq 1,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
(f_1^0(x,\xi,t), f_2^0(x,\xi,t), u^0(x,t)) &= (f_{10}(x,\xi), f_{20}(x,\xi), u_0(x)),\\
&\quad (x,\xi,t) \in \Omega \times \mathbb{R}^3 \times [0,T).
\end{aligned} \tag{3.3}$$

Then the following proposition is an immediate consequence of Theorem 2.1.

Proposition 3.1. *Suppose that the initial data (f_{10}, f_{20}, u) satisfies (2.2). Then there exists a unique solution (f_1^n, f_2^n, u^n) to the system (3.1)-(3.3)* *such that*

$$\begin{aligned}&(i)\ \ \|f_1^n\|_{L^\infty(0,T;W^{1,\infty})},\ \|f_2^n\|_{L^\infty(0,T;W^{1,\infty})} \le \varepsilon^\beta,\\ &(ii)\ \|u^n\|_{\mathcal{C}([0,T];H^2)} + \|u^n\|_{L^2(0,T;H^3)} \le \varepsilon^\alpha \quad \textit{and}\\ &\qquad \|u_t^n\|_{\mathcal{C}([0,T];L^2)} + \|u_t^n\|_{L^2(0,T;H^1)} \le \varepsilon^{\alpha^-},\end{aligned}$$

for all $n \ge 1$.

Proof. Since the smallness and regularity condition on the initial data (f_{10}, f_{20}, u_0) are assumed, our iteration scheme is well-defined by Theorem 2.1. □

We now provide the strong convergence of the approximated solutions $(f_1^n, f_2^n, u^n)_{n\ge1}$.

Lemma 3.1. *Let* (f_1^n, f_2^n, u^n) *be the solution to the system* (3.1)-(3.3) *obtained from Proposition* (3.1). *Then the approximate solutions* (f_1^n, f_2^n, u^n) *is Cauchy in* $L^\infty(\Omega\times\mathbb{R}^3\times(0,T))\times L^\infty(\Omega\times\mathbb{R}^3\times(0,T))\times L^\infty(0,T;H^1(\Omega))$.

Proof. These estimates are quite similar to the ones in the proof of Theorem 2.1. We postpone its proof to Appendix A. □

Proof of Theorem 1.1. ⋄ *Existence.*- From Lemma 3.1, we obtain that (f_1^n, $f_2^n)_{n\ge1}$ and $(u^n)_{n\ge1}$ are Cauchy sequences in $L^\infty(\Omega\times\mathbb{R}^3\times(0,T))$ and $L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))$, respectively. Then there exist the limit functions (f_1, f_2, u) such that

$$f_1^n \to f_1,\ \ f_2^n \to f_2 \quad \text{in } L^\infty(\Omega\times\mathbb{R}^3\times(0,T)),$$

and

$$u^n \to u \quad \text{in } L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega)).$$

Clearly, (f_1, f_2, u) is a weak solution to the system (1.1)-(1.4). Thus in order to complete the proof of existence, it remains to show the limit functions (f_1, f_2, u) are actually in $L^\infty(0,T;W^{1,\infty}(\Omega\times\mathbb{R}^3))\times L^\infty(0,T;W^{1,\infty}(\Omega\times\mathbb{R}^3))\times\mathcal{C}([0,T];H^2(\Omega))$. We briefly give our strategy for this proof.

- $(f_1, f_2, u) \in L^\infty(0,T;W^{1,\infty}(\Omega\times\mathbb{R}^3))\times L^\infty(0,T;W^{1,\infty}(\Omega\times\mathbb{R}^3))\times L^\infty(0,T;H^2(\Omega))$: We first notice from the estimates of uniform bounds in Proposition 3.1 that for each $t\in[0,T]$ there exists a convergent subsequence $(f_1^{n_k}, f_2^{n_k}, u^{n_k})$ such that

$$(f_1^{n_k}(t), f_2^{n_k}(t), u^{n_k}(t)) \rightharpoonup (\bar f_1(t), \bar f_2(t), \bar u(t)) \quad \text{as} \quad k\to\infty,$$

for some $(\bar{f}_1(t), \bar{f}_2(t), \bar{u}(t)) \in W^{1,\infty}(\Omega \times \mathbb{R}^3) \times W^{1,\infty}(\Omega \times \mathbb{R}^3) \times H^2(\Omega)$. On the other hand, the convergence-estimates in Lemma 3.1 yield that

$$(f_1^{n_k}(t), f_2^{n_k}(t), u^{n_k}(t)) \to (f_1(t), f_2(t), u(t))$$
$$\text{in } L^\infty(\Omega \times \mathbb{R}^3) \times L^\infty(\Omega \times \mathbb{R}^3) \times H^1(\Omega),$$

as $k \to \infty$. Hence we have

$$(\bar{f}_1(t), \bar{f}_2(t), \bar{u}(t)) \equiv (f_1(t), f_2(t), u(t))$$
$$\text{in } W^{1,\infty}(\Omega \times \mathbb{R}^3) \times W^{1,\infty}(\Omega \times \mathbb{R}^3) \times H^2(\Omega),$$

for each $t \in [0, T]$.

- $(u, u_t) \in \mathcal{C}([0,T]; H^2(\Omega)) \times \mathcal{C}([0,T]; L^2(\Omega))$: From the previous step, we have the existence of $u \in L^\infty(0,T; H^2(\Omega)) \cap L^2(0,T; H^3(\Omega))$. Then it follows from the momentum equations $(1.1)_3$ that $u_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$. We now apply the same arguments in Step E in the proof of Theorem 2.1 to have the desired regularity.

$\diamond$ *Uniqueness.-* Let (f_1, f_2, u) and $(\bar{f}_1, \bar{f}_2, \bar{u})$ be the two strong solutions obtained in the part of existence proof with the same initial data. We set the differences between two solutions:

$$\Delta(t) := \|f_1 - \bar{f}_1\|_{L^\infty}^2 + \|f_2 - \bar{f}_2\|_{L^\infty}^2 + \|u - \bar{u}\|_{H^1}^2.$$

Then it follows from the estimate in Appendix A that

$$\Delta(t) \le C \int_0^t \Delta(s) ds, \quad \Delta(0) = 0.$$

This yields $\Delta(t) = 0$ for all time $t \in [0, T]$, i.e.,

$$f_i \equiv \bar{f}_i \quad \text{in } L^\infty(\Omega \times \mathbb{R}^3 \times (0,T)) \quad \text{and} \quad u \equiv \bar{u} \quad \text{in } \mathcal{C}([0,T]; H^1(\Omega)).$$

Hence we easily conclude

$$f_i \equiv \bar{f}_i \quad \text{in } W^{1,\infty}(\Omega \times \mathbb{R}^3 \times (0,T)) \quad \text{and} \quad u \equiv \bar{u} \quad \text{in } \mathcal{C}([0,T]; H^2(\Omega)).$$

This completes the proof. $\square$

4. Large-time behavior of solutions

In this section, we explore the large-time behavior of solutions. In particular, we consider the periodic spatial domain $\mathbb{T}^3$.

Lemma 4.1. *Let (f_1, f_2, u) be the classical solutions to the system (1.1) satisfying*

(i) $\|f_{10}\|_{L^1},\ \|f_{20}\|_{L^1} < \infty$.
(ii) $\lim_{|\xi|\to\infty} |\xi|^2\big(f_1(x,\xi,t) + f_2(x,\xi,t)\big) = 0, \quad (x,t) \in \mathbb{T}^3 \times [0,T)$.

Then we have

$$(i)\quad \int_{\mathbb{T}^3\times\mathbb{R}^3} f_i(x,\xi,t)\,dxd\xi = \int_{\mathbb{T}^3\times\mathbb{R}^3} f_{i0}(x,\xi)\,dxd\xi, \quad i = 1,2.$$

$$(ii)\quad \frac{d}{dt}\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} \xi(f_1+f_2)\,dxd\xi + \int_{\mathbb{T}^3} u\,dx\right) = 0.$$

Lemma 4.2. *Let (f_1, f_2, u) be the classical solutions to the system (1.1) satisfying*

(i) $\|f_{10}\|_{L^1} = \|f_{20}\|_{L^1} = 1$.
(ii) $\lim_{|\xi|\to\infty} |\xi|^2\big(f_1(x,\xi,t) + f_2(x,\xi,t)\big) = 0, \quad (x,t) \in \mathbb{T}^3 \times [0,T)$.

Then we have

$$\begin{aligned}(i)\quad &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} |\xi - \xi_c^i| f_i\,dxd\xi \\ &\qquad \le -\psi_i^m \int_{\mathbb{T}^3\times\mathbb{R}^3} |\xi - \xi_c^i| f_i\,dxd\xi + \int_{\mathbb{T}^3\times\mathbb{R}^3} (\xi - \xi_c^i)\cdot(u-\xi) f_i\,dxd\xi,\end{aligned}$$

$$\begin{aligned}(ii)\quad &\frac{1}{2}\frac{d}{dt}\int_{T^3} |u - u_c|^2\,dx \\ &\qquad = -\int_{\mathbb{T}^3} |\nabla u|^2\,dx + \int_{\mathbb{T}^3\times\mathbb{R}^3} (u_c - u)\cdot(u-\xi)(f_1+f_2)\,dxd\xi,\end{aligned}$$

$$(iii)\quad \frac{1}{4}\frac{d}{dt}|u_c - \xi_c^1 - \xi_c^2|^2 = -\int_{\mathbb{T}^3\times\mathbb{R}^3} (u_c - \xi_c^1 - \xi_c^2)\cdot(u-\xi)(f_1+f_2)\,dxd\xi,$$

where ψ_i^m is a nonnegative constant defined by

$$\psi_i^m := \inf_{(x,y)\in\mathbb{T}^3\times\mathbb{T}^3} \psi_i(x,y).$$

Proof. Straightforward computations yield the results, so we omit here. $\square$

Proof of Theorem 1.2. Summing up all of the terms in Lemma 4.2, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigg(\int_{\mathbb{T}^3\times\mathbb{R}^3}|\xi-\xi_c^1|^2 f_1+|\xi-\xi_c^2|^2 f_2 dxd\xi\\
&\qquad+\int_{\mathbb{T}^3}|u-u_c|^2dx+\frac{1}{2}|u_c-\xi_c^1-\xi_c^2|^2\bigg)\\
&\quad\le -\psi_1^m\int_{\mathbb{T}^3\times\mathbb{R}^3}|\xi-\xi_c^1|f_1dxd\xi-\psi_2^m\int_{\mathbb{T}^3\times\mathbb{R}^3}|\xi-\xi_c^2|f_idxd\xi\\
&\qquad-\int_{\mathbb{T}^3}|\nabla u|^2dx-\int_{\mathbb{T}^3\times\mathbb{R}^3}|u-\xi|^2(f_1+f_2)dxd\xi.
\end{aligned}$$

On the other hand, we find

$$\begin{aligned}
&-\int_{\mathbb{T}^3\times\mathbb{R}^3}|u-\xi|^2(f_1+f_2)dxd\xi\\
&\le\int_{\mathbb{T}^3}(\rho_{f_1}+\rho_{f_2})|u-u_c|^2dx\\
&\quad-\frac{1}{2}|u_c-\xi_c^1-\xi_c^2|^2-\frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3}|\xi-\xi_c^1|f_1+|\xi-\xi_c^2|f_2dxd\xi\\
&\le C\|\rho_{f_1}+\rho_{f_2}\|_{L^{\frac{3}{2}}}\int_{\mathbb{T}^3}|\nabla u|^2dx\\
&\quad-\frac{1}{2}|u_c-\xi_c^1-\xi_c^2|^2-\frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3}|\xi-\xi_c^1|f_1+|\xi-\xi_c^2|f_2dxd\xi,
\end{aligned}$$

where we used

$$\begin{aligned}
\big\|\sqrt{\rho_{f_1}+\rho_{f_2}}(u-u_c)\big\|_{L^2}&\le\big\|\sqrt{\rho_{f_1}+\rho_{f_2}}\big\|_{L^3}\|u-u_c\|_{L^6}\\
&\le C\|\rho_{f_1}+\rho_{f_2}\|_{L^{\frac{3}{2}}}^{\frac{1}{2}}\|u\|_{H^1}\\
&\le C\|\rho_{f_1}+\rho_{f_2}\|_{L^{\frac{3}{2}}}^{\frac{1}{2}}\|\nabla u\|_{L^2}.
\end{aligned}$$

This yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigg(\int_{\mathbb{T}^3\times\mathbb{R}^3}|\xi-\xi_c^1|^2 f_1+|\xi-\xi_c^2|^2 f_2 dxd\xi\\
&\qquad+\int_{\mathbb{T}^3}|u-u_c|^2dx+\frac{1}{2}|u_c-\xi_c^1-\xi_c^2|^2\bigg)
\end{aligned}$$

$$\leq -\left(\psi_1^m + \frac{1}{2}\right)\int_{\mathbb{T}^3\times\mathbb{R}^3} |\xi - \xi_c^1| f_1 dx d\xi - \left(\psi_2^m + \frac{1}{2}\right)\int_{\mathbb{T}^3\times\mathbb{R}^3} |\xi - \xi_c^2| f_i dx d\xi$$
$$- C\left(1 - C\|\rho_{f_1} + \rho_{f_2}\|_{L^{\frac{3}{2}}}\right)\int_{\mathbb{T}^3} |u - u_c|^2 dx - \frac{1}{2}|u_c - \xi_c^1 - \xi_c^2|^2.$$

Hence if $\|\rho_{f_1} + \rho_{f_2}\|_{L^{\frac{3}{2}}}$ is small enough, then we conclude the desired result. □

Remark 4.1. *Even if $\psi_i^m = 0$, we still have the exponential alignment between the particles and fluid. The reason is that the drag forces in particle and fluid equations play a role as the alignment force between particles and fluid. On the other hand, we notice that if $\psi_i^m > 0$, then we can expect that the decay exponent for the alignment becomes large, and it makes them to align faster.*

5. Conclusion

In this paper, we presented a new particle-fluid equations which describes the interactions between the two-species flocking particles and incompressible viscous fluid. For this model, we proved the global existence of the unique strong solution for sufficiently small and regular initial data. We also established the large-time behavior of the classical solutions under suitable assumptions. It would be an interesting problem if we consider other interaction forces between two-species particle, such as repulsive, attractive, self-propulsion and friction forces. We will leave these interesting issues to our future work.

Acknowledgments

Y.-P. Choi was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A6A3A03039496). B. Kwon was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A1015116). This work was also supported by the year of 2012 Research Fund of the Ulsan National Institute of Science and Technology (UNIST).

Appendix A. Proof of Lemma 3.1

In this part, we provide the detailed proof of Lemma 3.1.

• Step A.- Estimate of the bound of $\|f_i^{n+1}(t) - f_i^n(t)\|_{L^\infty}$: Using the similar notations in the proof of Lemma 2.3, we find

$$\begin{aligned}\mathcal{N}_i\left(f_i^{n+1}-f_i^n\right) &= -\nabla_\xi\cdot\left(F_a^i[f_i^{n+1}]+F_d[u^n]\right)\left(f_i^{n+1}-f_i^n\right)\\ &\quad+\nabla_\xi\cdot\left(F_a^i[f_i^n]-F_a^i[f_i^{n+1}]\right)f_i^n\\ &\quad+\left(F_a^i[f_i^n]-F_a^i[f_i^{n+1}]\right)\cdot\nabla_\xi f_i^n-(u^n-u^{n-1})\cdot\nabla_\xi f_i^n\\ &=:\sum_{j=1}^4 I_j^i.\end{aligned}$$

Here $I_j^i, j=1,\cdots,4$ are estimated as follows.

$$\begin{aligned}I_1^i &= 3\left(\int_{\Omega\times\mathbb{R}^3}\psi_i(x,y)f_i^{n+1}dyd\xi_*\right)\left(f_i^{n+1}-f_i^n\right)\\ &\le 3\left(\|\psi_i\|_{L^\infty}M_{i0}+1\right)\|f_i^{n+1}-f_i^n\|_{L^\infty},\\ I_2^i &= 3\left(\int_{\Omega\times\mathbb{R}^3}\psi_i(x,y)\left(f_i^{n+1}-f_i^n\right)dyd\xi_*\right)f_i^n\\ &\le 3\pi(R_\xi^{i,\infty})\pi(R_x^{i,\infty})\|f_i^{n+1}-f_i^n\|_{L^\infty}\|\psi_i\|_{L^\infty}\|f_i^n\|_{L^\infty}\le C\|f_i^{n+1}-f_i^n\|_{L^\infty},\\ I_3^i &= \left(\int_{\Omega\times\mathbb{R}^3}\psi_i(x,y)(\xi-\xi_*)\left(f_i^{n+1}-f_i^n\right)dyd\xi_*\right)\cdot\nabla_\xi f_i^n\\ &\le 2\|\psi_i\|_{L^\infty}R_\xi^{i,\infty}\pi(R_\xi^{i,\infty})\pi(R_x^{i,\infty})\|f_i^{n+1}-f_i^n\|_{L^\infty}|\nabla_\xi f_i^n|\\ &\le C\|f_i^{n+1}-f_i^n\|_{L^\infty},\\ I_4^i &\le \|u^n-u^{n-1}\|_{L^\infty}|\nabla_\xi f_i^n|\le C\|u^n-u^{n-1}\|_{L^\infty}.\end{aligned}$$

This yields

$$\|f_i^{n+1}(t)-f_i^n(t)\|_{L^\infty}\le C\|u^n(t)-u^{n-1}(t)\|_{L^\infty}+C\int_0^t\|f_i^{n+1}(s)-f_i^n(s)\|_{L^\infty}ds,$$

in turn, we have

(A.1)
$$\|f_i^{n+1}(t)-f_i^n(t)\|_{L^\infty}\le C\int_0^t\|u^n(s)-u^{n-1}(s)\|_{H^2}ds\quad\text{for all}\quad t\in[0,T].$$

• Step B.- Estimate of the bound of $\|u^{n+1}-u^n\|_{H^1}^2$: It follows from $(3.1)_3$ that

$$\begin{aligned}&\partial_t(u^{n+1}-u^n)-\Delta(u^{n+1}-u^n)+\nabla(p^{n+1}-p^n)\\ &\quad=-u^n\cdot\nabla(u^{n+1}-u^n)-(u^n-u^{n-1})\cdot\nabla u^n\\ &\qquad-\int_{\mathbb{R}^3}(u^n-u^{n-1})\left(f_1^{n+1}+f_2^{n+1}\right)d\xi\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^3} u^{n-1} \left(f_1^{n+1} - f_1^n + f_2^{n+1} - f_2^n\right) d\xi \\
&+ \int_{\mathbb{R}^3} \xi \left(f_1^{n+1} - f_1^n + f_2^{n+1} - f_2^n\right) d\xi, \\
\nabla \cdot (u^{n+1} - u^n) = 0, \quad t > 0, \quad x \in \Omega.
\end{aligned}
$$

⋄ Substep B1.- Zeroth-order estimate: Similarly as in Step A for the proof of Theorem 2.1, we find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^{n+1} - u^n\|_{L^2}^2 + \|\nabla(u^{n+1} - u^n)\|_{L^2}^2 \\
&\quad \le C\|\nabla u^n\|_{L^2}\|u^n - u^{n-1}\|_{L^3}\|u^{n+1} - u^n\|_{L^6} \\
&\qquad + C\left(\|f_1^{n+1}\|_{L^\infty} + \|f_2^{n+1}\|_{L^\infty}\right)\|u^n - u^{n-1}\|_{L^2}\|u^{n+1} - u^n\|_{L^2} \\
&\qquad + C\left(\|f_1^{n+1} - f_1^n\|_{L^\infty} + \|f_2^{n+1} - f_2^n\|_{L^\infty}\right)\left(\|u^{n-1}\|_{L^2} + 1\right)\|u^{n+1} - u^n\|_{L^2} \\
&\quad \le C\varepsilon^\alpha\|u^n - u^{n-1}\|_{H^1}\|\nabla(u^{n+1} - u^n)\|_{L^2} \\
&\qquad + C\varepsilon^\beta\|u^n - u^{n-1}\|_{L^2}\|u^{n+1} - u^n\|_{L^2} \\
&\qquad + C(1 + \varepsilon^\alpha)\left(\|f_1^{n+1} - f_1^n\|_{L^\infty} + \|f_2^{n+1} - f_2^n\|_{L^\infty}\right)\|u^{n+1} - u^n\|_{L^2} \\
&\quad \le \frac{1}{2}\|\nabla(u^{n+1} - u^n)\|_{L^2}^2 + \frac{1}{2}\|u^{n+1} - u^n\|_{L^2}^2 + C\|u^n - u^{n-1}\|_{H^1}^2 \\
&\qquad + C\|f_1^{n+1} - f_1^n\|_{L^\infty}^2 + C\|f_2^{n+1} - f_2^n\|_{L^\infty}^2.
\end{aligned}
$$

This yields

$$
\begin{aligned}
&\frac{d}{dt}\|u^{n+1} - u^n\|_{L^2}^2 + \|\nabla(u^{n+1} - u^n)\|_{L^2}^2 \\
&\quad \le C\|u^{n+1} - u^n\|_{L^2}^2 + C\|u^n - u^{n-1}\|_{H^1}^2 \\
&\qquad + C\|f_1^{n+1} - f_1^n\|_{L^\infty}^2 + C\|f_2^{n+1} - f_2^n\|_{L^\infty}^2.
\end{aligned} \tag{A.2}
$$

⋄ Substep B2.- First-order estimate: We again use a similar argument in Step B for the proof of Theorem 2.1 to deduce

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla(u^{n+1} - u^n)\|_{L^2}^2 + \|\nabla^2(u^{n+1} - u^n)\|_{L^2}^2 \\
&\quad = -\int_\Omega \nabla(u^{n+1} - u^n) : \nabla\big(u^n \cdot \nabla(u^{n+1} - u^n)\big)dx \\
&\qquad - \int_\Omega \nabla(u^{n+1} - u^n) : \nabla\big((u^n - u^{n-1}) \cdot \nabla u^n\big)dx \\
&\qquad - \int_{\Omega\times\mathbb{R}^3} \nabla(u^{n+1} - u^n) : \nabla\left((u^n - u^{n-1})\left(f_1^{n+1} + f_2^{n+1}\right)\right) d\xi dx
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\Omega\times\mathbb{R}^3} \nabla(u^{n+1}-u^n) : \nabla \left(u^{n-1}(f_1^{n+1}-f_1^n+f_2^{n+1}-f_2^n) \right) d\xi dx \\
&+ \int_{\Omega\times\mathbb{R}^3} \nabla(u^{n+1}-u^n) : \xi \otimes \nabla\,(f_1^{n+1}-f_1^n+f_2^{n+1}-f_2^n) d\xi dx \\
=:& \sum_{j=1}^{5} J_j.
\end{aligned}
$$

where $J_j, j=1,\cdots,5$ are estimated by

$$
\begin{aligned}
J_1 &\le \|\nabla(u^{n+1}-u^n)\|_{L^3}\|\nabla u^n\|_{L^2}\|\nabla(u^{n+1}-u^n)\|_{L^6} \\
&\le C\|u^n\|_{H^1}\|\nabla(u^{n+1}-u^n)\|_{H^1}^2 \le C\varepsilon^\alpha\|\nabla(u^{n+1}-u^n)\|_{H^1}^2, \\
J_2 &\le \|\nabla(u^{n+1}-u^n)\|_{L^3}\Big(\|\nabla u^n\|_{L^6}\|\nabla(u^n-u^{n-1})\|_{L^2} \\
&\quad + \|\nabla^2 u^n\|_{L^2}\|u^n-u^{n-1}\|_{L^6}\Big) \\
&\le C\varepsilon^\alpha\|u^n-u^{n-1}\|_{H^1}\|\nabla(u^{n+1}-u^n)\|_{H^1}, \\
J_3 &\le C\|\nabla(u^{n+1}-u^n)\|_{L^2}\left(\|f_1^{n+1}\|_{L^\infty}+\|f_2^{n+1}\|_{L^\infty}\right)\|\nabla(u^n-u^{n-1})\|_{L^2} \\
&\quad + C\|\nabla(u^{n+1}-u^n)\|_{L^2}\left(\|\nabla f_1^{n+1}\|_{L^\infty}+\|\nabla f_2^{n+1}\|_{L^\infty}\right)\|u^n-u^{n-1}\|_{L^2} \\
&\le C\varepsilon^\beta\|u^n-u^{n-1}\|_{H^1}\|\nabla(u^{n+1}-u^n)\|_{L^2}, \\
J_4 &\le \int_{\Omega\times\mathbb{R}^3} |\nabla^2(u^{n+1}-u^n)||u^{n-1}||f_1^{n+1}-f_1^n+f_2^{n+1}-f_2^n| dx d\xi \\
&\le C\left(\|f_1^{n+1}-f_1^n\|_{L^\infty}+\|f_2^{n+1}-f_2^n\|_{L^\infty}\right)\|\nabla^2(u^{n+1}-u^n)\|_{L^2}\|u^{n-1}\|_{L^2} \\
&\le C\varepsilon^\alpha\left(\|f_1^{n+1}-f_1^n\|_{L^\infty}+\|f_2^{n+1}-f_2^n\|_{L^\infty}\right)\|\nabla(u^{n+1}-u^n)\|_{H^1}, \\
J_5 &\le \int_{\Omega\times\mathbb{R}^3} |\nabla^2(u^{n+1}-u^n)||\xi||f_1^{n+1}-f_1^n+f_2^{n+1}-f_2^n| d\xi dx \\
&\le C\left(\|f_1^{n+1}-f_1^n\|_{L^\infty}+\|f_2^{n+1}-f_2^n\|_{L^\infty}\right)\|\nabla(u^{n+1}-u^n)\|_{H^1}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \|\nabla^2(u^{n+1}-u^n)\|_{L^2}^2 \\
&\quad \le C\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \frac{1}{2}\|\nabla^2(u^{n+1}-u^n)\|_{L^2}^2 \\
&\qquad + C\left(\|u^n-u^{n-1}\|_{H^1}^2 + \|f_1^{n+1}-f_1^n\|_{L^\infty}^2 + \|f_2^{n+1}-f_2^n\|_{L^\infty}^2\right),
\end{aligned}
$$

and

$$
\text{(A.3)} \qquad \frac{d}{dt}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \|\nabla^2(u^{n+1}-u^n)\|_{L^2}^2
$$

$$\leq C\big(\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \|u^n - u^{n-1}\|_{H^1}^2 + \|f_1^{n+1} - f_1^n\|_{L^\infty}^2 + \|f_2^{n+1} - f_2^n\|_{L^\infty}^2\big).$$

We finally combine (A.1), (A.2) and (A.3) to find

$$\begin{aligned}
&\frac{d}{dt}\|u^{n+1}-u^n\|_{H^1}^2 + \|\nabla(u^{n+1}-u^n)\|_{H^1}^2 \\
&\quad\leq C\big(\|u^{n+1}-u^n\|_{H^1}^2 + \|u^n-u^{n-1}\|_{H^1}^2 + \|f_1^{n+1}-f_1^n\|_{L^\infty}^2 \\
&\qquad + \|f_2^{n+1}-f_2^n\|_{L^\infty}^2\big) \\
&\quad\leq C\left(\|u^{n+1}-u^n\|_{H^1}^2 + \|u^n-u^{n-1}\|_{H^1}^2 + \int_0^t \|u^n-u^{n-1}\|_{H^2}^2 ds\right).
\end{aligned}$$

This implies

$$\begin{aligned}
&\|u^{n+1}-u^n\|_{H^1}^2 + \int_0^t \|\nabla(u^{n+1}-u^n)\|_{H^1}^2 ds \\
&\leq C\bigg(\int_0^t \|u^{n+1}-u^n\|_{H^1}^2 ds + \int_0^t \|u^n-u^{n-1}\|_{H^1}^2 ds \\
&\quad + \int_0^t\int_0^s \|\nabla(u^n-u^{n-1})\|_{H^1}^2 d\tau ds\bigg).
\end{aligned}$$

Applying the Gronwall's inequality for $\|u^{n+1}-u^n\|_{H^1}^2$, and using the iteration argument for the resulting inequality, we obtain
(A.4)

$$\|u^{n+1}-u^n\|_{L^\infty(0,T;H^1)}+\|\nabla(u^{n+1}-u^n)\|_{L^2(0,T;H^1)} \leq \frac{(C(T))^{n+1}}{n!} \quad \text{for all } n \geq 0.$$

This together with (A.1) also implies that

$$\|f_1^{n+1}-f_1^n\|_{L^\infty} + \|f_2^{n+1}-f_2^n\|_{L^\infty} \leq \frac{(C(T))^{n+1}}{n!} \quad \text{for all } n \geq 0. \tag{A.5}$$

Here $C(T)$ denotes the positive constant depending only on $T > 0$. By (A.4) and (A.5), one can conclude that $\{u^n\}$ and $\{f_i^n\}$ are Cauchy sequences in the desired spaces. This completes the proof.

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Young-Pil Choi
Department of Mathematics
Imperial College London
London SW7 2AZ
United Kingdom
E-mail address: young-pil.choi@imperial.ac.uk

Bongsuk Kwon
Department of Mathematical Sciences
Ulsan National Institute of Science and Technology
Ulsan 689-798
Korea
E-mail address: bkwon@unist.ac.kr

Received August 30, 2013