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POLYNOMIAL CALCULATIONS IN DOPPLER TRACKING

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Abstract. Tracking a moving object by the Doppler effect is an important tool to locate the position and to measure the velocity of a moving object. In theory, the corresponding movement of the object can be formulated by a system of 12 quadratic polynomials in 12 unknowns. In this paper, we mainly propose a novel simplification to reduce the original system to a system of 4 polynomials of degrees 4, 3, 2 and 2 in 4 unknowns. Furthermore, we can also reduce the original system to a new system of only three quadratic polynomials in 3 unknowns when the 6 observation stations are located at the vertices and centre of a regular pentagon, numerical experiments show that the simplified polynomial system can be solved by the homotopy method efficiently and reliably. The method is much more robust than Newton's method when the initial vector is far from the solution. Also, the regular pentagon case outperforms the other configurations in terms of numerical accuracy.

1. Introduction. Tracking of moving objects is an important subject in many applications [2, 3, 9]. In general, the relative positions of moving objects are unknown and to be determined. Nevertheless, the relative speed of a moving object is known and can be measured by the Doppler effect by some observation stations. For tracking an object in the space, we are interested in finding its position $\mathbf{u}(t) = (x(t), y(t), z(t))^T$ and the associated velocity $\dot{\mathbf{u}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))^T$ in time t. Suppose we have N observation stations located at $\left\{\mathbf{s}_j \equiv (x_j, y_j, z_j)^T\right\}_{j=1}^N$. The distances between $\mathbf{u}(t)$ and $\mathbf{s}_j$, and the associated derivatives, can be formulated, for $j = 1, ..., N$, as

$$(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2 = r_j^2, \tag{1}$$

$$(x - x_j)\dot{x} + (y - y_j)\dot{y} + (z - z_j)\dot{z} = r_j\dot{r}_j. \tag{2}$$

In practice, the data $\dot{r}_j$ in (2) can be measured by the Doppler effect.

By substituting $r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2}$ of (1) into (2), we get

$$F_j(\mathbf{v}) = \frac{(x - x_j)\dot{x} + (y - y_j)\dot{y} + (z - z_j)\dot{z}}{\sqrt{(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2}} - \dot{r}_j = 0, \quad (j = 1, ..., N) \tag{3}$$

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where $\mathbf{v} = (x, y, z, \dot{x}, \dot{y}, \dot{z})^T$. Since $\mathbf{v}$ has 6 unknowns, it is natural to consider $N = 6$ in (3) with 6 measured data $\{\dot{r}_j\}_{j=1}^{6}$ and a system of nonlinear equations $F = (F_1, \cdots, F_6)^T : \mathbb{R}^6 \longrightarrow \mathbb{R}^6$. Newton's method can then be applied for its solution.

In practice, there are advantages and disadvantages for solving (3) by Newton-type methods. It is well-known that Newton's method converges locally and quadratically. However, to locate a "good" initial vector, in general, is crucial. It may become difficult if the time step in the observation process is too large (see the numerical experiments in Section 4). In addition, equations in (3) are fully nonlinear, thus the evaluation of the Jacobian matrix in each Newton step may be costly. Therefore, in the following sections, we are motivated to reconsider solving the original system of polynomials in (1) and (2) in the 12 unknowns $\{x, y, z, \dot{x}, \dot{y}, \dot{z}, r_1, \cdots, r_6\}$.

Historical Note. The problem of using six stations to track a moving object was proposed by the last author Zhu and his group. Newton's method were applied, with its associated problems of initial guesses. In December 2011, Zhu asked Yau whether the problem can be solved in closed form. Yau proposed the elimination of variables, reducing the problem to the intersection of a couple polynomials. The detail and nontrivial analysis was then carried out by the first three authors, and the calculations were implemented by the fast polynomial system solver developed by TY Li (Michigan State University).

2. Simplification of Systems of Polynomials.

The homotopy method [1, 6] is a very powerful tool for finding all solutions of systems of polynomial equations. We now consider the original system of quadratic polynomials of (1) and (2) with $N = 6$ in 12 unknowns $\{x, y, z, \dot{x}, \dot{y}, \dot{z}, r_1, \cdots, r_6\}$, and propose a novel simplification to reduce the 12 unknowns to three or four unknowns so that the homotopy method can be applied more efficiently.

For given 6 positions $\{(x_j, y_j, z_j)^T\}_{j=1}^{6}$ of observation stations, and the measured speeds $\{\dot{r}_j\}_{j=1}^{6}$ by the Doppler effect in (1) and (2), we denote the following simplifying notations. Let

$$(4)\quad V_0 = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} x_1^2 + y_1^2 + z_1^2 \\ x_2^2 + y_2^2 + z_2^2 \\ x_3^2 + y_3^2 + z_3^2 \\ x_4^2 + y_4^2 + z_4^2 \\ x_5^2 + y_5^2 + z_5^2 \\ x_6^2 + y_6^2 + z_6^2 \end{bmatrix}, \quad \dot{\mathbf{r}} = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{r}_4 \\ \dot{r}_5 \\ \dot{r}_6 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{bmatrix},$$

$$\dot{R} = \mathrm{diag}(\dot{\mathbf{r}}), \quad R = \mathrm{diag}(\mathbf{r}), \quad \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \dot{\mathbf{u}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \quad \mathbf{r} \circ \mathbf{r} = \begin{bmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \\ r_4^2 \\ r_5^2 \\ r_6^2 \end{bmatrix}. \tag{5}$$

Subtracting the 1st equation from the jth equation ($j \neq 1$) in (2) yields

$$CV_0\dot{\mathbf{u}} = -C\dot{R}\mathbf{r}, \quad C = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

Similar manipulations in (1) produce

$$CV_0\mathbf{u} = \frac{1}{2}C(\mathbf{n} - \mathbf{r} \circ \mathbf{r}). \tag{7}$$

Assuming that CV_0 is of full rank, the matrix $V_0^T C^T C V_0$ is invertible. Multiplying (6) and (7) by $V_0^T C^T$, respectively, we have

$$\dot{\mathbf{u}} = (V_0^T C^T C V_0)^{-1}(V_0^T C^T C)(-\dot{R}\mathbf{r}) \tag{8}$$

and

$$\mathbf{u} = \frac{1}{2}(V_0^T C^T C V_0)^{-1}(V_0^T C^T C)(\mathbf{n} - \mathbf{r} \circ \mathbf{r}). \tag{9}$$

It is easily seen that the unknowns $\mathbf{u} = (x, y, z)^T$ and $\dot{\mathbf{u}} = (\dot{x}, \dot{y}, \dot{z})^T$ in (8) and (9) can be represented in terms of $\{r_j\}_{j=1}^6$. Denote, for $j = 1, ..., 6$,

$$A = (V_0^T C^T C V_0)^{-1}(V_0^T C^T C), \quad \mathbf{u}_j = (x_j, y_j, z_j)^T, \tag{10}$$

$$\mathbf{p} = (-\dot{R}\mathbf{r}) \equiv (p_1, ..., p_6)^T, \quad \mathbf{q} = \frac{1}{2}(\mathbf{n} - \mathbf{r} \circ \mathbf{r}). \tag{11}$$

Substituting equations (8) and (9) into (2) and (1), we have, for $j = 1, ..., 6$, the cubic equations

$$\mathbf{q}^T A^T A\mathbf{p} - \mathbf{u}_j^T A\mathbf{p} = -p_j, \tag{12}$$

and the quartic equations

$$\mathbf{q}^T A^T A\mathbf{q} - 2\mathbf{u}_i^T A\mathbf{q} + \mathbf{u}_j^T \mathbf{u}_j = \dot{r}_j^{-2} p_j^2, \tag{13}$$

in the 12 unknown $\{x, y, z, \dot{x}, \dot{y}, \dot{z}, r_1, ..., r_6\}$.

We now subtract the 2nd equation of (12) from the jth equation for $j = 3, ..., 6$ and get four equations

$$(\mathbf{u}_2 - \mathbf{u}_j)^T A\mathbf{p} = p_2 - p_j. \tag{14}$$

Denote

$$\widehat{C} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

the equations in (14) can be rewritten in matrix form

$$\left(\begin{bmatrix} (\mathbf{u}_2 - \mathbf{u}_3)^T \\ (\mathbf{u}_2 - \mathbf{u}_4)^T \\ (\mathbf{u}_2 - \mathbf{u}_5)^T \\ (\mathbf{u}_2 - \mathbf{u}_6)^T \end{bmatrix} A + \widehat{C} \right) \dot{R}\mathbf{r} \equiv \widehat{A}\mathbf{r} = \mathbf{0}, \tag{16}$$

where $\widehat{A} \in \mathbb{R}^{4\times 6}$. In fact, from the original system in (1), the rank of $\widehat{A}$ can be shown generically to be 2 (see Appendix A). Hence, let $Q^T\widehat{A} = \begin{bmatrix} T_1 & T_2 \\ O & O \end{bmatrix}$ be the QR factorization [4] of $\widehat{A}$, where $Q \in \mathbb{R}^{4\times 4}$ is orthogonal, $T_2 \in \mathbb{R}^{2\times 4}$ and $T_1 \in \mathbb{R}^{2\times 2}$ is nonsingular. Then from (4) and (16), the unknowns $\{r_1, r_2\}$ can be represented in terms of r_3, r_4, r_5 and r_6:

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = -T_1^{-1}T_2 \begin{bmatrix} r_3 \\ r_4 \\ r_5 \\ r_6 \end{bmatrix}. \tag{17}$$

Therefore, from (8), (9) and (17), we can solve the cubic equations of (12) and the quartic equations of (13) on the (r_3, r_4, r_5, r_6)-plane.

Next, we simplify equations of (12) and (13) with $j = 1$. From (11) and the relation in (17), the 1st equation in (12) can be simplified to a cubic equations in r_3, r_4, r_5 and r_6:

$$(\mathbf{r} \circ \mathbf{r})^T \frac{1}{2} A^T A\dot{R}\mathbf{r} - \hat{\mathbf{u}}_1^T A\dot{R}\mathbf{r} = r_1\dot{r}_1, \quad \hat{\mathbf{u}}_1 = \frac{1}{2}A\mathbf{n} - \mathbf{u}_1. \tag{18}$$

Similarly, from (11), the 1st equation of (13) can be written as

$$\begin{aligned} &\mathbf{q}^T A^T A\mathbf{q} - 2\mathbf{u}_1^T A\mathbf{q} + \mathbf{u}_1^T\mathbf{u}_1 \\ &= \frac{1}{4}(\mathbf{r} \circ \mathbf{r})^T A^T A(\mathbf{r} \circ \mathbf{r}) - (\frac{1}{2}\mathbf{n}^T A^T - \mathbf{u}_1^T)A(\mathbf{r} \circ \mathbf{r}) \\ &\quad +(\frac{1}{4}\mathbf{n}^T A^T A\mathbf{n} - \mathbf{u}_1^T A\mathbf{n} + \mathbf{u}_1^T\mathbf{u}_1) \\ &= r_1^2. \end{aligned} \tag{19}$$

From the definition $\hat{\mathbf{u}}_1$ in (18) and the relation in (17), (19) can be simplified to a quartic equation in r_3, r_4, r_5 and r_6 :

$$\frac{1}{4}(\mathbf{r}\circ\mathbf{r})^T A^T A(\mathbf{r}\circ\mathbf{r}) - \hat{\mathbf{u}}_1^T A(\mathbf{r}\circ\mathbf{r}) + \hat{c} = r_1^2, \tag{20}$$

where $\hat{c} = (\frac{1}{4}\mathbf{n}^T A^T A\mathbf{n} - \mathbf{u}_1^T A\mathbf{n} + \mathbf{u}_1^T \mathbf{u}_1)$.

Finally, we subtract the 2nd equation of (13) from the jth equation for $j = 3, ..., 6$ and get four equations

$$2\,(\mathbf{u}_2 - \mathbf{u}_j)^T A\mathbf{q} + \mathbf{u}_j^T \mathbf{u}_j - \mathbf{u}_2^T \mathbf{u}_2 = r_j^2 - r_2^2. \tag{21}$$

From (16), (21) can be written in the matrix form

$$\left(\begin{bmatrix} (\mathbf{u}_2-\mathbf{u}_3)^T \\ (\mathbf{u}_2-\mathbf{u}_4)^T \\ (\mathbf{u}_2-\mathbf{u}_5)^T \\ (\mathbf{u}_2-\mathbf{u}_6)^T \end{bmatrix} A + \widehat{C}\right)\mathbf{r}\circ\mathbf{r} = \left(\begin{bmatrix} (\mathbf{u}_2-\mathbf{u}_3)^T \\ (\mathbf{u}_2-\mathbf{u}_4)^T \\ (\mathbf{u}_2-\mathbf{u}_5)^T \\ (\mathbf{u}_2-\mathbf{u}_6)^T \end{bmatrix} A + \widehat{C}\right)\mathbf{n} \tag{22}$$

$$\equiv \widehat{A}\mathbf{r}\circ\mathbf{r} = \widehat{A}\mathbf{n}$$

The matrix $\widehat{A}$ is generically of rank 2 (see Appendix A). Hence, it is sufficient to choose two linearly independent equations in (22).

For given positions of stations $\left\{(x_j, y_j, z_j)^T\right\}_{j=1}^6$ and 6 measured data $\{\dot{r}_j\}_{j=1}^6$ by the Doppler effect, the radii $\{r_j\}_{j=1}^6$ can be computed by solving the system of 4 polynomials of degree 3, 4, 2 and 2 in 4 unknowns r_3, r_4, r_5 and r_6 :

$$\begin{cases} \frac{1}{2}(\mathbf{r}\circ\mathbf{r})^T A^T A\dot{R}\mathbf{r} - \hat{\mathbf{u}}_1^T A\dot{R}\mathbf{r} - \dot{r}_1 r_1 = 0, \\ \frac{1}{4}(\mathbf{r}\circ\mathbf{r})^T A^T A(\mathbf{r}\circ\mathbf{r}) - \hat{\mathbf{u}}_1^T A(\mathbf{r}\circ\mathbf{r}) + \hat{c} - r_1^2 = 0, \\ (\mathbf{u}_2-\mathbf{u}_3)^T A(\mathbf{r}\circ\mathbf{r}) - r_2^2 + r_3^2 - \mathbf{u}_3^T\mathbf{u}_3 + \mathbf{u}_2^T\mathbf{u}_2 - (\mathbf{u}_2-\mathbf{u}_3)^T A\mathbf{n} = 0, \\ (\mathbf{u}_2-\mathbf{u}_4)^T A(\mathbf{r}\circ\mathbf{r}) - r_2^2 + r_4^2 - \mathbf{u}_4^T\mathbf{u}_4 + \mathbf{u}_2^T\mathbf{u}_2 - (\mathbf{u}_2-\mathbf{u}_4)^T A\mathbf{n} = 0, \end{cases} \tag{23}$$

where $\begin{bmatrix} r_1 & r_2 \end{bmatrix}^T = -T_1^{-1}T_2\begin{bmatrix} r_3 & r_4 & r_5 & r_6 \end{bmatrix}^T$. The system in (23) has 48 solutions. The position $(x, y, z)^T$ and the velocity $(\dot{x}, \dot{y}, \dot{z})^T$, respectively, of the moving object are immediately obtained by

$$\begin{cases} \mathbf{u} = (x, y, z)^T = \frac{1}{2}A(\mathbf{n} - \mathbf{r}\circ\mathbf{r}), \\ \dot{\mathbf{u}} = (\dot{x}, \dot{y}, \dot{z})^T = -A\dot{R}\mathbf{r}. \end{cases} \tag{24}$$

Furthermore, (23) can theoretically be simplified to a system of 2 polynomial equations of degrees 12 and 16 in 2 unknowns. Compared to (23), this simplification reduces the number of unknowns from 4 to 2. However, the number of potential solutions is increased from 48 to 192. In practice, the 2-variable polynomial system with higher degrees in much more troublesome than (23) when solved by the homotopy method. We leave the tedious derivation of this simplification in Appendix B.

3. Special Configurations of Stations. In this section we will consider several particular configurations: stations on level ground or on a sphere. It is obvious that if the configuration of stations is collinear or degenerate (with some stations at identical coordinates), the polynomial system for the moving objects is over-determined. Throughout this paper, we may assume that the stations are not collinear or degenerate.

3.1. Stations on level ground. With all stations on level ground, the original system in (1) and (2) becomes, for $j = 1, ..., 6$,

$$(x - x_j)^2 + (y - y_j)^2 + (z - 0)^2 = r_j^2, \tag{25}$$

$$(x - x_j)\dot{x} + (y - y_j)\dot{y} + (z - 0)\dot{z} = r_j\dot{r}_j, \tag{26}$$

where $\{(x_j, y_j)^T, \dot{r}_j\}_{j=1}^6$ are the given data and $\{x, y, z, \dot{x}, \dot{y}, \dot{z}, r_1, \cdots, r_6\}$ are the unknowns.

With this particular configuration we will show that the system of (25) and (26) can be simplified to a system of 3 quadratic polynomials in 3 unknowns. Furthermore, if all stations are on a circle, the polynomial equations will be degenerated. Later we also consider some special configurations, with 6 stations forming: (i) a regular pentagon and its centre, and (ii) two regular triangles. For convenience, we denote, for $j = 1, \cdots, 6$,

$$V_0 = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_6 & y_6 \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} x_1^2 + y_1^2 \\ x_2^2 + y_2^2 \\ x_3^2 + y_3^2 \\ x_4^2 + y_4^2 \\ x_5^2 + y_5^2 \\ x_6^2 + y_6^2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \dot{\mathbf{u}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}, \quad \mathbf{u}_j = \begin{bmatrix} x_j \\ y_j \end{bmatrix}. \tag{27}$$

The notations $\mathbf{r}$, $\mathbf{r} \circ \mathbf{r}$, R, $\dot{\mathbf{r}}$, $\dot{R}$ and C are the same as in Section 2.

As the derivation in (8) and (9), we also have

$$\begin{aligned} \dot{\mathbf{u}} &= -(V_0^T C^T C V_0)^{-1}(V_0^T C^T C)(\dot{R}\mathbf{r}) \\ &\equiv -A\dot{R}\mathbf{r} \end{aligned} \tag{28}$$

and

$$\mathbf{u} = \frac{1}{2}A(\mathbf{n} - \mathbf{r} \circ \mathbf{r}), \tag{29}$$

where V_0 and $\mathbf{n}$ are given in (27). From (28) and (29) we see that the unknowns $\{x, y, \dot{x}, \dot{y}\}$ can be represented in terms of $\{r_j\}_{j=1}^6$. Recall that $\mathbf{p} = (-\dot{R}\mathbf{r}) \equiv$

$(p_1, ..., p_6)^T$ and $\mathbf{q} = (\mathbf{n} - \mathbf{r} \circ \mathbf{r})/2$ as in (11). Substituting (28) and (29) into (25) and (26), we get, for $j = 1, ..., 6$, the cubic equations

$$\mathbf{q}^T A^T A\mathbf{p} - \mathbf{u}_j^T A\mathbf{p} + z\dot{z} = -p_j \tag{30}$$

and the quartic equations

$$\mathbf{q}^T A^T A\mathbf{q} - 2\mathbf{u}_j^T A\mathbf{q} + \mathbf{u}_j^T \mathbf{u}_j + z^2 = \dot{r}_j^{-2} p_j^2. \tag{31}$$

We now subtract the 1st equation of (30) from the rest, yielding for $j = 2, ..., 6$:

$$(\mathbf{u}_1 - \mathbf{u}_j)^T A\mathbf{p} = p_1 - p_j. \tag{32}$$

The equations in (32) can be rewritten in matrix form

$$\left(\begin{bmatrix} (\mathbf{u}_1 - \mathbf{u}_2)^T \\ (\mathbf{u}_1 - \mathbf{u}_3)^T \\ (\mathbf{u}_1 - \mathbf{u}_4)^T \\ (\mathbf{u}_1 - \mathbf{u}_5)^T \\ (\mathbf{u}_1 - \mathbf{u}_6)^T \end{bmatrix} A + C \right) \dot{R}\mathbf{r} \equiv \tilde{A}\mathbf{r} = \mathbf{0}, \tag{33}$$

where $\tilde{A} \in \mathbb{R}^{5\times 6}$. When the original system is of the form as in (25), $\tilde{A}$ is generically of rank 3. Hence, it is sufficient to choose three linearly independent equations in (33), say the first three equations, to from the system

$$\left(\begin{bmatrix} (\mathbf{u}_1 - \mathbf{u}_2)^T \\ (\mathbf{u}_1 - \mathbf{u}_3)^T \\ (\mathbf{u}_1 - \mathbf{u}_4)^T \end{bmatrix} A + \widetilde{C} \right) \dot{R}\mathbf{r} \equiv \widetilde{B}\mathbf{r} = \mathbf{0}, \tag{34}$$

where $\widetilde{C} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ and $\widetilde{B} \in \mathbb{R}^{3\times 6}$ is of full row rank. Let $Q^T \widetilde{B} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ be the QR decomposition of $\widetilde{B}$, where $Q \in \mathbb{R}^{3\times 3}$ is orthogonal, $T_2 \in \mathbb{R}^{3\times 3}$ and $T_1 \in \mathbb{R}^{3\times 3}$ is nonsingular. Thus, r_1, r_2, r_3 can be represented in terms of r_4, r_5 and r_6 as

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = -T_1^{-1} T_2 \begin{bmatrix} r_4 \\ r_5 \\ r_6 \end{bmatrix}. \tag{35}$$

Therefore, we can solve the cubic equations of (30) and quartic equations of (31) on the (r_4, r_5, r_6)-plane.

Next, we subtract the 1st equation of (31) from the jth equation for $j = 2, ..., 6$ and get five equations

$$2\left(\mathbf{u}_1 - \mathbf{u}_j\right)^T A\mathbf{q} + \mathbf{u}_j^T \mathbf{u}_j - \mathbf{u}_1^T \mathbf{u}_1 = r_j^2 - r_1^2. \tag{36}$$

The equations in (36) can be written in the matrix form

$$\widetilde{A}\mathbf{r} \circ \mathbf{r} = \widetilde{A}\mathbf{n}, \tag{37}$$

where $\widetilde{A}$ is defined by (33). As in (34) we also have $\widetilde{B}(\mathbf{r} \circ \mathbf{r}) = \widetilde{B}\mathbf{n}$.

Given the data $\{(x_j, y_j)^T, \dot{r}_j\}_{j=1}^6$, the radii $\{r_j\}_{j=1}^6$ can be computed by solving the system of 3 quadratic polynomials in 3 unknowns r_4, r_5 and r_6:

$$\begin{cases} (\mathbf{u}_6 - \mathbf{u}_1)^T A(\mathbf{r} \circ \mathbf{r}) - r_6^2 + r_1^2 - \mathbf{u}_1^T\mathbf{u}_1 + \mathbf{u}_6^T\mathbf{u}_6 - (\mathbf{u}_6 - \mathbf{u}_1)^T A\mathbf{n} = 0, \\ (\mathbf{u}_6 - \mathbf{u}_2)^T A(\mathbf{r} \circ \mathbf{r}) - r_6^2 + r_2^2 - \mathbf{u}_2^T\mathbf{u}_2 + \mathbf{u}_6^T\mathbf{u}_6 - (\mathbf{u}_6 - \mathbf{u}_2)^T A\mathbf{n} = 0, \\ (\mathbf{u}_6 - \mathbf{u}_3)^T A(\mathbf{r} \circ \mathbf{r}) - r_6^2 + r_3^2 - \mathbf{u}_3^T\mathbf{u}_3 + \mathbf{u}_6^T\mathbf{u}_6 - (\mathbf{u}_6 - \mathbf{u}_3)^T A\mathbf{n} = 0, \end{cases} \tag{38}$$

where $\begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^T = -T_1^{-1}T_2 \begin{bmatrix} r_4 & r_5 & r_6 \end{bmatrix}^T$ as in (35). The system (38) has potentially 8 solutions. The position $\mathbf{u} = (x, y, z)^T$ and the velocity $\dot{\mathbf{u}} = (\dot{x}, \dot{y}, \dot{z})^T$ of the moving object are immediately obtained, respectively, by (25) and (29), as well as, (26) and (28).

On the other hand, the system (38) can also be simplified to one equation of degree 8 with one unknown. Expanding the first 2 equations in (38) yields

$$\begin{cases} a_4r_4^2 + a_5r_5^2 + a_6r_6^2 + a_{45}r_4r_5 + a_{46}r_4r_6 + a_{56}r_5r_6 + a_0 = 0, \\ b_4r_4^2 + b_5r_5^2 + b_6r_6^2 + b_{45}r_4r_5 + b_{46}r_4r_6 + b_{56}r_5r_6 + b_0 = 0. \end{cases} \tag{39}$$

We can rewrite them in the quadratic form: $\widetilde{\mathbf{r}}^T A_0 \widetilde{\mathbf{r}} + a_0 = 0$ and $\widetilde{\mathbf{r}}^T B_0 \widetilde{\mathbf{r}} + b_0 = 0$, where

$$A_0 = \begin{bmatrix} a_4 & \frac{a_{45}}{2} & \frac{a_{46}}{2} \\ \frac{a_{45}}{2} & a_5 & \frac{a_{56}}{2} \\ \frac{a_{46}}{2} & \frac{a_{56}}{2} & a_6 \end{bmatrix}, \quad B_0 = \begin{bmatrix} b_4 & \frac{b_{45}}{2} & \frac{b_{46}}{2} \\ \frac{b_{45}}{2} & b_5 & \frac{b_{56}}{2} \\ \frac{b_{46}}{2} & \frac{b_{56}}{2} & b_6 \end{bmatrix},$$

and $\widetilde{\mathbf{r}} = \begin{bmatrix} r_4 & r_5 & r_6 \end{bmatrix}^T$. Since A_0 and B_0 are symmetric, we assume that there is a nonsingular V_0 such that

$$V_0^T A_0 V_0 = \begin{bmatrix} \widetilde{a}_4 & 0 & 0 \\ 0 & \widetilde{a}_5 & 0 \\ 0 & 0 & \widetilde{a}_6 \end{bmatrix}, \quad V_0^T B_0 V_0 = \begin{bmatrix} \widetilde{b}_4 & 0 & 0 \\ 0 & \widetilde{b}_5 & 0 \\ 0 & 0 & \widetilde{b}_6 \end{bmatrix}. \tag{40}$$

By changing variables

$$\begin{bmatrix} r_4 \\ r_5 \\ r_6 \end{bmatrix} = V_0 \begin{bmatrix} s_4 \\ s_5 \\ s_6 \end{bmatrix}, \tag{41}$$

the system (39) becomes

$$\begin{cases} \widetilde{a}_4 s_4^2 + \widetilde{a}_5 s_5^2 + \widetilde{a}_6 s_6^2 + a_0 = 0 \\ \widetilde{b}_4 s_4^2 + \widetilde{b}_5 s_5^2 + \widetilde{b}_6 s_6^2 + b_0 = 0. \end{cases} \tag{42}$$

Assume that $\begin{bmatrix} \widetilde{a}_4 & \widetilde{a}_5 \\ \widetilde{b}_4 & \widetilde{b}_5 \end{bmatrix}$ in (42) is invertible. Then we have the following simple relations

$$\begin{cases} s_4^2 = c_1 s_6^2 + c_2, \\ s_5^2 = c_3 s_6^2 + c_4, \end{cases} \tag{43}$$

where $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} \widetilde{a}_4 & \widetilde{a}_5 \\ \widetilde{b}_4 & \widetilde{b}_5 \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{a}_6 & a_0 \\ \widetilde{b}_6 & b_0 \end{bmatrix}$. Recall that the last equation in (38) is a quadratic polynomial in variables r_4, r_5 and r_6. Plugging (41) into (38), the degrees of new equations remain the same. In fact, each of them can be written as

$$s_4 g_1 + s_5 g_2 = s_4 s_5 g_3 + g_4, \tag{44}$$

where g_1, g_2, g_3 and g_4 are polynomials in s_4, s_5 and s_6, and the exponents of s_4 and s_5 in g_i's are even numbers. Squaring both sides yields

$$s_4^2 g_1^2 + s_5^2 g_2^2 + 2 g_1 g_2 s_4 s_5 = s_4^2 s_5^2 g_3^2 + 2 g_3 g_4 s_4 s_5 + g_4^2. \tag{45}$$

Collecting $s_4 s_5$ on one side, we have $2\left(g_1 g_2 - g_3 g_4\right) s_4 s_5 = s_4^2 s_5^2 g_3^2 + g_4^2 - s_4^2 g_1^2 - s_5^2 g_2^2$. Squaring both sides of the new equation results in an equation which has variables s_4, s_5 and s_6, and the exponents of s_4 and s_5 are even numbers in all terms. Using (43), the equation only involves the variable s_6. Therefore, the system (38) can be simplified to one equation of degree 8 with even power terms being nonzero.

Remark (i) For this case, we can reduce(30), (31) to one equation in one variable s_6 of degree 8 which can be solved by QR algorithm [4] efficiently. However, it has too many possibilities to transform the solutions s_6 back to r_4, r_5 and r_6 by (39) because of the repeated squaring of (44). This is the drawback of this simplification. The other one is the eigendecomposition in (38) may not always exist. In practice, we do not recommend the simplification of (40).

(ii) When all stations are on a circle, the vector $\mathbf{n}$ is parallel to $(1, \cdots, 1)^T$. In this case, the right hand side of (38) is a zero vector, and (41) is a system of 3 homogeneous equations of degree 2 which is degenerated.

Example (Regular Pentagon): Given five stations lying at

$$\left\{ (h \cos \frac{2\pi k}{5}, h \sin \frac{2\pi k}{5}, 0)^T | k = 0, 1, 2, 3, 4 \right\}$$

and the other one at origin $(0,0,0)^T$. The equation (36) gives the relations

$$\begin{cases} r_1 = \frac{5+\sqrt{5}}{10} \frac{\dot{r}_4}{\dot{r}_1} r_4 - \frac{\sqrt{5}}{5} \frac{\dot{r}_5}{\dot{r}_1} r_5 + \frac{5+\sqrt{5}}{10} \frac{\dot{r}_6}{\dot{r}_1} r_6, \\ \\ r_2 = \frac{\dot{r}_4}{\dot{r}_2} r_4 - \frac{1+\sqrt{5}}{2} \frac{\dot{r}_5}{\dot{r}_2} r_5 + \frac{1+\sqrt{5}}{2} \frac{\dot{r}_6}{\dot{r}_2} r_6, \\ \\ r_3 = \frac{1+\sqrt{5}}{2} \frac{\dot{r}_4}{\dot{r}_3} r_4 - \frac{1+\sqrt{5}}{2} \frac{\dot{r}_5}{\dot{r}_3} r_5 + \frac{\dot{r}_6}{\dot{r}_3} r_6, \end{cases} \tag{46}$$

and the equation of (38) are equivalent to

$$
(47) \qquad \begin{cases} f_1 \equiv \frac{1}{10}\left(10h^2 + 10r_1^2 - 6r_2^2 - r_3^2 + \sqrt{5}r_3^2 - r_4^2 - \sqrt{5}r_4^2 \right. \\ \qquad \left. -r_5^2 - \sqrt{5}r_5^2 - r_6^2 + \sqrt{5}r_6^2\right) = 0, \\ \\ f_2 \equiv \frac{1}{10}\left(10h^2 + 10r_1^2 - r_2^2 + \sqrt{5}r_2^2 - 6r_3^2 - r_4^2 + \sqrt{5}r_4^2 \right. \\ \qquad \left. -r_5^2 - \sqrt{5}r_5^2 - r_6^2 - \sqrt{5}r_6^2\right) = 0, \\ \\ f_3 \equiv \frac{1}{10}\left(10h^2 + 10r_1^2 - r_2^2 - \sqrt{5}r_2^2 - r_3^2 + \sqrt{5}r_3^2 - 6r_4^2 \right. \\ \qquad \left. -r_5^2 + \sqrt{5}r_5^2 - r_6^2 - \sqrt{5}r_6^2\right) = 0. \end{cases}
$$

Plugging (46) into (47), we have a system of 3 quadratic polynomials in variables r_4, r_5, and r_6:

$$
(48) \qquad [q_1, q_2, q_3]^T = 0,
$$

where

$$
\begin{aligned}
q_1 \equiv \frac{1}{10}(10h^2 &- r_4^2 - \sqrt{5}r_4^2 - r_5^2 - \sqrt{5}r_5^2 - r_6^2 + \sqrt{5}r_6^2 \\
&- \frac{\left(\left(1+\sqrt{5}\right)r_4\dot{r}_4 - \left(1+\sqrt{5}\right)r_5\dot{r}_5 + 2r_6\dot{r}_6\right)^2}{4\dot{r}_3^2} \\
&+ \frac{\sqrt{5}\left(\left(1+\sqrt{5}\right)r_4\dot{r}_4 - \left(1+\sqrt{5}\right)r_5\dot{r}_5 + 2r_6\dot{r}_6\right)^2}{4\dot{r}_3^2} \\
&+ \frac{\left(\left(5+\sqrt{5}\right)r_4\dot{r}_4 - 2\sqrt{5}r_5\dot{r}_5 + \left(5+\sqrt{5}\right)r_6\dot{r}_6\right)^2}{10\dot{r}_1^2} \\
&- \frac{3\left(-2r_4\dot{r}_4 + \left(1+\sqrt{5}\right)\left(r_5\dot{r}_5 - r_6\dot{r}_6\right)\right)^2}{2\dot{r}_2^2}),
\end{aligned}
$$

$$
\begin{aligned}
q_2 \equiv \frac{1}{10}(10h^2 &- r_4^2 + \sqrt{5}r_4^2 - r_5^2 - \sqrt{5}r_5^2 - r_6^2 - \sqrt{5}r_6^2 \\
&- \frac{3\left(\left(1+\sqrt{5}\right)r_4\dot{r}_4 - \left(1+\sqrt{5}\right)r_5\dot{r}_5 + 2r_6\dot{r}_6\right)^2}{2\dot{r}_3^2} \\
&+ \frac{\left(\left(5+\sqrt{5}\right)r_4\dot{r}_4 - 2\sqrt{5}r_5\dot{r}_5 + \left(5+\sqrt{5}\right)r_6\dot{r}_6\right)^2}{10\dot{r}_1^2} \\
&- \frac{\left(-2r_4\dot{r}_4 + \left(1+\sqrt{5}\right)\left(\mathrm{r5}\dot{r}_5 - r_6\dot{r}_6\right)\right)^2}{4\dot{r}_2^2} \\
&+ \frac{\sqrt{5}\left(-2r_4\dot{r}_4 + \left(1+\sqrt{5}\right)\left(r_5\dot{r}_5 - r_6\dot{r}_6\right)\right)^2}{4\dot{r}_2^2}),
\end{aligned}
$$

$$
\begin{aligned}
q_3 \equiv \frac{1}{10}(10h^2 - 6r_4^2 - r_5^2 + \sqrt{5}r_5^2 - r_6^2 - \sqrt{5}r_6^2 \\
-\frac{\left(\left(1+\sqrt{5}\right) r_4\dot{r}_4 - \left(1+\sqrt{5}\right) r_5\dot{r}_5 + 2r_6\dot{r}_6\right)^2}{4\dot{r}_3^2} \\
+\frac{\sqrt{5}\left(\left(1+\sqrt{5}\right) r_4\dot{r}_4 - \left(1+\sqrt{5}\right) r_5\dot{r}_5 + 2r_6\dot{r}_6\right)^2}{4\dot{r}_3^2} \\
+\frac{\left(\left(5+\sqrt{5}\right) r_4\dot{r}_4 - 2\sqrt{5}r_5\dot{r}_5 + \left(5+\sqrt{5}\right) r_6\dot{r}_6\right)^2}{10\dot{r}_1^2} \\
-\frac{\left(-2r_4\dot{r}_4 + \left(1+\sqrt{5}\right)\left(r_5\dot{r}_5 - r_6\dot{r}_6\right)\right)^2}{4\dot{r}_2^2} \\
-\frac{\sqrt{5}\left(-2r_4\dot{r}_4 + \left(1+\sqrt{5}\right)\left(r_5\dot{r}_5 - r_6\dot{r}_6\right)\right)^2}{4\dot{r}_2^2}).
\end{aligned}
$$

Given the Doppler data $\{\dot{r}_j | j = 1, ..., 6\}$, the system of (48) has 8 solutions in general and can be solved by the total degree homotopy method very fast (< 0.01 sec.).

Example (Two Regular Triangles): Given three stations lying at

$$\left\{(h\cos\frac{2\pi k}{3}, h\sin\frac{2\pi k}{3}, 0) | k = 0, 1, 2\right\}$$

and the other three lying at $\left\{(2h\cos\frac{2\pi k}{3}, 2h\sin\frac{2\pi k}{3}, 0) | k = 0, 1, 2\right\}$. The equation (35) gives the relations

$$
\text{(49)} \qquad \begin{cases}
r_1 = -2\frac{\dot{r}_4}{\dot{r}_1} r_4 + \frac{\dot{r}_5}{\dot{r}_1} r_5 + 2\frac{\dot{r}_6}{\dot{r}_1} r_6, \\
r_2 = 5\frac{\dot{r}_4}{\dot{r}_2} r_4 - 3\frac{\dot{r}_5}{\dot{r}_2} r_5 - \frac{\dot{r}_6}{\dot{r}_2} r_6, \\
r_3 = 8\frac{\dot{r}_4}{\dot{r}_3} r_4 - 5\frac{\dot{r}_5}{\dot{r}_3} r_5 - 2\frac{\dot{r}_6}{\dot{r}_3} r_6,
\end{cases}
$$

and the equations of (38) are equivalent to

$$
\text{(50)} \qquad \begin{cases}
f_1 \equiv \frac{1}{315}\left(-1035h^2 + 60r_1^2 - 107r_2^2 + 111r_3^2 - 233r_4^2 + 174r_5^2 - 5r_6^2\right) = 0, \\
f_2 \equiv \frac{1}{5}\left(-2r_2^2 + r_3^2 + 2r_4^2 - r_5^2\right) = 0, \\
f_3 \equiv \frac{1}{315}\left(-1305h^2 + 240r_1^2 - 113r_2^2 + 129r_3^2 + 13r_4^2 + 66r_5^2 - 335r_6^2\right) = 0.
\end{cases}
$$

Plugging (49) into (50), we have a system of 3 quadratic polynomials in variables

r_4, r_5, and r_6:

$$
\begin{aligned}
q_1 &\equiv -1035h^2 - 233r_4^2 + 174r_5^2 - 5r_6^2 - \frac{107(-5r_4\dot{r}_4 + 3r_5\dot{r}_5 + r_6\dot{r}_6)^2}{\dot{r}_2^2} \\
&\quad + \frac{60(-2r_4\dot{r}_4 + r_5\dot{r}_5 + 2r_6\dot{r}_6)^2}{\dot{r}_1^2} + \frac{111(-8r_4\dot{r}_4 + 5r_5\dot{r}_5 + 2r_6\dot{r}_6)^2}{\dot{r}_3^2} = 0, \\
q_2 &\equiv 2r_4^2 - r_5^2 - \frac{2(-5r_4\dot{r}_4 + 3r_5\dot{r}_5 + r_6\dot{r}_6)^2}{\dot{r}_2^2} + \frac{(-8r_4\dot{r}_4 + 5r_5\dot{r}_5 + 2r_6\dot{r}_6)^2}{\dot{r}_3^2} = 0, \\
q_3 &\equiv -1305h^2 + 13r_4^2 + 66r_5^2 - 335r_6^2 - \frac{113(-5r_4\dot{r}_4 + 3r_5\dot{r}_5 + r_6\dot{r}_6)^2}{\dot{r}_2^2} \\
&\quad + \frac{240(-2r_4\dot{r}_4 + r_5\dot{r}_5 + 2r_6\dot{r}_6)^2}{\dot{r}_1^2} + \frac{129(-8r_4\dot{r}_4 + 5r_5\dot{r}_5 + 2r_6\dot{r}_6)^2}{\dot{r}_3^2} = 0.
\end{aligned} \tag{51}
$$

Given the Doppler data $\{\dot{r}_j | j = 1, ..., 6\}$, the system of (51) has 8 solutions and can also be solved by the total degree homotopy method very efficient.

3.2. Stations on a Sphere. When stations lie on a sphere, the vector $\mathbf{n}$ is parallel to $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$. Hence, the original system can be simplified to the system

$$
\begin{cases}
\frac{1}{2}(\mathbf{r} \circ \mathbf{r})^T A^T A \dot{R} \mathbf{r} + \mathbf{u}_1^T A \dot{R} \mathbf{r} - \dot{r}_1 r_1 = 0, \\
\frac{1}{4}(\mathbf{r} \circ \mathbf{r})^T A^T A (\mathbf{r} \circ \mathbf{r}) + \mathbf{u}_1^T A (\mathbf{r} \circ \mathbf{r}) + \mathbf{u}_1^T \mathbf{u}_1 - r_1^2 = 0, \\
(\mathbf{u}_2 - \mathbf{u}_3)^T A (\mathbf{r} \circ \mathbf{r}) - r_2^2 + r_3^2 = 0, \\
(\mathbf{u}_2 - \mathbf{u}_4)^T A (\mathbf{r} \circ \mathbf{r}) - r_2^2 + r_4^2 = 0,
\end{cases} \tag{52}
$$

where $\begin{bmatrix} r_1 & r_2 \end{bmatrix}^T = -T_1^{-1} T_2 \begin{bmatrix} r_3 & r_4 & r_5 & r_6 \end{bmatrix}^T$ as in (17). Given the Doppler data $\{\dot{r}_j | j = 1, ..., 6\}$, the system of (52) has 16 solutions in general.

Example (Regular Octahedron): Given six stations lying at

$$\{(\pm h, 0, 0), (0, \pm h, 0), (0, 0, \pm h)\}.$$

The equation (17) gives the relations

$$
\begin{cases}
r_1 = \frac{\dot{r}_3}{\dot{r}_1} r_3 - \frac{\dot{r}_4}{\dot{r}_1} r_4 + \frac{\dot{r}_6}{\dot{r}_1} r_6 \\
r_2 = \frac{\dot{r}_3}{\dot{r}_2} r_3 - \frac{\dot{r}_5}{\dot{r}_2} r_5 + \frac{\dot{r}_6}{\dot{r}_2} r_6,
\end{cases} \tag{53}
$$

and the equations of (52) are equivalent to

(54)

$$\begin{cases} f_1 \equiv \frac{1}{128h^2}(36r_1^3\dot r_1 - 6r_1\left(r_2^2+r_3^2+2r_4^2+r_5^2+r_6^2\right)\dot r_1 + 17r_2^3\dot r_2 + r_2r_3^2\dot r_2 + 2r_2r_4^2\dot r_2 \\ \quad - 15r_2r_5^2\dot r_2 + r_2r_6^2\dot r_2 + r_2^2r_3\dot r_3 + 17r_3^3\dot r_3 + 2r_3r_4^2\dot r_3 + r_3r_5^2\dot r_3 - 15r_3r_6^2\dot r_3 \\ \quad + 2r_2^2r_4\dot r_4 + 2r_3^2r_4\dot r_4 + 4r_4^3\dot r_4 + 2r_4r_5^2\dot r_4 + 2r_4r_6^2\dot r_4 - 15r_2^2r_5\dot r_5 + r_3^2r_5\dot r_5 \\ \quad + 2r_4^2r_5\dot r_5 + 17r_5^3\dot r_5 + r_5r_6^2\dot r_5 + r_2^2r_6\dot r_6 - 15r_3^2r_6\dot r_6 + 2r_4^2r_6\dot r_6 + r_5^2r_6\dot r_6 \\ \quad + 17r_6^3\dot r_6 - 6r_1^2(r_2\dot r_2 + r_3\dot r_3 + 2r_4\dot r_4 + r_5\dot r_5 + r_6\dot r_6) \\ \quad - 16h^2(2r_1\dot r_1 + r_2\dot r_2 + r_3\dot r_3 + 2r_4\dot r_4 + r_5\dot r_5 + r_6\dot r_6)) = 0, \\ \\ f_2 \equiv \frac{1}{256h^2}(256h^4 + 36r_1^4 + 17r_2^4 + 2r_2^2r_3^2 + 17r_3^4 + 4r_2^2r_4^2 + 4r_3^2r_4^2 + 4r_4^4 - 30r_2^2r_5^2 \\ \quad + 2r_3^2r_5^2 + 4r_4^2r_5^2 + 17r_5^4 + 2r_2^2r_6^2 - 30r_3^2r_6^2 + 4r_4^2r_6^2 + 2r_5^2r_6^2 + 17r_6^4 \\ \quad - 12r_1^2\left(r_2^2+r_3^2+2r_4^2+r_5^2+r_6^2\right) \\ \quad - 32h^2\left(2r_1^2+r_2^2+r_3^2+2r_4^2+r_5^2+r_6^2\right)) = 0, \\ \\ f_3 \equiv \frac{r_2^2}{2} - \frac{r_3^2}{2} + \frac{r_5^2}{2} - \frac{r_6^2}{2} = 0, \\ \\ f_4 \equiv -\frac{3r_1^2}{4} + \frac{5r_2^2}{8} + \frac{r_3^2}{8} - \frac{3r_4^2}{4} + \frac{5r_5^2}{8} + \frac{r_6^2}{8} = 0. \end{cases}$$

Substituting (53) into (54), we have a system of 4 polynomials of degrees 3, 4, 2 and 2 in unknowns r_3, r_4, r_5, and r_6:

$$\begin{aligned} q_1 \equiv\ & 3r_3^3\left(2\dot r_2^2\dot r_3^3 + \dot r_1^2\dot r_3\left(\dot r_2^2+\dot r_3^2\right)\right) + 4r_4^3\dot r_1^2\dot r_2^2\dot r_4 + 2r_4r_5^2\dot r_1^2\dot r_2^2\dot r_4 + 2r_4r_6^2\dot r_1^2\dot r_2^2\dot r_4 - 12r_4^3\dot r_2^2\dot r_4^3 \\ & +8r_5^3\dot r_1^2\dot r_2^2\dot r_5 + 2r_4r_5^2\dot r_1^2\dot r_4\dot r_5^2 - 8r_5^3\dot r_1^2\dot r_5^3 - 2r_4^2r_6\dot r_1^2\dot r_2^2\dot r_6 - 5r_5^2r_6\dot r_1^2\dot r_2^2\dot r_6 + 3r_6^3\dot r_1^2\dot r_2^2\dot r_6 \\ & +30r_4^2r_6\dot r_2^2\dot r_4^2\dot r_6 - 4r_4r_5r_6\dot r_1^2\dot r_4\dot r_5\dot r_6 + 19r_5^2r_6\dot r_1^2\dot r_5^2\dot r_6 + 2r_4r_6^2\dot r_1^2\dot r_4\dot r_6^2 - 24r_4r_6^2\dot r_2^2\dot r_4\dot r_6^2 \\ & -14r_5r_6^2\dot r_1^2\dot r_5\dot r_6^2 + 3r_6^3\dot r_1^2\dot r_6^3 + 6r_6^3\dot r_2^2\dot r_6^3 - 16h^2\dot r_1^2\dot r_2^2(r_3\dot r_3 + r_6\dot r_6) \\ & +r_3^2\left(2r_4\left(-12\dot r_2^2\dot r_3^2 + \dot r_1^2\left(\dot r_2^2+\dot r_3^2\right)\right)\dot r_4\right. \\ & \left.-14r_5\dot r_1^2\dot r_3^2\dot r_5 + r_6\left(18\dot r_2^2\dot r_3^2 + \dot r_1^2\left(-5\dot r_2^2+9\dot r_3^2\right)\right)\dot r_6\right) \\ & -2r_4^2\dot r_2^2r_3\dot r_3\left(\dot r_1^2 - 15\dot r_4^2\right) + r_5^2\dot r_1^2r_3\dot r_3\left(-5\dot r_2^2+19\dot r_5^2\right) - 28r_5r_6\dot r_1^2\dot r_5\dot r_6r_3\dot r_3 \\ & -4r_4\dot r_4r_3\dot r_3\left(r_5\dot r_1^2\dot r_5 - r_6\left(\dot r_1^2 - 12\dot r_2^2\right)\dot r_6\right) + r_6^2r_3\dot r_3\left(18\dot r_2^2\dot r_6^2 + \dot r_1^2\left(-5\dot r_2^2+9\dot r_6^2\right)\right) = 0, \end{aligned}$$

$$\begin{aligned} q_2 \equiv\ & 256h^4 + 17r_3^4 + 4r_3^2r_4^2 + 4r_4^4 + 2r_3^2r_5^2 + 4r_4^2r_5^2 + 17r_5^4 \\ & -30r_3^2r_6^2 + 4r_4^2r_6^2 + 2r_5^2r_6^2 + 17r_6^4 \\ & +\frac{36(r_3\dot r_3 - r_4\dot r_4 + r_6\dot r_6)^4}{\dot r_1^4} + \frac{2r_3^2(r_3\dot r_3 - r_5\dot r_5 + r_6\dot r_6)^2}{\dot r_2^2} + \frac{4r_4^2(r_3\dot r_3 - r_5\dot r_5 + r_6\dot r_6)^2}{\dot r_2^2} \\ & -\frac{30r_5^2(r_3\dot r_3 - r_5\dot r_5 + r_6\dot r_6)^2}{\dot r_2^2} + \frac{2r_6^2(r_3\dot r_3 - r_5\dot r_5 + r_6\dot r_6)^2}{\dot r_2^2} + \frac{17(r_3\dot r_3 - r_5\dot r_5 + r_6\dot r_6)^4}{\dot r_2^4} \end{aligned}$$

$$- \frac{12(r_3\dot r_3 - r_4\dot r_4 + r_6\dot r_6)^2 \left(r_3^2 + 2r_4^2 + r_5^2 + r_6^2 + \frac{(r_3\dot r_3 - r_5\dot r_5 + r_6\dot r_6)^2}{\dot r_2^2}\right)}{\dot r_1^2}$$

$$- 32h^2 \left(r_3^2 + 2r_4^2 + r_5^2 + r_6^2 + \frac{2(r_3\dot r_3 - r_4\dot r_4 + r_6\dot r_6)^2}{\dot r_1^2} + \frac{(r_3\dot r_3 - r_5\dot r_5 + r_6\dot r_6)^2}{\dot r_2^2}\right) = 0,$$

$$q_3 \equiv -r_3^2 + r_5^2 - r_6^2 + \frac{(r_3\dot r_3 - r_5\dot r_5 + r_6\dot r_6)^2}{\dot r_2^2} = 0,$$

$$q_4 \equiv r_3^2 - 6r_4^2 + 5r_5^2 + r_6^2 - \frac{6(r_3\dot r_3 - r_4\dot r_4 + r_6\dot r_6)^2}{\dot r_1^2} + \frac{5(r_3\dot r_3 - r_5\dot r_5 + r_6\dot r_6)^2}{\dot r_2^2} = 0.$$

4. Numerical Experiments.

4.1. Numerical Accuracy of Homotopy Method. In this section we will show the numerical accuracy of solving the original systems and the corresponding simplified systems by the homotopy continuation method [7], implemented in HOM4PS-2.0 (available at

http://www.math.nsysu.edu.tw/~leetsung/works/HOM4PS_soft.htm).

(Regular Pentagon): Given five stations located at

$$\left\{(2\cos\frac{2\pi k}{5}, 2\sin\frac{2\pi k}{5}, 0)^T | k = 0,1,2,3,4\right\}$$

and the other one at origin $(0,0,0)^T$. We assume the object is at the position $(2,5,3)^T$ with velocity $(1,2,4)^T$. Then the measured Doppler data should be

$$\begin{aligned} \dot r_1 &= 3.893314107138301 & \dot r_4 &= 3.8397960099826884 \\ \dot r_2 &= 3.772968873135194 & \dot r_5 &= 3.6039686843251832 \\ \dot r_3 &= 4.32328941267566 & \dot r_6 &= 3.552946442309747. \end{aligned}$$

Hence, from (1) and (2) the original polynomial system is

$$(55) \quad \begin{cases} (x - 0)\dot x + (y - 0)\dot y + (z-0)\dot z = 3.893314107138301 \cdot r_1, \\ (x - 2\cos\frac{0\pi}{5})\dot x + (y - 2\sin\frac{0\pi}{5})\dot y + (z-0)\dot z = 3.772968873135194 \cdot r_2, \\ (x - 2\cos\frac{2\pi}{5})\dot x + (y - 2\sin\frac{2\pi}{5})\dot y + (z-0)\dot z = 4.32328941267566 \cdot r_3, \\ (x - 2\cos\frac{4\pi}{5})\dot x + (y - 2\sin\frac{4\pi}{5})\dot y + (z-0)\dot z = 3.8397960099826884 \cdot r_4, \\ (x - 2\cos\frac{6\pi}{5})\dot x + (y - 2\sin\frac{6\pi}{5})\dot y + (z-0)\dot z = 3.6039686843251832 \cdot r_5, \\ (x - 2\cos\frac{8\pi}{5})\dot x + (y - 2\sin\frac{8\pi}{5})\dot y + (z-0)\dot z = 3.552946442309747 \cdot r_6, \\ (x - 0)^2 + (y - 0)^2 + (z-0)^2 = r_1^2, \\ (x - 2\cos\frac{0\pi}{5})^2 + (y - 2\sin\frac{0\pi}{5})^2 + (z-0)^2 = r_2^2, \\ (x - 2\cos\frac{2\pi}{5})^2 + (y - 2\sin\frac{2\pi}{5})^2 + (z-0)^2 = r_3^2, \\ (x - 2\cos\frac{4\pi}{5})^2 + (y - 2\sin\frac{4\pi}{5})^2 + (z-0)^2 = r_4^2, \\ (x - 2\cos\frac{6\pi}{5})^2 + (y - 2\sin\frac{6\pi}{5})^2 + (z-0)^2 = r_5^2, \\ (x - 2\cos\frac{8\pi}{5})^2 + (y - 2\sin\frac{8\pi}{5})^2 + (z-0)^2 = r_6^2, \end{cases}$$

and from (48) the simplified system is

$$
(56) \quad \begin{cases}
4 - 0.18046568496886645 r_4^2 + 0.817451608915785 r_4 r_5 \\
-1.3605999657065038 r_5^2 - 0.6263091782107895 r_4 r_6 \\
+2.0051504885645226 r_5 r_6 - 0.7498101264961088 r_6^2 = 0, \\
\\
4 - 0.4781808395962457 r_4^2 + 1.3463107868155073 r_4 r_5 \\
-0.9485546803221847 r_5^2 - 0.09134819115874843 r_4 r_6 \\
+0.2012702178430671 r_5 r_6 - 0.0058130967013134555 r_6^2 = 0, \\
\\
4 - 0.17058796355164318 r_4^2 - 0.052044655877640844 r_4 r_5 \\
-0.2531518755495662 r_5^2 + 0.23087742111668474 r_4 r_6 \\
+0.7033756928598706 r_5 r_6 - 0.5553510918753413 r_6^2 = 0.
\end{cases}
$$

Solving (55) by the polyhedral homotopy continuation method [5, 6], HOM4PS-2.0 [7] takes 3.5 seconds and obtains solutions

$$
\begin{aligned}
x &= 1.9999999999999849 & r_1 &= 6.1644140029689547 \\
y &= 4.9999999999999849 & r_2 &= 5.8309518948452830 \\
z &= 2.9999999999999911 & r_3 &= 4.5284361228902368 \\
\dot{x} &= 0.9999999999999950 & r_4 &= 6.0594084619828887 \\
\dot{y} &= 2.0000000000000027 & r_5 &= 7.7606598302495318 \\
\dot{z} &= 3.9999999999999987 & r_6 &= 7.6517314622837613,
\end{aligned}
$$

which has 13 correct digits.

Solving (56) by the polyhedral homotopy continuation method, HOM4PS-2.0 takes less than 0.01 seconds and obtains the solution

$$
r_4 = 6.0594084619829323, \quad r_5 = 7.7606598302495700, \quad r_6 = 7.6517314622837826,
$$

which has 14 correct digits.

Note that HOM4PS-2.0 is implemented in a double precision floating point arithmetic system (i.e. 16 significant digits), and the exact solution of (55) is

$$
\begin{aligned}
x &= 2 & \dot{x} &= 1 & r_1 &= 6.164414002968976 & r_4 &= 6.059408461982912 \\
y &= 5 & \dot{y} &= 2 & r_2 &= 5.830951894845301 & r_5 &= 7.760659830249555 \\
z &= 3 & \dot{z} &= 4 & r_3 &= 4.528436122890257 & r_6 &= 7.651731462283781.
\end{aligned}
$$

Next, we consider the moving object is 10 times farther with the same station configuration:

We assume the object is at the position $(20, 50, 30)^T$ with the velocity $(1, 2, 4)^T$. Then the measured Doppler data should be

$$\begin{array}{ll} \dot r_1 = 3.893314107138301 & \dot r_4 = 3.906605581821401 \\ \dot r_2 = 3.9000674757995495 & \dot r_5 = 3.8641247950598454 \\ \dot r_3 = 3.9322710660446627 & \dot r_6 = 3.8598600586840734. \end{array}$$

The corresponding original polynomial system is

$$(57)\quad \begin{cases} (x-0\qquad)\dot x + (y-0\qquad)\dot y + (z-0)\dot z = 3.893314107138301\cdot r_1 \\ (x-2\cos\frac{0\pi}{5})\dot x + (y-2\sin\frac{0\pi}{5})\dot y + (z-0)\dot z = 3.9000674757995495\cdot r_2 \\ (x-2\cos\frac{2\pi}{5})\dot x + (y-2\sin\frac{2\pi}{5})\dot y + (z-0)\dot z = 3.9322710660446627\cdot r_3 \\ (x-2\cos\frac{4\pi}{5})\dot x + (y-2\sin\frac{4\pi}{5})\dot y + (z-0)\dot z = 3.906605581821401\cdot r_4 \\ (x-2\cos\frac{6\pi}{5})\dot x + (y-2\sin\frac{6\pi}{5})\dot y + (z-0)\dot z = 3.8641247950598454\cdot r_5 \\ (x-2\cos\frac{8\pi}{5})\dot x + (y-2\sin\frac{8\pi}{5})\dot y + (z-0)\dot z = 3.8598600586840734\cdot r_6 \\ (x-0\qquad)^2 + (y-0\qquad)^2 + (z-0)^2 = r_1^2 \\ (x-2\cos\frac{0\pi}{5})^2 + (y-2\sin\frac{0\pi}{5})^2 + (z-0)^2 = r_2^2 \\ (x-2\cos\frac{2\pi}{5})^2 + (y-2\sin\frac{2\pi}{5})^2 + (z-0)^2 = r_3^2 \\ (x-2\cos\frac{4\pi}{5})^2 + (y-2\sin\frac{4\pi}{5})^2 + (z-0)^2 = r_4^2 \\ (x-2\cos\frac{6\pi}{5})^2 + (y-2\sin\frac{6\pi}{5})^2 + (z-0)^2 = r_5^2 \\ (x-2\cos\frac{8\pi}{5})^2 + (y-2\sin\frac{8\pi}{5})^2 + (z-0)^2 = r_6^2 \end{cases}$$

and the simplified system is

$$(58)\quad \begin{cases} 4 - 0.07903587291009495 r_4^2 + 0.6505712626534441 r_4 r_5 \\ -1.3561074212644102 r_5^2 - 0.49301393079838474 r_4 r_6 \\ +2.057926246313345 r_5 r_6 - 0.7812486085909165 r_6^2 = 0, \\ \\ 4. - 0.7755657991400302 r_4^2 + 2.0255141276190507 r_4 r_5 \\ -1.3257723644882415 r_5^2 - 0.45514735753209457 r_4 r_6 \\ +0.6013776619641669 r_5 r_6 - 0.07009692145793234 r_6^2 = 0, \\ \\ 4. - 0.07810840035733123 r_4^2 - 0.23709857230188092 r_4 r_5 \\ -0.19856296380296357 r_5^2 + 0.3936762055096171 r_4 r_6 \\ +0.6388324706459287 r_5 r_6 - 0.519698425132617 r_6^2 = 0. \end{cases}$$

Solving (57) by the polyhedral homotopy continuation method, HOM4PS-2.0 takes 3.5 seconds and obtains a solution

$$\begin{array}{ll} x = 1.9999999999973671 & r_1 = 6.1644140029661536 \\ y = 4.9999999999976644 & r_2 = 6.1024585209541776 \\ z = 2.9999999999998462 & r_3 = 5.9908825202866510 \\ \dot x = 0.9999999999990583 & r_4 = 6.1246749375685148 \\ \dot y = 2.0000000000004783 & r_5 = 6.3136981318438636 \\ \dot z = 3.9999999999988072 & r_6 = 6.3003888322114669, \end{array}$$

which has 11 correct digits.

Solving (58) by the polyhedral homotopy continuation method, HOM4PS-2.0 takes less than 0.01 seconds and obtains the solution

$$r_4 = 61.246749375686868, \quad r_5 = 63.136981318440540, \quad r_6 = 63.003888322117241,$$

which has 11 correct digits.

Note that the exact solutions of (57) are

$$\begin{array}{llll} x = 20 & \dot{x} = 1 & r_1 = 61.644140029689765 & r_4 = 61.24674937571382 \\ y = 50 & \dot{y} = 2 & r_2 = 61.02458520956943 & r_5 = 63.13698131846731 \\ z = 30 & \dot{z} = 4 & r_3 = 59.908825202894555 & r_6 = 63.00388832214274. \end{array}$$

Table 1 lists the number of correct digits for the pentagon configuration. The first row of table is the position of the moving object. We can see that solving the original system and the simplified system by the homotopy method achieve similar accuracy. However, solving the simplified system saves a lot of cost.

TABLE 1
The number of correct digits for the pentagon configuration.

	$(2,5,3)$	$(20,50,30)$	$(2,5,3)\times 10^2$	$(2,5,3)\times 10^3$	$(2,5,3)\times 10^4$
original	13	11	9	8	5
simplified	14	11	10	8	5

(Two Regular Triangles): Given three stations located at $\{(2\cos\frac{2\pi k}{3}, 2\sin\frac{2\pi k}{3}, 0)|k = 0,1,2\}$ and the other three located at $\left\{(\cos\frac{2\pi k}{3}, \sin\frac{2\pi k}{3}, 0)|k = 0,1,2\right\}$. We assume the object is at the position $(200,500,300)^T$ with the velocity $(1,2,4)^T$. Then the measured Doppler data should be

$$\begin{array}{ll} \dot{r}_1 = 3.885974155004552 & \dot{r}_4 = 3.8961310021157534 \\ \dot{r}_2 = 3.8941504270435656 & \dot{r}_5 = 3.8989284653247402 \\ \dot{r}_3 = 3.894951626856594 & \dot{r}_6 = 3.8896389634680446. \end{array}$$

Hence, the original polynomial system is

$$
(59)\quad \begin{cases}
(x+2)\dot{x} + (y+2\sqrt{3})\dot{y} + (z-0)\dot{z} = 3.885974155004552 \cdot r_1 \\
(x-2)\dot{x} + (y-0)\dot{y} + (z-0)\dot{z} = 3.8941504270435656 \cdot r_2 \\
(x-4)\dot{x} + (y-0)\dot{y} + (z-0)\dot{z} = 3.894951626856594 \cdot r_3 \\
(x+1)\dot{x} + (y-\sqrt{3})\dot{y} + (z-0)\dot{z} = 3.8961310021157534 \cdot r_4 \\
(x+2)\dot{x} + (y-2\sqrt{3})\dot{y} + (z-0)\dot{z} = 3.8989284653247402 \cdot r_5 \\
(x+1)\dot{x} + (y+\sqrt{3})\dot{y} + (z-0)\dot{z} = 3.8896389634680446 \cdot r_6 \\
(x+2)^2 + (y+2\sqrt{3})^2 + (z-0)^2 = r_1^2 \\
(x-2)^2 + (y-0)^2 + (z-0)^2 = r_2^2 \\
(x-4)^2 + (y-0)^2 + (z-0)^2 = r_3^2 \\
(x+1)^2 + (y-\sqrt{3})^2 + (z-0)^2 = r_4^2 \\
(x+2)^2 + (y-2\sqrt{3})^2 + (z-0)^2 = r_5^2 \\
(x+1)^2 + (y+2\sqrt{3})^2 + (z-0)^2 = r_6^2
\end{cases}
$$

and the simplified system is

$$
(60)\quad \begin{cases}
-13.142857142857142 + 14.091546933694675 r_4^2 - 18.786077764507947 r_4 r_5 \\
+6.507002110304725 r_5^2 - 9.398837071638766 r_4 r_6 + 5.772125889899168 r_5 r_6 \\
+1.8142548595411274 r_6^2 = 0 \\
\\
3.1975781088237003 r_4^2 - 4.000351601820357 r_4 r_5 + 1.201375786990289 r_5^2 \\
-2.3958077727188263 r_4 r_6 + 1.5984616958701363 r_5 r_6 \\
+0.398745390848532 r_6^2 = 0 \\
\\
-16.571428571428573 + 20.352860593228623 r_4^2 - 25.09046689866869 r_4 r_5 \\
+7.999030346612654 r_5^2 - 15.622814521038018 r_4 r_6 + 9.095792878159973 r_5 r_6 \\
+3.2656081881260945 r_6^2 = 0.
\end{cases}
$$

Table 2 lists the number of correct digits for the two-triangle configuration. The first row of table is the position of the object. As in Table 2 we can see that solving the original system is slightly more accurate than solving the simplified system by the homotopy method. However, solving the simplified system is much cheaper than solving the original system.

TABLE 2
The number of correct digits for the two triangles configuration.

	$(2,5,3)$	$(20,50,30)$	$(2,5,3)\times 10^2$	$(2,5,3)\times 10^3$	$(2,5,3)\times 10^4$
original	14	12	10	8	6
simplified	13	11	9	7	5

(Regular Octahedron): Given six stations located at $\{(\pm 2,0,0),(0,\pm 2,0),(0,$

$0, \pm 2)\}$. We assume the moving object is at the position $(2000, 5000, 3000)^T$ with the velocity $(1, 2, 4)^T$. Then the measured Doppler data should be

$$\dot{r}_1 = 3.8933993124376407 \qquad \dot{r}_4 = 3.8932285531303217$$
$$\dot{r}_2 = 3.893689806539583 \qquad \dot{r}_5 = 3.8929384652559484$$
$$\dot{r}_3 = 3.8926308053089875 \qquad \dot{r}_6 = 3.8939968805122525.$$

Hence, the original polynomial system is

$$(61) \quad \begin{cases} (x-2)\dot{x} + (y-0)\dot{y} + (z-0)\dot{z} = 3.8933993124376407 \cdot r_1 \\ (x-0)\dot{x} + (y-2)\dot{y} + (z-0)\dot{z} = 3.893689806539583 \cdot r_2 \\ (x-0)\dot{x} + (y-0)\dot{y} + (z-2)\dot{z} = 3.8926308053089875 \cdot r_3 \\ (x+2)\dot{x} + (y-0)\dot{y} + (z-0)\dot{z} = 3.8932285531303217 \cdot r_4 \\ (x+0)\dot{x} + (y+2)\dot{y} + (z-0)\dot{z} = 3.8929384652559484 \cdot r_5 \\ (x+0)\dot{x} + (y-0)\dot{y} + (z+2)\dot{z} = 3.8939968805122525 \cdot r_6 \\ (x-2)^2 + (y-0)^2 + (z-0)^2 = r_1^2 \\ (x-0)^2 + (y-2)^2 + (z-0)^2 = r_2^2 \\ (x-0)^2 + (y-0)^2 + (z-2)^2 = r_3^2 \\ (x+2)^2 + (y-0)^2 + (z-0)^2 = r_4^2 \\ (x-0)^2 + (y+2)^2 + (z-0)^2 = r_5^2 \\ (x-0)^2 + (y-0)^2 + (z+2)^2 = r_6^2 \end{cases}$$

and the simplified system is shown as (62).

Solving (61) by the polyhedral homotopy continuation method, HOM4PS-2.0 takes 5.5 seconds, and solving (62) by the total degree homotopy continuation method, HOM4PS-2.0 takes about 0.01 seconds.

Table 3 lists the number of correct digits for the regular octahedron configuration. The first row of table is the position of the object, and "x" indicates the computation is out of the range that HOM4PS-2.0 can handle.

TABLE 3
The number of correct digits for the regular octahedrons configuration.

	$(2,5,3)$	$(20,50,30)$	$(2,5,3)\times 10^2$	$(2,5,3)\times 10^3$	$(2,5,3)\times 10^4$
original	13	11	10	8	5
simplified	14	12	9	x	x

(62)

$$
\left\{
\begin{array}{l}
-57253.72995800387r_3 + 10731.495952145619r_3^3 - 17887.03497970073r_3^2r_4 \\
+25046.15277794521r_3r_4^2 - 7156.873445635487r_4^3 - 12518.431025849644r_3^2r_5 \\
-3577.243811839096r_3r_4r_5 + 12517.694371289494r_3r_5^2 + 3578.21707777564r_4r_5^2 \\
+2.761926496344131r_5^3 - 57273.82251357924r_6 + 19677.13747668851r_3^2r_6 \\
-39366.78805717656r_3r_4r_6 + 25054.942444871875r_4^2r_6 \\
-25045.648458149586r_3r_5r_6 - 3578.4992055077487r_4r_5r_6 \\
+12522.08731599397r_5^2r_6 + 19687.182960489263r_3r_6^2 \\
-17900.84788994176r_4r_6^2 - 12527.218974045432r_5r_6^2 \\
+10740.913543932473r_6^3 = 0 \\
\\
4096 - 511.8293306215535r_3^2 + 47.968007553398124r_3^4 + 511.876486556012r_3r_4 \\
-95.93306310294683r_3^3r_4 - 511.9775448544891r_4^2 + 175.91172741865824r_3^2r_4^2 \\
-95.96421325697403r_3r_4^3 + 15.995789937200477r_4^4 + 255.88098811324448r_3r_5 \\
-47.950187618455416r_3^3r_5 - 47.96611126900205r_3^2r_4r_5 \\
+15.990457565858954r_3r_4^2r_5 - 255.9506060265429r_5^2 \\
+51.934831684462935r_3^2r_5^2 + 47.979161478789855r_3r_4r_5^2 \\
-15.994808112949734r_4^2r_5^2 - 7.970052529113161r_3r_5^3 + 3.998458969823517r_5^4 \\
-767.9280624563597r_3r_6 + 143.91521981580115r_3^3r_6 + 512.0561238785305r_4r_6 \\
-335.90545082603944r_3^2r_4r_6 + 367.9504324523441r_3r_4^2r_6 \\
-95.99789082330076r_4^3r_6 + 255.97078668144462r_5r_6 \\
-135.90195851620345r_3^2r_5r_6 - 95.96588887754612r_3r_4r_5r_6 \\
+15.996069237929111r_4^2r_5r_6 + 95.90485185739405r_3r_5^2r_6 \\
+47.995999228385884r_4r_5^2r_6 - 7.972849529823812r_5^3r_6 \\
-512.0987791064762r_6^2 + 195.9589737112044r_3^2r_6^2 \\
-336.0401770376238r_3r_4r_6^2 + 176.0408328536567r_4^2r_6^2 \\
-135.9468450894679r_3r_5r_6^2 \\
-47.99978351595655r_4r_5r_6^2 + 51.9684825149158r_5^2r_6^2 + 144.0302882281241r_3r_6^3 \\
-96.05094832072851r_4r_6^3 - 47.99788039887575r_5r_6^3 + 48.01852431706125r_6^4 = 0 \\
\\
-0.0005438837042719413r_3^2 - 1.9990702196347225r_3r_5 + 1.9996141095823665r_5^2 \\
+1.9996137284617352r_3r_6 - 1.999771770948786r_5r_6 \\
+0.00015773525706852531r_6^2 = 0 \\
\\
-0.0003510060768343592 5r_3^2 + 11.997105153656532r_3r_4 - 11.99947370752709r_4^2 \\
-9.995351098173611r_3r_5 + 9.998070547911833r_5^2 - 2.00140413612705r_3r_6 \\
+12.001315403403057r_4r_6 - 9.99885885474393r_5r_6 \\
-0.0010532532514888615r_6^2 = 0.
\end{array}
\right.
$$

From Table 1, 2 and 3 we see that simplified systems for the regular pentagon and two regular triangle configurations are strongly recommended for solving the original system (1) and (2). The simplified system for the regular octahedron is recommended only when the distances between the stations and the object are not too far (less than 100 meters).

4.2. Newton's Method vs Homotopy Method. In this subsection we will compare the reliability of Newton's method and the homotopy method. Suppose six stations form a regular pentagon around a centre with radius 20 m:

$$\begin{aligned} &\mathbf{u}_1 = (0,0,0) && \mathbf{u}_4 = \left(20\cos\tfrac{4\pi}{5}, 20\sin\tfrac{4\pi}{5}, 0\right) \\ &\mathbf{u}_2 = \left(20\cos\tfrac{0\pi}{5}, 20\sin\tfrac{0\pi}{5}, 0\right) && \mathbf{u}_5 = \left(20\cos\tfrac{6\pi}{5}, 20\sin\tfrac{6\pi}{5}, 0\right) \\ &\mathbf{u}_3 = \left(20\cos\tfrac{2\pi}{5}, 20\sin\tfrac{2\pi}{5}, 0\right) && \mathbf{u}_6 = \left(20\cos\tfrac{8\pi}{5}, 20\sin\tfrac{8\pi}{5}, 0\right). \end{aligned}$$

Suppose the trajectories of the moving object and the associated velocity from $t = 0$ to $t = 100$ (seconds) are, respectively, given by

$$\left\{\begin{array}{l} x = 10000\sin(0.05t-1) \\ y = 10000\cos(0.1t-10) \\ z = 10\exp(0.05t+3) \end{array}\right. \qquad \left\{\begin{array}{l} \dot{x} = 500\cos(0.05t-1) \\ \dot{y} = -1000\sin(0.1t-10) \\ \dot{z} = 0.5\exp(0.05t+3), \end{array}\right. \tag{63}$$

where the unit of the position is meter and the unit of the velocity is m/s = 3.6 km/hr. The objects of the moving object of (63) are shown in Figure 1.

Denote $\mathbf{v} = (x, y, z, \dot{x}, \dot{y}, \dot{z})$. For the given data $\{(x_j, y_j, z_j)\}_{j=1}^6$ and we have, for $j = 1, ..., 6$,

$$\dot{r}_j = \frac{(x-x_j)\dot{x} + (y-y_j)\dot{y} + (z-z_j)\dot{z}}{\sqrt{(x-x_j)^2 + (y-y_j)^2 + (z-z_j)^2}},$$

as in Section 2, thus Newton's method can be applied in solving (3).

Newton's iteration:

$$\left\{\begin{array}{l} \text{Given an approximate solution } \mathbf{v}^0, \\ \mathbf{v}^{i+1} = \mathbf{v}^i - DF\left(\mathbf{v}^i\right)^{-1} F\left(\mathbf{v}^i\right), \ \ i = 0, 1, 2, \cdots, \end{array}\right.$$

where $F = (F_1, \cdots, F_6)^T$ is given in (3) and DF is the Jacobian of F.

Experiment 1: Given initial solution $\mathbf{v}(0) = (x(0), y(0), z(0), \dot{x}(0), \dot{y}(0), \dot{z}(0))^T$, the Doppler radar responds measured data $\{\dot{r}_j\}_{j=1}^6$ every Δt seconds. The previous solution $\mathbf{v}(t_{i-1})$ is taken as an initial vector of Newton's iteration for computing $\mathbf{v}(t_i)$. Figure 2 shows the log scale plot of the relative error for $\Delta t = 0.1, 0.2, 0.3, 0.4,$ 0.5, and 1.0. Newton's method can trace the trajectory of the moving object for $\Delta t = 0.1, 0.2, 0.3$. For $\Delta t = 0.4$ and $\Delta t = 0.5$, $\mathbf{v}(21)$ is not in the convergent region of Newton's iteration. When $\Delta t = 1.0$, Newton's iteration converges to the other solution at $t = 4$.

Homotopy Method:

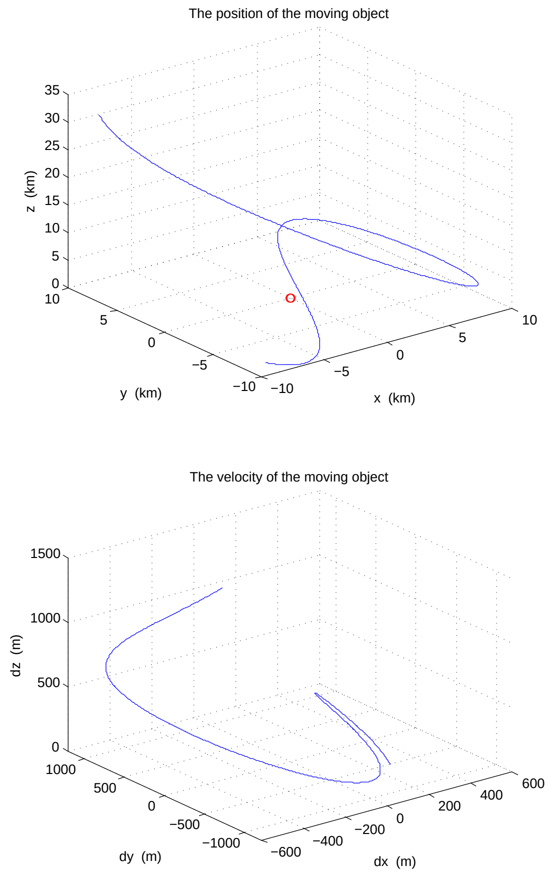

FIG. 1. *the orbit of the moving object.*

(i) Solving the polynomial system $\{q_j = 0\}_{j=1}^{3}$ as in (48) by the polyhedral homotopy continuation method (Hom4ps-2.0) and get 8 solutions in variables r_4, r_5 and r_6.

(ii) Compute the other variables $\{r_1, r_2, r_3, x, y, z, \dot{x}, \dot{y}, \dot{z}\}$ by

$$\left\{\begin{array}{l} r_1 = \frac{5+\sqrt{5}}{10}\frac{\dot{r}_4}{\dot{r}_1}r_4 - \frac{\sqrt{5}}{5}\frac{\dot{r}_5}{\dot{r}_1}r_5 + \frac{5+\sqrt{5}}{10}\frac{\dot{r}_6}{\dot{r}_1}r_6 \\ r_2 = \frac{\dot{r}_4}{\dot{r}_2}r_4 - \frac{1+\sqrt{5}}{2}\frac{\dot{r}_5}{\dot{r}_2}r_5 + \frac{1+\sqrt{5}}{2}\frac{\dot{r}_6}{\dot{r}_2}r_6 \\ r_3 = \frac{1+\sqrt{5}}{2}\frac{\dot{r}_4}{\dot{r}_3}r_4 - \frac{1+\sqrt{5}}{2}\frac{\dot{r}_5}{\dot{r}_3}r_5 + \frac{\dot{r}_6}{\dot{r}_3}r_6, \end{array}\right.$$

$$\left\{\begin{array}{l} \mathbf{u} = (x, y, z)^T = (V_0^T C^T C V_0)^{-1}(V_0^T C^T C)\frac{(\mathbf{n}-\mathbf{r}\circ\mathbf{r})}{2}, \\ \dot{\mathbf{u}} = (\dot{x}, \dot{y}, \dot{z})^T = (V_0^T C^T C V_0)^{-1}(V_0^T C^T C)(-\dot{R}\mathbf{r}). \end{array}\right.$$

Experiment 2: The Doppler radar responds measured data $\{\dot{r}_j\}_{j=1}^{6}$ every Δt seconds. We select the solution $\mathbf{v}(t)$ which is closest to $\mathbf{v}(t_{i-1})$. Figure 3 shows the log

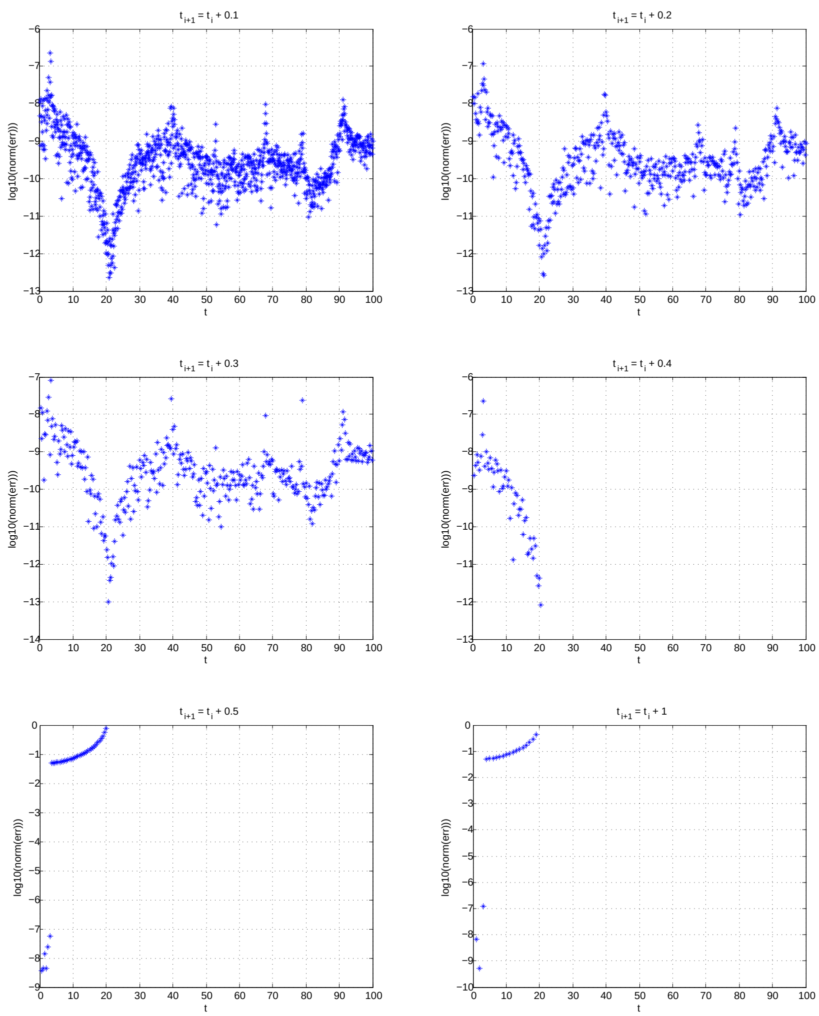

Fig. 2. *The log scale plot of the relative error for Newton's method.*

scale plot of the relative error for $\Delta t = 0.1, 0.2, 0.3, 0.4, 0.5$, and 1.0. The polynomial homotopy method can successfully trace the trajectory of the moving object in these cases.

Remark: Newton's method can quickly compute the position of the moving object if the Doppler radar can quickly responds measured data. Otherwise, we will not have a good initial vector to start with. The polynomial homotopy method does not need any initial vectors from users. The method solves a polynomial system and outputs all solutions. Users need to choose an appropriate solution among them. If stations sit on a plane but do not form a circle, the polynomial system can be reduced to 3 independent quadratic polynomials. The reduced system is numerically stable. Although the polynomial system can be reduced to a system of 4 polynomials of de-

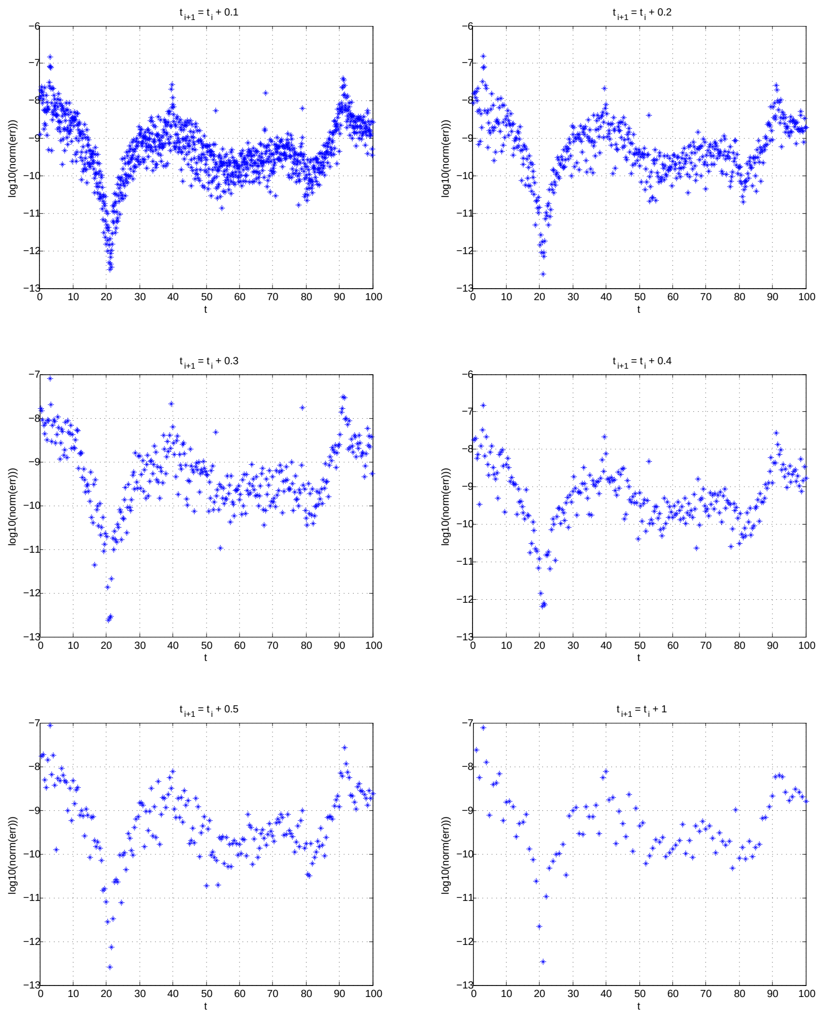

FIG. 3. *The log scale plot of the relative error for the polynomial homotopy method.*

grees 4, 3, 2 and 2 for the cases that stations do not sit on a plane, the reduced system may be numerical unstable. Therefore, we suggest that stations are deployed on a plane appropriately, and combine two methods into one algorithm: at the beginning, trace the position by Newton's method. Whenever the iteration fails, the polynomial homotopy method is invoked to get back solutions.

5. Conclusions. In this paper, we propose a novel simplification to reduce the original system of a moving object to a new system of only three polynomials all of degrees 2 in 3 unknowns when the 6 observation stations are located on a plane but not on a circle. Numerical results show that the homotopy method is robust and efficient for solving the simplified polynomial system when the 6 stations form a regular pentagon and its centre. One possible future work is to consider that whether this particular configuration of observation stations is optimal for the homotopy method.

Appendix A. For a generic V_0 we will show that $\widehat{A}$ in (16) has rank 2. WLOG we may assume V_0 and CV_0 are of full rank. For simplicity, the spectrum of $M \in \mathbb{R}^{n\times n}$ is denoted by $\sigma(M)$. Re-write $\widehat{A} = -\widehat{C}V_0A + \widehat{C} = \widehat{C}(I - V_0A)$. Since $AV_0 = (V_0^TC^TCV_0)^{-1}(V_0^TC^TC)V_0 = I_3$, $\sigma(AV_0) = \{1,1,1\}$. It can be shown that $\sigma(V_0A) = \{1,1,1,0,0,0\}$ (Theorem 1.3.20 in [8]). Therefore, $\sigma(I - V_0A) = \{0,0,0,1,1,1\}$. It is easily seen that three columns of V_0 and the vector $\mathbf{e} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$ are in the eigenspaces of $I - V_0A$ associated with eigenvalue 0 and 1, respectively. Moreover, they are all in the kernel of $\widehat{A}$. Therefore, $rank(\widehat{A}) \le 6 - 4 = 2$. Now we will show that $rank(\widehat{A}) < 2$ only when $\mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 + \mathbf{u}_5 + \mathbf{u}_6 = 5\mathbf{u}_1$. Let $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$ whose column space $col(X)$ is in the eigenspace of $I-V_0A$ associated with eigenvalue 1 but complemented with the space spanned by vector $\mathbf{e}$. Then (i) $X \ne O$ and (ii) $(I - V_0A)X = X$. From (ii) we have $V_0AX = O$; that is, $V_0(V_0^TC^TCV_0)^{-1}V_0^TC^TCX = O$. Since V_0 and $(V_0^TC^TCV_0)^{-1}$ are of full rank, we have $V_0^TC^TCX = O$. Let $K = CX \in \mathbb{R}^{5\times 2}$. Then $K \ne O_{5\times 2}$ due to $\mathbf{e} \notin col(X)$. In addition, $C\begin{bmatrix} \mathbf{0}^T \\ K \end{bmatrix} = K$ and $C\begin{bmatrix} a_1\mathbf{e} & a_2\mathbf{e} \end{bmatrix} = O$ for any $a_1, a_2 \in \mathbb{R}$, so we can write $X = \begin{bmatrix} \mathbf{0}^T \\ K \end{bmatrix} + \begin{bmatrix} a_1\mathbf{e} & a_2\mathbf{e} \end{bmatrix}$. Consider

$$\widehat{A}X = \widehat{C}(I - V_0A)X = \widehat{C}X = \widehat{C}\begin{bmatrix} \mathbf{0}^T \\ K \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} K.$$

From the above matrix equality, we can see that there exists a vector $\mathbf{x}_0 \in col(X)$ such that $\widehat{A}\mathbf{x}_0 = \mathbf{0}$ (hence, $rank(\widehat{A}) < 2$) if and only if $\hat{\mathbf{e}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$ is in the column space of K. In this case, we have $V_0^TC^T\hat{\mathbf{e}} = \mathbf{0}$, which is equivalent to $\mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 + \mathbf{u}_5 + \mathbf{u}_6 = 5\mathbf{u}_1$.

Appendix B. The system of (23) can be simplified to a system of 2 polynomials of degrees 12 and 16 with 2 unknowns. Expanding the last 2 equations in (23) yields

$$(64)\quad \begin{cases} a_3r_3^2 + a_4r_4^2 + a_5r_5^2 + a_6r_6^2 + a_{34}r_3r_4 + a_{35}r_3r_5 + \cdots + a_{56}r_5r_6 + a_0 = 0 \\ b_3r_3^2 + b_4r_4^2 + b_5r_5^2 + b_6r_6^2 + b_{34}r_3r_4 + b_{35}r_3r_5 + \cdots + b_{56}r_5r_6 + b_0 = 0. \end{cases}$$

We rewrite the equations of (64) in the quadratic form: $\widetilde{\mathbf{r}}^T A\widetilde{\mathbf{r}}+a_0=0$ and $\widetilde{\mathbf{r}}^T B\widetilde{\mathbf{r}}+b_0=0$, where

$$A_0=\begin{bmatrix} a_3 & \frac{a_{34}}{2} & \frac{a_{35}}{2} & \frac{a_{36}}{2} \\ \frac{a_{34}}{2} & a_4 & \frac{a_{45}}{2} & \frac{a_{46}}{2} \\ \frac{a_{35}}{2} & \frac{a_{45}}{2} & a_5 & \frac{a_{56}}{2} \\ \frac{a_{36}}{2} & \frac{a_{46}}{2} & \frac{a_{56}}{2} & a_6 \end{bmatrix}, \quad B_0=\begin{bmatrix} b_3 & \frac{b_{34}}{2} & \frac{b_{35}}{2} & \frac{b_{36}}{2} \\ \frac{b_{34}}{2} & b_4 & \frac{b_{45}}{2} & \frac{b_{46}}{2} \\ \frac{b_{35}}{2} & \frac{b_{45}}{2} & b_5 & \frac{b_{56}}{2} \\ \frac{b_{36}}{2} & \frac{b_{46}}{2} & \frac{b_{56}}{2} & b_6 \end{bmatrix},$$

and $\widetilde{\mathbf{r}}=\begin{bmatrix} r_3 & r_4 & r_5 & r_6 \end{bmatrix}^T$. Since A_0 and B_0 are symmetric, we assume there is a nonsingular V_0 such that

$$V_0^T A_0 V_0=\begin{bmatrix} \widetilde{a}_3 & 0 & 0 & 0 \\ 0 & \widetilde{a}_4 & 0 & 0 \\ 0 & 0 & \widetilde{a}_5 & 0 \\ 0 & 0 & 0 & \widetilde{a}_6 \end{bmatrix}, \quad V_0^T B_0 V_0=\begin{bmatrix} \widetilde{b}_3 & 0 & 0 & 0 \\ 0 & \widetilde{b}_4 & 0 & 0 \\ 0 & 0 & \widetilde{b}_5 & 0 \\ 0 & 0 & 0 & \widetilde{b}_6 \end{bmatrix}.$$

By changing variables $\begin{bmatrix} r_3 & r_4 & r_5 & r_6 \end{bmatrix}^T = V_0\begin{bmatrix} s_3 & s_4 & s_5 & s_6 \end{bmatrix}^T$, the system (64) becomes

$$\text{(65)} \qquad \begin{cases} \widetilde{a}_3 s_3^2+\widetilde{a}_4 s_4^2+\widetilde{a}_5 s_5^2+\widetilde{a}_6 s_6^2+a_0=0 \\ \widetilde{b}_3 s_3^2+\widetilde{b}_4 s_4^2+\widetilde{b}_5 s_5^2+\widetilde{b}_6 s_6^2+b_0=0. \end{cases}$$

Without loss of generality, we assume the matrix $\widetilde{C}\equiv\begin{bmatrix} \widetilde{a}_3 & \widetilde{a}_4 \\ \widetilde{b}_3 & \widetilde{b}_4 \end{bmatrix}$ is invertible. From (65) follows from

$$\text{(66)} \qquad \begin{cases} s_3^2=c_{35}s_5^2+c_{36}s_6^2+c_{30} \\ s_4^2=c_{45}s_5^2+c_{46}s_6^2+c_{40}, \end{cases}$$

where $\begin{bmatrix} c_{35} & c_{36} & c_{30} \\ c_{45} & c_{46} & c_{40} \end{bmatrix}=\widetilde{C}^{-1}\begin{bmatrix} \widetilde{a}_5 & \widetilde{a}_6 & a_0 \\ \widetilde{b}_5 & \widetilde{b}_6 & b_0 \end{bmatrix}$. Recall that the first two equations in (23) are polynomials in variables r_3, r_4, r_5 and r_6 of degree 3 and 4, respectively. By changing variables above in the 1st equation of (23), the degrees of new equations remain the same. In fact, each of them can be written as

$$s_3 g_1+s_4 g_2=s_3 s_4 g_3+g_4,$$

where g_1, g_2, g_3 and g_4 are polynomials in s_3, s_4, s_5 and s_6, and the exponents of s_3 and s_4 in g_i's are even numbers. Squaring both sides yields

$$\text{(67)} \qquad s_3^2 g_1^2+s_4^2 g_2^2+2g_1 g_2 s_3 s_4=s_3^2 s_4^2 g_3^2+2g_3 g_4 s_3 s_4+g_4^2.$$

Collecting $s_3 s_4$ on one side, we have $2\left(g_1 g_2-g_3 g_4\right) s_3 s_4=s_3^2 s_4^2 g_3^2+g_4^2-s_3^2 g_1^2-s_4^2 g_2^2$. Squaring both sides of the new equation results in one equation which has variables s_3, s_4, s_5 and s_6, and the exponents of s_3 and s_4 are even numbers in all terms. By using (66), the new equation only involves s_5 and s_6. Therefore, the system (23) can be simplified to a system of two polynomials of degree 12 and 16 in two unknowns.

REFERENCES

[1] E. L. Allgower and K. Georg, *Continuation and path following*, ACTA Numerica, (1993), pp. 1-64.

[2] J. W. Armstrong, *Low-frequency gravitational wave searches using spacecraft Doppler tracking*, Living Rev. Relativity, 9(2006).

[3] B. Bertotti, G. Comoretto, and I. Iess, *Doppler tracking of spacecraft with multi-frequency links*, Astron. Astrophys., 269(1993), pp. 608-616.

[4] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed., Johns Hopkins Univ. Press, Baltimore, MD, 1996.

[5] B. Huber and B. Sturmfels, *A polyhedral method for solving sparse polynomial systems*, Math. of Comp., 64(1995), pp. 1541-1555.

[6] T. Y. Li, *Solving polynomial systems by the homotopy continuation method*, Handbook of numerical analysis, Vol. XI, (pp. 209-304), North-Holland, Amsterdam (2003).

[7] T. L. Lee, T. Y. Li, and C, H. Tsai, *HOM4PS-2.0: A software package for solving polynomial systems by the polyhedral homotopy continuation method*, Computing, 83(2008), pp. 109-133.

[8] R. A. Horn and C. B. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[9] S. N. Salinger and J. J. Brandstatter, *Application of recursive estimation and Kalman filtering to Doppler tracking*, IEEE transactions on aerospace and electronic systems,6(1970).