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CONTROL AND STABILIZATION OF THE KAWAHARA EQUATION ON A PERIODIC DOMAIN*

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Abstract. In this paper, we study a class of distributed parameter control system described by the Kawahara equation posed on a periodic domain $\mathbb{T}$ (a unit circle in the plane) with an internal control acting on an arbitrary small nonempty subdomain of $\mathbb{T}$. Aided by the Bourgain smoothing property of the Kawahara equation on a periodic domain, we show that the system is locally exactly controllable and exponentially stabilizable with an arbitrarily large decay rate.

Keywords and phrases: Kawahara equation, well-posedness, exact controllability, feedback control, stabilizability.

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1. Introduction. Originally as a model for propagation of water waves with small amplitude, the Korteweg-de Vries (KdV) equation

$$u_t + \beta u_{3x} + uu_x = 0 \tag{1.1}$$

has been extremely intensively studied from various aspects since the discovery of solitons and inverse scattering method in the 1960's. The equation is now commonly viewed not only as a good model for some water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects [6, 9, 23].

The KdV equation (1.1) admits a compressive or rarefactive steady solitary solution depending on the sign of the coefficient of third-order dispersive term β in (1.1). However, under certain conditions, the coefficient β will become very small or even vanish. Consequently, a higher dispersive term has to be introduced to the equation to balance the dispersive effect and the nonlinear effect. For instance, Kakutani and Ono [17] demonstrated that if the angle between the propagation direction and the magnetic-acoustic wave in a cold collision-free plasma and the external magnetic field become critical value, then the third-order dispersive term vanishes and is replaced by the fifth-order dispersive term. Hasimoto[11] derived a fifth-order dispersive equation to describe the shallow water near the critical value of surface tension when an effect of the surface tension is taken into account. Kawahara [19] investigated the solitary

*Dedicated to Professor Hanfu Chen on the occasion of his 75th birthday.

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waves of the general fifth-order dispersive equation:

$$w_t + \alpha w_{5x} + \beta w_{3x} + 3\gamma w w_x = 0, \tag{1.2}$$

where α, $\beta \in R$ and $\alpha \neq 0$. He showed through numerical simulations that similar to the KdV equation, the fifth-order dispersive equation (1.2) also admits compressive or rarefactive steady solitary solution which is decided by the sign of the dispersive term. In order to understand the effect of surface tension on solitary water waves Hunter and Scheurle [14] studied the solitary behavior of (1.2). They found out that when Bond numbers is less than $\frac{1}{3}$, there are branches of traveling wave solutions to the water wave equations bifurcated from Froude number 1 and Bond number $\frac{1}{3}$, which are perturbations of supercritical elevation solitary waves. The equation (1.2) is now usually known as Kawahara equation, or the fifth-order KdV equation in the literature [1, 14, 18, 19].

Since the late 1980s, control theory of nonlinear dispersive wave equations have attracted a lot of attentions because of the rapid advances of the mathematical theory of nonlinear dispersive wave equations. Many new tools have been developed which enable us to attack the problems that seemed untouchable before. In particular, following the advances of mathematical theory of the KdV equation, control theory of the KdV equation has been intensively studied and significant progresses have been made through many people's work (see [4, 5, 7, 22, 26, 28, 29, 32] and the references therein). In contrast, there are relative few works on the Kawahara equation for its control theory (cf. [8, 27]).

In this paper we consider the Kawahara equation (1.2) posed on the periodic domain $\mathbb{T}$ (a unit circle in the plane)[1]. Without loss of generality, we may assume $\alpha = -1$, $\gamma = \frac{1}{3}$ and $\beta = -1$, 0 or 1. Furthermore, by the transformation $u = w - a$ with

$$a = [w_0] \equiv \frac{1}{2\pi}\int_{\mathbb{T}} w_0(x)dx, \; w_0(x) = w(x,0),$$

we find that if w solves (1.2), then $[w(\cdot,t)] \equiv a$, for any $t \in \mathbb{R}$ and u satisfies the following equation

$$u_t - u_{5x} + \beta u_{3x} + a u_x + u u_x = 0, \; x \in \mathbb{T}, \; t \in \mathbb{R} \tag{1.3}$$

with $[u(\cdot,t)] = 0$ for any $t \in \mathbb{R}$. The Cauchy problem of (1.3) has been intensively studied for its well-posedness in the space $H^s(\mathbb{T})$ following the footsteps of the study

[1] This is equivalent to impose the periodic boundary conditions over the interval $(0, 2\pi)$:

$$w_{jx}(0,t) = w_{jx}(2\pi,t), \quad j = 0,1,2,3,4.$$

of the KdV equation in the literature (see [12, 13, 20, 3, 10, 16] and the references therein). The best known result [16] so far is that the Cauchy problem is locally well-posed in the space $H_0^s(\mathbb{T}) := \{v \in H^s(\mathbb{T}) : [v] = 0\}$ for any $s \geq -\frac{3}{2}$ and is globally well-posed in the space $H_0^s(\mathbb{T})$ for $s \geq -1$.

The equation (1.3) will be studied in this paper from control point of view with a forcing term $f = f(x,t)$ added to the equation as a control input:

$$u_t - u_{5x} + \beta u_{3x} + au_x + uu_x = f, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{1.4}$$

where f is assumed to be supported in a given open set $\omega \subset \mathbb{T}$. The following exact control problem and stabilization problem are fundamental in control theory.

Exact control problem: *Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input f so that the equation (1.4) admits a solution u which satisfies*

$$u|_{t=0} = u_0, \qquad u|_{t=T} = u_1?$$

Stabilization problem: *Can one find a feedback control law: $f = Ku$ so that the resulting closed-loop system*

$$u_t - u_{5x} + \beta u_{3x} + au_x + uu_x = Ku, \quad x \in \mathbb{T},\ t \in \mathbb{R}^+$$

is asymptotically stable as $t \to +\infty$?

Note that for solution u of the system (1.4) for the Kawahara equation, its mass $\int_{\mathbb{T}} u(x,t)dx$ is conserved:

$$\frac{d}{dt}\int_{\mathbb{T}} u(x,t)dx = \int_{\mathbb{T}} f(x,t)dx = 0$$

for any $t \in \mathbb{R}$ when no control is in action ($f \equiv 0$). In applications, one would also like to keep the mass conserved while conducting control. For that purpose, it is sufficient to put the following constrain on our control input f:

$$\int_{\mathbb{T}} f(x,t)dx = 0, \qquad \forall\ t \in \mathbb{R}.$$

Thus, as in [28], the control input $f(x,t)$ is chosen to be of the form

$$f(x,t) = [Gh](x,t) := g(x)\left(h(x,t) - \int_{\mathbb{T}} g(y)h(y,t)dy\right) \tag{1.5}$$

where h is considered as a new control input, and $g(x)$ is a given nonnegative smooth function such that $\{g > 0\} = \omega$ and

$$2\pi[g] = \int_{\mathbb{T}} g(x)dx = 1.$$

The resulting control system is of the form

$$u_t - u_{5x} + \beta u_{3x} + au_x + uu_x = Gh, \quad x \in \mathbb{T},\ t \in \mathbb{R}. \tag{1.6}$$

The following two theorems, which address the exact controllability problem and stabilizability problem for the system (1.6), are main results of this paper.

THEOREM 1.1 (Exact controllability). *Let $T > 0$ and $s \geq -1$ be given. There exists a $\delta > 0$. For any u_0, $u_1 \in H^s(\mathbb{T})$ with*

$$\|u_0\|_{H^s(\mathbb{T})} \leq \delta, \ \ \|u_1\|_{H^s(\mathbb{T})} \leq \delta,$$

one can find a control function $h \in L^2([0,T]; H^s(\mathbb{T}))$ such that the system:

$$u_t - u_{5x} + \beta u_{3x} + au_x + uu_x = Gh,\ x \in \mathbb{T},\ t \in (0,T)$$

admits a solution $u \in C([0,T]; H^s(\mathbb{T}))$ satisfying

$$u|_{t=0} = u_0, \ \ u|_{t=T} = u_1.$$

THEOREM 1.2 (Stabilizability). *Let $s \geq 0$ and $\lambda > 0$ be given. There exists bounded linear operator $M_\lambda : H^s(\mathbb{T}) \to H^s(\mathbb{T})$ such that if one chooses the feedback control*

$$h = K_\lambda u$$

in the system (1.6), then the resulting closed-loop system

$$\begin{cases} u_t - u_{5x} + \beta u_{3x} + au_x + uu_x = GK_\lambda u, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{T} \end{cases} \tag{1.7}$$

is locally exponentially stable in the space $H^s(\mathbb{T})$:

There exists $\delta > 0$ such that for any $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s(\mathbb{T})} < \delta$, the corresponding solution u of (1.7) satisfies

$$\|u(\cdot,t) - [u_0]\|_{H^s(\mathbb{T})} \leq Ce^{-\lambda t}\|u_0 - [u_0]\|_{H^s(\mathbb{T})},$$

for any $t > 0$.

The paper is organized as follows. In Section 2, we consider the associated linear system (dropping the nonlinear term uu_x). The controllability of the linear open loop system is established through solving a moment problem. Then the linear system is shown to be exponentially stabilizable with arbitrarily large decay rate. In Section 3 we show the nonlinear system is locally exactly controllable in the space $H^s(\mathbb{T})$ for any $s \geq -1$. The Bourgain smoothing properties of the Kawahara equation on a periodic domain will play a key role in the proof. In Section 4, the nonlinear feedback system is first shown to be globally well-posed in the space $H^s(\mathbb{T})$ for any $s \geq 0$ and then it is shown to be locally exponentially stabilizable with arbitrarily large decay rate.

2. Linear systems. Consideration is first given to the associated linear open loop control system

$$v_t - v_{5x} + \beta v_{3x} + a v_x = Gh, \quad v(x,0) = v_0(x), \quad x \in \mathbb{T},\ t \in \mathbb{R} \tag{2.1}$$

where the operator G is as defined in Section 1 and $h = h(x,t)$ is the applied control input.

Let A denote the operator

$$Aw = \frac{d^5 w}{dx^5} - \beta \frac{d^3 w}{dx^3} - a \frac{dw}{dx}$$

with the domain $\mathcal{D}(A) = H^5(\mathbb{T})$. It generates a strongly continuous group $W(t)$ on the space $L^2(\mathbb{T})$ and its eigenfunctions are simply the orthonormal Fourier basis functions in $L^2(\mathbb{T})$,

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx},\ k = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \cdots .$$

The corresponding eigenvalue of ϕ_k is

$$\lambda_k = (-k^5 + \beta k^3 - ak)i, \quad k = 0, \pm 1, \pm 2, \cdots .$$

For any $l \in \mathbb{Z}$, let

$$m(l) = \#\{k \in \mathbb{Z}; \quad \lambda_k = \lambda_l\}.$$

Then $m(l) \leq 5$ for any l and $m(l) = 1$ if l is large enough. Moreover,

$$\lim_{|k| \to \infty} |\lambda_k - \lambda_{k+1}| = \infty.$$

The solution v of the system (2.1) can be expressed in the form

$$v(x,t) = \sum_{k=-\infty}^{\infty} \left(e^{\lambda_k t} v_{0,k} + \int_0^t e^{\lambda_k (t-\tau)} G_k[h](\tau) d\tau \right) \phi_k(x) \tag{2.2}$$

where $v_{0,k}$ and $G_k[h]$ are the Fourier coefficients of v_0 and $G[h]$, respectively,

$$v_{0,k} = (v_0, \phi_k)_{L^2(\mathbb{T})}, \quad G_k[h] = (Gh, \phi_k)_{L^2(\mathbb{T})} = (h, G\phi_k)_{L^2(\mathbb{T})}$$

for $k = 0, \pm 1, \pm 2, \cdots$. Furthermore, for given $s \in \mathbb{R}$, if $v_0 \in H^s(\mathbb{T})$ and $h \in L^2(0,T; H^s(\mathbb{T}))$, the function given by (2.2) belongs to the space $C([0,T]; H^s(\mathbb{T}))$.

We have the following exact controllability result for the system (2.1).

THEOREM 2.1. *Let $T > 0$ and $s \in \mathbb{R}$ be given. There exists a bounded linear operator*

$$\Phi :\ H^s(\mathbb{T}) \times H^s(\mathbb{T}) \to L^2(0,T; H^s(\mathbb{T}))$$

such that for any $v_0,\ v_1 \in H^s(\mathbb{T})$, if one chooses $h = \Phi(v_0, v_1)$ in (2.1), then the system (2.1) admits a solution $v \in C([0,T]; H^s(\mathbb{T}))$ satisfying

$$v|_{t=0} = v_0, \qquad v|_{t=T} = v_1.$$

In the sequel we will denote by C a constant which may be different from line to line. Moreover,

$$\|f\|_s := \|f\|_{H^s(\mathbb{T})} \text{ for any } f \in H^s(\mathbb{T})$$

and

$$\|f\| := \|f\|_0.$$

Proof. For given $v_0,\ v_1 \in H^s(\mathbb{T})$, we need to find $h \in L^2(0,T; H^s(\mathbb{T}))$ such that

$$v_1(x) = \sum_{k=-\infty}^{\infty} \left(e^{\lambda_k T} v_{0,k} + \int_0^T e^{\lambda_k (T-\tau)} G_k[h](\tau) d\tau \right) \phi_k(x)$$

or

$$\sum_{k=-\infty}^{\infty} \left(e^{-\lambda_k T} v_{1,k} - v_{0,k} \right) \phi_k = \sum_{k=-\infty}^{\infty} \int_0^T e^{-\lambda_k \tau} G_k[h](\tau) d\tau \phi_k(x)$$

which is equivalent to

$$e^{-\lambda_k T} v_{1,k} - v_{0,k} = \int_0^T e^{-\lambda_k \tau} G_k[h](\tau) d\tau \tag{2.3}$$

for $k = \pm 1, \pm 2, \cdots$.

If we define $p_k = e^{\lambda_k t}$, then $\mathcal{P} \equiv \{p_k | -\infty < k < \infty\}$ will form a Riesz basis for its closed span $\mathcal{P}_T$ in $L^2(0,T)$. We let $\mathcal{Q} \equiv \{q_k | -\infty < k < \infty\}$ be the unique dual Riesz basis for $\mathcal{P}$ in $\mathcal{P}_T$ such that

$$\int_0^T q_j(t) \overline{p_k(t)} dt = \delta_{jk}, \ \ -\infty < j,\ k < \infty. \tag{2.4}$$

We take the control h in (2.3) to have the form

$$h(x,t) = \sum_{j=-\infty}^{\infty} h_j q_j(t) (G\phi_j)(x), \tag{2.5}$$

where the coefficients h_j are to be determined so that, among other things, the series (2.5) is appropriately convergent. Substituting (2.5) into (2.3) yields, using the biorthogonality (2.4), that

$$\begin{aligned} e^{-\lambda_k T} v_{1,k} - v_{0,k} &= \sum_{-\infty}^{\infty} h_j \int_0^T e^{-\lambda_k t} q_j(t) \int_{\mathbb{T}} G(G\phi_j)(x) \overline{\phi_k(x)} dx dt \\ &= h_k \int_{\mathbb{T}} G(G\phi_k)(x) \overline{\phi_k(x)} dx dt. \end{aligned} \tag{2.6}$$

for $-\infty < k < \infty$. As G is a self-adjoint operator in $L^2(\mathbb{T})$,

$$\int_{\mathbb{T}} G(G\phi_k)(x)\overline{\phi_k(x)}dxdt = \|G\phi_k\|^2, \ -\infty < k < \infty.$$

We have

$$\begin{aligned}\|G\phi_k\|^2 &= \int_{\mathbb{T}} \left| g(x)\Big(\phi_k(x) - \int_{\mathbb{T}} g(s)\phi_k(s)ds\Big)\right|^2 dx \\ &= \frac{1}{4\pi^2}\int_{\mathbb{T}} g^2(x)dx - 2\left|\int_{\mathbb{T}} g(x)\phi_k(x)dx\right|^2 + \int_{\mathbb{T}} g^2(x)dx\left|\int_{\mathbb{T}} g(x)\phi_k(x)dx\right|^2 \\ &=: \beta_k.\end{aligned}$$

It is easy to see that $\beta_0 = 0$ and $\beta_k \neq 0$ if $k \neq 0$. Moreover, the familiar Lebesgue lemma together with the second identity above shows that

$$\lim_{k\to\infty} \beta_k = \frac{1}{2\pi}\int_{\mathbb{T}} g^2(x)dx \neq 0.$$

It follows that there is a $\delta > 0$ such that

$$\beta_k > \delta, \text{ for } k \neq 0.$$

Setting $h_0 = 0$ and

$$h_k = \frac{e^{-\lambda_k T}v_{1,k} - v_{0,k}}{\beta_k}, \ k \neq 0. \tag{2.7}$$

It remains to show that h defined by (2.5) and (2.7) is in $L^2([0,T];H^s(\mathbb{T}))$ provided that v_0, $v_1 \in H^s(\mathbb{T})$. To this end, let us write

$$G\phi_j(x) = \sum_{k=-\infty}^{\infty} a_{jk}\phi_k(x),$$

where

$$a_{jk} = \int_{\mathbb{T}} G\phi_j(x)\overline{\phi_k(x)}dx = (G\phi_j(x), \phi_k(x))_{L^2(\mathbb{T})}, \ -\infty < j, \ k < \infty.$$

Thus

$$h(x,t) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h_j a_{jk} q_j(t)\phi_k(x), \tag{2.8}$$

and

$$\begin{aligned}\|h\|^2_{L^2([0,T];H^s(\mathbb{T}))} &= \int_0^T \sum_{k=-\infty}^{\infty} < k >^{2s} \Big| \sum_{j=-\infty}^{\infty} h_j a_{jk} q_j(t)\Big|^2 dt \\ &= \sum_{k=-\infty}^{\infty} < k >^{2s} \int_0^T \Big| \sum_{j=-\infty}^{\infty} h_j a_{jk} q_j(t)\Big|^2 dt \\ &\leq C \sum_{k=-\infty}^{\infty} < k >^{2s} \sum_{j=-\infty}^{\infty} |h_j|^2 |a_{jk}|^2 \\ &= C \sum_{j=-\infty}^{\infty} |h_j|^2 \sum_{k=-\infty}^{\infty} < k >^{2s} |a_{jk}|^2,\end{aligned}$$

where the constant C comes from the Riesz basis property of $\mathcal{Q}$ in $\mathcal{P}_T$. However

$$\begin{aligned}|a_{jk}| &= (G\phi_j, \phi_k)_{L^2(\mathbb{T})} \\ &= |(g\phi_j, \phi_k)_{L^2(\mathbb{T})} - (g, \phi_j)_{L^2(\mathbb{T})}(g, \phi_k)_{L^2(\mathbb{T})}| \\ &= \Big| \sum_{m=-\infty}^{\infty} g_m (\phi_m \phi_j, \phi_k)_{L^2(\mathbb{T})} - \Big(\sum_{m=-\infty}^{\infty} g_m (\phi_m, \phi_j)_{L^2(\mathbb{T})}\Big) \\ &\quad \times \Big(\sum_{m=-\infty}^{\infty} g_m (\phi_m, \phi_k)_{L^2(\mathbb{T})}\Big)\Big| \\ &= \Big| \frac{1}{2\pi} g_{k-j} - g_j g_k \Big| \\ &\leq \frac{1}{2\pi} |g_{k-j}| + |g_j||g_k|\end{aligned}$$

where

$$g(x) = \sum_{j=-\infty}^{\infty} g_m \phi_m(x).$$

Hence

$$|a_{jk}|^2 \leq C(|g_{k-j}|^2 + |g_k|^2 |g_j|^2)$$

and

$$\begin{aligned}\sum_{k=-\infty}^{\infty} < k >^{2s} |a_{jk}|^2 &\leq C \sum_{k=-\infty}^{\infty} < k >^{2s} |g_{k-j}|^2 + C \sum_{k=-\infty}^{\infty} < k >^{2s} |g_j|^2 |g_k|^2 \\ &= C \sum_{k=-\infty}^{\infty} < k+j >^{2s} |g_k|^2 + C|g_j|^2 \sum_{k=-\infty}^{\infty} < k >^{2s} |g_k|^2.\end{aligned}$$

Thus, in the case of $s \geq 0$,

$$\begin{aligned}\sum_{k=-\infty}^{\infty} < k >^{2s} |a_{jk}|^2 &\leq C \sum_{k=-\infty}^{\infty} < k >^{2s} < j >^{2s} |g_k|^2 + C|g_j|^2 \sum_{k=-\infty}^{\infty} < k >^{2s} |g_k|^2 \\ &= C(< j >^{2s} + |g_j|^2) \|g\|_s^2.\end{aligned}$$

We have, according to (2.7), that

$$\begin{aligned}
\|h\|^2_{L^2([0,T];H^s(\mathbb{T}))} &\le C\Big\{\sum_{j=-\infty}^{\infty}(< j >^{2s} + |g_j|^2)|h_j|^2\Big\}\|g\|_s^2 \\
&\le C\Big\{\sum_{j=-\infty}^{\infty}(< j >^{2s} + |g_j|^2)\frac{|e^{-\lambda_j T}v_{1,j} - v_{0,j}|^2}{\beta_j^2}\Big\}\|g\|_s^2 \\
&\le C\max_{j\neq 0}|\beta_j|^{-2}(1+\|g\|_0^2)\sum_{j=-\infty}^{\infty} < j >^{2s} (|v_{1,j}|^2 + |v_{0,j}|^2) \\
&\le C\max_{j\neq 0}\frac{1}{|\beta_j|^2}(1+\|g\|_0^2)\|g\|_s^2\Big(\|v_1\|_s^2 + \|v_0\|_s^2\Big).
\end{aligned}$$

In the case of $s < 0$, as for any $-\infty < k,\ j < \infty$,

$$< j >^{-2s} < k + j >^{2s} \le < k >^{-2s}, \quad < j >^{-2s} |g_j|^2 \le \|g\|^2_{-s},$$

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} < j >^{-2s} < k >^{2s} |a_{jk}|^2 &\le C\sum_{k=-\infty}^{\infty} < j >^{-2s} < k + j >^{2s} |g_k|^2 \\
&\quad + C < j >^{-2s} |g_j|^2 \sum_{k=-\infty}^{\infty} < k >^{2s} |g_k|^2 \\
&\le C(1+\|g\|_s^2)\|g\|^2_{-s}
\end{aligned}$$

and therefore

$$\begin{aligned}
\|h\|^2_{L^2([0,T];H^s(\mathbb{T}))} &\le C\sum_{j=-\infty}^{\infty}|h_j|^2\sum_{k=-\infty}^{\infty} < k >^{2s} |a_{jk}|^2 \\
&\le C\sum_{j=-\infty}^{\infty} < j >^{2s} |h_j|^2 \sum_{k=-\infty}^{\infty} < j >^{-2s} < k >^{2s} |a_{jk}|^2 \\
&\le C(1+\|g\|_s^2)\|g\|^2_{-s}\sum_{j=-\infty}^{\infty} < j >^{2s} |h_j|^2 \\
&\le C\max_{j\neq 0}\frac{1}{|\beta_j|^2}(1+\|g\|_s^2)\|g\|^2_{-s}\Big(\|v_1\|_s^2 + \|v_0\|_s^2\Big).
\end{aligned}$$

□

Now we turn to consider feedback stabilization problem of the linear system (2.1). We show that it is possible to choose an appropriate linear feedback law such that the decay rate of the resulting closed-loop system is as large as one desires.

For any $\lambda > 0$, define

$$L_\lambda\phi = \int_0^1 e^{-2\lambda\tau}W(-\tau)GG^*W^*(-\tau)\phi d\tau$$

for any $\phi \in H^s(\mathbb{T})$. Clearly, L_λ is a bounded linear operator from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$. Moreover, L_λ is a self-adjoint positive operator on $L^2_0(\mathbb{T}) := \{\phi \in L^2(\mathbb{T}) : [\phi] = 0\}$ and so is its inverse L_λ^{-1}. Indeed, the following result holds for the operator L_λ.

LEMMA 2.2. *For any $s \geq 0$, the operator L_λ is an isomorphism from $H^s(\mathbb{T})$ onto $H^s(\mathbb{T})$ for all $s \geq 0$.*

Proof. See Lemma 2.4 in [22]. □

According to Lemma 2.2, L_λ has bounded inverse in $H^s(\mathbb{T})$. Taking the control function $h(x,t) = -G^* L_\lambda^{-1} v(x,t)$, employing the following feedback control law,

$$K_\lambda v(x,t) \equiv -GG^* L_\lambda^{-1} v(x,t),$$

we obtain the following closed-loop system, which is exponentially stable,

$$v_t - v_{5x} + \beta v_{3x} + a v_x = -K_\lambda v, \quad v(x,0) = v_0(x), \quad x \in \mathbb{T}. \tag{2.9}$$

PROPOSITION 2.3. *Let $s \geq 0$ and $\lambda > 0$ be given. Then for any $v_0 \in H^s(\mathbb{T})$, the system (2.9) admits a unique solution $v \in C([0,T]; H^s(\mathbb{T}))$. Moreover there exist positive constants M_s depending only on s such that*

$$\|v(\cdot,t)\|_s \leq M_s e^{-\lambda t} \|v_0\|_s \tag{2.10}$$

for any $t > 0$.

Proof. The existence of the solution v follows from the standard semigroup theory [24]. For the decay estimate (2.10), it suffices to provide the proof for the cases $s = 0$ and $s = 5$. The case of $0 < s < 5$ follows by interpolation. The other cases of s can be proved similarly.

The case of $s = 0$ follows from [30]. For $s = 5$, let $w = v_t$. Then w solves

$$w_t - w_{5x} + \beta w_{3x} + a w_x = -K_\lambda w, \quad w(x,0) = w_0(x), \quad x \in \mathbb{T}$$

where $w_0(x) = v_0^{(5)}(x) - \beta v_0^{(3)}(x) - a v_0'(x) - K_\lambda v_0(x)$. Thus

$$\|w(\cdot,t)\| = \|v_t(\cdot,t)\| \leq C e^{-\lambda t} \|w_0\|$$

for any $t \geq 0$. It then follows from

$$v_{5x} - \beta v_{3x} - a v_x - K_\lambda v = w$$

that

$$\|v(\cdot,t)\|_5 \leq e^{-\lambda t} \|v_0\|_5$$

for any $t \geq 0$. □

3. Exact controllability. In this section, we study the exact controllability for the open loop nonlinear control system

$$u_t - u_{5x} + \beta u_{3x} + a u_x + u u_x = Gh, \quad u(x,0) = u_0(x), \quad x \in \mathbb{T} \tag{3.1}$$

and prove Theorem 1.1; the system (3.1) is locally exactly controllable in the space $H_0^s(\mathbb{T})$ for any $s \geq -1$. Some technical preparations are needed before presenting our proof for Theorem 1.1.

For given $b, s \in \mathbb{R}$, and a function $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, define the quantities

$$||u||_{X_{b,s}} := \left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k \rangle^{2s} \langle \tau - p(k) \rangle^{2b} |\widehat{u}(k,\tau)|^2 \, d\tau \right)^{\frac{1}{2}},$$

$$||u||_{Y_{b,s}} := \left(\sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} \langle k \rangle^{s} \langle \tau - p(k) \rangle^{b} |\widehat{u}(k,\tau)| \, d\tau \right)^{2} \right)^{\frac{1}{2}}$$

where $\widehat{u}(k,\tau)$ denotes the Fourier transform of u with respect to the space variable x and the time variable t, $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$ and $p(k) = k^5 - \beta k^3 + ak$. The Bourgain space $X_{b,s}$ (resp. $Y_{b,s}$) associated to the Kawahara equation on $\mathbb{T}$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm $\|u\|_{X_{b,s}}$ (resp. $\|u\|_{Y_{b,s}}$). Note that for any $u \in X_{b,s}$,

$$\|u\|_{X_{b,s}} = \|W(-t)u\|_{H^b(\mathbb{R}, H^s(\mathbb{T}))}.$$

For given $b, s \in \mathbb{R}$, let

$$Z_{b,s} = X_{b,s} \cap Y_{b-\frac{1}{2},s}$$

be endowed with the norm

$$||u||_{Z_{b,s}} = ||u||_{X_{b,s}} + ||u||_{Y_{b-\frac{1}{2},s}}.$$

For a given interval I, let $X_{b,s}(I)$ (resp. $Z_{b,s}(I)$) be the restriction space of $X_{b,s}$ to the interval I with the norm

$$\|u\|_{X_{b,s}(I)} = \inf \left\{ \|\widetilde{u}\|_{X_{b,s}} \mid \widetilde{u} = u \text{ on } \mathbb{T} \times I \right\}$$

$$(\text{resp. } \|u\|_{Z_{b,s}(I)} = \inf \left\{ \|\widetilde{u}\|_{Z_{b,s}} \mid \widetilde{u} = u \text{ on } \mathbb{T} \times I \right\}).$$

For simplicity, we denote $X_{b,s}(I)$ (resp. $Z_{b,s}(I)$) by $X_{b,s}^T$ (resp. $Z_{b,s}^T$) if $I = (0,T)$. In addition, let

$$\mathbb{Z}_{\frac{1}{2},s}^T := Z_{\frac{1}{2},s}^T \cap C([0,T]; H^s(\mathbb{T})).$$

The following estimates related to the Bourgain space $X_{b,s}^T$ and $Z_{b,s}^T$ will play important roles in establishing the exact controllability and stabilizability of the nonlinear Kawahara equation.

LEMMA 3.1. *Let $b, s \in \mathbb{R}$ and $T > 0$ be given. There exists a constant $C > 0$ such that*

(i) for any $\phi \in H^s(\mathbb{T})$,

$$\|W(t)\phi\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \leq C\|\phi\|_s;$$

(ii) for any $f \in Z^T_{-\frac{1}{2},s}$,

$$\left\| \int_0^t W(t-\tau)f(\tau)\, d\tau \right\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \leq C\|f\|_{Z^T_{-\frac{1}{2},s}}.$$

Proof. See [12]. □

LEMMA 3.2. *Let $s \geq -1$ and $T > 0$ be given. There exist a constant C such that the following bilinear estimate*

$$\|(uv)_x\|_{Z^T_{-\frac{1}{2},s}} \leq C\|u\|_{Z^T_{\frac{1}{2},s}} \|v\|_{Z^T_{\frac{1}{2},s}}$$

holds.

Proof. Let $\lambda = 1$ in Theorem 1.3 of [12], we obtain the result on $Z^1_{\frac{1}{2},s}$. Furthermore, taking $\psi(\frac{t}{T})$ as the cut-off function, we obtain Lemma 3.2 for bilinear estimate on $Z^T_{\frac{1}{2},s}$. □

Now we turn to prove Theorem 1.1.

Proof of Theorem 1.1. Rewrite the system (3.1) in its equivalent integral equation form:

$$u(t) = W(t)u_0 + \int_0^t W(t-\tau)(Gh)(\tau)d\tau - \int_0^t W(t-\tau)(uu_x)(\tau)d\tau. \tag{3.2}$$

Define

$$\omega(T, u) := \int_0^T W(T-\tau)(uu_x)(\tau)d\tau.$$

According to Theorem 2.1, for given u_0, $u_1 \in H^s_0(\mathbb{T})$, if one chooses

$$h = \Phi(u_0, u_1 + \omega(T, u))$$

in the equation (3.2), then

$$u(t) = W(t)u_0 + \int_0^t W(t-\tau)(G\Phi(u_0, u_1 + \omega(T, u)))(\tau)d\tau - \int_0^t W(t-\tau)(uu_x)(\tau)d\tau.$$

and

$$u|_{t=0} = u_0,\ u|_{t=T} = u_1.$$

This leads us to consider the map

$$\Gamma u(t) = W(t)u_0 + \int_0^t W(t-\tau)(G\Phi(u_0, u_1 + \omega(T,u)))(\tau)d\tau - \int_0^t W(t-\tau)(uu_x)(\tau)d\tau.$$

If we can prove that Γ is a contraction mapping in an appropriate space, then its fixed point u is a solution of (3.2) with $h = \Phi(u_0, u_1 + \omega(T,u))$ and satisfies $u|_{t=T} = u_1$.

Applying Lemma 3.1-3.2 yields that

$$\begin{aligned}\|\Gamma u\|_{\mathbb{Z}^T_{\frac{1}{2},s}} &\le C\|u_0\|_s + C\Big\|\int_0^t W(t-\tau)(G\Phi(u_0, u_1 + w(T,u)))(\tau)d\tau\Big\|_{Z^T_{\frac{1}{2},s}} \\ &\quad + C\|uu_x\|_{Z^T_{-\frac{1}{2},s}} \\ &\le C\|u_0\|_s + C\|G\Phi(u_0, u_1+\omega(T,u))\|_{L^2([0,T];H^s_0(\mathbb{T}))} + C\|u\|^2_{\mathbb{Z}^T_{\frac{1}{2},s}} \\ &\le C\|u_0\|_s + C\Big[\|u_1\|_s + \|u_0\|_s + \|\omega(T,u)\|_s\Big] + C\|u\|^2_{\mathbb{Z}^T_{\frac{1}{2},s}}.\end{aligned}$$

Notice that

$$\begin{aligned}\|w(T,u)\|_s &= \Big\|\int_0^T W(T-\tau)(uu_x)(\tau)d\tau\Big\|_s \\ &\le C \sup_{t\in(0,T)} \Big\|\int_0^t W(t-\tau)(uu_x)(\tau)d\tau\Big\|_s \\ &\le C\|u\|^2_{\mathbb{Z}^T_{\frac{1}{2},s}}.\end{aligned}$$

Consequently,

$$\|\Gamma(u)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C(\|u_0\|_s + \|u_1\|_s) + C\|u\|^2_{\mathbb{Z}^T_{\frac{1}{2},s}}.$$

For $R > 0$, let B_R be a bounded subset of $\mathbb{Z}^T_{\frac{1}{2},s}$:

$$B_R = \{v \in \mathbb{Z}^T_{\frac{1}{2},s} |\ [v] = 0,\ \|v\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le R\}.$$

Then, for any $u \in B_R$

$$\|\Gamma(u)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C(\|u_0\|_s + \|u_1\|_s) + CR^2.$$

We choose $\delta > 0$ and $R > 0$ such that

$$2C\delta + CR^2 \le R,\ \ CR < \frac{1}{2}.$$

Then, $\|\Gamma(u)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le R$, that is Γ map B_R into itself. In addition, for any u, $v \in B_R$, similarly, we have

$$\|\Gamma(u) - \Gamma(v)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le \frac{1}{2}\|u-v\|_{\mathbb{Z}^T_{\frac{1}{2},s}}.$$

Γ is thus a contracting map on B_R. By the Banach fixed point theorem, there is a unique solution to the integral equation (3.2) which is the desired solution of (3.1).□

4. Stabilizability. In this section, we investigate stability properties of the closed-loop system

$$\begin{cases} u_t - u_{5x} + \beta u_{3x} + a u_x + u u_x = -K_\lambda u, \ x \in \mathbb{T}, \ t > 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{T}. \end{cases} \tag{4.1}$$

First we consider the associated linear system

$$\begin{cases} u_t - u_{5x} + \beta u_{3x} + a u_x = -K_\lambda u, \ x \in \mathbb{T}, \ t > 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{T}. \end{cases} \tag{4.2}$$

Its solution can be written as

$$u(t) = W_\lambda(t) u_0$$

where W_λ is the C_0-semigroup associated to the linear system (4.2).

LEMMA 4.1. *Let $s \in \mathbb{R}$ and $T > 0$ be given. There exists a constant $C > 0$ such that*

(i)

$$\|W_\lambda(t)\phi\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C\|\phi\|_s$$

for any $\phi \in H_0^s(\mathbb{T})$;

(ii)

$$\left\| \int_0^t W_\lambda(t-\tau) f(\tau) d\tau \right\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C\|f\|_{Z^T_{-\frac{1}{2},s}}$$

for any $f \in Z^T_{-\frac{1}{2},s}$.

Proof. For given $\phi \in H_0^s(\mathbb{T})$ and $f \in Z^T_{-\frac{1}{2},s}$, let

$$u(t) = W_\lambda(t)\phi + \int_0^t W_\lambda(t-\tau) f(\tau) d\tau.$$

Then it u solves

$$\begin{cases} u_t - u_{5x} + \beta u_{3x} + a u_x = -K_\lambda u + f, \ x \in \mathbb{T}, \ t > 0, \\ u(x,0) = \phi(x), \ x \in \mathbb{T}. \end{cases} \tag{4.3}$$

Consequently,

$$u(t) = W(t)\phi + \int_0^t W(t-\tau) f(\tau) d\tau - \int_0^t W(t-\tau)[K_\lambda u](\tau) d\tau$$

and for any $0 < T' \le T$,

$$\|u\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C\left(\|\phi\|_s + \|f\|_{Z^T_{-\frac{1}{2},s}} \right) + C\|K_\lambda u\|_{Z^{T'}_{-\frac{1}{2},s}}$$

where $C > 0$ depends only on s and T. As

$$\|K_\lambda u\|_{Z^{T'}_{-\frac{1}{2},s}} \le C_1 \|K_\lambda u\|_{X^{T'}_{-\frac{1}{2}+\epsilon,s}} \le C_1 (T')^\nu \|u\|_{Z^T_{\frac{1}{2},s}}$$

for some $\nu > 0$ and C_1 depending only on s and T. Thus if T' is chosen small enough, we have

$$\|u\|_{\mathbb{Z}^{T'}{}_{\frac{1}{2},s}} \le C \left(\|\phi\|_s + \|f\|_{Z^T_{-\frac{1}{2},s}} \right).$$

It then follows from the semigroup property of the system (4.3) that

$$\|u\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C \left(\|\phi\|_s + \|f\|_{Z^T_{-\frac{1}{2},s}} \right).$$

The proof is complete. □

We first show the system (4.1) is well-posed in the space $H^s(\mathbb{T})$ for any $s \ge -1$.

PROPOSITION 4.2. *Let $T > 0$ and $s \ge -1$ be given. Then there exists a $\delta > 0$ such that for any $u_0 \in H^s(\mathbb{T})$ with*

$$\|u_0\|_s \le \delta,$$

the system (4.1) admits a unique solution $u \in \mathbb{Z}^T_{\frac{1}{2},s}$. Moreover, the corresponding solution map is Lipschitz continuous.

Proof. Rewrite the system (4.1) in its equivalent integral equation form:

$$u(t) = W_\lambda(t)u_0 - \int_0^t W_\lambda(t-\tau)(uu_x)(\tau)d\tau. \tag{4.4}$$

Then define the map

$$\Gamma u(t) = W_\lambda(t)u_0 - \int_0^t W_\lambda(t-\tau)(uu_x)(\tau)d\tau.$$

Applying Lemma 4.1 yields

$$\|\Gamma u\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C\|u_0\|_s + \left\| \int_0^t [W_\lambda(t-\tau)(u^2)_x](\tau)d\tau \right\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C\|u_0\|_s + C\|u\|^2_{\mathbb{Z}^T_{\frac{1}{2},s}}.$$

For $R > 0$, let B_R be a bounded subset of $\mathbb{Z}^T_{\frac{1}{2},s}$:

$$B_R = \{v \in \mathbb{Z}^T_{\frac{1}{2},s} |\ [v] = 0,\ \|v\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le R\}.$$

Then, for any $u \in B_R$

$$\|\Gamma(u)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C\|u_0\|_s + CR^2.$$

We choose $\delta > 0$ and $R > 0$ such that

$$C\delta + CR^2 \leq R, \quad CR < \frac{1}{2}.$$

Then, $\|\Gamma(u)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \leq R$, which suggests Γ maps B_R into itself. In addition, for any u, $v \in B_R$, similarly, we have

$$\|\Gamma(u) - \Gamma(v)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \leq \frac{1}{2}\|u - v\|_{\mathbb{Z}^T_{\frac{1}{2},s}},$$

the map Γ is thus a contracting mapping on B_R whose fixed point is the desired solution of the system (4.1). □

REMARK 4.3. *The local well-posedness result presented in Proposition 4.2 can be restated as follows.*

Let $s \geq -1$ and $r > 0$ be given. There exists a $T > 0$ such that for any $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_s \leq r$, the system (4.1) admits a unique solution $u \in \mathbb{Z}^T_{\frac{1}{2},s}$.

Next we show that the system (4.1) is globally well-posed in the space $H^s(\mathbb{T})$ for any $s \geq 0$.

THEOREM 4.4. *Let $s \geq 0$ and $T > 0$ be given. For any $u_0 \in H^s(\mathbb{T})$, the system (4.1) admits a unique solution $u \in \mathbb{Z}^T_{\frac{1}{2},s}$. Furthermore, the following estimate holds*

$$\|u\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \leq \alpha_{T,s}(\|u\|_0)\|u_0\|_s,$$

where $\alpha_{T,s} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending only on T and s.

Proof. The proof is very much similar to that of Theorem 4.7 in [22] and is therefore omitted. □

Now we present the proof of Theorem 1.2 showing that the closed-loop system (4.1) is locally exponentially stable in the space $H^s(\mathbb{T})$.

Proof of Theorem 1.2. For given $s \geq 0$ and $\lambda > 0$, by Proposition 2.3, there exists positive constant C such that

$$\|W_\lambda(t)u_0\|_s \leq Ce^{-\lambda t}\|u_0\|_s, \ \forall\, t \geq 0.$$

For any given $0 < \lambda' < \lambda$, pick $T > 0$ such that

$$2Ce^{-\lambda T} \leq e^{-\lambda' T}.$$

We seek a solution u to the integral equation (4.4) as a fixed point of the map

$$\Gamma u(t) = W_\lambda(t)u_0 - \int_0^t W_\lambda(t-\tau)(uu_x)(\tau)d\tau$$

in some closed ball $B_R(0)$ in the function space $\mathbb{Z}^T_{\frac{1}{2},s}$. This will be done provided that $\|u_0\|_s \leq \delta$ where δ is a small number to be determined. Furthermore, to ensure the exponential stability with the claimed decay rate, the numbers δ and R will be chosen in such a way that

$$\|u(T)\|_s \le e^{-\lambda' T}\|u_0\|_s.$$

By Lemma 4.1, there exist some positive constant C_1, C_2 (independent of δ and R) such that

$$\|\Gamma(u)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C_1\|u_0\|_s + C_2\|u\|^2_{\mathbb{Z}^T_{\frac{1}{2},s}}$$

and

$$\|\Gamma(u_1)-\Gamma(u_2)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le C_2(\|u_1\|_{\mathbb{Z}^T_{\frac{1}{2},s}} + \|u_2\|_{\mathbb{Z}^T_{\frac{1}{2},s}})\|u_1-u_2\|_{\mathbb{Z}^T_{\frac{1}{2},s}}.$$

On the other hand, we have for some constant $C' > 0$ and all $u \in B_R(0)$

$$\begin{aligned}\|\Gamma(u)(T)\|_s &\le C_1\|W_\lambda(T)u_0\|_s + C_2\Big\|\int_0^T W_\lambda(T-\tau)(uu_x)(\tau)d\tau\Big\|_s \\ &\le e^{-\lambda T}\delta + C'R^2.\end{aligned}$$

Pick $\delta = C_4R^2$, where C_4 and R are chosen so that

$$\frac{C'}{C_4} \le Ce^{-\lambda T},\ (C_1C_4+C_2)R^2 \le R,\ 2C_2R \le \frac{1}{2}.$$

Then we have

$$\|\Gamma(u)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le R,\ \forall u \in B_R(0)$$

and

$$\|\Gamma(u_1)-\Gamma(u_2)\|_{\mathbb{Z}^T_{\frac{1}{2},s}} \le \frac{1}{2}\|u_1-u_2\|_{\mathbb{Z}^T_{\frac{1}{2},s}},\ \forall u_1,\ u_2 \in B_R(0).$$

Therefore, Γ is a contraction in $B_R(0)$. Furthermore, its unique fixed point $u \in B_R(0)$ fulfills

$$\|u(T)\|_s \le \|\Gamma(u)(T)\|_s \le e^{-\lambda' T}\delta.$$

Assume now that $0 < \|u_0\|_0 < \delta$. Changing δ into $\delta' \equiv \|u_0\|_s$ and R into $R' \equiv (\delta'/\delta)^{\frac{1}{2}}R$, we infer that

$$\|u(T)\|_s \le e^{-\lambda' T}\|u_0\|_s,$$

and an obvious induction yields

$$\|u(nT)\|_s \le e^{-\lambda' nT}\|u_0\|_s$$

for any $n \ge 0$. We infer by the semigroup property that there exists some positive constant $C > 0$ such that

$$\|u(t)\|_s \le Ce^{-\lambda' t}\|u_0\|_s$$

if $\|u_0\|_s \le \delta$. The proof is complete. □

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