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A SIMPLE EXAMPLE OF AN ADAPTIVE CONTROL SYSTEM*

A. S. MORSE†

Abstract. This paper describes and analyzes a switched adaptive control system which on the one hand is simple enough to allow for a fairly transparent analysis while on the other, is formidable enough to explain how one might deal with two of the most challenging attributes associated with any adaptive control problem, namely noise and un-modeled dynamics.

1. Introduction. With the development of many clever and innovative ideas, adaptive feedback control has come a very long way over the past forty years. Despite this, it is fair to say that the methodology has not achieved widespread acceptance in practice or within the broader control research community. Two of the main reasons for this are (1) the methodology is not articulated clearly enough so that non-experts can easily grasp key concepts and (2) there is no convincing performance theory upon which to base designs. In this paper we try to address the first of these issues by focusing on an example which on the one hand is simple enough to allow for a fairly transparent analysis, while on the other is formidable enough to exemplify how one might deal with two of the most challenging attributes associated with any adaptive control problem, namely noise and un-modeled dynamics. Although all of the ideas in the paper can be found in earlier work, especially in [1] and [2], it is hoped that this simple example will stimulate much needed further research addressing the aforementioned issues.

We begin in Section 2 with a detailed description of the overall system to be considered. The system consists of the process to be controlled, a multi-controller, a multi-estimator, a monitor and a dwell-time switching logic. For the sake of conciseness, we spend little time on motivation. Readers interested in a broader picture are referred to [2] and the references therein. In Section 3 we carry out a complete analysis of the system with the ultimate goal of obtaining an explicit bound on the response of the system to a bounded noise input. The bound appears in the inequality in (3.29) at the end of Section 3. Simulation results are presented in Section 4. The simulations suggest that an improvement in performance can be achieved with smaller dwell times. A partial explanation of why this is so is given in Section 5.

2. The System. The system to be considered consists of the process to be controlled $\mathbb{P}$, a linear multi-controller $\mathbb{C}$, a linear multi-estimator $\mathbb{E}$, a monitor $\mathbb{M}$ and

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†Department of Electrical Engineering, Yale University, New Haven, Ct., 06520-1968. E-mail: as.morse@yale.edu

a dwell time switching logic $\mathbb{S}$. In this section we define each of these sub-systems.

2.1. The Process $\mathbb{P}$. The *process* $\mathbb{P}$ is assumed to be accurately modeled by a linear system of the form

$$\dot{y} = .1y + bu + \mathbf{n} + \delta \circ y \quad b \in \{-1, 1\} \tag{2.1}$$

where b is an uncertain parameter whose value might be either 1 or -1, $\mathbf{n}$ is $\mathcal{L}^\infty$ bounded noise and δ is a linear operator representing un-modeled dynamics. We assume that δ has a rational transfer function whose poles have real parts smaller than the value -0.05. It will be convenient to parameterize the system in (2.1) by setting $b = 3 - 2q$ where q's value is in the parameter space $\{1, 2\}$. Thus with the new parameterization

$$\dot{y} = .1y + (3 - 2q)u + \mathbf{n} + \delta \circ y, \tag{2.2}$$

for some $q \in \{1, 2\}$.

2.2. The Multi-Controller $\mathbb{C}$. The *multi-controller* to be considered is described by the switched output-feedback law

$$u = -3(3 - 2\sigma)y \tag{2.3}$$

where σ is a piecewise-constant switching signal taking values in the parameter space $\{1, 2\}$. Note that if the "correct" value of q were known and σ were set equal to q, then this control would result in the closed-loop model $\dot{y} = -2.9y$ of the nominal open-loop subsystem $\dot{y} = .1y + (3 - 2q)u$. Since the correct value of q is not assumed to be known, an alternative procedure must be developed to generate σ. Towards this end, let we will consider a supervisory control system consisting of three subsystems which we describe next.

2.3. The Multi-Estimator $\mathbb{E}$. The first subsystem, called a *multi-estimator* $\mathbb{E}$, is described by the linear system

$$\dot{x}_1 = -2.9x_1 + 3y + u, \qquad x_1(0) = 0 \tag{2.4}$$

$$\dot{x}_2 = -2.9x_2 + 3y - u, \qquad x_2(0) = 0 \tag{2.5}$$

together with the *output estimation errors*

$$e_1 = x_1 - y \qquad\qquad e_2 = x_2 - y \tag{2.6}$$

Recognize that the multi-estimator consists of two observers, one for the nominal model $\dot{y} = .1y + u$ and the other for the nominal model $\dot{y} = .1y - u$. Thus if δ and $\mathbf{n}$ were both zero, the qth output estimator error would satisfy $\dot{e}_q = -2.9e_q$, and thus would tend to zero exponentially fast. Of course with $\mathbf{n} \neq 0$, convergence of e_q to zero cannot be concluded. Nonetheless the multi-estimator is useful because it provides guidance for selecting the value of σ.

2.4. The Monitor $\mathbb{M}$. In order to keep track of the relative sizes of the output estimation errors, use is made of the two "monitoring" signals μ_1 and μ_2 which are generated by the equations

$$\dot{\mu}_1 = -2\lambda\mu_1 + e_1^2, \quad \mu_1(0) = 0 \qquad \dot{\mu}_2 = -2\lambda\mu_2 + e_2^2, \quad \mu_2(0) = 0 \tag{2.7}$$

where

$$\lambda = 0.05 \tag{2.8}$$

Here λ is a design parameter which has been chosen to be smaller than the assumed stability margin of δ.

2.5. The Switching Logic $\mathbb{S}$. The switching logic to be considered is a simplified version of the dwell-time switching logic discussed in [2]. The logic is a hybrid dynamical system whose inputs are μ_1 and μ_2, and whose state is the ordered pair $\{\tau, \sigma\}$ where τ is a continuous-time variable called a *timing signal*. τ takes values in the closed interval $[0, \tau_D]$ where τ_D is a pre-specified positive design parameter called a *dwell time*. In the sequel we will discuss what needs to be considered in picking τ_D. The function of $\mathbb{S}$ is described by the computer flow diagram shown in Figure 2.1.

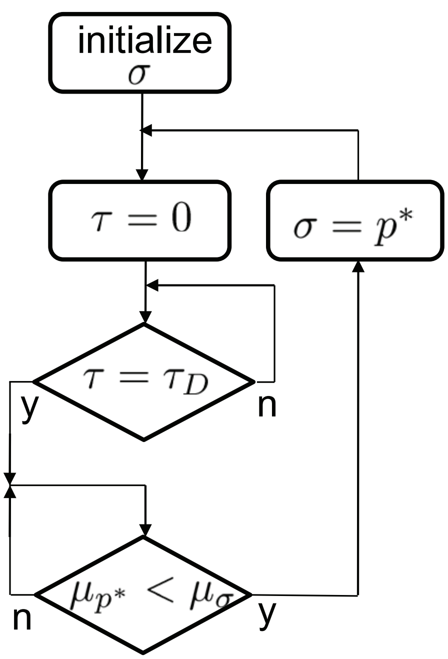

FIG. 2.1. *Dwell-Time Switching Logic*

Here $p^*(t)$ is a value of $p \in \{1, 2\}$ which minimizes $\mu_p(t)$. The diagram can be explained as follows. Initially τ is set equal to zero and σ is set equal to one of the values in $\{1,2\}$. Over the next τ_D time units, τ increases linearly to τ_D and σ is held fixed at its initial value. After τ_D time units have elapsed, μ_{p^*} is compared with μ_σ and τ is held fixed at τ_D. A switch in the value of σ occurs only if a time t_s is reached at which the value of the smallest $\mu_p(t_s),\ p \in \{1,2\}$ drops below the value of $\mu_\sigma(t_s)$. If this occurs $\sigma(t_s)$ is set equal to $p^*(t_s)$ and held fixed at this value while τ is reset to zero and then increased linearly to τ_D; τ is then held constant until σ again switches which is just when the value of the smallest $\mu_p,\ p \in \{1,2\}$ again drops below the value of $\mu_{\sigma(t_s^-)}(t_s)$ where t_s^- is the time instant just before t_s. And so on.

By the switching times of $\mathbb{S}$ are meant the times at which σ changes value. It will be convenient to regard time $t = 0$ as a switching time. We say that $\mathbb{S}$ is *dwelling* at time t if $t \in (t_s, t_s + \tau_D)$ where t_s is a time at which $\mathbb{S}$ switches. Note that if t is such a time, it is impossible to draw a conclusion about the relative sizes of $\mu_1(t)$ and $\mu_2(t)$ from properties of $\mathbb{S}$. On the other hand, if t is *not* a time at which $\mathbb{S}$ is dwelling, then as a consequence of the definition of $\mathbb{S}$,

$$\mu_{\sigma(t)}(t) \leq \mu_k(t),\ k \in \{1,2\} \tag{2.9}$$

Let us note that the outputs which $\mathbb{S}$ produces are piecewise constant signals taking values in $\{1,2\}$, which have the property that the time between any two successive switching times is bounded below by τ_D. We write $\mathcal{S}$ for the set of all such switching signals.

3. Analysis. The interconnection of the process $\mathbb{P}$, the multi-controller $\mathbb{C}$, the multi-estimator $\mathbb{E}$, the monitor $\mathbb{M}$ and the dwell-time switching logic $\mathbb{S}$ can be viewed as a hybrid dynamical system with input $\mathbf{n}$ and output y. In this section we analyze the behavior of this system.

3.1. The Injected System. Observe that application of the multi-controller described by (2.3) to the multi-estimator equations (2.4) and (2.5) results in the system

$$\dot{x}_1 = -2.9x_1 - 6(1-\sigma)y \tag{3.1}$$

$$\dot{x}_2 = -2.9x_2 + 6(\sigma-2)y \tag{3.2}$$

Note that

$$x_\sigma = (2-\sigma)x_1 + (\sigma-1)x_2 \tag{3.3}$$

and that

$$e_\sigma = x_\sigma - y \tag{3.4}$$

These two equations show that y can be written as

$$y = \begin{bmatrix} (2-\sigma) & (\sigma-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - e_\sigma \tag{3.5}$$

Substitution of this expression for y into (3.1) and (3.2) thus yields the equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2.9 & 6(1-\sigma)^2 \\ 6(2-\sigma)^2 & -2.9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 6 \begin{bmatrix} 1-\sigma \\ 2-\sigma \end{bmatrix} e_\sigma \tag{3.6}$$

In deriving (3.6) we have used the fact that $(1-\sigma)(\sigma-2) = 0$ for $\sigma \in \{1, 2\}$. By the *injected system* for the problem under consideration is meant the system described by (3.5) and (3.6). The injected system is thus a switched linear system depending on $\sigma \in \mathcal{S}$, with input e_σ, output y and input-output map h_I; i.e. $y = h_I \circ e_\sigma$. This system will play a central role in describing the behavior of the overall hybrid system under consideration.

3.2. Choosing τ_D. A key consideration in the selection of τ_D it to make sure that it is large enough so that for any switching signal $\sigma \in \mathcal{S}$, the injected system is exponentially stable. In this section we explain how this might be carried out.

We begin with the observation that the state equation (3.6) is a switched linear system of the form $\dot{x} = A_\sigma x + b_\sigma e_\sigma$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -2.9 & 0 \\ 6 & -2.9 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2.9 & 6 \\ 0 & -2.9 \end{bmatrix}$$

A simple calculation shows that

$$\left|e^{A_p t}\right| = (6t+1)e^{-2.9t}, \ \ t \geq 0 \quad\ \ p = 1, 2$$

where $|\cdot|$ denotes the induced infinity norm. From this it is easy to verify that

$$\left|e^{A_p t}\right| \leq e^{(.466 - 1.0t)}, \ \ p \in \{1, 2\}$$

Lemma 2 of [1] states that if $\tau_D > \frac{.466}{1.0}$, then for all $\sigma \in \mathcal{S}$, the state transition matrix of A_σ must satisfy $|\phi(t, \tau)| \leq e^{.466 - \omega t}, \ t \geq \tau \geq 0$ where $\omega = 1.0 - \frac{.466}{\tau_D}$. Thus if we pick

$$\tau_D = .5 \tag{3.7}$$

then

$$|\phi(t, \tau)| \leq e^{\{.466 - .068(t-\tau)\}}, \ t \geq \tau \geq 0, \quad \sigma \in \mathcal{S} \tag{3.8}$$

3.3. Analyzing Dwell-time Switching. In the sequel we view the combination of the monitor $\mathbb{M}$ and the dwell time switching logic $\mathbb{S}$ as a hybrid dynamical system with continuous inputs e_1, e_2 and piecewise constant output σ. It will be convenient to introduce the notation

$$||s||_T = \sqrt{\int_0^T \left(e^{\lambda t} s(t)\right)^2 dt}, \ \ T \geq 0$$

where $s : [0, \infty) \to \mathbb{R}^m$ is a piecewise continuous signal and λ is the design parameter defined by (2.8) ; for $T > 0$, $|| \cdot ||_T$ is thus an exponentially weighted $\mathcal{L}^2[0, T]$ norm. Note that because of the definition of μ_p in (2.7) and the variation of constants formula

$$\mu_p(t) = e^{-2\lambda t} ||e_p||_t^2, \;\; t \geq 0, \;\; p \in \{1, 2\}$$

This and (2.9) imply that

$$||e_{\sigma(T)}||_T \leq ||e_k||_T, \;\; k \in \{1, 2\} \tag{3.9}$$

for every value of T at which $\mathbb{S}$ is not dwelling. It is worth emphasizing at this point that $||e_\sigma||_T$ and $||e_{\sigma(T)}||_T$ are different quantities. The former is the norm of the signal $e_{\sigma(t)}(t), \; t \in [0, T)$ while the latter is the norm of the signal $e_{\sigma(T)}(t), \; t \in [0, T)$.

The following lemma is the key to deriving a relationship between e_σ and e_q. The lemma is a direct consequence of (3.9) and the fact that σ is constant when $\mathbb{S}$ is dwelling.

Lemma 3.1. *Let σ be the response of the interconnection of $\mathbb{M}$ and $\mathbb{S}$ to inputs e_1 and e_2. For any time $T > 0$ and any $p \in \{1, 2\}$, there is a piecewise constant signal $\psi : [0, \infty) \to \{0, 1\}$ such that*

$$||e_\sigma - \psi(e_\sigma - e_p)||_T \leq \sqrt{2} ||e_p||_T, \tag{3.10}$$

where

$$\int_0^\infty |\psi| dt \leq \tau_D \tag{3.11}$$

Proof. Note first that if $\sigma(t) = p$ for all $t \in [0, T)$, then (3.10) and (3.11) both hold with any $\psi(t)$. To complete the proof, it is enough to consider the case when $\sigma(t) = \bar{p}$ for some $t \in [0, T)$ where $\bar{p}$ is the complement of p in $\{1, 2\}$. We do this next.

Suppose $\sigma(t) = \bar{p}$ for some $t \in [0, T)$ and let $\bar{t}$ be the supremum of the values of $t \leq T$ for which $\sigma(t) = \bar{p}$. For $k \in \{1, 2\}$ let $\mathcal{I}_k$ denote the union of the intervals within $[0, T)$ on which $\sigma = k$. We claim that ψ can be chosen so that (3.11) holds, and in addition so that

$$||(1 - \psi)e_{\bar{p}}||_{\bar{t}} \leq ||e_p||_{\bar{t}} \tag{3.12}$$

and

$$\mathcal{I}^* \subset \mathcal{I}_{\bar{p}} \tag{3.13}$$

where $\mathcal{I}^*$ is the set of values of t for which $\psi(t) = 1$. To prove that this is so, first consider the case when $||e_{\bar{p}}||_{\bar{t}} \leq ||e_p||_{\bar{t}}$. In this case, (3.11) - (3.13) clearly hold with $\psi = 0, \; t \in [0, \infty)$. Consider next the case when $||e_{\bar{p}}||_{\bar{t}} > ||e_p||_{\bar{t}}$; in other words

$||e_{\sigma(\bar{t})}||_{\bar{t}} > ||e_p||_{\bar{t}}$. In this case (3.9) implies that $\mathbb{S}$ must be dwelling at $\bar{t}$; therefore $\sigma(t_s) = \bar{p}$ where t_s is the last switching time before $\bar{t}$. Thus $||e_{\bar{p}}||_{t_s} \leq ||e_p||_{t_s}$ because of (3.9). Thus if we define $\psi = 1$ on the interval $[t_s, \bar{t})$ and $\psi = 0$ elsewhere, then (3.11) and (3.13) hold and $||(1-\psi)e_{\bar{p}}||_{\bar{t}} = ||(1-\psi)e_{\bar{p}}||_{t_s}$. Therefore $||(1-\psi)e_{\bar{p}}||_{\bar{t}} \leq ||e_p||_{t_s}$. But $||e_p||_{t_s} \leq ||e_p||_{\bar{t}}$ because $t_s \leq \bar{t}$ so (3.12) is true. Thus the claim is proved.

To proceed note that $||e_\sigma - \psi(e_\sigma - e_p)||_T^2 = ||(1-\psi)e_\sigma||_T^2 + ||\psi e_p||_T^2$ because $\psi(1-\psi) = 0$. Thus

$$\begin{aligned} ||e_\sigma - \psi(e_\sigma - e_p)||_T^2 &= \int_{\mathcal{I}_{\bar{p}}} (e^{\lambda t}(1-\psi)e_{\bar{p}})^2 dt + \int_{\mathcal{I}_p} (e^{\lambda t}(1-\psi)e_p)^2 dt + \int_0^T (e^{\lambda t}\psi e_p)^2 dt \\ &= \int_{\mathcal{I}_{\bar{p}}} (e^{\lambda t}(1-\psi)e_{\bar{p}})^2 dt + \int_{\mathcal{I}_p} (e^{\lambda t} e_p)^2 dt + \int_{\mathcal{I}^*} (e^{\lambda t} e_p)^2 dt \end{aligned}$$

But $\mathcal{I}_p$ and $\mathcal{I}^*$ are disjoint because of (3.13) so

$$||e_\sigma - \psi(e_\sigma - e_p)||_T^2 = \int_{\mathcal{I}_{\bar{p}}} (e^{\lambda t}(1-\psi)e_{\bar{p}})^2 dt + \int_{\mathcal{I}_p \cup \mathcal{I}^*} (e^{\lambda t} e_p)^2 dt$$

Therefore

$$||e_\sigma - \psi(e_\sigma - e_p)||_T^2 \leq \int_0^{\bar{t}} (e^{\lambda t}(1-\psi)e_{\bar{p}})^2 dt + \int_0^T (e^{\lambda t} e_p)^2 dt$$

so $||e_\sigma - \psi(e_\sigma - e_p)||_T^2 \leq ||(1-\psi)e_{\bar{p}}||_{\bar{t}}^2 + ||e_p||_T^2$. But $||(1-\psi)e_{\bar{p}}||_{\bar{t}}^2|| \leq ||e_p||_{\bar{t}}^2$ because of (3.12) and $||e_p||_{\bar{t}}^2 \leq ||e_p||_T^2$ so $||e_\sigma - \psi(e_\sigma - e_p)||_T^2 \leq 2||e_p||_T^2$. It follows that (3.10) is true. □

3.4. The Block Diagram. In analyzing the overall hybrid system, we will make use of a block diagram which captures the essential features of the hybrid system. Towards this end let us note that the definitions of the output estimation errors in (2.6), together with the process model equation (2.2) and the multi-estimator equations (2.4) and (2.5) imply that e_q must satisfy

$$\dot{e}_q = -2.9e_q - \mathbf{n} - \delta \circ y \qquad (3.14)$$

This equation plus the injected system lead to the partially complete block diagram shown in Figure 3.1. Observe that each of the three subsystems shown in the block diagram is exponentially stable with stability margin greater than the value of λ given by (2.8). Thus if e_σ were related to e_q by for example a gain {e.g., $e_\sigma = ge_q$}, then one would have an exponentially stable closed loop system provided δ were small in some suitably defined norm. Although no such gain exists, it is nonetheless possible to derive a relationship between e_q and e_σ which serves much the same purpose. In particular we will show that

$$||e_\sigma||_t \leq 2e^{\gamma \tau_D}||e_q||_t, \qquad t \geq 0 \qquad (3.15)$$

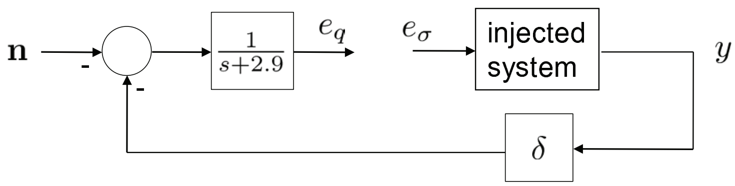

FIG. 3.1. *Block Diagram*

where

$$\gamma = \sup_{\sigma\in\mathcal{S}} \sup_{t>0} \sup_{p\in\{1,2\}} \int_0^t \left(e^{\lambda(t-\tau)} h_p(t,\tau)\right)^2 d\tau, \tag{3.16}$$

$$h_p(t,\tau) = (p-\sigma(t))\begin{bmatrix}1 & -1\end{bmatrix}\phi(t,\tau)6\begin{bmatrix}1-\sigma(\tau)\\ 2-\sigma(\tau)\end{bmatrix} \tag{3.17}$$

and ϕ is the state transition matrix of A_σ. Note that γ is finite because $\lambda I + A_\sigma$ is a stability matrix.

To derive the inequality in (3.15) we will need to first develop a relationship between the signal $e_\sigma - e_q$ which appears on the left side of the inequality in (3.10), and the signal e_σ. To do this, first note that $e_\sigma - e_q = x_\sigma - x_q$ because of (2.6) and (3.4). Using the expression for x_σ in (3.3) we thus obtain the formula

$$e_\sigma - e_q = (q-\sigma)(x_1 - x_2) \tag{3.18}$$

The signal $e_\sigma - e_1$ is thus an output of the injected system. In particular

$$e_\sigma - e_1 = h_q \circ e_\sigma \tag{3.19}$$

where h_p is as in (3.17). Fix $T > 0$ and $\sigma \in \mathcal{S}$ and let ψ be as in Lemma 3.1. In view of the preceding we may therefore write

$$\begin{aligned} ||\psi(e_\sigma - e_q)||_t^2 &= \int_0^t \left(e^{\lambda s}\psi(s)\int_0^s h_q(s,\tau)e_{\sigma(\tau)}(\tau)d\tau\right)^2 ds \\ &= \int_0^t \psi^2(s)\left(\int_0^s e^{\lambda(s-\tau)}h_q(s,\tau)e^{\lambda\tau}e_{\sigma(\tau)}(\tau)d\tau\right)^2 ds \end{aligned}$$

From this, the Cauchy-Schwartz inequality and the fact that ψ is idempotent it follows that

$$||\psi(e_\sigma - e_q)||_t^2 \leq \int_0^t \psi(s)\left\{\left(\int_0^s e^{\lambda(s-\tau)}h_q(s,\tau)\right)^2 d\tau \int_0^s \left(e^{\lambda\tau}e_{\sigma(\tau)}(\tau)d\tau\right)^2\right\} ds$$

Thus

$$||\psi(e_\sigma - e_q)||_t^2 \leq \gamma \int_0^t \psi(s)||e_\sigma||_s^2 ds, \quad t\in[0,T] \tag{3.20}$$

where γ is given by (3.16).

To proceed, suppose we set

$$\bar{e} = e_\sigma - \psi(e_\sigma - e_q) \tag{3.21}$$

in which case

$$||\bar{e}||_T \leq \sqrt{2}||e_q||_T \tag{3.22}$$

because of (3.10). But if we re-write (3.21) as $e_\sigma = \bar{e} + \psi(e_\sigma - e_q)$ we obtain

$$||e_\sigma||_t^2 \leq 2||\bar{e}||_t^2 + 2||\psi(e_\sigma - e_q)||_t^2, \ \ t \in [0, T]$$

Therefore

$$||e_\sigma||_t^2 \leq 2||\bar{e}||_T^2 + 2||\psi(e_\sigma - e_q)||_t^2, \ \ t \in [0, T]$$

so by (3.22)

$$||e_\sigma||_t^2 \leq 4||e_q||_T^2 + 2||\psi(e_\sigma - e_q)||_t^2, \ \ t \in [0, T]$$

Hence by (3.20)

$$||e_\sigma||_t^2 \leq 4||e_q||_T^2 + 2\gamma \int_0^t \psi||e_\sigma||_s^2 ds, \ t \in [0, T]$$

Therefore by the Bellman-Gronwall Lemma

$$||e_\sigma||_t^2 \leq \left(4e^{2\gamma \int_0^t \psi ds}\right) ||e_q||_T^2, \ \ t \in [0, T]$$

so

$$||e_\sigma||_T^2 \leq 4e^{2\gamma \int_0^T \psi ds}||e_q||_T^2$$

Hence by (3.11)

$$||e_\sigma||_T^2 \leq 4e^{2\gamma \tau_D}||e_q||_T^2$$

so finally

$$||e_\sigma||_T \leq 2e^{\gamma \tau_D}||e_q||_T$$

Since T is arbitrary, this inequality holds for all T and so (3.15) is true.

3.5. Completing the Analysis. Let us define the gains g_1, g_2, g_3

$$g_1 = \sup_{\sigma \in \mathcal{S}} ||h_I||_\infty \qquad g_2 = ||\delta||_\infty \qquad g_3 = 2e^{\gamma \tau_D}||h_E||_\infty \tag{3.23}$$

where $||\cdot||_\infty$ is the induced exponentially weighted $\mathcal{L}^2[0, \infty]$ norm and h_I and h_E are respectively the input-output maps of the injected system and the linear system with transfer function $\frac{1}{s+2.9}$. These gains are all finite because the subsystems to which they correspond all have stability margins larger than λ. In view of (3.15) and the block diagram in Figure 3.1, we have the inequalities

$$||e_\sigma||_t \leq g_3(||\mathbf{n}||_t + g_2||y||_t) \quad t \geq 0$$
$$||y||_t \leq g_1||e_\sigma||_t, \quad t \geq 0$$

Thus

$$||e_\sigma||_t \leq g_3(||\mathbf{n}||_t + g_2 g_1||e_\sigma||_t) \quad t \geq 0$$

At this point we make the *small gain assumption*

$$g_3 g_2 g_1 < 1 \tag{3.24}$$

which provides a bound on how large the un-modeled dynamics can be. In other words, the "gain" around the loop in the "feedback" system described by the block diagram and (3.15) is less than 1. As a result

$$||e_\sigma||_t \leq \frac{g_3}{(1 - g_1 g_2 g_3)}||\mathbf{n}||_t, \quad t \geq 0 \tag{3.25}$$

Our next goal is to show that the $\mathcal{L}_\infty[0,\infty]$ norm of y, namely $|y|_\infty$, is bounded above by a constant times the $\mathcal{L}_\infty[0,\infty]$ norm of $\mathbf{n}$. To do this we will make use of the following easily verified facts. First, for any piecewise-continuous signal $w : [0,\infty) \to \mathbb{R}$

$$e^{-\lambda t}||w||_t \leq \frac{1}{\sqrt{2\lambda}}|w|_\infty, \quad t \geq 0 \tag{3.26}$$

Second, if h is the input-output map of a *strictly causal* linear system with stability margin larger than λ, then the gain

$$\langle h \rangle \triangleq \sup_{t>0} \sup_{w} \frac{|\{h \circ w\}_t|}{e^{-\lambda t}||w||_t} \tag{3.27}$$

is finite; here $\{h \circ w\}_t$ denotes

$$\{h \circ w\}_t = \int_0^t h(t,\tau)w(\tau)d\tau$$

The definition of $\langle h \rangle$ implies that

$$|\{h \circ w\}_t| \leq \langle h \rangle e^{-\lambda t}||w||_t \quad t \geq 0$$

To proceed note first that

$$|e_\sigma - e_q|_t \leq \sup_{\tau \leq t} \langle h_q \rangle e^{-\lambda \tau}||e_\sigma||_\tau \quad t \geq 0$$

because of (3.19). This, (3.25) and (3.26) imply that

$$|e_\sigma - e_q|_\infty \leq \frac{g_3 g_4}{\sqrt{2\lambda}(1 - g_1 g_2 g_3)} |\mathbf{n}|_\infty \tag{3.28}$$

where

$$g_4 = \max_{p \in \{1,2\}} \langle h_p \rangle$$

Next observe from the block diagram that

$$e_q = h_E \circ (\delta \circ y - \mathbf{n})$$

Therefore

$$|e_q|_t \leq \sup_{\tau \leq t} \langle h_E \rangle e^{-\lambda\tau} (||\delta \circ y - \mathbf{n}||_\tau$$

so

$$|e_q|_t \leq \sup_{\tau \leq t} \langle h_E \rangle e^{-\lambda\tau} (g_2 ||y||_\tau + ||\mathbf{n}||_t)$$

But $||y||_t \leq g_1 ||e_\sigma||_t$ and $||e_\sigma||_t \leq \frac{g_3}{1 - g_1 g_2 g_3} ||\mathbf{n}||_t$ so

$$|e_q|_t \leq \sup_{\tau \leq t} \langle h_E \rangle e^{-\lambda\tau} \left(g_1 g_2 \frac{g_3}{1 - g_1 g_2 g_3} + 1 \right) ||\mathbf{n}||_\tau)$$

which simplifies to

$$|e_q|_t \leq \sup_{\tau \leq t} e^{-\lambda\tau} \left(\frac{\langle h_E \rangle}{1 - g_3 g_2 g_1} \right) ||\mathbf{n}||_\tau)$$

This and (3.26) imply that

$$|e_q|_\infty \leq \frac{\langle h_E \rangle}{\sqrt{2\lambda}(1 - g_3 g_2 g_1)} |\mathbf{n}|_\infty$$

Combining this with (3.28) one thus obtains

$$|e_\sigma|_\infty \leq \frac{\langle h_E \rangle + g_3 g_4}{\sqrt{2\lambda}(1 - g_3 g_2 g_1)} |\mathbf{n}|_\infty$$

Therefore we arrive finally at the inequality

$$\boxed{|y|_\infty \leq |h_I|_\infty \frac{\langle h_E \rangle + g_3 g_4}{\sqrt{2\lambda}(1 - g_3 g_2 g_1)} |\mathbf{n}|_\infty} \tag{3.29}$$

where $|h_I|_\infty$ is the induced infinity norm of h_I. This bound, while undoubtedly quite conservative, proves that the $\mathcal{L}^\infty[0, \infty]$ norm of y is bounded by a finite constant times the $\mathcal{L}^\infty[0, \infty]$ norm of $\mathbf{n}$.

4. Simulation Results. Below are graphs[1] showing the response y to a sinusoidal noise input $\mathbf{n} = \sin 5t$ assuming that $q = 1$, and that the transfer function of δ is $\frac{3}{s+5}$. The graphs on the left are for a dwell time of $\tau_D = 0.5$ while those on the right are for $\tau_D = 0.1$. The simulation on the right shows a modest improvement in performance when compared with the simulation for $\tau_D = .5$. Although the theory we have developed so far does not predict this, performance improvement with faster switching has been observed before.

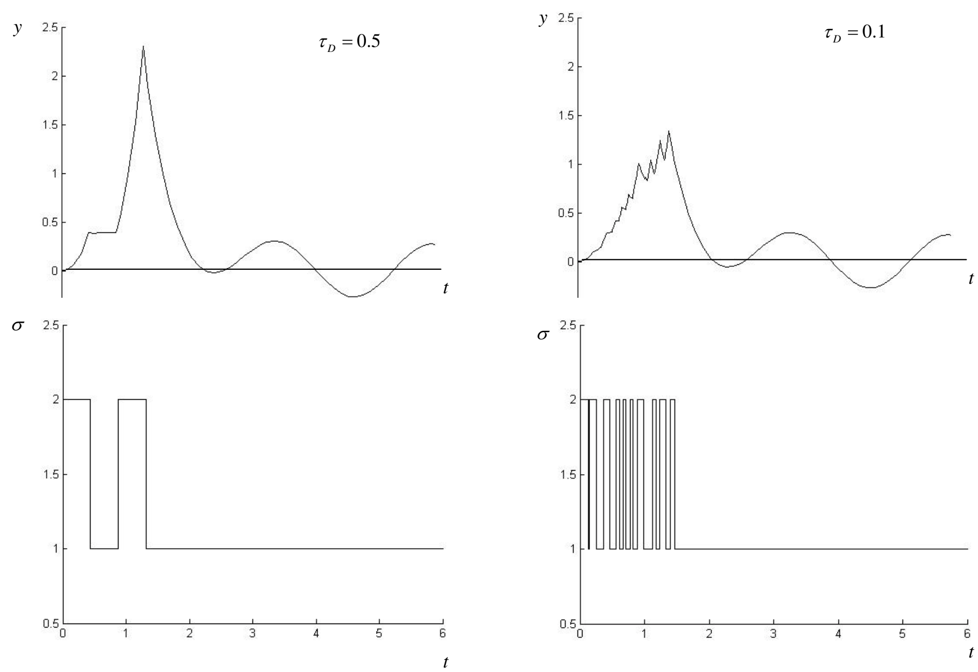

FIG. 4.1. *Simulation Graphs*

5. Fast Switching. The theory developed in the Section 3.2 demands that $\tau_D >$ .446 in order for the injected system to be stable. So why doesn't switching with a value of τ_D significantly smaller than this destabilize the overall system? The reason for this is to some extent because the lower bound of .446 on τ_D is conservative. But there is another more compelling reason which we now explain. For simplicity, we do this assuming that the uncertain process model parameter $q = 1$.

Recall that

$$e_\sigma - e_1 = (\sigma - 1)(x_2 - x_1) \tag{5.1}$$

Note that for any 2×1 matrix $k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ it is possible to re-write the state equation

[1]These graphs document the results of simulations performed by Shaoshuai Mou.

of the injected system (3.6) as

$$\dot{x} = \bar{A}_\sigma x + b_\sigma e_\sigma + k(e_\sigma - e_1) \tag{5.2}$$

where

$$\bar{A}_1 = A_1 = \begin{bmatrix} -2.9 & 0 \\ 6 & -2.9 \end{bmatrix} \qquad \bar{A}_2 = A_2 + kc = \begin{bmatrix} -2.9 & 6 \\ 0 & -2.9 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} [\, 1 \quad -1 \,]$$

and $c = [\,1 \quad -1\,]$. Note that the matrix pair (c, A_2) is observable as is the pair $(c, \lambda I + A_2)$. The "squashing lemma" in [1] states that for any given $\tau_D > 0$ and any positive number α, it is possible to find a matrix k and a positive number ω so that

$$\left| e^{(\lambda I + A_2 + kc)t} \right| \leq \alpha e^{-\omega(t - \tau_D)}$$

or equivalently so that

$$\left| e^{(A_2 + kc)t} \right| \leq \alpha e^{-(\lambda+\omega)(t - \tau_D)}$$

Using this fact plus the fact that $\lambda I + A_1$ is a stability matrix, it is not difficult to shown that for *any* $\tau_D > 0$, it is possible to choose k so that $\bar{A}_\sigma$ is exponentially stable with stability margin greater than λ.

Suppose that for $\tau_D = .1$, k is so chosen. Recall the readout equation (3.5) for y:

$$y = [\,(2 - \sigma) \quad (\sigma - 1)\,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - e_\sigma$$

This plus the differential equation (5.2) define an exponentially stable system with stability margin larger than λ whose inputs are e_1 and e_σ and whose output is y. We call this the *squashed system*. The relevant block diagram is as follows.

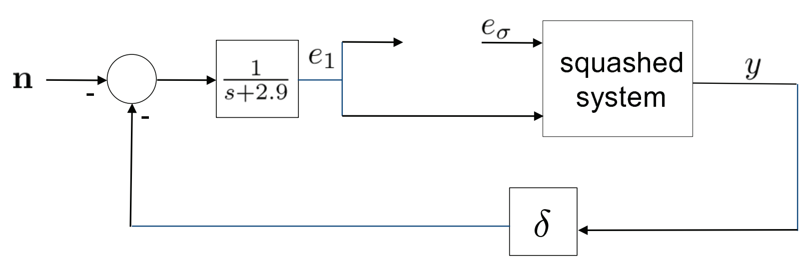

FIG. 5.1. *Modified Block Diagram*

It is important to keep in mind that we have not changed the original system in any way other than allowing τ_D to be a smaller dwell time than that which is known to stabilize the original injected system.

Note that the relationship between e_q and e_σ in (3.15) still holds with the original values of γ and τ_D. In addition, each of the subsystems depicted in the block diagram in Figure 4.1 have stability margin greater than λ. Thus the same type of analysis as in Section 3.5 can be used here to obtain a relationship between $|y|_\infty$ and $|\mathbf{n}|_\infty$ provided $||\delta||_\infty$ is sufficiently small. In this regard, "squashing" the original injected system with the output injection $k(e_\sigma - e_1)$ almost certainly increases the loop gain of the system shown in Figure 4.1, suggesting that there is a natural tradeoff between the size of τ_D {or transient performance} on the one hand, and how much un-modeled dynamics the system can tolerate on the other. Indeed these observation are consistent with the simulation results we have observed.

6. Concluding Remarks. In this paper we have sought to explain in relatively uncluttered terms how a particular type of switched adaptive control system works. A reading of the paper reveals one of adaptive control's main weaknesses, namely that there are very few available guidelines for choosing the many design parameters needed to actually design such a system. In order to address this weakness, a considerably more transparent explanation of adaptive control concepts is needed. This paper takes a step in this direction.

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