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OPTIMAL NATURAL FRAMES*

ERIC W. JUSTH† AND P. S. KRISHNAPRASAD‡

Abstract. Problems of optimal control on Lie groups are of broad interest and application dating back to the early days of geometric control theory. We study a class of such problems defined on the special Euclidean group and demonstrate by appealing to reduction methods that the extremals in these problems admit special structure associated to the nonlinear Schrödinger equation.

Key words: natural frames, optimal control on Lie groups, Lie-Poisson reduction, nonlinear Schrödinger equation, vortex filament

1. Intoduction. In this paper, we discuss a class of optimal control problems on Lie groups, focusing on the special Euclidean groups $SE(n)$. The models we discuss arise in the dynamics and control of unmanned aerial vehicles (Justh & Krishnaprasad 2002, 2004, 2005), in continuum models in biophysics (Wiggins 2001) and in fluid mechanics (Newton 2001). In his Ph.D. thesis (Baillieul 1975, 1978), John Baillieul investigated optimal control problems on matrix Lie groups using the Maximum Principle of Pontryagin and methods from the calculus of variations, emphasizing in Chapter 3 of the thesis the details of the special case of $SO(3)$. The present paper is somewhat in the spirit of that chapter, seeking explicit solutions while finding common patterns in a wider class of problems.

In Section 2 of this paper we set up a notation explaining the natural framing of twice differentiable curves in $\mathbb{R}^3$. In Section 3 we discuss optimal control problems with symmetry on a Lie group and state the reduction of the Maximum Principle to the dual of the Lie algebra. With this as the starting point we show, with the help of examples, that in $SE(n)$ the extremals are stationary solutions to a nonlinear Schrödinger equation. Connections to vortex filament equations associated with singular solutions in fluid mechanics (Newton 2001, Ricca 1996) are also indicated. In the work of Langer and Perline (1991, 1996) connections between the vortex filament equation and the nonlinear Schrödinger equation are discussed using natural frames and generalizations.

2. Natural frame representation of curves. Given a curve $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^3$ which is at least twice continuously differentiable, we can associate with it a natural

*Dedicated to John Baillieul on the Occasion of His 65th Birthday.

†Naval Research Laboratory, Washington, DC 20375, USA. E-mail: eric.justh@nrl.navy.mil

‡Institute for Systems Research, and Dept. of Electrical and Computer Eng., University of Maryland, College Park, MD 20742, USA. E-mail: krishna@umd.edu

Frenet frame $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ which evolves according to (Bishop 1975)

$$\begin{aligned} \boldsymbol{\gamma}' &= \mathbf{T}, \\ \mathbf{T}' &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \mathbf{M}_1' &= -k_1\mathbf{T}, \\ \mathbf{M}_2' &= -k_2\mathbf{T}, \end{aligned} \tag{2.1}$$

where prime denotes differentiation with respect to the arc length parameter s, $\mathbf{T}$ is the unit tangent vector to the curve, $\mathbf{M}_1$ and $\mathbf{M}_2$ are unit normal vectors, and $[\mathbf{T}\ \mathbf{M}_1\ \mathbf{M}_2] \in SO(3)$. The natural curvatures k_1 and k_2 can be thought of as controls: given an initial position $\boldsymbol{\gamma}(0)$, velocity $\boldsymbol{\gamma}'(0)$, and initial choice $\mathbf{M}_1(0)$ and $\mathbf{M}_2(0)$, the functions $k_1 : \mathbb{R} \to \mathbb{R}$ and $k_2 : \mathbb{R} \to \mathbb{R}$ completely determine the curve.

The natural Frenet frame equations (2.1) can be written as a left-invariant system on $SE(3)$ as

$$g' = g\xi, \tag{2.2}$$

with

$$\begin{aligned} g &= \begin{bmatrix} \mathbf{T} & \mathbf{M}_1 & \mathbf{M}_2 & \boldsymbol{\gamma} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3), \\ \xi &= \begin{bmatrix} 0 & -k_1 & -k_2 & 1 \\ k_1 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in se(3). \end{aligned} \tag{2.3}$$

Curves and natural Frenet frames have proved useful for designing coordinated vehicle trajectories for tasks such as pursuit (Justh & Krishnaprasad 2006; Reddy, Justh & Krishnaprasad 2006), formation flight (Justh & Krishnaprasad 2004, 2005), and boundary tracking (Zhang, Justh & Krishnaprasad 2004). They have also been applied to gather evidence for particular pursuit strategies in nature (Reddy, 2007).

An alternative framing of the curve $\boldsymbol{\gamma}$ (when $\boldsymbol{\gamma}$ is three times continuously differentiable and $\boldsymbol{\gamma}'' \neq 0$) is the Frenet-Serret frame

$$\begin{aligned} \boldsymbol{\gamma}' &= \mathbf{T}, \\ \mathbf{T}' &= \kappa\mathbf{N}, \\ \mathbf{N}' &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \mathbf{B}' &= -\tau\mathbf{N}, \end{aligned} \tag{2.4}$$

where κ is the curvature, τ is the torsion, $\mathbf{N}$ is the unit normal vector, and $\mathbf{B}$ is the unit binormal vector. The Frenet-Serret equations (2.4) can also be expressed as a left-invariant system on $SE(3)$, and there are formulas which relate κ and τ to the natural curvatures k_1 and k_2. For example,

$$\kappa^2 = k_1^2 + k_2^2, \quad \tau = \frac{d}{ds}\left[\arg(k_1, k_2)\right] = \frac{d}{ds}\left[\tan^{-1}(k_2/k_1)\right]. \tag{2.5}$$

Circular helical curves are easily described using the Frenet-Serret frame, because κ and τ are constant. The corresponding natural curvatures k_1 and k_2 are sinusoidal functions of s (in phase quadrature). A right circular helix with radius ρ, pitch $2\pi r$, and initial point $\boldsymbol{\gamma}(0) = (\rho, 0, 0)$ is described by

$$\boldsymbol{\gamma}(s) = \begin{bmatrix} \rho\cos(s/\sqrt{\rho^2+r^2}) \\ \rho\sin(s/\sqrt{\rho^2+r^2}) \\ rs/\sqrt{\rho^2+r^2} \end{bmatrix}, \tag{2.6}$$

so that

$$\mathbf{T}(s) = \boldsymbol{\gamma}' = \frac{1}{\sqrt{\rho^2+r^2}} \begin{bmatrix} -\rho\sin(s/\sqrt{\rho^2+r^2}) \\ \rho\cos(s/\sqrt{\rho^2+r^2}) \\ r \end{bmatrix}, \tag{2.7}$$

and corresponding explicit formulas for $\mathbf{M}_1(s)$ and $\mathbf{M}_2(s)$ can also be derived. The curvature and torsion are given by

$$\kappa = |\boldsymbol{\gamma}''| = \rho/(\rho^2 + r^2) \tag{2.8}$$

and

$$\tau = \boldsymbol{\gamma}' \cdot (\boldsymbol{\gamma}'' \times \boldsymbol{\gamma}''')/\kappa^2 = r/(\rho^2 + r^2). \tag{2.9}$$

3. Maximum Principle and Poisson Reduction. The use of the Maximum Principle and Lie-Poisson Reduction for left-invariant systems on finite dimensional Lie groups is discussed in (Krishnaprasad 1993). Here we summarize the main points, as they apply to the problems analyzed below. Given a controlled left-invariant system

$$\dot{g} = g\xi_u, \tag{3.1}$$

where $g \in G$, a Lie group, and $\xi_u \in \mathfrak{g}$, the corresponding Lie algebra, we assume that ξ_u is affine in the control vector u, i.e.,

$$\xi_u = \xi_0 + \sum_{i=1}^{m} u_i(t)\xi_i, \quad i = 0, 1, ..., m. \tag{3.2}$$

In (3.2), $u = (u_1, u_2, ..., u_m) \in \mathbb{R}^m$, $\xi_i \in \{X_1, X_2, ..., X_n\}$, a set of basis vectors for $\mathfrak{g}$, and $m < n$ where n is the dimension of G. Thus, the system is underactuated. In particular, we assume that

$$\begin{aligned} &\xi_0 = \sigma_0 X_q \text{ for some } q \in \{m+1, ..., n\}, \quad \sigma_0 \in \{-1, 0, 1\}, \\ &\xi_i = \sigma_i X_i, \quad \sigma_i \in \{-1, 1\}, \quad i = 1, ..., m, \end{aligned} \tag{3.3}$$

and we take L to have the form

$$L(u) = \frac{1}{2}\sum_{i=1}^{m} u_i^2. \tag{3.4}$$

If $\sigma_0 = 0$ then the system is drift-free; otherwise, the system has drift. We consider the fixed-endpoint problem

$$\min_u \int_0^T L(u(t))dt \tag{3.5}$$

subject to $g(0) = g_0$ and $g(T) = g_T$, $T > 0$, and note that L given by (3.4) is clearly G-invariant (i.e., L depends only on u and not on g). Assuming that controls exist which will drive the system from g_0 at $t = 0$ to g_T at $t = T$, and restricting attention to regular extremals of the fixed endpoint problem, we define the pre-hamiltonian

$$H(g, p, u) = \langle p, T_eL_g \cdot \xi_u\rangle - L(u), \tag{3.6}$$

where $p \in T_g^*G$, and T_eL_g denotes the tangent map of left translation by g on G. The Maximum Principle then states that for optimal $u(\cdot) = u^{opt}(\cdot)$, the trajectory in G is a base integral curve of the canonical hamiltonian system on T^*G with hamiltonian

$$\mathrm{H}(g, p) = \sup_u H(g, p, u), \text{ for a.e. } t \in [0, T]. \tag{3.7}$$

If H is differentiable with respect to u, we have

$$\frac{\partial H}{\partial u_i}(g, p, u)\Big|_{u_i = u_i^{opt}} = \left(\frac{\partial}{\partial u_i}\langle p, T_eL_g \cdot \xi_u\rangle - \frac{\partial L}{\partial u_i}\right)\Big|_{u_i = u_i^{opt}} = 0, \quad i = 1, ..., m. \tag{3.8}$$

Defining $\mu \in \mathfrak{g}^*$, the linear dual of $\mathfrak{g}$, by $\mu = T_eL_g^* \cdot p$, and substituting (3.4) for L, (3.8) becomes

$$\begin{aligned}\left(\frac{\partial}{\partial u_i}\langle \mu, \xi_u\rangle - \frac{\partial L}{\partial u_i}\right)\Big|_{u_i = u_i^{opt}} &= \left\langle \mu, \frac{\partial}{\partial u_i}\left(\xi_0 + \sum_{i=1}^m u_i\xi_i\right)\Big|_{u_i = u_i^{opt}}\right\rangle - u_i^{opt} \\ &= \sum_{i=1}^m \langle \mu, \xi_i\rangle - u_i^{opt} \\ &= \sum_{i=1}^m \sigma_i\langle \mu, X_i\rangle - u_i^{opt} \\ &= \sigma_i\mu_i - u_i^{opt} = 0,\end{aligned} \tag{3.9}$$

where μ is expressed in the dual basis $\{X_1^\flat, X_2^\flat, ..., X_n^\flat\}$ to that of $\mathfrak{g}$, as

$$\mu = \sum_{i=1}^n X_i^\flat \mu_i. \tag{3.10}$$

Thus, optimal controls u_i^{opt} satisfy

$$u_i^{opt} = \sigma_i\mu_i, \quad i = 1, ..., m. \tag{3.11}$$

Substituting the optimal controls back into the hamiltonian then gives

$$\begin{aligned} \mathrm{H}(g,p) &= \langle p, T_e L_g \cdot \xi_{u^{opt}} \rangle - L(u^{opt}) \\ &= \langle \mu, \xi_{u^{opt}} \rangle - L(u^{opt}) \\ &= \left\langle \mu, \xi_0 + \sum_{i=1}^{m} \sigma_i \mu_i \xi_i \right\rangle - \frac{1}{2}\sum_{i=1}^{m} \mu_i^2 \\ &= \sigma_0 \langle \mu, X_q \rangle + \sum_{i=1}^{m} \sigma_i^2 \langle \mu, X_i \mu_i \rangle - \frac{1}{2}\sum_{i=1}^{m} \mu_i^2 \\ &= \sigma_0 \mu_q + \frac{1}{2}\sum_{i=1}^{m} \mu_i^2. \end{aligned} \tag{3.12}$$

Clearly H is independent of g, and thus permits reduction. In fact, we are able to use Lie-Poisson reduction to take the original system on T^*G and reduce it to a system on $\mathfrak{g}^*$, with the reduced variables defined as $\mu_1, ..., \mu_n$. The machinery of Lie-Poisson reduction (technical details may be found in Krishnaprasad, 1993) then allows us to write the reduced hamiltonian

$$h = \sigma_0 \mu_q + \frac{1}{2}\sum_{i=1}^{m} \mu_i^2, \tag{3.13}$$

along with the dynamics for μ,

$$\dot{\mu}_i = -\sum_{j=1}^{n}\sum_{k=1}^{n} \mu_k \Gamma_{ij}^k \frac{\partial h}{\partial \mu_j}, \quad i = 1, 2, ..., n, \tag{3.14}$$

where Γ_{ij}^k are structure constants of $\mathfrak{g}$. We can express (3.14) in the form

$$\dot{\mu} = \Lambda(\mu)\nabla h = -\left(\sum_{k=1}^{n} \mu_k \begin{bmatrix} \Gamma_{11}^k & \Gamma_{12}^k & \cdots & \Gamma_{1n}^k \\ \Gamma_{21}^k & \Gamma_{22}^k & \cdots & \Gamma_{2n}^k \\ \vdots & \vdots & & \vdots \\ \Gamma_{n1}^k & \Gamma_{n2}^k & \cdots & \Gamma_{nn}^k \end{bmatrix} \right) \begin{bmatrix} \partial h/\partial \mu_1 \\ \partial h/\partial \mu_2 \\ \vdots \\ \partial h/\partial \mu_n \end{bmatrix}, \tag{3.15}$$

which turns out to be more illuminating.

Remark: We focus on extremal solutions, i.e., solutions which satisfy the necessary condition (3.8) for optimality. Because we have not considered (second-derivative-based) sufficient conditions, or analyzed conjugate points, we cannot assert that these solutions are truly optimal without doing more work. However, we shall continue to refer to the controls u_i^{opt} defined by (3.11) and (3.15) as the "optimal controls."

4. Optimal framing. We apply the Maximum Principle and Lie-Poisson reduction to fixed-endpoint optimal control problems on $SO(3)$, $SE(3)$, $SE(4)$, and generalize to $SE(n)$, $n > 4$. For $SO(3)$, we obtain explicit equations for not only the optimal controls, but also the corresponding base integral curves. For $SE(3)$, we show that the optimal controls are stationary solutions to a nonlinear Schrödinger (NLS) equation. For $SE(4)$, the optimal controls are stationary solutions to a three-dimensional generalization of the stationary NLS (SNLS) equation, a pattern which also holds for $SE(n)$, $n > 4$.

4.1. An optimal control problem on SO(3). Consider the left-invariant system

$$\dot{g} = g\xi_u, \quad g \in SO(3), \quad \xi_u = X_1 u_1 - X_2 u_2, \tag{4.1}$$

where

$$X_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{4.2}$$

Suppose that the endpoints for a fixed-endpoint optimal control problem are given by $g(0) = g_0$, $g(T) = g_T$, and we seek (regular extremals which)

$$\min_{(u_1,u_2)\in\mathbb{R}^2} \int_0^T L(u_1(t), u_2(t))dt, \tag{4.3}$$

where

$$L(u_1, u_2) = \frac{1}{2}\left(u_1^2 + u_2^2\right). \tag{4.4}$$

This problem clearly fits within the class of problems described in the previous section, with $\sigma_0 = 0$ since there is no drift term in ξ_u. In the absence of drift, by the Lie Algebra Rank Condition (LARC) (Jurdjevic, 1997), it is easily verified that the system is controllable, and therefore controls exist which will steer the system from g_0 at $t = 0$ to g_T at $t = T$. Thus, we may apply the Maximum Principle and Poisson reduction to obtain the reduced hamiltonian

$$h = \frac{1}{2}\left(\mu_1^2 + \mu_2^2\right), \tag{4.5}$$

and the reduced dynamics (3.14), where $n = 3$. The structure constants are easily found from

$$[X_1, X_2] = -X_3, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1, \tag{4.6}$$

to be

$$\Gamma_{12}^3 = -1, \quad \Gamma_{13}^2 = 1, \quad \Gamma_{23}^1 = -1, \tag{4.7}$$

with $\Gamma_{ij}^k = -\Gamma_{ji}^k$ and $\Gamma_{ii}^k = 0$ for all $1 \le i, j, k \le 3$. The reduced dynamics are thus

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\mu}_2 \\ \dot{\mu}_3 \end{bmatrix} = -\begin{bmatrix} 0 & -\mu_3 & \mu_2 \\ \mu_3 & 0 & -\mu_1 \\ -\mu_2 & \mu_1 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \mu_2\mu_3 \\ -\mu_1\mu_3 \\ 0 \end{bmatrix}. \tag{4.8}$$

It is clear that the reduced hamiltonian (4.5) is conserved, as is the Casimir

$$c = \frac{1}{2}\left(\mu_1^2 + \mu_2^2 + \mu_3^2\right). \tag{4.9}$$

Conservation of h and c imply that $2(c-h) = \mu_3^2$ is also conserved, which is easily seen from (4.8). The optimal controls thus take the form

$$u_1(t) = \sqrt{2h}\cos(\omega t + \phi), \quad u_2(t) = \sqrt{2h}\sin(\omega t + \phi), \tag{4.10}$$

where $\omega \in \mathbb{R}$ and $\phi \in S^1$ (the circle group).

It is possible to write down explicit solutions for the corresponding trajectories in $SO(3)$ by identifying time t with arc length parameter s, noting that the sinusoidal optimal controls (4.10) are in phase quadrature, and interpreting the optimal controls as the natural curvatures for a circular helix in $\mathbb{R}^3$. If we assume that $\omega > 0$ (the $\omega < 0$ case may be treated similarly), we can use the formulas for a right circular helix from section 2 to derive

$$\tilde{g} = \begin{bmatrix} \mathbf{T} & \mathbf{M}_1 & \mathbf{M}_2 \end{bmatrix}, \tag{4.11}$$

with

$$\begin{aligned}
\mathbf{T} &= \omega_1 \begin{bmatrix} -\rho\sin(\omega_1 t) \\ \rho\cos(\omega_1 t) \\ r \end{bmatrix}, \\
\mathbf{M}_1 &= -\omega_1\sqrt{2h} \begin{bmatrix} \frac{\rho}{2(\omega_1+\omega)}\cos(\omega_1 t + \omega t + \phi) + \frac{\rho}{2(\omega_1-\omega)}\cos(\omega_1 t - \omega t - \phi) \\ \frac{\rho}{2(\omega_1+\omega)}\sin(\omega_1 t + \omega t + \phi) + \frac{\rho}{2(\omega_1-\omega)}\sin(\omega_1 t - \omega t - \phi) \\ \frac{r}{\omega}\sin(\omega t + \phi) \end{bmatrix}, \\
\mathbf{M}_2 &= -\omega_1\sqrt{2h} \begin{bmatrix} \frac{\rho}{2(\omega_1+\omega)}\sin(\omega_1 t + \omega t + \phi) - \frac{\rho}{2(\omega_1-\omega)}\sin(\omega_1 t - \omega t - \phi) \\ -\frac{\rho}{2(\omega_1+\omega)}\cos(\omega_1 t + \omega t + \phi) + \frac{\rho}{2(\omega_1-\omega)}\cos(\omega_1 t - \omega t - \phi) \\ -\frac{r}{\omega}\cos(\omega t + \phi) \end{bmatrix},
\end{aligned} \tag{4.12}$$

where $\omega_1 = 1/\sqrt{\rho^2 + r^2}$, $\rho/(\rho^2 + r^2) = \sqrt{2h}$ and $r/(\rho^2 + r^2) = \omega$. Thus, (4.11) and (4.12) can be written as functions of h, ω, ϕ, and t alone. The parameters h, ω, and ϕ need to be chosen to satisfy $\tilde{g}(T) = \tilde{g}(0)g_0^{-1}g_T$.

4.2. An optimal control problem on SE(3). Now consider the fixed endpoint problem $\dot{g} = g\xi_u$, where ξ_u is given by

$$\xi_u = X_4 + X_1 u_1 - X_2 u_2, \tag{4.13}$$

where

(4.14)
$$X_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $m = 2$, $n = \dim(SE(3)) = 6$. The fixed endpoints in $SE(3)$ are denoted by $g(0) = g_0$ and $g(T) = g_T$, $L(u)$ is given by (4.4), and we note that ξ_u does have a drift term. We can view $\dot{g} = g\xi_u$ as the natural Frenet frame system (2.1), where we identify the controls (u_1, u_2) with the natural curvatures (k_1, k_2). Since the coefficient of X_4 is 1 in (4.13), we are dealing with unit-speed curves, and hence time is equal to arc length parameter.

Remark: We assume, but do not verify, that controls (u_1, u_2) exist which will take the system from g_0 at $t = 0$ to g_T at $t = T$. Because drift is present in this system, proving the existence of such controls is more complicated than in the $SO(3)$ example above, even though the local strong accessibility condition is met (Jurdjevic, 1997). For example, we must have

$$|(g_0^{-1} g_T)\mathbf{e}_4|^2 - 1 \leq T, \tag{4.15}$$

where $\mathbf{e}_4 = [\ 0\ \ 0\ \ 0\ \ 1\ ]^T$, or it will be impossible to find any (u_1, u_2) which satisfies the endpoint conditions. In terms of curves, (4.15) simply states that the distance (in arc length) between the initial and final position (in $\mathbb{R}^3$) must not exceed the time T (multiplied by unit speed).

We now compute the Lie-Poisson reduced equations. Using the completion of (4.14) to a basis for $SE(3)$, we compute the structure constants

$$\begin{aligned} &\Gamma_{12}^3 = -1,\ \ \Gamma_{13}^2 = 1,\ \ \Gamma_{23}^1 = -1, \\ &\Gamma_{14}^5 = \Gamma_{26}^4 = \Gamma_{35}^6 = 1, \\ &\Gamma_{15}^4 = \Gamma_{24}^6 = \Gamma_{36}^5 = -1, \end{aligned} \tag{4.16}$$

$\Gamma_{ji}^k = -\Gamma_{ij}^k$, and $\Gamma_{ij}^k = 0$ for all other $1 < i, j, k < 6$. The dynamics for μ_j, $j = 1, ..., 6$, found from (3.14) are thus

(4.17)
$$\begin{bmatrix} \dot\mu_1 \\ \dot\mu_2 \\ \dot\mu_3 \\ \dot\mu_4 \\ \dot\mu_5 \\ \dot\mu_6 \end{bmatrix} = -\left[\begin{array}{ccc|ccc} 0 & -\mu_3 & \mu_2 & \mu_5 & -\mu_4 & 0 \\ \mu_3 & 0 & -\mu_1 & -\mu_6 & 0 & \mu_4 \\ -\mu_2 & \mu_1 & 0 & 0 & \mu_6 & -\mu_5 \\ \hline -\mu_5 & \mu_6 & 0 & 0 & 0 & 0 \\ \mu_4 & 0 & -\mu_6 & 0 & 0 & 0 \\ 0 & -\mu_4 & \mu_5 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \mu_1 \\ \mu_2 \\ 0 \\ \hline 1 \\ 0 \\ 0 \end{array}\right] = \begin{bmatrix} \mu_3\mu_2 - \mu_5 \\ -\mu_3\mu_1 + \mu_6 \\ 0 \\ \mu_5\mu_1 - \mu_6\mu_2 \\ -\mu_4\mu_1 \\ \mu_4\mu_2 \end{bmatrix}.$$

Conserved quantities include the (reduced) hamiltonian,

$$h = \mu_4 + \frac{1}{2}(\mu_1^2 + \mu_2^2), \tag{4.18}$$

and the two Casimir functions

$$\begin{aligned} c_1 &= \frac{1}{2}\left(\mu_4^2 + \mu_5^2 + \mu_6^2\right), \\ c_2 &= \mu_1\mu_6 + \mu_2\mu_5 + \mu_3\mu_4. \end{aligned} \tag{4.19}$$

As in the optimal control problem on $SO(3)$ described in the previous subsection, μ_3 is constant, as can be seen directly from (4.17).

4.2.1. Connection to the nonlinear Schrödinger equation. Defining

$$\mathbf{a} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -\mu_5 \\ \mu_6 \end{bmatrix}, \quad \omega = \mu_3 = \text{constant}, \tag{4.20}$$

we have

$$\begin{aligned} h &= \mu_4 + \frac{|\mathbf{a}|^2}{2}, \\ c_1 &= \frac{1}{2}\left(\mu_4^2 + |\mathbf{b}|^2\right), \\ c_2 &= \omega\mu_4 + \mathbf{b}^T J\mathbf{a}, \end{aligned} \tag{4.21}$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4.22}$$

The dynamics (4.17) become

$$\begin{aligned} \dot{\mathbf{a}} &= -\omega J\mathbf{a} + \mathbf{b}, \\ \dot{\mathbf{b}} &= \left(h - \frac{|\mathbf{a}|^2}{2}\right)\mathbf{a}, \end{aligned} \tag{4.23}$$

which can be written in terms of $\mathbf{a}$ alone as

$$\ddot{\mathbf{a}} + \omega J\dot{\mathbf{a}} - \left(h - \frac{|\mathbf{a}|^2}{2}\right)\mathbf{a} = 0. \tag{4.24}$$

We can now use a change of variables to eliminate the $\dot{\mathbf{a}}$ term from (4.24). Defining $\tilde{\mathbf{a}}$ by

$$\mathbf{a} = \exp\left(-\frac{1}{2}\omega Jt\right)\tilde{\mathbf{a}}, \tag{4.25}$$

we have

$$\dot{\mathbf{a}} = -\exp\left(-\frac{1}{2}\omega Jt\right)\frac{1}{2}\omega J\tilde{\mathbf{a}} + \exp\left(-\frac{1}{2}\omega Jt\right)\dot{\tilde{\mathbf{a}}}, \tag{4.26}$$

and

$$\ddot{\mathbf{a}} = \exp\left(-\frac{1}{2}\omega J t\right)\frac{1}{4}\omega^2 J^2 \tilde{\mathbf{a}} - \exp\left(-\frac{1}{2}\omega J t\right)\omega J \dot{\tilde{\mathbf{a}}} + \exp\left(-\frac{1}{2}\omega J t\right)\ddot{\tilde{\mathbf{a}}}. \tag{4.27}$$

Substituting into (4.24) and multiplying through by $\exp\left(\frac{1}{2}\omega J t\right)$, we obtain

$$\begin{aligned}\frac{1}{4}\omega^2 J^2 \tilde{\mathbf{a}} - \omega J \dot{\tilde{\mathbf{a}}} + \ddot{\tilde{\mathbf{a}}} + \omega J\left[-\frac{1}{2}\omega J \tilde{\mathbf{a}} + \dot{\tilde{\mathbf{a}}}\right] - \left(h - \frac{|\tilde{\mathbf{a}}|^2}{2}\right)\tilde{\mathbf{a}} \\ = \ddot{\tilde{\mathbf{a}}} + \frac{1}{4}\omega^2 \tilde{\mathbf{a}} - \left(h - \frac{|\tilde{\mathbf{a}}|^2}{2}\right)\tilde{\mathbf{a}} = 0.\end{aligned} \tag{4.28}$$

Defining $\tilde{h} = h - \frac{1}{4}\omega^2$, we finally obtain

$$\ddot{\tilde{\mathbf{a}}} - \tilde{h}\tilde{\mathbf{a}} + \frac{1}{2}|\tilde{\mathbf{a}}|^2 \tilde{\mathbf{a}} = 0. \tag{4.29}$$

Consider the (focusing cubic) nonlinear Schrödinger equation (NLS) (with an additional linear term),

$$-i\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} - \tilde{h}\psi + \frac{1}{2}|\psi|^2\psi, \tag{4.30}$$

where $\psi(t, x)$ is a complex function of time t and position $x \in \mathbb{R}$. Identifying the variable x in (4.30) with t in (4.29) and $[\ \mathrm{Re}(\psi)\ \ \mathrm{Im}(\psi)\ ]^T$ in (4.30) with $\tilde{\mathbf{a}}$ in (4.29), we see that (4.29) can be viewed as a stationary NLS (SNLS).

4.2.2. Special solutions of SNLS. Among the solutions to (4.29) are two easily obtained classes: sinusoidal solutions and (Jacobi) elliptic function solutions. For the sinusoidal solutions, we have

$$\tilde{\mathbf{a}} = \alpha \begin{bmatrix} \cos(\tilde{\omega}t + \phi) \\ \sin(\tilde{\omega}t + \phi) \end{bmatrix}, \tag{4.31}$$

where $\alpha = |\tilde{\mathbf{a}}|$ is a constant amplitude, and ϕ is a constant phase. Then α must satisfy

$$-\alpha\tilde{\omega}^2 - \tilde{h}\alpha + \frac{1}{2}\alpha^3 = 0, \tag{4.32}$$

or

$$\alpha = \sqrt{2(\tilde{\omega}^2 + \tilde{h})}, \tag{4.33}$$

where $\tilde{\omega}$, $\tilde{h}$, and ϕ are determined by the specified endpoints g_0 and g_T (assuming that there exist curves in $SE(3)$ corresponding to these sinusoidal solutions which actually interpolate g_0 and g_T).

To describe the elliptic function solutions, we represent $\tilde{\mathbf{a}}$ as

$$\tilde{\mathbf{a}} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \tilde{a}, \tag{4.34}$$

where θ is constant and $\tilde{a}$ is a scalar-valued function. For this class of solutions, (4.29) implies

$$\ddot{\tilde{a}} - \tilde{h}\tilde{a} + \frac{1}{2}\tilde{a}^3 = 0. \tag{4.35}$$

Equation (4.35) has solutions of the form

$$\tilde{a}(t) = (2\nu\sqrt{m})\mathrm{cn}(\nu(t-\eta), m), \quad m = \frac{1}{2}\left(1 + \frac{\tilde{h}}{\nu^2}\right), \tag{4.36}$$

where η, ν, and m are constant, and ν satisfies $\nu^2 \geq |\tilde{h}|$ (Davis, 1962). To connect back to the original optimization problem, these elliptic function solutions correspond to optimal controls for $\dot{g} = g\xi_u$, with ξ_u given by (4.13), of the form

$$\begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} = \mathbf{a} = \exp\left(-\frac{1}{2}\omega J t\right)\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\tilde{a}, \tag{4.37}$$

where $\tilde{a}$ given by (4.36), $\tilde{h} = h - \frac{1}{4}\omega^2$, and the constants ω, h, θ, η, and ν are determined by the endpoints g_0 and g_T (if such constants exists). Of course, we could have written an analogous expression for the optimal u_1 and u_2 corresponding to the sinusoidal solutions, as well. The point is that we can work with the Lie-Poisson reduced equations to ultimately obtain explicit formulas (for the controls) for certain extremal solutions to the fixed-endpoint optimal control problem on $SE(3)$, and a key step is recognizing that the reduced equations can actually be re-cast into the form of a stationary NLS equation.

4.3. An optimal control problem on SE(4). It turns out that the techniques used above for the fixed-endpoint problem on $SE(3)$ can be generalized to higher dimensions. We first use $SE(4)$ for illustration, and then proceed to the general case. Consider the fixed endpoint problem $\dot{g} = g\xi_u$ for $g \in SE(4)$, where

$$\xi_u = X_7 + X_1 u_1 + X_2 u_2 + X_3 u_3, \tag{4.38}$$

with

$$X_1 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.39}$$

and

$$L(u) = \frac{1}{2}\left(u_1^2 + u_2^2 + u_3^2\right). \tag{4.40}$$

Extending (4.39) to a particular basis for the Lie algebra $se(4)$ (in particular, $\{X_1, ..., X_6\}$ correspond to infinitesimal rotation while $\{X_7, ..., X_{10}\}$ correspond to translation), we have

$$h = \mu_7 + \frac{1}{2}\left(\mu_1^2 + \mu_2^2 + \mu_3^2\right), \tag{4.41}$$

and the Lie-Poisson reduced equations are computed to be

$$\begin{bmatrix} \dot\mu_1 \\ \dot\mu_2 \\ \dot\mu_3 \\ \dot\mu_4 \\ \dot\mu_5 \\ \dot\mu_6 \\ \dot\mu_7 \\ \dot\mu_8 \\ \dot\mu_9 \\ \dot\mu_{10} \end{bmatrix} = \Lambda(\mu) \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\mu_4\mu_2 - \mu_5\mu_3 - \mu_8 \\ \mu_4\mu_1 - \mu_6\mu_3 - \mu_9 \\ \mu_5\mu_1 + \mu_6\mu_2 - \mu_{10} \\ 0 \\ 0 \\ 0 \\ \mu_8\mu_1 + \mu_9\mu_2 + \mu_{10}\mu_3 \\ -\mu_7\mu_1 \\ -\mu_7\mu_2 \\ -\mu_7\mu_3 \end{bmatrix}, \tag{4.42}$$

where

(4.43)

$$\Lambda(\mu) = -\left[\begin{array}{cccccc|cccc} 0 & \mu_4 & \mu_5 & -\mu_2 & -\mu_3 & 0 & \mu_8 & -\mu_7 & 0 & 0 \\ -\mu_4 & 0 & \mu_6 & \mu_1 & 0 & -\mu_3 & \mu_9 & 0 & -\mu_7 & 0 \\ -\mu_5 & -\mu_6 & 0 & 0 & \mu_1 & \mu_2 & \mu_{10} & 0 & 0 & -\mu_7 \\ \mu_2 & -\mu_1 & 0 & 0 & \mu_6 & -\mu_5 & 0 & \mu_9 & -\mu_8 & 0 \\ \mu_3 & 0 & -\mu_1 & -\mu_6 & 0 & \mu_4 & 0 & \mu_{10} & 0 & -\mu_8 \\ 0 & \mu_3 & -\mu_2 & \mu_5 & -\mu_4 & 0 & 0 & 0 & \mu_{10} & -\mu_9 \\ \hline -\mu_8 & -\mu_9 & -\mu_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_7 & 0 & 0 & -\mu_9 & -\mu_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_7 & 0 & \mu_8 & 0 & -\mu_{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_7 & 0 & \mu_8 & \mu_9 & 0 & 0 & 0 & 0 \end{array}\right].$$

Defining

$$\mathbf{a} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}, \quad \mathbf{b} = -\begin{bmatrix} \mu_8 \\ \mu_9 \\ \mu_{10} \end{bmatrix}, \quad \hat\Omega = \begin{bmatrix} 0 & \mu_4 & \mu_5 \\ -\mu_4 & 0 & \mu_6 \\ -\mu_5 & -\mu_6 & 0 \end{bmatrix}, \tag{4.44}$$

we have from (4.42)

$$\begin{aligned} \dot{\mathbf{a}} &= -\hat\Omega\mathbf{a} + \mathbf{b}, \\ \dot{\mathbf{b}} &= \left(h - \frac{|\mathbf{a}|^2}{2}\right)\mathbf{a}, \end{aligned} \tag{4.45}$$

where the optimal controls are given by

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{a}. \tag{4.46}$$

Note that the form of (4.45) is analogous to (4.23), except that now $\mathbf{a}$ and $\mathbf{b}$ are 3-dimensional vectors. Indeed, (4.45) leads to a generalization of the SNLS equation (see the next subsection). Among the solutions to (4.45) are sinusoidal and elliptic function solutions analogous to those obtained in the $SE(3)$ example of the previous subsection.

4.4. An optimal control problem on SE(n). More generally, we can consider the fixed endpoint problem $\dot{g} = g\xi_u$ for $g \in SE(n)$, where $se(n)$ is parameterized such that $\{X_1, ..., X_{n(n-1)/2}\}$ correspond to infinitesimal rotation and $\{X_{[n(n-1)/2+1]}, ..., X_{n(n+1)/2}\}$ correspond to translation, and

$$\xi_u = X_{[n(n-1)/2+1]} + X_1 u_1 + X_2 u_2 + \cdots + X_{n-1} u_{n-1}. \tag{4.47}$$

In particular, $X_1, ..., X_{n-1}$ take the following form: the $[1, (1+k)]$ component of X_k is -1, the $[(1+k), 1]$ component of X_k is 1, and all other components of X_k are zero, $k = 1, ..., n-1$. Furthermore, the $[1, (1+n)]$ component of $X_{[n(n-1)/2+1]}$ is 1, and all other components of $X_{[n(n-1)/2+1]}$ are zero. We then extend $X_1, ..., X_{n-1}$ and $X_{[n(n-1)/2+1]}$ to a basis for $se(n)$ in a manner analogous to the $se(4)$ case of the previous subsection. We consider the cost function

$$L(u) = \frac{1}{2}\left(u_1^2 + u_2^2 + ... + u_{n-1}^2\right), \tag{4.48}$$

and we have

$$h = \mu_{[n(n-1)/2+1]} + \frac{1}{2}\left(\mu_1^2 + \mu_2^2 + ... + \mu_{n-1}^2\right). \tag{4.49}$$

We define

$$\mathbf{a} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n-1} \end{bmatrix}, \quad \mathbf{b} = -\begin{bmatrix} \mu_{[n(n-1)/2+2]} \\ \mu_{[n(n-1)/2+3]} \\ \vdots \\ \mu_{n(n+1)/2} \end{bmatrix}, \quad \hat{\Omega} = \hat{\Omega}(\mu_n, ..., \mu_{n(n-1)/2}) \tag{4.50}$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors of length $n-1$ and $\hat{\Omega}$ is an $(n-1)\times(n-1)$ skew-symmetric matrix. That $\hat{\Omega}$ is constant follows from a calculation which makes use of the form of the gradient of h, as well as the form of $\Lambda(\mu)$, which in turn involves the structure constants for $se(n)$. We then have, analogously to the calculation for $SE(4)$, that $\mathbf{a}$

and **b** obey (4.45) with optimal controls given by

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix} = \mathbf{a}. \tag{4.51}$$

Writing (4.45) In terms of **a** alone, we have

$$\ddot{\mathbf{a}} + \hat{\Omega}\dot{\mathbf{a}} - \left(h - \frac{|\mathbf{a}|^2}{2}\right)\mathbf{a} = 0. \tag{4.52}$$

Defining $\tilde{\mathbf{a}}$ by

$$\mathbf{a} = \exp\left(-\frac{1}{2}\hat{\Omega}t\right)\tilde{\mathbf{a}}, \tag{4.53}$$

we use a calculation analogous to the one for $SE(3)$ to obtain

$$\ddot{\tilde{\mathbf{a}}} - \tilde{H}\tilde{\mathbf{a}} + \frac{1}{2}|\tilde{\mathbf{a}}|^2\tilde{\mathbf{a}} = 0, \tag{4.54}$$

where

$$\tilde{H} = h\mathbb{I}_{(n-1)\times(n-1)} + \frac{1}{4}\hat{\Omega}^2 \tag{4.55}$$

is a constant, symmetric matrix. Equation (4.54) can be considered a vector version of the stationary NLS, and among its solutions are classes of solutions analogous to the sinusoidal and elliptic function solutions identified in the analysis for $SE(3)$.

Remark: In place of (4.13) and X_2 given by (4.14) for the analysis in Section 4.2 of the optimal control problem on $SE(3)$, we could have used

$$\xi_u = X_4 + X_1u_1 + X_2u_2, \tag{4.56}$$

with

$$X_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.57}$$

in order to adhere to the pattern followed in Section 4.3 for $SE(4)$ and Section 4.4 for $SE(n)$, $n > 4$. The choice (4.56) with (4.57) is more natural for framed curves, but (4.13) with (4.14) is used in Section 4.2 to more smoothly segue from $SO(3)$ to $SE(3)$.

5. Physics. The nonlinear Schrödinger equation appears in the study of the DaRios-Betchov, or vortex filament equation (Betchov, 1965). Vortex filaments are persistent slender filamentary structures observed in three-dimensional fluid flows, and they exhibit self-induced motion (and shape change) due to the curvature distribution along the filament. The identification of vortex filament shapes and their stability has been studied extensively over the past half-century due to the aeronautical importance of vortices in fluid flow over airfoils.

Vortex filaments can be modeled using curves and natural Frenet frames, where the vortex filament is modeled as a curve in three-dimensional space which does not change length, but which can move and change shape with time. In place of (2.1), we have the partial differential equation

$$\begin{aligned} \boldsymbol{\gamma}_s &= \mathbf{T}, \\ \mathbf{T}_s &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \mathbf{M}_{1s} &= -k_1\mathbf{T}, \\ \mathbf{M}_{2s} &= -k_2\mathbf{T}, \end{aligned} \tag{5.1}$$

where $\boldsymbol{\gamma}$ and $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ are functions of both time t and arc-length parameter s. The DaRios-Betchov equation (or vortex filament equation) is

$$\boldsymbol{\gamma}_t = \boldsymbol{\gamma}_s \times \boldsymbol{\gamma}_{ss}, \tag{5.2}$$

and it describes the evolution of the curve representing the vortex filament evolving in three-dimensional space. Note that (5.2) is length-preserving, i.e.,

$$\frac{d}{dt}|\boldsymbol{\gamma}_s| = \frac{\boldsymbol{\gamma}_s \cdot \boldsymbol{\gamma}_{st}}{|\boldsymbol{\gamma}_s|} = \frac{\boldsymbol{\gamma}_s \cdot \boldsymbol{\gamma}_{ts}}{|\boldsymbol{\gamma}_s|} = \frac{\boldsymbol{\gamma}_s \cdot (\boldsymbol{\gamma}_{ss} \times \boldsymbol{\gamma}_{ss} + \boldsymbol{\gamma}_s \times \boldsymbol{\gamma}_{sss})}{|\boldsymbol{\gamma}_s|} = 0. \tag{5.3}$$

Equation (5.2) possesses even more structure: the Hasimoto transformation shows that the natural curvatures (k_1, k_2) in (5.1), evolving according to (5.2), are given by the nonlinear Schrödinger equation (Hasimoto, 1972). Analysis of the DaRios-Betchov equation from a geometric point of view, using natural frames and the nonlinear Schrödinger equation, can be found in the work of Langer and Perline (1991, 1996).

5.1. DaRios-Betchov equation and NLS. Using the fact that mixed partial derivatives commute, i.e., $\boldsymbol{\gamma}_{st} = \boldsymbol{\gamma}_{ts}$ and $\mathbf{T}_{ts} = \mathbf{T}_{st}$, after some calculation we obtain

$$k_{2t} = \left(\frac{1}{2}\left(k_1^2 + k_2^2\right) - A(t)\right) k_1 + k_{1ss}, \tag{5.4}$$

where $A(t)$ is a constant of integration. An analogous calculation gives

$$k_{1t} = \left(-\frac{1}{2}\left(k_1^2 + k_2^2\right) + A(t)\right) k_2 - k_{2ss}. \tag{5.5}$$

To show the connection between (5.4), (5.5) and the nonlinear Schrödinger equation, we let

$$\psi = k_1 + ik_2, \tag{5.6}$$

where $i = \sqrt{-1}$, so that

$$\begin{aligned}\psi_t = k_{1t} + ik_{2t} &= i\left(k_1 + ik_2\right)_{ss} + i\left[\frac{1}{2}\left(k_1^2 + k_2^2\right) - A(t)\right]\left(k_1 + ik_2\right)\\ &= i\left[\Delta + \left(\frac{1}{2}|\psi|^2 - A(t)\right)\right]\psi,\end{aligned} \tag{5.7}$$

where Δ denotes the Laplacian operator with respect to s.

In fact, (5.7) can be simplified somewhat by introducing a change of variables based on a phase factor, as observed by Hasimoto (1972). We let

$$\tilde{\psi} = \exp\left(i\int_0^t A(\sigma)d\sigma\right)\psi, \tag{5.8}$$

so that

$$\begin{aligned}\tilde{\psi}_t &= iA(t)\exp\left(i\int_0^t A(\sigma)d\sigma\right)\psi + \exp\left(i\int_0^t A(\sigma)d\sigma\right)\psi_t\\ &= iA(t)\tilde{\psi} + \exp\left(i\int_0^t A(\sigma)d\sigma\right)\left[i\Delta\psi + i\left(\frac{1}{2}|\psi|^2 - A(t)\right)\psi\right]\\ &= \exp\left(i\int_0^t A(\sigma)d\sigma\right)\left(i\Delta\psi + i\frac{1}{2}|\psi|^2\psi\right)\\ &= i\left(\Delta + \frac{1}{2}|\tilde{\psi}|^2\right)\tilde{\psi}.\end{aligned} \tag{5.9}$$

We have thus obtained

$$i\tilde{\psi}_t = -\left(\Delta + \frac{1}{2}|\tilde{\psi}|^2\right)\tilde{\psi}, \tag{5.10}$$

the focusing cubic nonlinear Schrödinger equation (without the additional term involving $A(t)$ in (5.7).

5.2. Steady vortex filaments. The NLS (5.7) thus describes how the natural curvatures of the vortex filament evolve with time. There are a variety of solutions to (5.7), including ones which move (i.e., rotate about their axis or translate) without changing their form - such solutions are traveling wave solutions of (5.7). One such class of solutions, whose stability properties have been extensively analyzed, are helical vortex filaments (Widnall, 1972). Others include circular rings, closed coils, planar sinusoidal curves, and curves described by elliptic functions (Kida, 1981).

Thus, the solutions to the fixed-endpoint optimal control problem on $SE(3)$ in Section 4.2 are related to the steady vortex filament solutions, in the sense that both satisfy restricted versions of the NLS equation. There are also cosmetic differences between (5.7) and (4.30), such as the role of fixed endpoint conditions versus the DaRios-Betchov equation for determining the constants which appear in the solutions. Nevertheless, there is clearly a close relationship in terms of the form of the solutions. A more subtle question is to establish direct links between the underlying variational principle for the vortex filament equation and optimal control that explains the connection between these problems at a deeper level.

6. Conclusion. The present paper is inspired by earlier work of Baillieul on optimal control problems. We have shown that the equations governing extremals have interesting structure that appears to be previously unknown. Furthermore, the extremal solutions for a particular optimal control problem on $SE(n)$ are associated with stationary solutions to the usual nonlinear Schrödinger equation (for $n = 3$) and its higher-dimensional analogs (for $n \geq 4$). Connections to similar solutions which appear in the study of vortex filament equations have also been described. A generalization of the methods used here to multiple particles in $SE(2)$ interacting through a fixed communication graph can be found in Justh & Krishnaprasad (2010).

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REFERENCES

[1] J. Baillieul (1975), *Some Optimization Problems in Geometric Control Theory*, Ph.D. Thesis, Harvard University.

[2] J. Baillieul (1978), *Geometric methods for nonlinear optimal control problems*, J. Opt. Theory Appl., 25:4, pp. 519-548.

[3] R. Betchov (1965), *On the curvature and torsion of an isolated vortex filament*, J. Fluid Mech., 22, pp. 471 - 479.

[4] R. L. Bishop (1975), *There is more than one way to frame a curve*, The American Mathematical Monthly, 82:3, pp. 246-251.

[5] H. T. Davis (1962), *Introduction to Nonlinear Differential and Integral Equations*, Dover (Reprint).

[6] H. Hasimoto (1972), *A soliton on a vortex filament*, J. Fluid Mech. 51:3, pp. 477-485.

[7] V. Jurdjevic (1997), *Geometric Control Theory*, Cambridge Univ. Press.

[8] E. W. Justh and P. S. Krishnaprasad (2002), *A simple control law for UAV formation flying*, Institute for Systems Research Technical Report TR 2002-38.

[9] E. W. Justh and P. S. Krishnaprasad (2004), *Equilibria and steering laws for planar formations*, Syst. and Control Lett., 52:1, pp. 25-38.

[10] E. W. Justh and P. S. Krishnaprasad (2005), *Natural frames and interacting particles in three dimensions*, Proc. 44th IEEE Conf. Decision and Control, pp. 2841-2846 (see also arXiv:math.OC/0503390v1).

[11] E. W. Justh and P. S. Krishnaprasad (2006), *Steering laws for motion camouflage*, Proc. R. Soc. A, Vol. 462, No. 2076, pp. 3629-3643 (see also arXiv:math.OC/0508023).

[12] E. W. Justh and P. S. Krishnaprasad (2010), *Extremal Collective Behavior*, Proc. 49th IEEE Conf. Decision and Control, to appear.

[13] S. Kida (1981), *A vortex filament moving without change of form*, J. Fluid Mech., 112, pp. 397-409.

[14] P. S. Krishnaprasad (1993), *Optimal control and Poisson reduction*, Institute for Systems Research Technical Report T.R. 93-87.

[15] J. Langer and R. Perline (1991), *Poisson geometry of the filament equation*, J. Nonlin. Sci., 1:1, pp. 71-93.

[16] J. Langer and R. Perline (1996), *The localized induction equation, the Heisenberg chain, and the nonlinear Schrodinger equation*, in W. F. Shadwick, P. S. Krishnaprasad and T. S. Ratiu (eds.), Mechanics Day, Fields Institute Communications, Vol. 7, pp. 181-187.

[17] P. K. Newton (2001), *The N-Vortex Problem: Analytical Techniques*, Springer-Verlag, New York.

[18] P. V. Reddy (2007), *Steering laws for Pursuit*, M. S. Thesis, Department of Electrical and Computer Engineering, Univ. of Maryland.

[19] P. V. Reddy, E. W. Justh, and P. S. Krishnaprasad (2006), *Motion camouflage in three dimensions*, Proc. 45th IEEE Conf. Decision and Control, pp. 3327-3332 (see also arXiv:math.OC/0603176v1).

[20] R. L. Ricca (1996), *The contributions of Da Rios and Levi-Civita to asymptotic potential theory and vortex filament dynamics*, Fluid Dynamics Research, Vol. 18, pp. 245-268.

[21] S. E. Widnall (1972), *The stability of a helical vortex filament*, J. Fluid Mech. 54:4, pp. 641-663.

[22] C. Wiggins (2001), *Biopolymer mechanics: stability, dynamics and statistics*, Mathematical Methods in the Applied Sciences, 24, pp. 1325-1335.

[23] F. Zhang, E. W. Justh, and P. S. Krishnaprasad (2004), *Boundary following using gyroscopic control*, Proc. 43rd IEEE Conf. Decision and Control, pp. 5204-5209.