NONHOLONOMIC SYSTEMS AND SUB-RIEMANNIAN GEOMETRY*

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Abstract. This paper presents several classical mechanical systems with nonholonomic constraints from the point of view of sub-Riemannian geometry. For those systems that satisfy the bracket generating condition the system can move continuously between any two given states. However, the paper provides a counterexample to show that the bracket generating condition is not also a sufficient condition for connectivity. All possible motions of the system correspond to curves tangent to the distribution defined by the nonholonomic constraints. Among the connecting curves we distinguish an optimal one which minimizes a certain energy induced by a natural sub-Riemannian metric on the non-integrable distribution. The paper discusses several classical problems such as the knife edge, the skater, the rolling disk and the nonholonomic bicycle.

Key words: sub-Riemannian distribution, nonholonomic constraint, Euler-Lagrange equations

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1. Introduction. The goal of this expository paper is to explore some classical mechanical systems with constraints from the point of view of the sub-Riemannian geometry. In this framework we have control over the velocities of the system, which belong to a certain non-integrable distribution defined by the nonholonomic constraints. One of the main problems is to determine whether there is a continuous curve tangent to the distribution and connecting any two given points. It was shown by Chow [3] that if the bracket generating condition holds, then the system can move continuously, piece-wise differentiable, between any two given states. The bracket generating condition says that the vector fields which generate the distribution, together with their iterated brackets, span the tangent space of the coordinate space at each point. If the number of brackets needed to generate all the missing directions at a point p is denoted by k, then it is said that the distribution is of step $k + 1$ at the point p. This way, the step 1 corresponds to the case when the distribution is the entire tangent bundle of a Riemannian manifold.

The distribution is endowed with a metric which enables us to measure the distance along the curves tangent to the distribution and to define the variational problems related to the length and energy of the trajectories. Most of the time these equations are not the same as the classical variational nonholonomic equations considered by other authors. This is partially due to the fact that the Lagrangians which

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describe the dynamics of the system are constructed only by pure geometric means, with the kinetic energy along the motion defined by the underlining sub-Riemannian metric, and no other exterior fields like mass, momenta of inertia, gravitation, etc., are added to the Lagrangian. This is also consistent with the Riemannian geometry case, where the velocity is defined using the Riemannian metric on the tangent bundle. In general, the variational equations are hard or almost impossible to be solved explicitly. However, this paper presents closed form integration formulas for some nonholonomic systems such as the rolling coin or the skating knife edge.

Since the sub-Riemannian metric determines the structure of the Lagrangian which implicitly determines the optimal solution, one natural question is how one chooses the natural metric among the infinitely many sub-Riemannian metrics which exist on a given distribution? Assume there is a distinguished basis $\{X_1, \ldots, X_k\}$ of vector fields on the non-integrable distribution $\mathcal{H}$, with the property that each vector field has a certain physical meaning (such as rotation, translation, angular momentum, acceleration etc). Then the natural sub-Riemannian metric is the unique metric with respect to the aforementioned basis becomes orthonormal.

The present paper deals with nonholonomic mechanical systems such as the knife edge, nonholonomic skater, rolling disk, nonholonomic bicycle and elastic curves. The states of each of the aforementioned systems can be viewed as points on a certain coordinate space, and their motion corresponds, due to the constraints, to a curve tangent to the distribution defined by the constraints. In all of the cases presented here the constraints are given by non-integrable one-forms, with non-integrable induced distributions. The global connectivity by curves tangent to the distribution is either implied by the bracket generating property of the distribution or is shown by some direct ad-hoc methods. This can be interpreted as steering the system from a given initial position to a final position; for instance, in the case of the bicycle this becomes a common parking problem.

Another important problem is to find the optimal energy curve between two given states. In other words, "how do you park your bicycle" with a minimum energy effort? This problem is more difficult than the connectivity problem and it might have multiple solutions, depending on how the "optimality" is defined. If the optimal curves are solutions of the variational problem given by a Lagrangian with constraints, then they are regular sub-Riemannian geodesics of the associated sub-Riemannian geometry.

The paper is organized as follows. Section 2 deals with skating as a 3-dimensional sub-Riemannian problem endowed with a distribution of rank 2. Section 3 characterizes the motion of a skater with two knife edges which slides on a horizontal plane. The associated sub-Riemannian geometry satisfies the connectivity property but it is not bracket generating; this is a counterexample for the converse of Chow's theorem. In section 4 we recall a variational approach of elastic curves, which are related to

the optimal trajectory of a skater. Section 5 deals with the nonholonomic problem of the rolling disk and its optimal solution. Finally, Section 6 associates a nonholonomic geometry to the motion of a bicycle.

2. Skating. Consider a convex blade tangent to a horizontal plane at the point (x, y) and making an angle θ with the x-axis, see Fig.1(a). This kind of nonholonomic system is known in the literature under the name of *knife edge*, see [1]. We shall treat this problem from the sub-Riemannian point of view only. This means that the inertia momenta, gravitational forces or damping forces are not taken into account, and the only control we have is on the velocity given by some constraint imposed in the tangent bundle.

The knife edge can be parameterized by three parameters $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$. On this 3-dimensional manifold we shall introduce a rank 2 distribution $\mathcal{H}$ defined by the velocity constraint. Let $\gamma(s) = \big(x(s), y(s)\big)$ be the trace of the contact point. The relations $\dot{x} = |\dot{\gamma}| \cos\theta$, $\dot{y} = |\dot{\gamma}| \sin\theta$ lead to the constraint $\dot{x} \sin\theta - \dot{y} \cos\theta = 0$. This induces the one-form $\omega = \sin\theta \, dx - \cos\theta \, dy$, which defines the distribution $\mathcal{H} = \ker\omega$. One may check the condition

$$\omega \wedge d\omega = -dx \wedge dy \wedge d\theta \neq 0,$$

implies that $\mathcal{H}$ is a contact distribution.

The trajectory of the knife edge on the space $\mathbb{R}^2 \times \mathbb{S}^1$ corresponds to a curve tangent to the distribution $\mathcal{H}$. Consider the vector fields

$$X = \cos\theta \, \partial_x + \sin\theta \, \partial_y, \qquad Y = \partial_\theta.$$

Since $\omega(X) = \omega(Y) = 0$, then the pair $\{X, Y\}$ forms a basis for the distribution $\mathcal{H}$. The first vector field describes the position in the plane while the later describes the rotation. Their bracket is the angular momentum

$$Z = [X, Y] = \sin\theta \, \partial_x - \cos\theta \, \partial_y = \dot{y} \, \partial_x - \dot{x} \, \partial_y \neq 0.$$

Since $\{X, Y, Z\}$ generate the tangent space of $\mathbb{R}^2 \times \mathbb{S}^1$ at each point, then the bracket generating condition holds and the distribution is of step 2. Since $\omega(Z) = 1$, then Z is a contact vector field for the contact distribution $\mathcal{H}$. By Chow's connectivity theorem, there is a piece-wise smooth curve tangent to $\mathcal{H}$ joining any two given points. However, we shall prove next that the attribute "piece-wise" can be removed.

PROPOSITION 2.1. *Given any two points* $(x_0, y_0, \theta_0), (x_1, y_1, \theta_1) \in \mathbb{R}^2 \times \mathbb{S}^1$, *there is a smooth trajectory of the knife edge joining these points.*

Proof. The previous result can be restated in the following slightly stronger version: *Given two points* $A = (x_0, y_0), B = (x_1, y_1) \in \mathbb{R}^2$ *and two unit vectors* $v_A \in T_A\mathbb{R}^2$, $v_B \in T_B\mathbb{R}^2$, *there is a plane curve that joins the points A and B, having the initial and final velocities v_A and v_B, respectively, see Fig.1(b).*
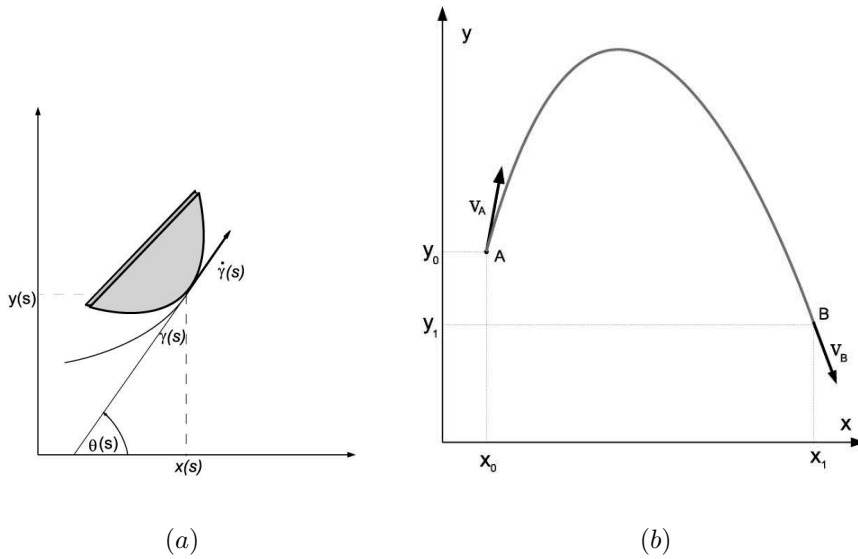
FIG. 1. (a) *The parameter space of a skate is* $\mathbb{R}^2 \times \mathbb{S}^1$; (b) *Connectivity by smooth curves.*

Let s denote the arc length parameter. We shall construct a smooth curve $\big(x(s), y(s), \theta(s)\big)$ in $\mathbb{R}^2 \times \mathbb{S}^1$ tangent to the distribution $\mathcal{H}$ satisfying the boundary conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad \theta(0) = \theta_0, \quad x(\tau) = x_1, \quad y(\tau) = y_1, \quad \theta(\tau) = \theta_1.$$

Using a translation, we may assume that all the values x_0, y_0, x_1, y_1 are non-zero. Let the curve be parameterized by the interval $[0, \tau]$. Choose the first two components of the curve to be given by

$$(2.1) \qquad\qquad x(s) = x_0 \left(\frac{\tau - s}{\tau} \right)^\alpha + x_1 \left(\frac{s}{\tau} \right)^\beta$$

$$(2.2) \qquad\qquad y(s) = y_0 \left(\frac{\tau - s}{\tau} \right)^{\alpha'} + y_1 \left(\frac{s}{\tau} \right)^{\beta'}.$$

The curve joins the points A and B in time τ and depends on the real parameters $\alpha, \alpha', \beta,$ and β'. Differentiating yields

$$\dot{x}(0) = -\frac{\alpha x_0}{\tau}, \qquad \dot{x}(\tau) = \frac{\beta x_1}{\tau}$$

$$\dot{y}(0) = -\frac{\alpha' y_0}{\tau}, \qquad \dot{y}(\tau) = \frac{\beta' y_1}{\tau}.$$

Comparing with

$$\dot{x}(0) = \cos\theta_0, \qquad \dot{x}(\tau) = \cos\theta_1, \qquad \dot{y}(0) = \sin\theta_0, \qquad \dot{y}(\tau) = \sin\theta_1$$

we determine the following values for the parameters

$$\alpha = -\frac{\tau \cos\theta_0}{x_0}, \qquad \alpha' = -\frac{\tau \sin\theta_0}{y_0}, \qquad \beta = \frac{\tau \cos\theta_1}{x_1}, \qquad \beta' = \frac{\tau \sin\theta_1}{y_1}.$$

Substituting in (2.1–2.2) yields the expression for the components $x(s)$ and $y(s)$. The component $\theta(s)$ is obtained from the velocity constraint $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$, and it is given by $\theta(s) = \tan^{-1}\frac{\dot{y}(s)}{\dot{x}(s)}$, with $x(s)$ and $y(s)$ considered previously. The conditions $\theta(0) = \theta_0$ and $\theta(\tau) = \theta_1$ are automatically satisfied by construction. $\qquad\square$

The sub-Riemannian metric. If $c(s) = \big(x(s), y(s), \theta(s)\big)$ is a curve tangent to the contact distribution $\mathcal{H}$, then its velocity can be written as

$$\dot{c} = \dot{x}\partial_x + \dot{y}\partial_y + \dot{\theta}\partial_\theta = (\dot{x}\cos\theta + \dot{y}\sin\theta)X + \dot{\theta}Y.$$

Choosing a metric on $\mathcal{H}$ in which the vector fields $\{X, Y\}$ are orthonormal, the length of the velocity becomes

$$|\dot{c}| = \sqrt{(\dot{x}\cos\theta + \dot{y}\sin\theta)^2 + \dot{\theta}^2}.$$

The kinetic energy along a curve measured with respect to the previous sub-Riemannian metric, given that the curve is tangent to the distribution $\mathcal{H}$, leads to the following Lagrangian with constraint

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}(\dot{x}\cos\theta + \dot{y}\sin\theta)^2 + \lambda(\dot{x}\sin\theta - \dot{y}\cos\theta),$$

where λ is a Lagrange multiplier function.

Solving the Euler-Lagrange system. Let $\xi = \dot{x}\cos\theta + \dot{y}\sin\theta$. Then the derivative with respect to θ is $\xi' = -(\dot{x}\sin\theta - \dot{y}\cos\theta)$. Since $\frac{\partial L}{\partial\dot{x}} = \xi\cos\theta + \lambda\sin\theta$, $\frac{\partial L}{\partial\dot{y}} = \xi\sin\theta - \lambda\cos\theta$ and $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0$, the corresponding Euler-Lagrange equations become

$$\xi\cos\theta + \lambda\sin\theta = C_1$$
$$\xi\sin\theta - \lambda\cos\theta = C_2,$$

with C_1, C_2 constants. Solving for ξ and λ yields

$$\xi = C_1\cos\theta + C_2\sin\theta$$
$$\lambda = C_1\sin\theta - C_2\cos\theta,$$

which implies $\xi' = -\lambda$. Then $\frac{\partial L}{\partial\theta} = \xi\xi' + \lambda\xi = 0$. The variational equation corresponding to θ becomes $\ddot{\theta} = 0$, with the linear solution $\theta(s) = as + b = \frac{\theta_1 - \theta_0}{\tau}s + \theta_0$. Writing ξ in two equivalent ways and using the nonholonomic constraint, we arrive at the following system

$$\dot{x}\cos\theta + \dot{y}\sin\theta = C_1\cos\theta + C_2\sin\theta$$
$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0.$$

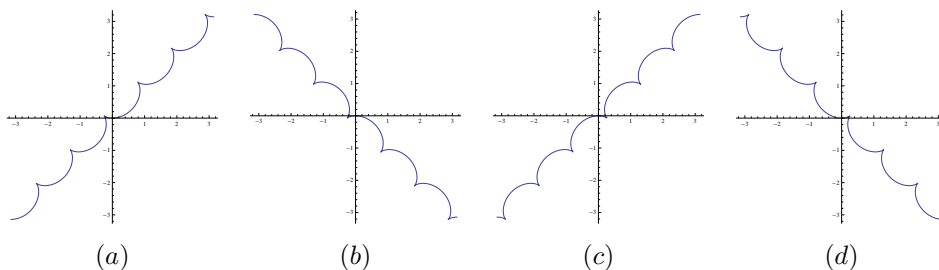(a)   (b)   (c)   (d)

FIG. 2. *The trace of a skate in optimal motion with $a = 1.5$, $b = 2$, $s \in [-2\pi, 2\pi]$: (a) The case $C_1 = 1, C_2 = 1$; (b) The case $C_1 = -1, C_2 = 1$; (c) The case $C_1 = -1, C_2 = -1$; (d) The case $C_1 = 1, C_2 = -1$;*

Solving for the velocity components yields

$$\dot{x} = C_1 \cos^2 \theta + C_2 \sin \theta \cos \theta$$
$$\dot{y} = C_2 \sin^2 \theta + C_1 \sin \theta \cos \theta.$$

Integrating we get

$$x(s) = x_0 + \frac{1}{2}C_1 s + \frac{1}{2a} \sin(as)\big(C_1 \cos \theta + C_2 \sin \theta\big)$$
$$y(s) = y_0 + \frac{1}{2}C_2 s + \frac{1}{4a}C_2 \sin(2b) - \frac{C_2}{4a} \sin(2\theta) + \frac{1}{2a}C_1 \sin(as) \sin \theta.$$

The values of the constants C_1 and C_2 are determined from the boundary conditions $x(\tau) = x_1$, $y(\tau) = y_1$. The parametric plot of the solution is represented in Fig.2 for different signs of the constants C_1 and C_2.

3. The nonholonomic skater. A skater is a bicycle-type vehicle with skates instead of wheels that slide on a horizontal plane, see Fig.3(a). Both the front and rear skates are convex sharp blades tangent to the plane. The skater is vertical all the time to the skating plane. The distance between the contact points is constant, equal to a. Just the front skate can rotate in a vertical plane, while the rear one is fixed. The contact points describe two curves:

- the *front curve $c(s)$*, which is the trace of the contact point of the front skate and
- the *rear curve $\gamma(s)$*, which is the trace of the contact point of the rear skate.

We shall start the analysis of the motion by dealing with the relationship between the rear and the front curves.

PROPOSITION 3.1. *(i) Given the rear curve $\gamma(s)$, there is a unique front curve $c(s)$ associated with γ. This curve starts at the point $c(0) = \gamma(0) + a\dot{\gamma}(0)$ with the velocity $\dot{c}(0) = \dot{\gamma}(0) + a\ddot{\gamma}(0)$.*
(ii) Given the front curve $c(s)$ and a point P in the plane, with $\mathrm{dist}(P, c(0)) = a$, there is a unique rear curve $\gamma(s)$ associated with $c(s)$, starting at P.
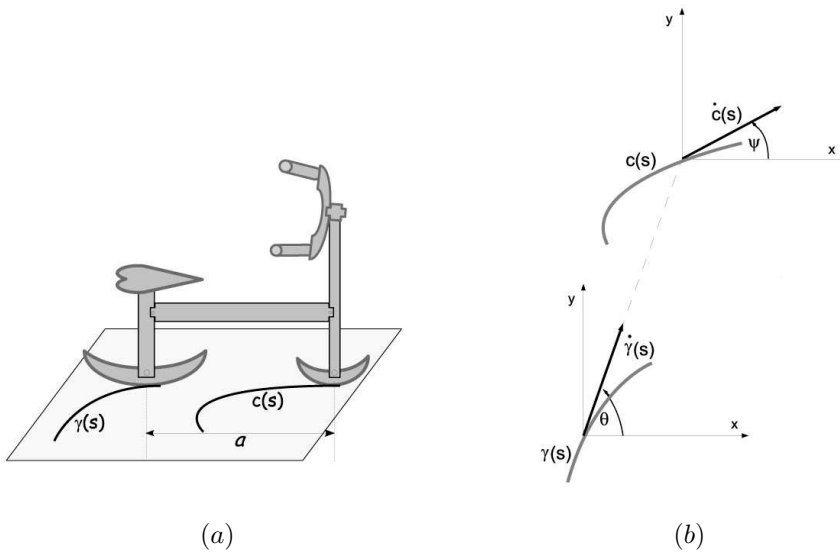
(a) (b)

FIG. 3. (a) *The front curve $c(s)$ and the rear curve $\gamma(s)$ of a skater; (b) The angular parameters θ and ψ.*

Proof. (i) Let s denote the arc length parameter along the rear curve γ. The contact point $c(s)$ of the front blade can be obtained by a shift in the direction $\dot{\gamma}(s)$ of the rear contact point

(3.1) $$c(s) = \gamma(s) + a\dot{\gamma}(s).$$

(ii) The components of the curve $\gamma = (\gamma^1, \gamma^2)$ satisfy the linear ODE

$$a\dot{\gamma}^k(s) + \gamma^k(s) = c^k(s), \qquad k = 1, 2,$$

with the solution

$$\gamma^k(s) = \frac{1}{a}e^{-\frac{s}{a}}\int_0^s e^{\frac{w}{a}}c^k(w)\,dw + \gamma^k(0)e^{-\frac{s}{a}}, \qquad k = 1, 2,$$

where $P = \gamma(0)$.

$\square$

Particular cases. If the front curve $c(s)$ is a line, then the rear curve $\gamma(s)$ is a tractrix curve.

When the front curve $c(s)$ is an arc of circle of radius R, then the rear curve is an arc of circle of radius $r = \sqrt{R^2 - a^2}$, see Fig.4(b). If $R = a$, then the rear curve degenerates to a point.

The total energy. In this subsection the total energy of the skater is defined to be the sum of the energies of the front and rear curves

$$(3.2) \qquad E = \frac{1}{2} \int_0^\tau \left(|\dot{c}(s)|^2 + |\dot{\gamma}(s)|^2 \right) ds.$$

The *optimum motion* is defined to be the motion for which the total energy is minimum. If s denotes the arc length along the rear curve, then (3.1) yields

$$(3.3) \qquad |\dot{c}(s)|^2 = |\dot{\gamma}(s) + a\ddot{\gamma}(s)|^2 = 1 + a^2 \kappa_\gamma^2(s),$$

where we used that $\dot{\gamma} \perp \ddot{\gamma}$ and we denoted the curvature along γ by κ_γ. Then the total energy (3.2) becomes

$$(3.4) \qquad E = \tau + \frac{a^2}{2} \int_0^\tau \kappa_\gamma^2(s) \, ds.$$

As it was first inferred by D. Bernoulli, the elastic potential of a lamina is given by the integral of the square of its curvature. Then the quantity (3.4) has the mean of potential elastic energy. We shall come back to this problem shortly.

Boundary values. Consider the endpoints

$$(3.5) \qquad c(0) = A_1, \quad \gamma(0) = B_1, \quad c(\tau) = A_2, \quad \gamma(\tau) = B_2$$

given such that

$$|c(0) - c(\tau)| = |\gamma(0) - \gamma(\tau)| = a.$$

Since

$$\dot{\gamma}(0) = \frac{1}{a}\left(c(0) - \gamma(0) \right)$$
$$\dot{\gamma}(\tau) = \frac{1}{a}\left(c(\tau) - \gamma(\tau) \right),$$

the previous conditions are equivalent with providing the positions and velocities at the end points of the rear curve γ

$$(3.6) \qquad \gamma(0) = B_1, \quad \gamma(\tau) = B_2, \quad \dot{\gamma}(0) = v_1, \quad \dot{\gamma}(\tau) = v_2,$$

with $|v_1| = |v_2| = 1$. Finding the optimum solution for the variational problem (3.2) with the boundary values (3.5) is equivalent with finding the minimum of the energy (3.4) subject to conditions (3.6). This can be also stated as finding the minimum elastic potential $\int_0^\tau \frac{1}{2}\kappa^2(s) \, ds$ for a curve passing through two given points with prescribed end-point velocities and having a fixed length τ. These curves are known in the literature under the name of *elastica curves* and they have been characterized by Euler. For the sake of completeness a variational approach of elastica will be included in the section 4.
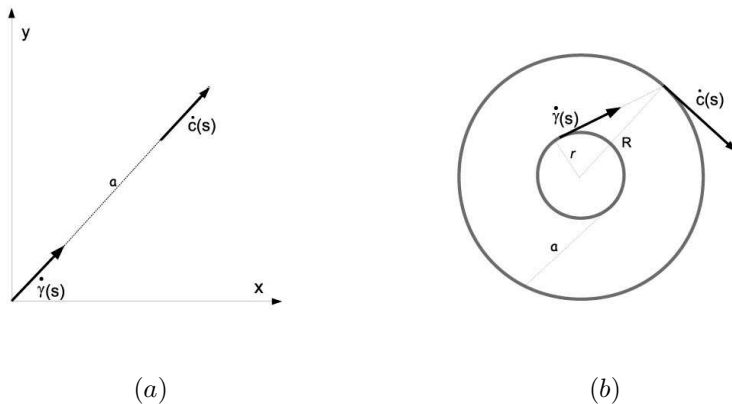
(a) (b)

FIG. 4. (a) *The case* $\theta = \psi$ (b) *The case* $\theta = \psi - \frac{\pi}{2}$.

The length of the front curve. Using (3.3) we can write a formula for the length of the front curve

$$\ell(c) = \int_0^\tau |\dot{c}(s)|\, ds = \int_0^\tau \sqrt{1 + a^2 \kappa_\gamma^2(s)}\, ds \geq \tau = \ell(\gamma),$$

with equality if and only if $\kappa_\gamma = 0$, i.e. when γ is a line. It follows that in this case c is also a line. The aforementioned inequality implies that the front curve is at least as long as the rear curve.

The nonholonomic constraints. Consider the front and rear curves having the following components in a Cartesian system of coordinates

$$c(s) = \big(u(s), v(s)\big), \quad \gamma(s) = \big(x(s), y(s)\big),$$

where s is the arc length along γ. Let θ and ψ be the angles made by the velocities $\dot\gamma$ and $\dot{c}$ with the x-axis, see Fig.3(b). From $\dot{u} = |\dot{c}|\cos\psi$, $\dot{v} = |\dot{c}|\sin\psi$ and $\dot{x} = \cos\theta$, $\dot{y} = \sin\theta$ yields the following constraints on the front and rear velocities:

(3.7) $\dot{u}\sin\psi = \dot{v}\cos\psi, \qquad \dot{x}\sin\theta = \dot{y}\cos\theta.$

Writing the relation $c = \gamma + a\dot\gamma$ on components

$$u = x + a\dot{x} = x + a\cos\theta$$
$$v = y + a\dot{y} = y + a\sin\theta,$$

and differentiating with respect to s yields

$$(3.8) \qquad\qquad \dot{u} = \dot{x} - a \sin\theta\, \dot{\theta}$$

$$(3.9) \qquad\qquad \dot{v} = \dot{y} + a \cos\theta\, \dot{\theta}.$$

Substituting in the first relation of (3.7) we obtain

$$(3.10) \qquad\qquad \sin\psi\, \dot{x} - \cos\psi\, \dot{y} = a \cos(\theta - \psi)\, \dot{\theta}.$$

It follows that the motion of the skater corresponds to the curve $\big(x(s), y(s), \theta(s), \psi(s)\big)$ on the parameters space $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ satisfying the following Pfaff system

$$(3.11) \qquad\qquad \sin\psi\, dx - \cos\psi\, dy = a \cos(\theta - \psi)\, d\theta$$

$$(3.12) \qquad\qquad \sin\theta\, dx - \cos\theta\, dy = 0.$$

The associated sub-Riemannian geometry. Consider the one-forms

$$(3.13) \qquad\qquad \omega_1 = \sin\psi\, dx - \cos\psi\, dy - a \cos(\theta - \psi)\, d\theta$$

$$(3.14) \qquad\qquad \omega_2 = \sin\theta\, dx - \cos\theta\, dy,$$

and define the rank 2 distribution $\mathcal{H} = \ker\omega_1 \cap \ker\omega_2$. The motion of the skater corresponds to curves in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ tangent to the distribution $\mathcal{H}$.

The case $\theta = \psi$. In this case the tangent vectors $\dot{c}$ and $\dot{\gamma}$ are collinear, so $\dot{c} = \lambda\dot{\gamma}$, with $\lambda \neq 0$. Since $\dot{c} = \dot{\gamma} + a\ddot{\gamma}$, then

$$a|\ddot{\gamma}|^2 = \langle \dot{c}, \ddot{\gamma}\rangle - \langle \dot{\gamma}, \ddot{\gamma}\rangle = 0,$$

where s is the arc length along γ. It follows that $\gamma(s)$ is a line. Then $c(s)$ is also a line with the same support as γ, see Fig.4(a).

The case $\theta \neq \psi$. Multiplying the first equation of (3.11) by $\sin\theta$ and the second equation by $\sin\psi$, subtracting yields

$$dy = a \sin\theta\, \cot(\psi - \theta)\, d\theta,$$

as long as $\psi \neq \theta$. Multiplying the first equation of (3.11) by $\cos\theta$ and the second equation by $\cos\psi$, subtracting we obtain

$$dx = a \cos\theta\, \cot(\psi - \theta)\, d\theta.$$

Then the distribution can be written equivalently as $\mathcal{H} = \ker \eta_1 \cap \ker \eta_2$, with the one-forms

$$(3.15) \qquad\qquad\qquad \eta_1 = dx - a \cos \theta \cot(\psi - \theta) \, d\theta$$

$$(3.16) \qquad\qquad\qquad \eta_2 = dy - a \sin \theta \cot(\psi - \theta) \, d\theta.$$

In order to find a basis $\{X_1, X_2\}$ of the distribution $\mathcal{H}$, we may chose $X_1 = \partial_\psi$ since $\eta_i(X_1) = 0$, $i = 1, 2$. Let the second vector field be $X_2 = \alpha \partial_x + \beta \partial_y + \partial_\theta$, with the components α and β determined by the conditions $\eta_1(X_2) = \eta_2(X_2) = 0$. Then

$$\alpha = a \cos \theta \cot(\psi - \theta), \qquad \beta = a \sin \theta \cot(\psi - \theta).$$

Hence the distribution $\mathcal{H}$ is spanned by the linear independent vector fields

$$(3.17) \qquad\qquad X_1 = \partial_\psi$$

$$(3.18) \qquad\qquad X_2 = a \cot(\psi - \theta)\big(\cos \theta \partial_x + \sin \theta \partial_y \big) + \partial_\theta.$$

The non-integrability of distribution $\mathcal{H}$. Let $V = \cos \theta \partial_x + \sin \theta \partial_y$. Since X_1 commutes with V and ∂_θ, we have

$$X_3 = [X_1, X_2] = [X_1, a \cot(\psi - \theta) V + \partial_\theta] = X_1\big(a \cot(\psi - \theta) \big) V$$
$$= -a \csc^2(\psi - \theta) V \neq 0,$$

so $\mathcal{H}$ is not integrable. If the vector fields X_1, X_2 and their iterated brackets span the tangent space of $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ at each point, then the distribution satisfies the bracket generating condition. In our case $\{X_1, X_2, X_3\}$ are linear independent vector fields, but further iterations lead to vector fields proportional to X_3

$$[X_1, [X_1, X_2]] = 2a \csc^3(\phi - \theta) V$$
$$[X_1, \ldots [X_1, [X_1, X_2]] \ldots] = -a \partial_\psi^k (\csc^2(\psi - \theta)) V := f(\psi - \theta) V$$
$$[X_2, [X_1, \ldots [X_1, [X_1, X_2]] \ldots]] = [a \cot(\psi - \theta) V + \partial_\theta, f V] = (\partial_\theta f) V.$$

It follows that the distribution is not bracket generating, and hence Chow's theorem cannot be applied in this case. However, the connectivity property might still hold. It suffices to consider the parking motion of the skater between any two given positions. If this can be always done, then this provides a counterexample for the converse of Chow's theorem (i.e. there are distributions satisfying the connectivity property which are not bracket generating). Since the literature is short on these type of examples, we shall include next an ad-hoc construction of such a piece-wise smooth curve joining any two given points in the parameters space.

The parking problem. Let A_0, A_1, B_0, B_1 be points in the plane such that $|A_0 B_0| = |A_1 B_1| = a$. The parking problem answers the following question: *Is it possible for*

the skater to start from a given initial position defined by the contact points A_0, B_0 and move continuously into the final position given by the contact points A_1, B_1?

The answer is positive. Let θ_0 and θ_1 be the angles made by the rays $A_0 B_0$ and $A_1 B_1$ with the x-axis. Giving the points (A_0, B_0) and (A_1, B_1) is equivalent with providing the boundary values (x_0, y_0, θ_0) and (x_1, y_1, θ_1). From Proposition 2.1 there is a rear curve $\gamma(s) = \big(x(s), y(s)\big)$ and an angular function $\theta(s)$ which join the given points (x_0, y_0, θ_0) and (x_1, y_1, θ_1). Proposition 3.1 assures the existence and uniqueness of the front curve $c(s)$ between the points B_0 and B_1. This shows that the parking problem has always solutions.

The only parameter we did not take care of yet is ψ, which is the angle between the front blade and the x-axis. This parameter can be adjusted by keeping the contact points of both front and rear blades fixed and steering the front blade by the desired angle such that at the beginning and at the end the angle has the two given values ψ_0 and ψ_1. This way, one obtains a piece-wise smooth curve $\big(x(s), y(s), \theta(s), \psi(s)\big)$ with end points $(x_0, y_0, \theta_0, \psi_0)$ and $(x_1, y_1, \theta_1, \psi_1)$. The associated sub-Riemannian manifold is a counterexample for the converse of Chow's theorem.

The sub-Riemannian metric. The distribution $\mathcal{H}$ will be endowed with a metric in which the vector fields X_1, X_2 are orthonormal. This will be the sub-Riemannian metric associated with our problem. In order to see how this metric acts, we consider a curve $\lambda(s) = \big(x(s), y(s), \theta(s), \psi(s)\big)$ on the coordinate space $\mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{S}^1$, which is tangent to the distribution $\mathcal{H}$. Using the constraints (3.15–3.16), the velocity vector becomes

$$\dot{\lambda} = \dot{x}\partial_x + \dot{y}\partial_y + \dot{\theta}\partial_\theta + \dot{\psi}\partial_\psi$$
$$= a\cos\theta\cot(\psi - \theta)\dot{\theta}\partial_x + a\sin\theta\cot(\psi - \theta)\dot{\theta}\partial_y + \dot{\psi}\partial_\psi$$
$$= \dot{\theta}X_2 + \dot{\psi}X_1,$$

so the controls of the curve λ are the angular velocities $\dot{\theta}$ and $\dot{\psi}$. Using the orthonormality of $\{X_1, X_2\}$ we have $|\dot{\lambda}(s)|^2 = \dot{\theta}^2 + \dot{\psi}^2$. This is used to introduce the notion of the sub-Riemannian energy along the trajectory λ by

$$(3.19) \qquad I = \frac{1}{2}\int_0^\tau |\dot{\lambda}(s)|^2 \, ds = \frac{1}{2}\int_0^\tau \big(\dot{\theta}(s)^2 + \dot{\psi}(s)^2\big) \, ds.$$

The optimal parking problem. Among all the parking motions with the initial points A_1, B_1 and final points A_2, B_2, find the one which minimizes the action integral (3.19)!

Since the motion occurs along curves tangent to the distribution $\mathcal{H}$ the dynamics are described by the following Lagrangian with constraints

$$L = \frac{1}{2}\big(\dot{\theta}(s)^2 + \dot{\psi}(s)^2\big) + \mu_1\big(\dot{x} - a\cos\theta\,\cot(\psi - \theta)\dot{\theta}\big) + \mu_2\big(\dot{y} - a\sin\theta\cot(\psi - \theta)\dot{\theta}\big).$$

Since $\dfrac{\partial L}{\partial x} = \dfrac{\partial L}{\partial y} = 0$, the Lagrange multipliers μ_1 and μ_2 are constant along the solutions of the Euler-Lagrange equations. The variational equations are complicated and they do not obviously seem to be integrable by quadratures.

4. Elastic curves. As it was for the first time noticed by D. Bernoulli, the elastic potential energy of a curve $c : [0, \tau] \to \mathbb{R}^2$ is given by the integral of the square of its curvature $\int_0^\tau \kappa^2(s)\,ds$. A curve which minimizes the previous integral is called an *elastic curve.* Finding all the planar elastic curves of a fixed length passing through two given points and having prescribed endpoint velocities was first asked by Bernoulli and then after solved by Euler, who provided a complete classification of planar elastic curves into 9 distinct types, see [6].

We came across elastic curves when we discussed the optimum motion of a skater on a plane. The rear curve of the skater in optimal motion describes an elastic curve. We shall provide next a variational method approach involving nonholonomic constraints. If the curve $c(s) = \big(x(s), y(s)\big)$ is considered unit speed and $\theta(s)$ is the angle made by the velocity $\dot{c}(s)$ with the x-axis, then the curvature satisfies $\kappa^2(s) = \dot{\theta}^2(s)$. Considering the velocity constraints $\dot{x} = \cos\theta$ and $\dot{y} = \sin\theta$, the problem of finding the elastic curves of length τ can be solved by considering the integral action $\int_0^\tau L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})\,ds$ with the Lagrangian

$$(4.1) \qquad L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + \lambda_1(\dot{x} - \cos\theta) + \lambda_2(\dot{y} - \sin\theta),$$

with λ_1, λ_2 Lagrange multipliers. The Euler-Lagrange equations

$$(4.2) \qquad \ddot{\theta} = \lambda_1 \sin\theta - \lambda_2 \cos\theta$$

$$(4.3) \qquad \dot{\lambda}_1 = 0$$

$$(4.4) \qquad \dot{\lambda}_2 = 0$$

lead to the standard pendulum equation

$$(4.5) \qquad \ddot{u} + a^2 \sin u = 0$$

by substituting $a^2 = \sqrt{\lambda_1^2 + \lambda_2^2}$ and $u = \theta + \pi - \alpha$, with $\alpha = \tan^{-1}(\lambda_2/\lambda_1)$. The solution of the equation (4.5) can be represented in terms of elliptic functions (see [4], vol. I, p.142)

$$\sin\frac{u(s)}{2} = k\,\mathrm{sn}(as, k),$$

where $k = \sin\frac{\gamma}{2}$ is the elliptic modulus and $\gamma = \max u(s)$ is the maximal amplitude of the swing. The value of the curvature is given by

$$\kappa(s) = \frac{d\theta}{ds} = \frac{du}{ds} = 2k\,\mathrm{cn}(s, k),$$

and then the minimum value of the elastic potential becomes

$$\int_0^\tau \frac{1}{2}\kappa^2(s)\,ds = 2k^2 \int_0^\tau \left(1 - \mathrm{sn}^2(s,k)\right) ds = 2\tau(k^2 - 1) + 2E(\tau, k)$$

(4.6)
$$= 2E(\tau, k) - 2\tau k'^2,$$

where we used the following formula of [5], p.62:

$$k^2 \int \mathrm{sn}^2(s,k)\,ds = s - E(s,k),$$

where $E(\cdot, k)$ is the Jacobi epsilon function defined by

$$E(s, k) = \int_0^s \mathrm{dn}^2(u, k)\,du, \qquad k \in (0,1),$$

and $k^2 + k'^2 = 1$. Integrating in the constraints $\dot{x} = \cos\theta$, $\dot{y} = \sin\theta$ yields the plane elastic curve $c(s) = \big(x(s), y(s)\big)$, see [6]

(4.7)
$$x(s) = s - 2E(am(s,k), k)$$

(4.8)
$$y(s) = -2k\,\mathrm{cn}(s, k),$$

where $am(s, k)$ is defined as the value of ϕ such that

$$s = \int_0^\phi \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}.$$

This variational treatment of plane elastic curves can be found also in [4], vol I, p.142.

Elastica curves appear naturally in other domains of Physics, as it was noticed in 1800's by Laplace and Maxwell who showed that the shape of the cross section of the capillary surface in a cylindrical tube is an elastic curve, see [7].

Elastic curves in the space. A similar approach can be applied to find elastic curves in $\mathbb{R}^3$. The curvature of a unit speed space curve $c(s) = \big(x(s), y(s), z(s)\big)$ can be written in terms of spherical coordinates as

$$\kappa^2(s) = \ddot{x}^2(s) + \ddot{y}^2(s) + \ddot{z}^2(s)$$

(4.9)
$$= \dot{\theta}^2(s) + \sin^2\theta(s)\dot{\psi}^2(s),$$

where θ and ψ are the angles made by the velocity $\dot{c}$ with the x-axis and the yz-plane, respectively. Using the constraints

(4.10)
$$\dot{x} = \cos\theta, \qquad \dot{y} = \sin\theta\cos\psi, \qquad \dot{z} = \sin\theta\sin\psi,$$

we set up the following Lagrangian

$$L = \frac{1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + \lambda_1(\dot{x} - \cos\theta) + \lambda_2(\dot{y} - \sin\theta\cos\psi) + \lambda_3(\dot{z} - \sin\theta\sin\psi).$$

Since $d\sigma^2 = d\theta^2 + \sin^2\theta\, d\psi^2$ is the canonical Riemannian metric on the sphere $\mathbb{S}^2$, the first term of the aforementioned Lagrangian is the kinetic energy induced by the natural metric of $\mathbb{S}^2$.

The Euler-Lagrange equations. Since L does not depend explicitly on x, y and z, it follows that the Lagrange multipliers λ_1, λ_2, and λ_3 are constants. The variational equations satisfied by θ and ψ are

$$\ddot{\theta} = \sin\theta(\cos\theta\dot{\psi}^2 + \lambda_1) - \cos\theta(\lambda_2\cos\psi + \lambda_3\sin\psi)$$

$$\frac{d}{ds}(\sin^2\theta\,\dot{\psi}) = \sin\theta(\lambda_2\sin\psi - \lambda_3\cos\psi).$$

Let $\Lambda = \sqrt{\lambda_2^2 + \lambda_3^2}$ and $\xi = \tan^{-1}(\lambda_3/\lambda_2)$. Then the previous system takes the form

$$\ddot{\theta} = \sin\theta(\cos\theta\dot{\psi}^2 + \lambda_1) - \Lambda\cos\theta\cos(\psi - \xi)$$

$$\frac{d}{ds}(\sin^2\theta\,\dot{\psi}) = \Lambda\sin\theta\sin(\psi - \xi),$$

whose method of exact integration is still unknown to the authors. The curve (x, y, z) is obtained by integrating in the constraints (4.10).

5. The rolling disk. The rolling coin problem is a classical problem of nonholonomic mechanics. In the following we shall investigate this problem from the sub-Riemannian geometry point of view by following the reference [2].

A disk of radius $R = 1$ rolls on a horizontal plane and is constrained to be vertical all the time. The position of the disk can be parameterized by four coordinates $(x, y, \psi, \phi) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$, where

- (x, y) are the coordinates of the center of the disk;
- ϕ is the angle made by some fixed radius on the disk makes with the vertical;
- ψ is the angle made by the plane of the disk with the x-axis, see Fig.5(a).

The motion of the disk can be described by a curve on the space $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ which is tangent to a certain non-integrable distribution as we shall describe in the following. In order to find the nonholonomic constraints which define the distribution, we shall equate the components of the instantaneous velocity $v(t) = R\dot{\phi}(t)$ against the velocity components of the trace curve $(x(t), y(t))$ to get

$$\dot{x} = v\cos\psi = R\dot{\phi}\cos\psi, \qquad \dot{y} = v\sin\psi = R\dot{\phi}\sin\psi.$$

These constraints define the rank 2 distribution $\mathcal{H} = \ker\omega_1 \cap \ker\omega_2$ in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$, where

$$\omega_1 = dx - R\cos\psi\, d\phi, \qquad \omega_2 = dy - R\sin\psi\, d\phi$$

are two independent one-forms. One may check that the linear independent vector fields

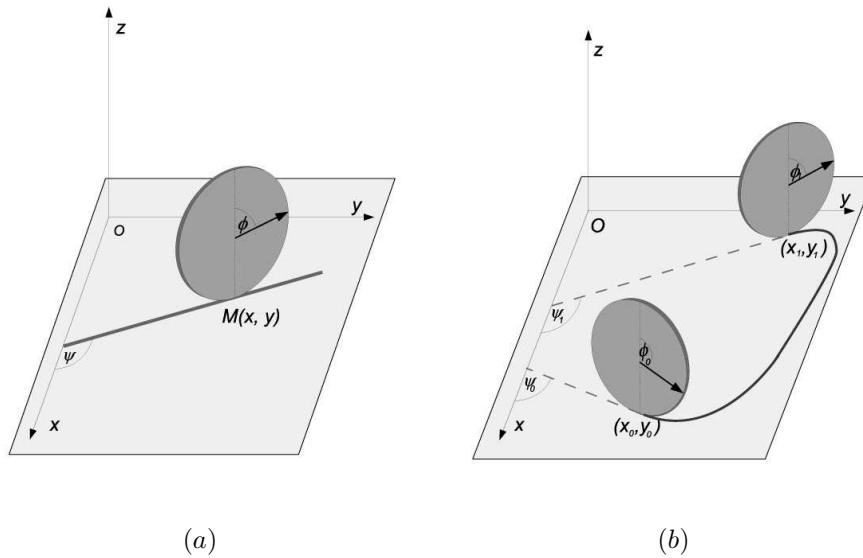$$X_1 = \partial_\psi, \qquad X_2 = R(\cos\psi\,\partial_x + \sin\psi\,\partial_y) + \partial_\phi$$

(a) (b)

FIG. 5. (a) *The parameters (x, y, ϕ, ψ) of a disk rolling on a plane; (b) By Chow's theorem there is a curve connecting $(x_0, y_0, \phi_0, \psi_0)$ and $(x_1, y_1, \phi_1, \psi_1)$.*

generate the distribution $\mathcal{H}$. Recall the assumption made on the disk radius that $R = 1$. In order to check the bracket generating condition, we introduce two other vector fields

$$X_3 = -\sin\psi\, \partial_x + \cos\psi\, \partial_y, \qquad X_4 = -(\cos\psi\, \partial_x + \sin\psi\, \partial_y)$$

which satisfy the following commutation relations

(5.1) $[X_1, X_2] = -[X_1, X_4] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_2, X_3] = [X_2, X_4] = 0.$

Since X_1, X_2, X_3, X_4 are linearly independent, they span the tangent space of $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ at each point, so the bracket generating condition holds, and by Chow's theorem we have the following connectivity result, see Fig.5(b):

Given two points $(x_0, y_0, \phi_0, \psi_0), (x_1, y_1, \phi_1, \psi_1)$ in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$, there is at least one piece-wise smooth trajectory of the disk that starts with the contact point (x_0, y_0) and initial angles ψ_0, ϕ_0 and ends at the contact point (x_1, y_1) having the final angles ψ_1, ϕ_1, respectively.

Like in the case of the knife edge, the attribute "piece-wise" can be omitted. One can construct a smooth trajectory between the end points in the following way. The angular velocity can be chosen to be constant, i.e $\phi = as + b$, with $b = \phi_0$, $a = \dfrac{\phi_1 - \phi_0}{\tau}$. Then the nonholonomic constraints become $\dot{x} = a\cos\psi$, $\dot{y} = a\sin\psi$,
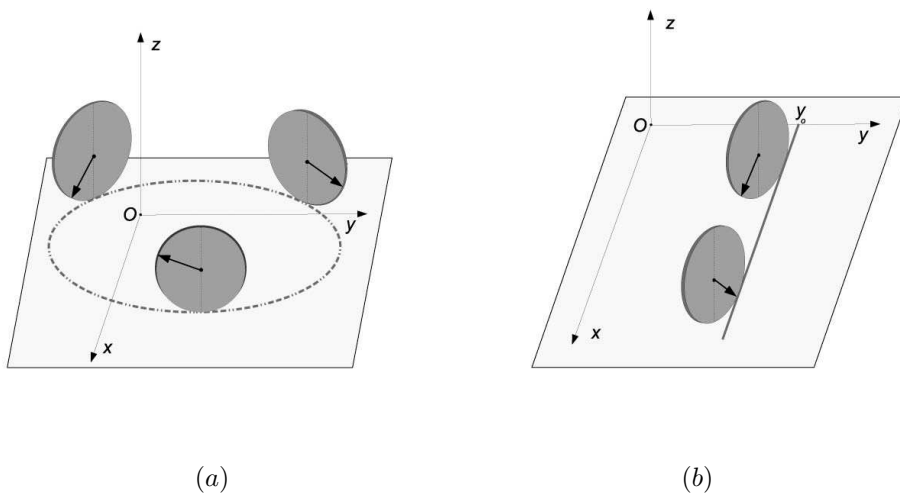
(a) (b)

FIG. 6. *The disk trajectory for* $|\mu| = 0$*: (a) The case* $\xi_2 \neq 0$*; (b) The case* $\xi_2 = 0$*.*

and hence $\frac{dy}{dx} = \tan \psi$. It suffices to choose any smooth curve (x, y) with $x(0) = x_0$, $y(0) = y_0$, $x(\tau) = x_1$, $y(\tau) = y_1$, with the initial and final slopes $\tan \psi_0$ and $\tan \psi_1$, respectively, see Fig.1(b). One construction is suggested by formulas (2.1–2.2). The function ψ can be chosen as $\psi(s) = \tan^{-1} \frac{\dot{y}(s)}{\dot{x}(s)}$.

Since two brackets are needed to generate all the missing directions, the commutation relations (5.1) imply that the distribution $\mathcal{H}$ is step 3 at each point.

We shall endow $\mathcal{H}$ with the sub-Riemannian metric in which the vector fields X_1 and X_2 are orthonormal. Suppose $c(t) = \big(x(t), y(t), \psi(t), \phi(t)\big)$ is a curve tangent to the distribution $\mathcal{H}$. Then its velocity can be written as

$$\dot{c}(t) = \dot{\psi} X_1 + \dot{\phi} X_2 + (\dot{x} - \dot{\phi} \cos \psi) \partial_x + (\dot{y} - \dot{\phi} \sin \psi) \partial_y$$
$$= \dot{\psi} X_1 + \dot{\phi} X_2.$$

It follows that the length of the velocity vector of the curve $c : [0, T] \to \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ with respect to the sub-Riemannian metric is $|\dot{c}(t)| = \sqrt{\dot{\psi}^2(t) + \dot{\phi}^2(t)}$. Then the dynamics of the disk is described by the following Lagrangian with constraints

(5.2) $$L = \frac{1}{2}\big(\dot{\psi}^2(t) + \dot{\phi}^2(t)\big) + \mu_1(\dot{x} - \dot{\phi} \cos \psi) + \mu_2(\dot{y} - \dot{\phi} \sin \psi).$$

In order to find the motion of the disk with the smallest energy one needs to solve

the Euler-Lagrange system of equations

$$\frac{\partial L}{\partial \dot{x}} = 0 \Longrightarrow \mu_1 = \text{constant}$$

$$\frac{\partial L}{\partial \dot{y}} = 0 \Longrightarrow \mu_2 = \text{constant}$$

$$\ddot{\psi} = \dot{\phi}(\mu_1 \sin \psi - \mu_2 \cos \psi)$$

$$\frac{d}{ds}(\dot{\phi} - \mu_1 \cos \psi - \mu_2 \sin \psi) = 0 \Longrightarrow \dot{\phi} = \xi_1 + \mu_1 \cos \psi + \mu_2 \sin \psi,$$

with ξ_1 constant. Eliminating $\dot{\phi}$ from the last two equations yields the following nonlinear equation in ψ

$$\ddot{\psi} = (\xi_1 + \mu_1 \cos \psi + \mu_2 \sin \psi)(\mu_1 \sin \psi - \mu_2 \cos \psi).$$

This equation can be integrated by using elliptic functions. We shall follow the integration presented in [2]. Let $|\mu| = \sqrt{\mu_1^2 + \mu_2^2}$. We can write the formula for the optimal solutions in the cases $|\theta| \neq 0$ and $|\theta| = 0$. In the first case, elliptic function machinery is used, while in the later just usual trigonometric functions will suffice to express the solution.

The case $|\mu| \neq 0$. In order to write down the optimal solution, we need first to review a list of notations used in the aforementioned reference.

$$\xi_1 = \text{constant}, \quad \omega = \tan^{-1}(\theta_1/\theta_2), \quad A = \frac{\xi_1}{|\theta|}, \quad B = \frac{1+\xi_1}{|\theta|}, \quad T = \frac{1}{|\theta|}$$

$$F = \int_0^{A+\sin \omega} \frac{d\beta}{\sqrt{1-(\beta-A)^2}\sqrt{T^2-\beta^2}}$$

$$D = \frac{-2|\theta|}{\sqrt{(1+|\theta|)^2 - \xi_1^2}}, \quad k = 2\sqrt{\frac{T}{(1+T^2)^2 - A^2}}$$

$$L(t) = \frac{F + |\theta|t}{D}, \quad \alpha = \frac{2}{1+A+T}, \quad \Omega = \frac{1-B^2}{2snL(t)-(1+B)},$$

where the elliptic sine of modulus k is defined by

$$sn^{-1}w = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Then the components ψ and ϕ are given by the following explicit formulas

$$\psi(t) = \sin^{-1}\left(\frac{1-B^2}{2 \, sn \, L(t) - (1+B)} - B\right) - \omega$$

$$\phi(t) = (\xi_1 - |\theta|B)t + D(B-1)\int_{F/D}^{(F+|\theta|t)/D} \frac{du}{1 - \alpha \, sn \, u}.$$

The integral in the last term can be expressed using the elliptic function cd and theta functions, see reference [5], p. 92.

The other two components of the solution are obtained by integrating in the nonholonomic constraints

$$x(t) = \int_0^t \dot{\phi} \cos\psi = \int_0^t \left(\xi_1 + |\theta|(\Omega(s) - B) \right) \left(\sqrt{1 - \Omega(s)^2} \cos\omega + \Omega(s) \sin\omega \right) ds,$$

$$y(t) = \int_0^t \dot{\phi} \sin\psi = \int_0^t \left(\xi_1 + |\theta|(\Omega(s) - B) \right) \left(\Omega(s) \cos\omega - \sqrt{1 - \Omega(s)^2} \sin\omega \right) ds.$$

Unfortunately this is the best we can do in the matter of finding the components.

The case $|\mu| = 0$. Since $\mu_1 = \mu_2 = 0$, integrating in the Euler-Lagrange equations yields

$$\phi(s) = \xi_1 s,$$
$$\psi(s) = \xi_2 s,$$
$$x(s) = x_0 + \frac{\xi_1}{\xi_2} \sin(\xi_2 s),$$
$$y(s) = y_0 - \frac{\xi_1}{\xi_2} \cos(\xi_2 s),$$

with ξ_1, ξ_2 constants. In this case the trajectory of the contact point $M\big(x(s), y(s)\big)$ is a circle of radius $|\xi_1/\xi_2|$, centered at (x_0, y_0), see Fig.6(a). If $\xi_2 = 0$, then $\phi(s) = \xi_1 s$ and $\psi(s) = 0$, case in which the disk roles along the following straight line, see Fig.6(b)

$$x(s) = x_0 + \xi_1 s, \quad y(s) = y_0.$$

REMARK 5.1. *If choose $\dot{\phi} = 1$ in the Lagrangian (5.2) we obtain the Lagrangian (4.1) up to an additive constant. This might state a relationship between optimum constant angular velocity rolling curves and plane elastic curves.*

The vertical rolling oval wheel. This is a generalization of the rolling disk problem to a closed, simple, plane, convex curve, i.e. an oval curve, which rolls without slipping on a horizontal plane. Let s denote the arc length along the oval curve. The velocity vector along the contact curve of the oval with the plane is given by $v = (\dot{x}, \dot{y}) = |v|(\cos\psi, \sin\psi)$, where ψ is the angle made by the plane of the oval π with the x-axis, see Fig.7. Since the speed is given by $|v| = \frac{ds}{dt}$, projecting on components yields the following rolling constraints

$$dx = \cos\psi \, ds, \qquad dy = \sin\psi \, ds.$$

The central angle parameter ϕ from the rolling disk is replaced here by the angle
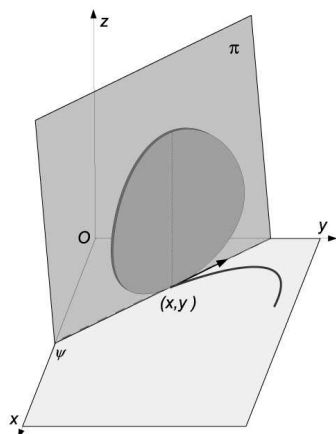
FIG. 7. *An oval wheel rolling on a horizontal plane.*

parameter θ which defines the curvature along the oval by $\kappa = \dfrac{d\theta}{ds}$. Then the afore-mentioned rolling constraints become

$$\kappa dx = \cos\psi\, d\theta, \qquad \kappa dy = \sin\psi\, d\theta.$$

The rolling motion of the oval curve can be considered as a curve on the space $(x, y, \theta, \psi) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$, tangent to the distribution $\mathcal{H} = \ker\omega_1 \cap \omega_2$, with

$$\omega_1 = \kappa dx - \cos\psi\, d\theta, \qquad \omega_2 = \kappa dy - \sin\psi\, d\theta.$$

The following vector fields

$$X_1 = \kappa\partial_\psi, \qquad X_2 = \cos\psi\,\partial_x + \sin\psi\,\partial_y + \kappa\partial_\theta.$$

generate the distribution $\mathcal{H}$. It worth to note that the oval property $\kappa > 0$ is essential here. The aforementioned vector fields have the bracket generating property, since the commutation relations (5.1) hold with

$$X_3 = \kappa\big(-\sin\psi\,\partial_x + \cos\psi\,\partial_y\big), \qquad X_4 = -\kappa\big(\cos\psi\,\partial_x + \sin\psi\,\partial_y\big),$$

and $\{X_1, X_2, X_3, X_4\}$ are linearly independent. Hence by Chow's theorem, the oval wheel can be moved between any two given positions $A_0 = (x_0, y_0, \theta_0, \psi_0)$ and $A_1 = (x_1, y_1, \theta_1, \psi_1)$ in the space $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$.

6. The nonholonomic bicycle. This section deals with the motion of a bicycle treated as a problem with nonholonomic constraints. This is a mechanical system similar to a skater with the knife edges replaced by circular wheels of radii r and R. Besides the parameters $(x, y, \theta, \psi) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ that describe the dynamics of a
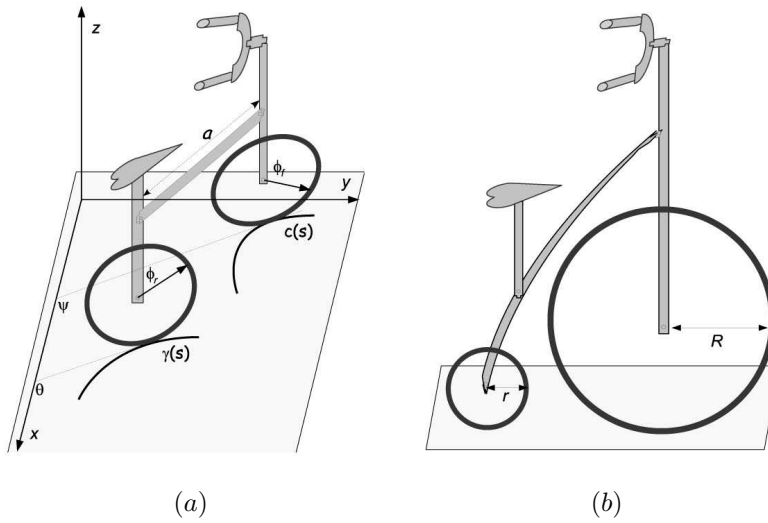
(a) (b)

FIG. 8. (a) *The parametrization of a bicycle; (b) bicycle with unequal radii.*

skater, in the case of a bicycle two additional angle parameters are needed $(\phi_f, \phi_r) \in$ $\mathbb{S}^1 \times \mathbb{S}^1$. They are angular parameters which parameterize the rotation of the front and rear wheels, respectively, see Fig.8(a). Hence the motion of a bicycle is described by a curve $\big(x(s), y(s), \theta(s), \psi(s), \phi_f(s), \phi_r(s)\big)$ on the 6-dimensional space $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1 \times$ $\mathbb{S}^1 \times \mathbb{S}^1$. The distance between the centers of the wheels is constant and is denoted by a. The formulas from the nonholonomic skater regarding the front and rear curves $c = (u, v)$ and $\gamma = (x, y)$ hold also in the case of a bicycle, but in this case they are the traces of the contact points of the front and the rear wheels, respectively

$$\dot{u} \sin \psi = \dot{v} \cos \psi, \qquad \dot{x} \sin \theta = \dot{y} \cos \theta.$$

Differentiating in $(u, v) = (x, y) + a(\cos \theta, \sin \theta)$ yields

(6.1) $$\dot{u} = \dot{x} - a \sin \theta \, \dot{\theta}, \qquad \dot{v} = \dot{y} + a \cos \theta \, \dot{\theta}.$$

The rolling constraints for the rear and front wheels are

(6.2) $$\dot{x} = r \cos \theta \, \dot{\phi}_r, \qquad \dot{y} = r \sin \theta \, \dot{\phi}_r$$

(6.3) $$\dot{u} = R \cos \psi \, \dot{\phi}_f, \qquad \dot{v} = R \sin \psi \, \dot{\phi}_f.$$

Since u and v are not among the parameters we shall eliminate them by substituting (6.1) in (6.3)

$$R \cos \psi \, \dot{\phi}_f = \dot{x} - a \sin \theta \, \dot{\theta}$$
$$R \sin \psi \, \dot{\phi}_f = \dot{y} + a \cos \theta \dot{\theta}.$$

The bicycle motion can be described by the following 3 pairs of constraints: one pair for each of the trace curves c and γ, and one for describing the rotation of the wheels. After an easy algebraic manipulation we obtain the following equivalencies:

$$(6.4) \qquad \begin{cases} \dot{x} &= r\cos\theta\,\dot{\phi}_r \\ \dot{y} &= r\sin\theta\,\dot{\phi}_r \end{cases} \Longleftrightarrow \begin{cases} \dot{x}\cos\theta + \dot{y}\sin\theta &= r\,\dot{\phi}_r \\ \dot{x}\sin\theta - \dot{y}\cos\theta &= 0. \end{cases}$$

$$(6.5) \qquad \begin{cases} \dot{u} &= R\cos\psi\,\dot{\phi}_f \\ \dot{y} &= R\sin\psi\,\dot{\phi}_f \end{cases} \Longleftrightarrow \begin{cases} \dot{x}\cos\psi + \dot{y}\sin\psi + a\dot{\theta}\sin(\psi-\theta) &= R\,\dot{\phi}_f \\ \dot{x}\sin\psi - \dot{y}\cos\psi - a\dot{\theta}\cos(\psi-\theta) &= 0. \end{cases}$$

$$(6.6) \qquad \begin{cases} \dot{u}\sin\psi &= \dot{v}\cos\psi \\ \dot{x}\sin\theta &= \dot{y}\cos\theta \end{cases} \Longleftrightarrow \begin{cases} \dot{x}\sin\psi - \dot{y}\cos\psi - a\dot{\theta}\cos(\psi-\theta) &= 0 \\ \dot{x}\sin\theta - \dot{y}\cos\theta &= 0. \end{cases}$$

The equivalent systems of the right side are obtained by adding and subtracting the equations of the left side, after they have been previously multiplied by a sine or a cosine function. The advantage of this transformation is to show that the initial 6 constraints were not linearly independent. We easily see that some of the constraints on the right side are the same:

- the second equation of (6.4) and the second equation of (6.6).

- the second equation of (6.5) and the first equation of (6.6).

Removing the duplications, we obtain 4 constraints associated with the following one-forms

$$\omega_1 = \sin\psi\,dx - \cos\psi\,dy - a\cos(\psi-\theta)\,d\theta$$
$$\omega_2 = \sin\theta\,dx - \cos\theta\,dy$$
$$\omega_3 = \cos\psi\,dx + \sin\psi\,dy + a\sin(\psi-\theta)\,d\theta - R\,d\phi_f$$
$$\omega_4 = \cos\theta\,dx + \sin\theta\,dy - r\,d\phi_r.$$

It is not hard to show that the aforementioned one-forms are linearly independent as long as $\psi \neq \theta$. They define a rank 2 distribution $\mathcal{D} = \bigcap_{j=1}^{4}\ker\omega_j$. We note that the one-forms ω_1 and ω_2 had defined the distribution $\mathcal{H}$ in the case of the skater. In order to find a basis of the rank 2 distribution $\mathcal{D}$ we shall add to the basis $\{X_1, X_2\}$ of $\mathcal{H}$ given by $(3.17 - 3.18)$ a linear combination of vector fields ∂_{ϕ_r} and ∂_{ϕ_f}

$$Y_1 = X_1 + \alpha_1\partial_{\phi_r} + \alpha_2\partial_{\phi_f} = \partial_\psi + \alpha_1\partial_{\phi_r} + \alpha_2\partial_{\phi_f}$$
$$Y_2 = X_2 + \beta_1\partial_{\phi_r} + \beta_2\partial_{\phi_f} = a\cot(\psi-\theta)\big(\cos\theta\partial_x + \sin\theta\partial_y\big) + \partial_\theta + \beta_1\partial_{\phi_r} + \beta_2\partial_{\phi_f}.$$

The coefficients α_i and β_i are determined from the conditions $\omega_3(Y_i) = \omega_4(Y_i) = 0$, $i = 1, 2$

$$\alpha_1 = \alpha_2 = 0, \qquad \beta_1 = \frac{a}{r}\cot(\psi-\theta), \qquad \beta_2 = \frac{2a}{R}\sin(\psi-\theta).$$

Hence the vector fields which generate the distribution $\mathcal{D}$ are

$$Y_1 = \partial_\psi$$
$$Y_2 = a\cot(\psi - \theta)\big(\cos\theta\partial_x + \sin\theta\partial_y\big) + \partial_\theta + \frac{a}{r}\cot(\psi - \theta)\partial_{\phi_r} + \frac{2a}{R}\sin(\psi - \theta)\partial_{\phi_f}.$$

It worth noting that when $r, R \to \infty$ we get $Y_i \to X_i$, i.e. we recuperate the case of a skater.

The commutation relations. Since

$$[Y_1, Y_2] = a\partial_\psi\big(\cot(\psi - \theta)\big)V + \frac{a}{r}\partial_\psi(\cot(\psi - \theta))\partial_{\phi_r} + \frac{2a}{R}\partial_\psi(\sin(\psi - \theta))\partial_{\phi_f},$$

$$[Y_1, [Y_1, Y_2]] = a\partial_\psi^2\big(\cot(\psi - \theta)\big)V + \frac{a}{r}\partial_\psi^2(\cot(\psi - \theta))\partial_{\phi_r} + \frac{2a}{R}\partial_\psi^2(\sin(\psi - \theta))\partial_{\phi_f},$$

$$[Y_2, [Y_1, Y_2]] = (\partial_{\theta\psi}^2\cot(\psi - \theta))V + \frac{a}{r}(\partial_{\theta\psi}^2\cot(\psi - \theta))\partial_{\phi_r} + \frac{2a}{R}\partial_{\theta\psi}^2\sin(\psi - \theta)\partial_{\phi_f},$$

it follows that the iterated brackets will be a linear combination of the vector fields

$$V = \cos\theta\partial_x + \sin\theta\partial_y, \qquad \partial_{\phi_r}, \qquad \partial_{\phi_f},$$

and hence it cannot generate the entire tangent space of $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$.

Future developments. The study of problems involving rolling nonholonomic constraints led to non-integrable distributions which are not bracket generating, and hence the Chow's connectivity result cannot be applied here. However, ad-hoc constructions of curves tangent to the distribution can be obtained, so these type of problems produce a class of distributions which are counterexamples for the converse of the theorem of Chow. The problem involving multiple wheels or blades, as well as finding their optimal trajectories constitute a further development of the problem. The dynamics of a bicycle with oval wheels constitutes also an open problem.

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