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# OPTIMAL CONTROL OF VARIATIONAL INEQUALITIES*

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**Abstract.** We consider control problems for the variational inequality describing a single degree of freedom elasto-plastic oscillator. We are particularly interested in finding the "critical excitation", i.e., the lowest energy input excitation that drives the system between the prescribed initial and final states within a given time span. This is a control problem for a state evolution described by a variational inequality. We obtain Pontryagin's necessary condition of optimality. An essential difficulty lies with the non continuity of adjoint variables.

## 1. Introduction.

We showed in [4], [5], and [6] that the models used in the literature for non-linear elasto-plastic oscillators ( see e.g., [8], [11], [12], [15], [16], [17], [18] and the references therein) are equivalent to stochastic variational inequalities. The main objective of this paper is to develop a framework to study control problems for these variational inequalities. We are particularly interested in finding the critical excitation of the system, which can be defined as the input excitation with the lowest energy that connects prescribed states of the system in a given time interval. The study of critical excitation has an extensive literature (see e.g., [1], [2], [9], [10], [14], [19]), and in case of nonlinear hysteretic systems, it has relevance in the understanding of nonlinear response of structures under severe loads (like earthquakes).

In this paper, we present a complete solution to the optimal control problem for the variational inequality describing the single degree of freedom elasto-plastic oscillator. First, we derive Pontryagin's necessary condition for optimality using a penalized problem and limiting arguments. Then we formulate conditions on the non continuity of adjoint variables at instances of phase changes and obtain a two point boundary value problem with additional internal boundary conditions at the phase changes for the state and adjoint variables. The solution of this problem gives an expression for the optimal control.

## 2. Control of the Variational Inequality.

Consider the variational inequality describing a single degree of freedom (sdof) elasto-plastic oscillator

$$\dot{y} + c_0 y + kz = v, (\dot{z} - y)(\zeta - z) \geq 0, |\zeta| \leq Y, |z(t)| \leq Y, \tag{2.1}$$

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where, $y = \dot{x}$, $z = x - \tilde{x}$, $x$ is the displacement of the oscillator, $\tilde{x}(t)$ is the total plastic deformation accumulated by time $t$ by the oscillator, $Y > 0$ represents the size of the elastic region, and $v$ denotes the control input. We shall assume zero initial conditions

$$y(0) = 0, z(0) = 0. \tag{2.2}$$

With zero initial conditions we seek to minimize the control input energy which takes the system to a prescribed state at time $T > 0$, i.e.,

$$\text{Min } J(v(\cdot)) = \frac{1}{2}\int_0^T v^2 dt \text{ subject to } x(T) = \bar{x}, y(T) = \bar{y}.$$

Introducing Lagrange multipliers $\lambda, \mu$ to satisfy the constraints and noting that

$$x(t) = \int_0^t y ds$$

we have the optimal control problem

$$\text{Min } J(v(\cdot)) = \frac{1}{2}\int_0^T v^2 dt + \lambda \int_0^T y(t)dt + \mu y(T), \tag{2.3}$$

where $y, z$ satisfy (2.1) - (2.2).

**2.1. The Penalized Problem.** Let $\varepsilon > 0$. The penalized problem corresponding to (2.1) is

$$\begin{aligned} &\dot{y} + c_0 y + kz = v, \\ &\dot{z} = y - \frac{1}{\varepsilon}(z - Y)^+ + \frac{1}{\varepsilon}(z + Y)^- \\ &y(0) = 0, z(0) = 0 \end{aligned} \tag{2.4}$$

where $y = y_\varepsilon(v(\cdot))$ and $z = z_\varepsilon(v(\cdot))$ and $v(\cdot)$ minimizes the functional

$$\text{Min } J_\varepsilon(v(\cdot)) = \frac{1}{2}\int_0^T v^2 dt + \lambda \int_0^T y_\varepsilon(v(\cdot))(t)dt + \mu y_\varepsilon(v(\cdot))(T). \tag{2.5}$$

Note that the penalized problem has a solution $u_\varepsilon(\cdot)$ such that

$$J_\varepsilon(u_\varepsilon(\cdot)) \leq J_\varepsilon(v(\cdot)).$$

Indeed $J_\varepsilon(v(\cdot))$ is continuous in $L^2(0,T)$ and $J_\varepsilon(v(\cdot)) \to \infty$ as $\|v\|_{L^2(0,T)} \to \infty$.

**2.2. Necessary Conditions for the Penalized Problem.** When we replace the variational inequality by the penalized system, we can apply standard techniques of control theory to obtain the necessary conditions of optimality. We shall use the notation $u(\cdot) = u_\varepsilon(\cdot)$ for the optimal control of (2.4)-(2.5). Let $y(t), z(t)$ and $\xi(t), \eta(t)$ be solutions of (2.4)-(2.5) with controls $u(\cdot)$ and $u(\cdot) + \theta v(\cdot)$ , respectively. Then

$$\tilde{y} = \frac{1}{\theta}(\xi - y), \text{ and } \tilde{z} = \frac{1}{\theta}(\eta - z)$$

satisfy the equations

$$\dot{\tilde{y}} + c_0\tilde{y} + k\tilde{z} = v,$$

$$\dot{z} + \theta\dot{\tilde{z}}_\theta = y + \theta\tilde{y}_\theta - \frac{1}{\varepsilon}(z + \theta\tilde{z}_\theta - Y)^+ + \frac{1}{\varepsilon}(z + \theta\tilde{z}_\theta + Y)^-$$

with initial conditions

$$\tilde{y}(0) = 0 \text{ and } \tilde{z}(0) = 0.$$

It follows that

$$\theta\dot{\tilde{z}}_\theta = \theta\tilde{y}_\theta + \frac{1}{\varepsilon}(z - Y)^+ - \frac{1}{\varepsilon}(z + Y)^- - \frac{1}{\varepsilon}(z + \theta\tilde{z}_\theta - Y)^+ + \frac{1}{\varepsilon}(z + \theta\tilde{z}_\theta + Y)^-.$$

Then we have

$$|\dot{\tilde{z}}_\theta - \tilde{y}_\theta| \leq \frac{1}{\varepsilon}|\tilde{z}_\theta|.$$

It is easy to check that $\tilde{y}_\theta, \tilde{z}_\theta$ are bounded in $H^1(0,T)$ as $\theta \to 0$ ($\varepsilon$ is fixed here). Let us extract a subsequence such that

$$\tilde{y}_\theta \to \tilde{y}, \tilde{z}_\theta \to \tilde{z} \text{ in } H^1(0,T) \text{ weakly and } C^0(0,T) \text{ strongly.} \tag{2.6}$$

We claim that

(2.7)
$$\frac{1}{\theta}\left(-\frac{1}{\varepsilon}(z(t) + \theta\tilde{z}_\theta(t) - Y)^+ + \frac{1}{\varepsilon}(z(t) - Y)^+)\right) \to -\frac{1}{\varepsilon}\mathbb{1}_{z(t)-Y>0}\tilde{z}(t) \text{ in } L^2(0,T)$$
weakly, as $\theta \to 0$.

It is sufficient to prove the convergence *a.e.t.* But *a.e.t* $z(t) > Y$ or $z(t) < Y$. Since

$$z(t) + \theta\tilde{z}_\theta(t) - Y \to 0,$$

necessarily

$$z(t) + \theta\tilde{z}_\theta(t) - Y > 0 \text{ if } z(t) - Y > 0 \text{ or } z(t) + \theta\tilde{z}_\theta(t) - Y < 0 \text{ if } z(t) - Y < 0$$

for $\theta$ sufficiently small, depending on $t$. Therefore the left hand side of (2.7) is equal to

$$-\frac{1}{\varepsilon}\mathbb{1}_{z(t)-Y>0}\tilde{z}_\theta$$

for $\theta$ sufficiently small, depending on $t$. Since

$$\tilde{z}_\theta(t) \to \tilde{z}(t) \text{ a.e.t,}$$

we obtain (2.7).

It then easily follows that the limit $\tilde{y}, \tilde{z}$ is the solution of the system

$$
\begin{aligned}
&\dot{\tilde{y}} + c_0\tilde{y} + k\tilde{z} = v, \\
&\dot{\tilde{z}} = \tilde{y} - \frac{1}{\varepsilon}\tilde{z}1\!\!1_{z(t)-Y>0} - \frac{1}{\varepsilon}\tilde{z}1\!\!1_{z(t)+Y<0} \\
&\tilde{y}(0) = \tilde{z}(0) = 0.
\end{aligned}
\tag{2.8}
$$

We next compute

$$
J_\varepsilon(u(\cdot)+\theta v(\cdot)) = \frac{1}{2}\theta^2\int_0^T v^2(t)dt + \theta\int_0^T v(t)u(t)dt + \int_0^T u^2(t)dt
$$

$$
+\lambda\int_0^T y(t)dt + \theta\lambda\int_0^T \tilde{y}_\theta(t)dt + \mu y(T) + \theta\mu\tilde{y}_\theta(T)
$$

$$
= J_\varepsilon(u(\cdot)) + \theta[\int_0^T v(t)u(t)dt + \lambda\int_0^T \tilde{y}_\theta(t)dt + \mu\tilde{y}_\theta(T)] + \frac{1}{2}\theta^2\int_0^T v^2(t)dt
$$

and

$$
\frac{1}{\theta}(J_\varepsilon(u(\cdot)+\theta v(\cdot)) - J_\varepsilon(u(\cdot))) \to \int_0^T v(t)u(t)dt + \lambda\int_0^T \tilde{y}(t)dt + \mu\tilde{y}(T)
$$

and from the optimality of $u(\cdot)$ we deduce

$$
\int_0^T v(t)u(t)dt + \lambda\int_0^T \tilde{y}(t)dt + \mu\tilde{y}(T) = 0. \tag{2.9}
$$

**2.3. Adjoint System.** Introduce $(p(t), q(t)) = (p_\varepsilon(t), q_\varepsilon(t))$ solution of the corresponding adjoint system

$$
\begin{aligned}
&-\dot{p} = -c_0 p + q + \lambda \\
&-\dot{q} = -kp - \frac{q}{\varepsilon}(1\!\!1_{z-Y>0} + 1\!\!1_{z+Y<0}) \\
&\text{ with } p(T) = \mu,\ q(T) = 0.
\end{aligned}
\tag{2.10}
$$

Then straightforward calculations yield that

$$
\int_0^T (-\dot{p}(t) + c_0 p(t) - q(t))\tilde{y}(t)dt + p(T)\tilde{y}(T)
$$

$$
= \int_0^T [p(t)(\dot{\tilde{y}}(t) + c_0\tilde{y}(t)) - q(t)\tilde{y}(t))dt
$$

$$
= \int_0^T (p(t)(v(t) - k\tilde{z}(t)) - q(t)\tilde{y}(t))dt
$$

$$
= \int_0^T p(t)v(t)dt + \int_0^T (\tilde{z}(t)(-\dot{q}(t)) - q(t)\tilde{y}(t))dt
$$

$$= \int_0^T p(t)v(t)dt + \int_0^T q(t)(\dot{\tilde{z}}(t) - \tilde{y}(t))dt$$

$$= \int_0^T p(t)v(t)dt.$$

Substituting into the Euler condition we get for all $v$,

$$\int_0^T (u(t) + p(t))v(t)dt = 0.$$

It follows that

$$u(t) + p(t) = 0.$$

Hence we have obtained the following set of necessary conditions for the penalized problem:

$$\begin{aligned} &\dot{y}_\varepsilon + c_0 y_\varepsilon + k z_\varepsilon + p_\varepsilon = 0, \\ &\dot{z}_\varepsilon = y_\varepsilon - \frac{1}{\varepsilon}(z_\varepsilon - Y)^+ + \frac{1}{\varepsilon}(z_\varepsilon + Y)^-, \\ &-\dot{p}_\varepsilon = -c_0 p_\varepsilon + q_\varepsilon + \lambda, \\ &-\dot{q}_\varepsilon = -k p_\varepsilon - \frac{q_\varepsilon}{\varepsilon}(\mathbb{1}_{z_\varepsilon - Y > 0} + \mathbb{1}_{z_\varepsilon + Y < 0}), \\ &\text{ with } y_\varepsilon(0) = 0,\ z_\varepsilon(0) = 0,\ p_\varepsilon(T) = \mu,\ q_\varepsilon(T) = 0. \end{aligned} \tag{2.11}$$

**3. Estimates and Convergence.** To the control $v(\cdot) \equiv 0$ correspond the trajectories $y(\cdot) = z(\cdot) \equiv 0$ and therefore, for the optimal control $u(\cdot)_\varepsilon$ of the penalized problem we have

$$\frac{1}{2}\int_0^T (u_\varepsilon(t))^2 dt + \lambda \int_0^T y_\varepsilon(t)dt + \mu y_\varepsilon(T) \le 0. \tag{3.1}$$

From the state equations we get

$$\frac{1}{2}\frac{d}{dt}|y_\varepsilon|^2 + c_0(y_\varepsilon)^2 + k z_\varepsilon y_\varepsilon = u_\varepsilon y_\varepsilon,$$

$$\frac{1}{2}\frac{d}{dt}|z_\varepsilon|^2 = y_\varepsilon z_\varepsilon - \frac{1}{\varepsilon} z_\varepsilon (z_\varepsilon - Y)^+ + \frac{1}{\varepsilon} z_\varepsilon (z_\varepsilon + Y)^- \le y_\varepsilon z_\varepsilon,$$

hence we have the estimate

$$\frac{1}{2}(y_\varepsilon(t))^2 + \frac{1}{2}k(z_\varepsilon(t))^2 + c_0 \int_0^t (y_\varepsilon(s))^2 ds \le \frac{c_0}{2}\int_0^t (y_\varepsilon(s))^2 ds + \frac{1}{2c_0}\int_0^t (u_\varepsilon(s))^2 ds.$$

Using (3.1) we obtain

$$\frac{1}{2}(y_\varepsilon(t))^2 + \frac{1}{2}k(z_\varepsilon(t))^2 + \frac{c_0}{2}\int_0^t (y_\varepsilon(s))^2 ds \le -\frac{\lambda}{c_0}\int_0^T y_\varepsilon(t)dt - \frac{\mu}{c_0} y_\varepsilon(T).$$

Applying this inequality with $t = T$ yields

$$\frac{1}{2}(y_\varepsilon(T))^2 + \frac{\mu}{c_0} y_\varepsilon(T) + \frac{1}{2}k(z_\varepsilon(T))^2 + \frac{c_0}{2}\int_0^T (y_\varepsilon(s))^2 ds + \frac{\lambda}{c_0}\int_0^T y_\varepsilon(t)dt \leq 0$$

from which we get immediately for some constant $C$ that

$$\int_0^T (y_\varepsilon(t))^2 dt \leq C, (y_\varepsilon(T))^2 \leq C, \text{ and } (z_\varepsilon(T))^2 \leq C. \tag{3.2}$$

Hence also,

$$|\int_0^T y_\varepsilon(t)dt| \leq C, \text{ and } |y_\varepsilon(T)| \leq C. \tag{3.3}$$

Going back to the previous inequality we also get that

$$|y_\varepsilon(t)| \leq C, |z_\varepsilon(t)| \leq C, \forall t \in [0, T], \text{ and } \int_0^T (u_\varepsilon(t))^2 dt| \leq C. \tag{3.4}$$

Considering the state equations

$$\begin{aligned} &\dot{y_\varepsilon} + c_0 y_\varepsilon + k z_\varepsilon = u_\varepsilon, \\ &\dot{z}_\varepsilon = y_\varepsilon - \frac{1}{\varepsilon}(z_\varepsilon - Y)^+ + \frac{1}{\varepsilon}(z_\varepsilon + Y)^- \\ &y_\varepsilon(0) = 0, z_\varepsilon(0) = 0, \end{aligned} \tag{3.5}$$

we get easily

$$\int_0^T (\dot{y}_\varepsilon(t))^2 dt \leq C.$$

Using (3.5) we get

$$\int_0^t (\dot{z}_\varepsilon(t))^2 dt = \int_0^t y_\varepsilon \dot{z}_\varepsilon(t)dt - \frac{1}{2\varepsilon}((z_\varepsilon(t) - Y)^+)^2 - \frac{1}{2\varepsilon}((z_\varepsilon(t) + Y)^-)^2$$

and hence

$$\int_0^T (\dot{z}_\varepsilon(t))^2 dt \leq C,\ \frac{1}{\varepsilon}((z_\varepsilon(t) - Y)^+)^2 \leq C \text{ , and } \frac{1}{\varepsilon}((z_\varepsilon(t) + Y)^-)^2 \leq C. \tag{3.6}$$

Note that for all $\zeta$ with $|\zeta| \leq Y$ we have

$$(\dot{z}_\varepsilon(t)) - y_\varepsilon(t))(\zeta - z_\varepsilon(t)) \geq 0. \tag{3.7}$$

We extract a subsequence such that

$$\begin{aligned} &u_\varepsilon \to u \text{ in } L^2(0, T) \text{ weakly}, \\ &y_\varepsilon \to y \text{ in } H^1(0, T) \text{ weakly and } \forall t \text{ uniformly}, \\ &z_\varepsilon \to z \text{ in } H^1(0, T) \text{ weakly and } \forall t \text{ uniformly}. \end{aligned} \tag{3.8}$$

From (3.6) we get $-Y \leq z(t) \leq Y$, and we see easily that

$$\begin{aligned} &\dot{y} + c_0 y + kz = u \\ &(\dot{z} - y)(\zeta - z) \geq 0, \forall \zeta \text{ with } |\zeta| \leq Y. \end{aligned} \tag{3.9}$$

Let us prove that $u$ is an optimal control for the original problem.

THEOREM 3.1. *Let* $u(\cdot)$, $y(\cdot)$ *and* $z(\cdot)$ *obtained as in* (3.8). *Then* $u(\cdot)$ *is an optimal control for* (2.1)-(2.3).

*Proof.* Clearly,

$$J(u(\cdot)) \leq \liminf [\frac{1}{2}\int_0^T (u_\varepsilon(t))^2 dt + \lambda \int_0^T y_\varepsilon(t) dt + \mu y_\varepsilon(T)]$$

$$\leq \frac{1}{2}\int_0^T v^2(t) dt + \lambda \int_0^T y_\varepsilon(t; v(\cdot)) dt + \mu y_\varepsilon(T; v(\cdot)),$$

where $y_\varepsilon(t; v(\cdot))$ is the solution of

$$\begin{aligned} &\dot{y}_\varepsilon + c_0 y_\varepsilon + k z_\varepsilon = v, \\ &\dot{z}_\varepsilon = y_\varepsilon - \frac{1}{\varepsilon}(z_\varepsilon - Y)^+ + \frac{1}{\varepsilon}(z_\varepsilon + Y)^-, \\ &\text{with } y_\varepsilon(0) = 0, z_\varepsilon(0) = 0. \end{aligned} \tag{3.10}$$

As easily seen

$$\begin{aligned} &y_\varepsilon(t; v(\cdot)) \to y(t; v(\cdot)) \text{ in } H^1(0,T) \text{ weakly and } \forall t, \\ &z_\varepsilon(t; v(\cdot)) \to z(t; v(\cdot)) \text{ in } H^1(0,T) \text{ weakly and } \forall t, \end{aligned} \tag{3.11}$$

which is the solution of the variational inequality corresponding to the control $v(\cdot)$. Therefore, we have obtained

$$J(u(\cdot)) \leq J(v(\cdot)) \ , \forall v(\cdot)$$

which proves that $u(\cdot)$ is optimal. □

Consider the adjoint equation

$$\begin{aligned} &-\dot{p}_\varepsilon = -c_0 p_\varepsilon + q_\varepsilon + \lambda, \\ &-\dot{q}_\varepsilon = -k p_\varepsilon - \frac{q_\varepsilon}{\varepsilon}(\mathbb{1}_{z_\varepsilon - Y > 0} + \mathbb{1}_{z_\varepsilon + Y < 0}), \\ &\text{with } p_\varepsilon(T) = \mu, q_\varepsilon(T) = 0. \end{aligned} \tag{3.12}$$

We obtain by straightforward calculations

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}(p_\varepsilon(t))^2 = -c_0(p_\varepsilon)^2 + q_\varepsilon p_\varepsilon + \lambda p_\varepsilon, \\ &-\frac{1}{2}\frac{d}{dt}(q_\varepsilon(t))^2 = -k p_\varepsilon q_\varepsilon - \frac{(q_\varepsilon)^2}{\varepsilon}(\mathbb{1}_{z_\varepsilon - Y > 0} + \mathbb{1}_{z_\varepsilon + Y < 0}), \end{aligned} \tag{3.13}$$

and then by integrating these equations over the interval $(t,T)$ we get

$$\frac{k}{2}(p_\varepsilon(t))^2 - \frac{k}{2}\mu^2 + c_0 k \int_t^T (p_\varepsilon)^2 ds + \frac{1}{2}(q_\varepsilon(t))^2$$

$$+\frac{1}{2}\int_t^T (q_\varepsilon(s))^2 (\mathbb{1}_{z_\varepsilon - Y>0} + \mathbb{1}_{z_\varepsilon+Y<0}) ds = \lambda k \int_t^T p_\varepsilon ds.$$

Therefore

$$|p_\varepsilon(t)| \leq C, |q_\varepsilon(t)| \leq C \text{ and } \frac{1}{\varepsilon}\int_0^T (q_\varepsilon(t))^2 (\mathbb{1}_{z_\varepsilon - Y\geq 0} + \mathbb{1}_{z_\varepsilon+Y\leq 0}) dt \leq C, \tag{3.14}$$

where we have used the fact that $\mathbb{1}_{z_\varepsilon - Y=0} = 0, \mathbb{1}_{z_\varepsilon+Y=0} = 0$ a.e.t.
Next from the adjoint equations

$$\begin{aligned} &|\dot{p}_\varepsilon| \leq C, \\ &-\frac{d}{dt}|q_\varepsilon(t)| = -kp_\varepsilon \operatorname{sign} q_\varepsilon(t) - \frac{|q_\varepsilon(t)|}{\varepsilon}(\mathbb{1}_{z_\varepsilon - Y\geq 0} + \mathbb{1}_{z_\varepsilon+Y\leq 0}), \end{aligned} \tag{3.15}$$

hence

$$\frac{1}{\varepsilon}\int_0^T |q_\varepsilon(t)| (\mathbb{1}_{z_\varepsilon - Y\geq 0} + \mathbb{1}_{z_\varepsilon+Y\leq 0}) dt \leq C. \tag{3.16}$$

Therefore

$$\dot{q}_\varepsilon \text{ is bounded in } L^1. \tag{3.17}$$

We can extract a subsequence such that

$$\begin{aligned} &p_\varepsilon \to p \text{ in } H^1(0,T) \text{ weakly and } \forall t \text{ uniformly}, \\ &q_\varepsilon \to q \text{ in } L^2(0,T) \text{ weakly}, \\ &\dot{q}_\varepsilon \to \dot{q} \text{ in the space of measures on } (0,T), \\ &q_\varepsilon \to q \text{ in } BV(0,T) \text{ weakly}. \end{aligned} \tag{3.18}$$

**4. Study of the System Governed by $(y,z,p,q)$.** We derive relations for the system governed by $(y,z,p,q)$.
We have clearly

$$\begin{aligned} &\dot{y} + c_0 y + kz + p = 0, \\ &(\dot{z} - y)(\zeta - z) \geq 0, , \forall \zeta, |\zeta| \leq Y, |z(t)| \leq Y, \\ &-\dot{p} = -c_0 p + q + \lambda. \end{aligned} \tag{4.1}$$

The equation for $q$ is the difficult part.
In the sequel it is convenient to extend $u(t) = u(T)$ and $u_\varepsilon(t) = u_\varepsilon(T)$, for $t > T$ and to consider $y, z$, $y_\varepsilon, z_\varepsilon$ extended accordingly for $t > T$. We assume initial conditions

$y(0) = 0, z(0) = 0$ and $y_\varepsilon(0) = 0, z_\varepsilon(0) = 0$ and define the sequences $t_0 = 0 < t_1 \leq t_2 < t_3 \leq t_4 < ...$ and $t_0^\varepsilon < t_1^\varepsilon < t_2^\varepsilon < ...$ with

$$\begin{aligned} &t_1 = \inf\{t > 0 : |z(t)| = Y\}, t_1^\varepsilon = \inf\{t > 0 : |z_\varepsilon| > Y\} \\ &t_2 = \inf\{t > t_1 : |z(t)| < Y\}, t_2^\varepsilon = \inf\{t > t_1^\varepsilon : |z_\varepsilon(t)| < Y\} \\ &\delta_1 = \text{sign} z(t_1), \text{ if } t_1 < \infty, \delta_1^\varepsilon = \text{sign} z_\varepsilon(t_1^\varepsilon), \text{ if } t_1^\varepsilon < \infty, \end{aligned} \tag{4.2}$$

and more generally

$$\begin{aligned} &t_{2j+1} = \inf\{t > t_{2j} : |z(t)| = Y\}, t_{2j+1}^\varepsilon = \inf\{t > t_{2j}^\varepsilon : |z_\varepsilon(t)| > Y\} \\ &t_{2j+2} = \inf\{t > t_{2j+1} : |z(t)| < Y\}, t_{2j+2}^\varepsilon = \inf\{t > t_{2j+1}^\varepsilon : |z_\varepsilon(t)| < Y\} \\ &\delta_{2j+1} = \text{sign} z(t_{2j+1}), \text{ if } t_{2j+1} < \infty \text{ and } \delta_{2j+1}^\varepsilon = \text{sign} z_\varepsilon(t_{2j+1}^\varepsilon), \text{ if } t_{2j+1}^\varepsilon < \infty. \end{aligned} \tag{4.3}$$

We also have

$$\begin{aligned} &z(t_{2j+1}) = z(t_{2j+2}) = Y\delta_{2j+1}, z_\varepsilon(t_{2j+1}^\varepsilon) = z_\varepsilon(t_{2j+2}^\varepsilon) = Y\delta_{2j+1}^\varepsilon \\ &\text{sign} y(t_{2j+1}) = \delta_{2j+1}, \text{ if } y(t_{2j+1}) \neq 0 \\ &\text{sign} y_\varepsilon(t_{2j+1}^\varepsilon) = \delta_{2j+1}^\varepsilon \text{ and } sign y_\varepsilon(t_{2j+2}^\varepsilon) = -\delta_{2j+1}^\varepsilon. \end{aligned} \tag{4.4}$$

Next, for $j \geq 1$, $t_{2j-1} < t < t_{2j}$ we have that

$$\dot{z} = 0, \text{sign} y(t) = \delta_{2j-1}, \text{ and } z(t_{2j-1}) = z(t_{2j}) = Y\delta_{2j-1}.$$

We necessarily have

$$y(t_{2j}) = 0, \text{ if } t_{2j} < T\ ,\ j \geq 1. \tag{4.5}$$

Indeed suppose that $y(t_{2j}) \neq 0$, then $\text{sign} y(t_{2j}) = \delta_{2j-1}$, by the continuity of the function $y(t)$. Also, for sufficiently small $\varepsilon > 0$ we have $\text{sign} y(t) = \delta_{2j-1}$ for $t \in (t_{2j}, t_{2j} + \varepsilon)$. But for $t \in (t_{2j}, t_{2j} + \varepsilon)$, $y(t) = \dot{z}$, which implies $\text{sign} \dot{z}(t) = \delta_{2j-1}$, which is impossible.
The function $\dot{z}$ is continuous at $t_{2j}$. At $t_{2j+1}$ it satisfies the relations

$$\dot{z}(t_{2j+1} + 0) = 0 \text{ and } \dot{z}(t_{2j+1} - 0) = y(t_{2j+1}).$$

Proposition 4.1. *We have for all $j \geq 1$ that*

$$t_{2j-1}^\varepsilon \to t_{2j-1} \text{ and } t_{2j}^\varepsilon \to t_{2j}.$$

*Proof.* Let us prove that

$$t_1^\varepsilon \to t_1. \tag{4.6}$$

Assume $t_1 < \infty$. Let $\delta > 0$, then

$$\sup_{0 \leq t \leq t_1 - \delta} |z(t)| < Y.$$

Since $z_\varepsilon$ converges to $z$ in $C^0([0, t_1 - \delta])$, we can assert that for $\varepsilon$ sufficiently small

$$\sup_{0\le t\le t_1-\delta} |z_\varepsilon(t)| < Y.$$

Therefore $t_1^\varepsilon > t_1 - \delta$ for $\varepsilon$ sufficiently small depending on $\delta$. Hence $\liminf_{\varepsilon\to 0} t_1^\varepsilon \geq t_1 - \delta$. Since $\delta$ is arbitrary we get $\liminf_{\varepsilon\to 0} t_1^\varepsilon \geq t_1$.
Suppose $\limsup_{\varepsilon\to 0} t_1^\varepsilon = t_1^* > t_1$. Pick a sequence $t_1^\varepsilon \to t_1^*$. Let $\delta$ be sufficiently small with $t_1 + \delta < t_1^*$. For $\varepsilon$ sufficiently small depending on $\delta$ we $t_1^\varepsilon > t_1 + \delta$. Therefore $\dot z_\varepsilon = y_\varepsilon$ on $(0, t_1 + \delta)$. Going to the limit we would have $\dot z = y$ on $(0, t_1 + \delta)$. This contradicts the fact that $\dot z = 0$ on $(t_1, t_1 + \delta)$.
If $t_1 = +\infty$, then $\sup_{0\le t\le T} |z(t)| < Y$, $\forall T > 0$. Therefore $\sup_{0\le t\le T} |z_\varepsilon(t)| < Y$ for $\varepsilon$ sufficiently small. Hence $t_1^\varepsilon > T$, and hence $\liminf t_1^\varepsilon > T$. Since $T$ is arbitrary $t_1^\varepsilon \to +\infty$.
Let us next prove that

$$t_2^\varepsilon \to t_2. \tag{4.7}$$

The case $t_1 = t_2$ is trivial. Assume $t_2 < \infty$ and consider the case $t_1 < t_2 < \infty$. For $\delta$ sufficiently small $|z(t_2 + \delta)| < Y$. Therefore for $\varepsilon$ sufficiently small depending on $\delta$ we have $|z_\varepsilon(t_2 + \delta)| < Y$. Since $t_1^\varepsilon \to t_1 < t_2 + \delta$, we can assume that $t_1^\varepsilon < t_2 + \delta$. Therefore $t_2 + \delta > t_2^\varepsilon$. It follows that $\limsup t_2^\varepsilon \le t_2 + \delta$. Since $\delta$ is arbitrary, we get

$$\limsup t_2^\varepsilon \le t_2. \tag{4.8}$$

Let us check that

$$t_2 \le \liminf t_2^\varepsilon. \tag{4.9}$$

Without loss of generality we can assume that $\delta_1^\varepsilon = 1$, so $z_\varepsilon(t_1^\varepsilon) = z_\varepsilon(t_2^\varepsilon) = Y$. Necessarily $z(t_1) = z(t_2) = Y$. Recall that $y(t_1) > 0$ and $y(t) > 0$ for $t_1 \le t < t_2$ with $y(t_2) = 0$. From the uniform convergence of $y_\varepsilon(t)$ to $y(t)$ on compact intervals, we deduce that $y_\varepsilon(t) > 0$ for $t \in [T_1, t_2 - \delta]$, for $\varepsilon$ sufficiently small depending on $\delta$. Since $t_1^\varepsilon \to t_1$, we can also assume that $y_\varepsilon(t) > 0$ for $t \in [t_1^\varepsilon, t_2 - \delta]$.
Now, on $(t_1^\varepsilon, t_2^\varepsilon)$ we have

$$\dot z_\varepsilon = y_\varepsilon - \frac{1}{\varepsilon}(z_\varepsilon - Y)$$

hence

$$\frac{d}{dt}[(z_\varepsilon - Y)e^{\frac{t}{\varepsilon}}] = y_\varepsilon e^{\frac{t}{\varepsilon}}.$$

Therefore $z_\varepsilon(t) - Y > 0$ as long as $y_\varepsilon(t) > 0$, $t > t_1^\varepsilon$. This implies that $t_2^\varepsilon > t_2 - \delta$. Therefore $\liminf t_2^\varepsilon > t_2 - \delta$. Since $\delta$ is arbitrary, we have obtained (4.9).
If $t_2 = +\infty$, we have $z(t) = Y$ , $\forall t \ge t_1$, and $y(t) > 0$ , $\forall t \ge t_1$. Also $y_\varepsilon(t) > 0$ for

$t \in [t_1, t_1+T]$ , $\forall T$, and $\varepsilon$ sufficiently small. This implies $t_2^\varepsilon > t_1+T$, for $\varepsilon$ sufficiently small. Hence $t_2^\varepsilon \to +\infty$.

Suppose that the property is proven for $t_{2j-1}^\varepsilon, t_{2j}^\varepsilon$. We want to prove it for $t_{2j+1}^\varepsilon, t_{2j+2}^\varepsilon$. Assume $t_{2j+1}, t_{2j+2} < \infty$.

We begin with $t_{2j+1}^\varepsilon$. The situation is different from that of $t_1^\varepsilon$, (j=0), since $z(t_{2j}) = Y\delta_{2j}$ (and not 0). To fix the ideas, suppose that $\delta_{2j} = 1$. We have $y(t_{2j}) = 0$ and $y(t) > 0$, for $t \in [t_{2j-1}, t_{2j})$. Moreover $\dot y(t_{2j}) < 0$, since for $t_{2j} \le t < t_{2j+1}$

$$z(t) = z(t_{2j}) + \int_{t_{2j}}^t (t-s)\dot y(s)ds$$

and $z(t) < z(t_{2j}) = Y$ for $t_{2j} < t < t_{2j} + \delta$, $\delta$ sufficiently small. Hence

$$\int_{t_{2j}}^t (t-s)\dot y(s)ds < 0 \text{ for } t_{2j} < t < t_{2j} + \delta.$$

Since $\dot y(s)$ is a continuous function, necessarily $\dot y(t_{2j}) < 0$. Since $\dot y_\varepsilon \to \dot y$ in $C^0[0,T]$, we can assert that

$$\dot y_\varepsilon(t) \le -C_\delta \text{ for } t_{2j} - \delta < t < t_{2j} + \delta$$

for $\varepsilon$ sufficiently small depending on $\delta$. Moreover since $t_{2j}^\varepsilon \to t_{2j}$, we can also assert that $t_{2j} - \delta < t_{2j}^\varepsilon < t_{2j} + \delta$ for $\varepsilon$ sufficiently small depending on $\delta$. Therefore

$$\dot y_\varepsilon(t) \le -C_\delta \text{ for } t \in [t_{2j}^\varepsilon, t_{2j} + \delta].$$

Also, $y_\varepsilon(t_{2j}^\varepsilon) \le 0$. This is because $z_\varepsilon(t_{2j}^\varepsilon) = Y$, $z_\varepsilon(t) < Y$ for $t > t_{2j}^\varepsilon$ sufficiently close to $t_{2j}^\varepsilon$ and

$$\dot z_\varepsilon(t) = y_\varepsilon(t) \text{ for } t_{2j}^\varepsilon < t < t_{2j+1}^\varepsilon.$$

Therefore

$$y_\varepsilon(t) \le -C_\delta(t - t_{2j}^\varepsilon) \text{ for } t_{2j}^\varepsilon \le t \le t_{2j} + \delta.$$

Hence also

$$z_\varepsilon(t) \le Y - \frac{C_\delta}{2}(t - t_{2j}^\varepsilon)^2 \text{ for } t_{2j}^\varepsilon \le t \le t_{2j} + \delta \tag{4.10}$$

and $\varepsilon$ sufficiently small depending on $\delta$.

Next on $t_{2j} + \delta \le t \le t_{2j+1} - \delta$ we have $|z(t)| < Y$. Therefore for $\varepsilon$ sufficiently small depending on $\delta$

$$|z_\varepsilon(t)| < Y \text{ for } t_{2j} + \delta \le t \le t_{2j+1} - \delta.$$

From (4.10) and this property we can assert that $t_{2j+1}^\varepsilon > t_{2j+1} - \delta$. Therefore $\liminf t_{2j+1}^\varepsilon \ge t_{2j+1}$. Suppose $\limsup_{\varepsilon\to 0} t_{2j+1}^\varepsilon = t_{2j+1}^* > t_{2j+1}$. Pick a sequence

$t^{\varepsilon}_{2j+1} \to t^{*}_{2j+1}$. We then proceed as in the case of $j = 0$ to obtain a contradiction. Hence

$$\limsup_{\varepsilon\to 0} t^{\varepsilon}_{2j+1} \leq t_{2j+1}$$

and then

$$t^{\varepsilon}_{2j+1} \to t_{2j+1}. \tag{4.11}$$

We finally prove that

$$t^{\varepsilon}_{2j+2} \to t_{2j+2}. \tag{4.12}$$

We assume $t_{2j+2} < \infty$. Just as we did for (4.8) we prove that $\limsup t^{\varepsilon}_{2j+2} \leq t_{2j+2}$. We then prove that

$$t_{2j+2} \leq \limsup t^{\varepsilon}_{2j+2}.$$

We suppose to fix the ideas that $\delta^{\varepsilon}_{2j+1} = 1$, then $z_{\varepsilon}(t^{\varepsilon}_{2j+1}) = z_{\varepsilon}(t^{\varepsilon}_{2j+2}) = Y$, and $z(t_{2j+1}) = z(t_{2j+2}) = Y$. Also,

$$y(t_{2j+1}) > 0,\ y(t) > 0 \text{ for } t_{2j+1} \leq t < t_{2j+2} \text{ and } y(t_{2j+2}) = 0.$$

We deduce that $y_{\varepsilon}(t) > 0$ for $t \in [t_{2j+1}, t_{2j+2} - \delta]$ for $\varepsilon$ sufficiently small depending on $\delta$. Since $t^{\varepsilon}_{2j+1} \to t_{2j+1}$, we can also assert that $y_{\varepsilon}(t) > 0$ for $t \in [t^{\varepsilon}_{2j+1}, t_{2j+2} - \delta]$. We have

$$\dot{z}_{\varepsilon} = y_{\varepsilon} - \frac{1}{\varepsilon}(z_{\varepsilon} - Y) \text{ for } t \in (t^{\varepsilon}_{2j+1}, t^{\varepsilon}_{2j+2})$$

hence

$$\frac{d}{dt}[(z_{\varepsilon} - Y)e^{\frac{t}{\varepsilon}}] = y_{\varepsilon}e^{\frac{t}{\varepsilon}}.$$

Therefore $z_{\varepsilon}(t) - Y > 0$ as long as $y_{\varepsilon}(t) > 0$ , $t > t^{\varepsilon}_{2j+1}$. This implies that $t^{\varepsilon}_{2j+2} > t_{2j+2} - \delta$, and we conclude as in the case $j = 0$.
The proposition is proven. □

Let us prove that $q(t)$ is zero for $t \in (t_{2j+1}, t_{2j+2})$

PROPOSITION 4.2. *Assume that $t_{2j+1} < t_{2j+2}$. Then we have*

$$q(t) = 0 \textit{ on } (t_{2j+1}, t_{2j+2}). \tag{4.13}$$

*Proof.* Assume without loss of generality that $\delta_{2j+1} = 1$, and therefore $y(t) > 0$ on $[t_{2j+1}, t_{2j+2})$. By (4.5) $y(t_{2j+2}) = 0$. So $y(t) \geq c_{\delta} > 0$ on $[t_{2j+1}, t_{2j+2} - \delta]$, for $\delta$ sufficiently small. Since $y_{\varepsilon} \to y(t)$ on $C^0([0, T])$ we have

$$y_{\varepsilon}(t) \geq \tilde{c}_{\delta} > 0 \text{ on } [t_{2j+1}, t_{2j+2} - \delta]. \tag{4.14}$$

Now since $z_\varepsilon(t_{2j+1}) \to z(t_{2j+1}) = Y$ there exists a point $\theta_{\varepsilon,2j+1} \in [t_{2j+1}, t_{2j+2} - \delta]$ such that $z_\varepsilon(\theta_{\varepsilon,2j+1}) \geq Y$.
If $z_\varepsilon(t_{2j+1}) \geq Y$, then obviously $\theta_{\varepsilon,2j+1} = t_{2j+1}$.
Suppose that $z_\varepsilon(t_{2j+1}) < Y$ and since we can assume that $z_\varepsilon(t_{2j+1}) > -Y$, we have

$$\dot{z}_\varepsilon(t) = y_\varepsilon(t) \geq \tilde{c}_\delta$$

as long as $z_\varepsilon(t) < Y$. So we have

$$z_\varepsilon(t) \geq z_\varepsilon(t_{2j+1}) + (t - t_{2j+1})\tilde{c}_\delta.$$

Since

$$z_\varepsilon(t_{2j+1}) + (t_{2j+2} - \delta - t_{2j+1})\tilde{c}_\delta > Y,$$

for $\varepsilon > 0$ sufficiently small, the point $\theta_{\varepsilon,2j+1}$ exists for $\varepsilon$ sufficiently small. Also

$$\theta_{\varepsilon,2j+1} \to t_{2j+1} \text{ as } \varepsilon \to 0. \tag{4.15}$$

This is because

$$z_\varepsilon(t_{2j+1}) + (\theta_{\varepsilon,2j+1} - t_{2j+1})\tilde{c}_\delta = Y.$$

Consider the initial value problem

$$\dot{z}_\varepsilon = y_\varepsilon - \frac{1}{\varepsilon}(z_\varepsilon - Y),\ z_\varepsilon(\theta_{\varepsilon,2j+1}) = Y,\ t > \theta_{\varepsilon,2j+1},\ t \text{ is close to } \theta_{\varepsilon,2j+1}. \tag{4.16}$$

This implies

$$\frac{d}{dt}[(z_\varepsilon - Y)e^{\frac{t}{\varepsilon}}] = y_\varepsilon e^{\frac{t}{\varepsilon}},$$

and thus $z_\varepsilon - Y > 0$ as long as $y_\varepsilon > 0$, which is true up to $t = t_{2j+2} - \delta$.
Now using (3.16) we have

$$\int_{\theta_{\varepsilon,2j+1}}^{t_{2j+2}-\delta} |q_\varepsilon(t)|dt \leq C\varepsilon$$

and

$$\int_{t_{2j+1}}^{t_{2j+2}-\delta} |q_\varepsilon(t)|dt \leq C\varepsilon + \int_{t_{2j+1}}^{\theta_{\varepsilon,2j+1}} |q_\varepsilon(t)|dt$$

$$\leq C\varepsilon + C(\theta_{\varepsilon,2j+1} - t_{2j+1}) \to 0, \text{ as } \varepsilon \to 0.$$

Therefore

$$q(t) = 0 \text{ on } (t_{2j+1}, t_{2j+2} - \delta)$$

and since $\delta$ is arbitrarily small we have (4.13).

$\square$

Let us next prove that $q(t)$ satisfies the following differential equation on $(t_{2j}, t_{2j+1})$.

PROPOSITION 4.3.

$$\dot{q} = kp \text{ on } (t_{2j}, t_{2j+1}). \tag{4.17}$$

*Proof.* Indeed, let $\varphi(t)$ be a smooth function on $(t_{2j}, t_{2j+1})$ with compact support on $(t_{2j}, t_{2j+1})$. We have

$$\int_{t_{2j}}^{t_{2j+1}} \dot{q}_\varepsilon \varphi dt = \int_{t_{2j}}^{t_{2j+1}} kp\varphi dt + \int_{t_{2j}}^{t_{2j+1}} \varphi \frac{q_\varepsilon}{\varepsilon}(\mathbb{1}_{z_\varepsilon - Y > 0} + \mathbb{1}_{z_\varepsilon + Y < 0})dt.$$

On the domain of $\varphi$ we have $|z(t)| < Y$, hence $|z_\varepsilon(t)| < Y$ for $\varepsilon$ sufficiently small, therefore

$$\int_{t_{2j}}^{t_{2j+1}} \dot{q}_\varepsilon \varphi dt = \int_{t_{2j}}^{t_{2j+1}} kp\varphi dt$$

for $\varepsilon$ sufficiently small. Since $\varphi$ is arbitrary (4.17) follows. $\square$

The function $q(t)$ is discontinuous. We will next argue the continuity of $q(t)$ at $t_{2j+1}$ if $t_{2j+1} \neq t_{2j+2}$.

PROPOSITION 4.4. *Consider the case $t_{2j} < t_{2j+1} < t_{2j+2}$ (we assume strict inequality). Then*

$$q(t_{2j+1} - 0) = 0 = q(t_{2j+1} + 0).$$

*Proof.* We have already established the weak convergence in $L^2$. We also know that $t^\varepsilon_{2j} \to t_{2j}, t^\varepsilon_{2j+1} \to t_{2j+1}$, and $t^\varepsilon_{2j+2} \to t_{2j+2}$. Assume, to fix the ideas, that $\delta_{2j+1} = 1$, and hence $\delta^\varepsilon_{2j+1} = 1$ for $\varepsilon$ sufficiently small. Therefore for $\delta$ sufficiently small we have the relations

$$t_{2j}, t^\varepsilon_{2j} < t_{2j+1} - \delta < t^\varepsilon_{2j+1} < t_{2j+1} + \delta.$$

For $t \in [t^\varepsilon_{2j}, t_{2j+1} + \delta)$ we have

$$-\dot{q}_\varepsilon + \frac{q_\varepsilon}{\varepsilon} = -kp_\varepsilon$$

and then

$$-\frac{d}{dt}(q_\varepsilon e^{-\frac{t}{\varepsilon}}) = -kp_\varepsilon e^{-\frac{t}{\varepsilon}}.$$

Hence

$$q_\varepsilon(t^\varepsilon_{2j+1})e^{-\frac{t^\varepsilon_{2j+1}}{\varepsilon}} - q_\varepsilon(t_{2j+1} + \delta)e^{-\frac{t_{2j+1}+\delta}{\varepsilon}} = -k \int_{t^\varepsilon_{2j+1}}^{t_{2j+1}+\delta} p_\varepsilon(s) e^{-\frac{s}{\varepsilon}} ds$$

It follows that

$$q_\varepsilon(t^\varepsilon_{2j+1}) = q_\varepsilon(t_{2j+1}+\delta)e^{\frac{t^\varepsilon_{2j+1}-t_{2j+1}+\delta}{\varepsilon}} - k\int_{t^\varepsilon_{2j+1}}^{t_{2j+1}+\delta} p_\varepsilon(s)e^{-\frac{s-t^\varepsilon_{2j+1}}{\varepsilon}}ds.$$

Since $|q_\varepsilon(s)|, |p_\varepsilon(s)| \leq C$ we have

$$|q_\varepsilon(t^\varepsilon_{2j+1}| \leq Ce^{\frac{t^\varepsilon_{2j+1}-t_{2j+1}+\delta}{\varepsilon}} + C\int_{t^\varepsilon_{2j+1}}^{t_{2j+1}+\delta} p_\varepsilon(s)e^{-\frac{s-t^\varepsilon_{2j+1}}{\varepsilon}}ds.$$

$$\leq Ce^{\frac{t^\varepsilon_{2j+1}-t_{2j+1}+\delta}{\varepsilon}} + C\delta \to 0, \text{ as } \varepsilon \to 0.$$

Therefore

$$q_\varepsilon(t^\varepsilon_{2j+1}) \to 0.$$

But on $(t_{2j+1}-\delta, t^\varepsilon_{2j+1})$ we have

$$\dot{q}_\varepsilon = kp_\varepsilon$$

hence

$$q_\varepsilon(t^\varepsilon_{2j+1}) - q_\varepsilon(t_{2j+1}-\delta) = k\int_{t_{2j+1}-\delta}^{t^\varepsilon_{2j+1}} p_\varepsilon(t)dt$$

from which it follows that

$$q_\varepsilon(t_{2j+1}-\delta) \to -k\int_{t_{2j+1}-\delta}^{t^\varepsilon_{2j+1}} p(t)dt.$$

But

$$q_\varepsilon(t_{2j+1}-\delta) \to q(t_{2j+1}-\delta)$$

hence

$$q(t_{2j+1}-\delta) = -k\int_{t_{2j+1}-\delta}^{t^\varepsilon_{2j+1}} p(t)dt$$

and letting $\delta \to 0$ we obtain

$$q(t_{2j+1}-0) = 0.$$

□

We have established the following result on system $y, z, p, q$.

THEOREM 4.5. *Let $y_\varepsilon, z_\varepsilon, p_\varepsilon, q_\varepsilon$ satisfy the necessary conditions given in* (2.11) *and $y, z, p, q$ be obtained through the limiting relations* (3.8) *and* (3.18). *The optimal control, $u(t)$, for the variational inequality* (2.1)-(2.2) *is given by $u(t) = -p(t)$ (see*

*Theorem 3.1). Then there exists a sequence* $t_0 = 0 < t_1 \leq t_2 < t_3 \leq t_4 < \ldots$ *of switching times with* $t_{2J} \leq T < t_{2j+1}$ *or* $t_{2J+1} < T \leq t_{2J+2}$*, where* $J \geq 0$*, such that* $y(\cdot), z(\cdot), p(\cdot), q(\cdot)$ *satisfy*

$$\begin{aligned}
&\dot{y} + c_0 y + kz + p = 0 \text{ for } t \in (0,T) \\
&\dot{z} = y \text{ if } t_{2j} < t < t_{2j+1} \\
&z = Y\delta_{2j+1} \text{ if } t_{2j+1} < t < t_{2j+2} \text{ void if } t_{2j+1} = t_{2j+2} \\
&-\dot{p} = -c_0 p + q + \lambda \text{ for } t \in (0,T) \\
&\dot{q} = kp \text{ if } t_{2j} < t < t_{2j+1} \\
&q = 0 \text{ if } t_{2j+1} < t < t_{2j+2} \text{ void if } t_{2j+1} = t_{2j+2}.
\end{aligned} \tag{4.18}$$

*Moreover* $y(\cdot), z(\cdot), p(\cdot)$ *are continuous on* $[0,T]$, $q(\cdot)$ *is piecewise continuous on* $[0,T]$ *with possible jumps at* $t_{2j+2}$, $j = 0, 1, ..,$ *and satisfy the initial and boundary conditions*

$$y(0) = z(0) = 0,\ p(T) = \mu\ ,\ q(T) = 0. \tag{4.19}$$

*In addition,* $y(t_{2j}) = 0$ *and* $q(t_{2j+1} - 0) = q(t_{2j+1} + 0) = 0$ *if* $t_{2j+1} < t_{2j+2}$.

□

Remark 4.6. *The function* $q(t)$ *satisfies the following relations for* $j \geq 1$*:*

$$\begin{aligned}
&q(t_{2j} - 0) = 0 \neq q(t_{2j} + 0) \text{ if } t_{2j-1} < t_{2j}, \text{ and} \\
&q(t_{2j} - 0) \neq 0 \text{ and } q(t_{2j} - 0) \neq q(t_{2j} + 0) \text{ if } t_{2j-1} = t_{2j}.
\end{aligned} \tag{4.20}$$

□

## 5. Computation of the Cost Function.

We have

$$J(u) = \frac{1}{2}\int_0^T p^2(t)dt + \lambda \int_0^T y(t)dt + \mu y(T). \tag{5.1}$$

However from (4.18) we get easily

$$\int_0^T p^2(t)dt + \mu y(T) + \lambda \int_0^T y(t)dt + \int_0^T (qy + kzp)dt = 0,$$

and hence

$$J(u) = \frac{1}{2}\lambda \int_0^T y(t)dt + \frac{1}{2}\mu y(T) - \frac{1}{2}\int_0^T (qy + kzp)dt. \tag{5.2}$$

We have that

$$\begin{aligned}
&\int_0^T (qy + kzp)dt = \sum_{j=0}^{J-1} [\int_{t_{2j}}^{t_{2j+1}} (q\dot{z} + \dot{q}z)dt + \delta_{2j+1}\int_{t_{2j+1}}^{t_{2j+2}} kYpdt] \\
&+ \mathbb{1}_{t_{2J} < T < t_{2J+1}} \int_{t_{2J}}^T (q\dot{z} + \dot{q}z)dt \\
&+ \mathbb{1}_{t_{2J+1} < T < t_{2J+2}} [\int_{t_{2J}}^{t_{2J+1}} (q\dot{z} + \dot{q}z)dt + \delta_{2J+1}\int_{t_{2J+1}}^T kYpdt].
\end{aligned} \tag{5.3}$$

Note that in (5.3) the integrals over $[t_{2j+1}, t_{2j+2}]$ disappear if $t_{2j+1} = t_{2j+2}$ and the last term disappears if $t_{2J+1} = t_{2J+2}$.
Easy calculations yield the following expressions:

$$
\begin{aligned}
&\int_0^{t_1} (q\dot{z} + \dot{q}z)dt + \delta_1 \int_{t_1}^{t_2} kYpdt = q(t_2 - 0)Y\delta_1 \mathbb{1}_{t_1=t_2} + \delta_1 \int_{t_1}^{t_2} kYpdt, \\
&\int_{t_{2j}}^{t_{2j+1}} (q\dot{z} + \dot{q}z)dt + \delta_{2j+1} \int_{t_1}^{t_2} kYpdt = q(t_{2j+2} - 0)Y\delta_{2j+1} \mathbb{1}_{t_{2j+1}=t_{2j+2}} \\
&-q(t_{2j} + 0)Y\delta_{2j-1} + \delta_{2j+1} \int_{t_{2j+1}}^{t_{2j+2}} kYpdt, \text{ for } j = 1, .., J-1, \\
&\int_{t_{2J}}^{T} (q\dot{z} + \dot{q}z)dt = -q(t_{2J} + 0)Y\delta_{2J-1}, \text{ if } t_{2J} < T < t_{2J+1}, \\
&\int_{t_{2J}}^{t_{2J+1}} (q\dot{z} + \dot{q}z)dt + \delta_{2J+1} \int_{t_{2J+1}}^{T} kYpdt = -q(t_{2J} + 0)Y\delta_{2J-1} \\
&+\delta_{2J+1} \textstyle\int_{t_{2J+1}}^{T} kYpdt.
\end{aligned}
\tag{5.4}
$$

Therefore

$$
\begin{aligned}
\int_0^T (qy + kzp)dt = &\sum_{j=1}^{J} Y\delta_{2j-1}(q(t_{2j} - 0)\mathbb{1}_{t_{2j-1}=t_{2j}} - q(t_{2j} + 0)) \\
&+ \sum_{j=0}^{J} \delta_{2j+1} \int_{t_{2j+1}}^{(t_{2j+2})\wedge T} kYpdt.
\end{aligned}
\tag{5.5}
$$

If $J = 0$, then

$$
\int_0^T (qy + kzp)dt = 0. \tag{5.6}
$$

Hence

$$
\begin{aligned}
&J(u(\cdot)) = \frac{1}{2}\lambda \int_0^T y(t)dt + \frac{1}{2}\mu y(T) \\
&+\frac{Y}{2} \sum_{j=0}^{J-1} \delta_{2j+1}[q(t_{2j+2} + 0) - q(t_{2j+2} - 0)\mathbb{1}_{t_{2j+1}=t_{2j+2}} - k \int_{t_{2j+1}}^{t_{2j+2}} p(t)dt] \\
&-\frac{Y}{2} k\delta_{2J+1} \int_{(t_{2J+1})\wedge T}^{(t_{2J+2})\wedge T} pdt.
\end{aligned}
\tag{5.7}
$$

**6. Conclusions.** In Theorem 4.5 and Remark 4.6 we obtained a two point boundary value problem with additional internal boundary conditions for the state and adjoint variables, i.e., (4.18), with (4.19) and (4.20). For the solution of this problem an iterative process is needed because the phase changing instances $t_j$, $j = 1, 2, ...$ are only defined implicitly. The computation proceeds segment by segment fashion, where a segment is a pair of consecutive elastic and plastic phases including the possibility of a one point plastic excursion (i.e., when the oscillator just touches on the plastic boundary). A detailed description of the corresponding algorithm is discussed in [3] and [7].

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