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LOWER BOUNDED CONTROL-LYAPUNOV FUNCTIONS*

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Abstract. The well known Brockett condition - a topological obstruction to the existence of smooth stabilizing feedback laws - has engendered a large body of work on discontinuous feedback stabilization. The purpose of this paper is to introduce a class of control-Lyapunov function from which it is possible to specify a (possibly discontinuous) stabilizing feedback law. For control-affine systems with unbounded controls Sontag has described a Lyapunov pair which gives rise to an explicit stabilizing feedback law smooth away from the origin - Sontag's "universal construction" of Artstein's Theorem. In this work we introduce the more general "lower bounded control-Lyapunov function" and a "universal formula" for nonaffine systems. Our "universal formula" is a static state feedback which is measurable and locally bounded but possibly discontinuous. Thus, for the corresponding closed loop system, the classical notion of solution need not apply. To deal with this situation we use the generalized solution due to Filippov.
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1. Introduction. In his 1983 paper [1] Artstein proves that, for smooth control-affine systems evolving on $\mathbb{R}^n$ with controls in $\mathbb{R}^m$, there exists a globally stabilizing feedback control law continuous on $\mathbb{R}^n \setminus \{0\}$ if the system has a *smooth* control-Lyapunov function. In 1989 Sontag provided an explicit proof of Artstein's Theorem which uses a "universal formula" for the stabilizing controller (cf [14]). Our goal here is to introduce the more general LB-CLF and a "universal formula" for a stabilizing feedback controller for possibly non-affine nonlinear systems for which a lower-bounded CLF exists.

Consider a system together with a control-Lyapunov function (CLF). Corresponding to each state x are control values for which the rate of change of the Lyapunov function along a system trajectories at x is negative (cf. [14]). In particular the rate of change of the CLF along system trajectories is upper bounded. For control-affine systems with unconstrained controls and the *small control property* a CLF is automatically lower-bounded as well. For a more general class of systems we call a CLF with such a lower bound a lower bounded control-Lyapunov function (LB-CLF). One consequence of having this lower bound is that one can always find control values for which the "rate of change" of our LB-CLF $v(x)$ is precisely $-w(x)$ for some positive definite function $w(x)$. For control-affine systems this control value is unique, and we show that, for an appropriate choice of w, this controller is Sontag's "universal formula". For more general systems the control values at a state x for which the "rate of change" of a lower-bounded CLF $v(x)$ is precisely $-w(x)$ need not be unique,

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and a controller enforcing "$\dot{v}(x) = -w(x)$" need not stay bounded along the resulting state trajectories. We introduce a "universal formula" for a feedback controller which enforces "$\dot{v}(x) = -w(x)$" and which stays bounded along the resulting state trajectories.

Sontag and Sussmann examine connections between *continuous* but possibly nondifferentiable CLFs and null controllability in [13]. The nondifferentiabilty of the CLF necessitates the use of generalized derivatives to measure rates of change of the CLF along system trajectories and the introduction of the Lyapunov pair. These ideas are also employed in the present work. Finally, in [11], Smirnov considers the problem of designing feedback regulators for possibly nonaffine nonlinear systems where the set of controls can be a subset of $\mathbb{R}^m$ - i.e. constrained controls. His major result is the construction of a Lyapunov function and an explicit feedback stabilizer for linear systems with convex constraints on the control set. Our work provides an alternate approach to this particular controller design problem.

The paper is organized as follows: in Section 2 we introduce notations used, define our systems, and discuss the "basic conditions" for differential inclusions. In Section 3 we introduce the notion of lower bounded Lyapunov pairs and examine some of their basic properties. In Section 4 we define our stabilizing feedback controller and present our main result, and in Section 5 several examples are presented.

2. Preliminaries. We will consider systems of the form

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

where the state $x(t)$ evolves in $\mathbb{R}^n$ and the controls $u(t)$ take values in a subset $U \subset \mathbb{R}^m$ containing the origin. For simplicity we assume that $U = \mathbb{R}^m$. The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz at (x, u) and $f(0, 0) = 0$. To stabilize this system to $x = 0$ we will use an abstract energy function v which can be made to decrease along system trajectories. A function $v : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is *positive definite* if $v(0) = 0$, $v(\xi) > 0$ for $\xi \neq 0$, and *proper* if $v(\xi) \to \infty$ as $\|\xi\| \to \infty$. For $\xi \in \mathbb{R}^n$ we denote by $\|\xi\|$ the Euclidean norm. For a locally Lipschitz continuous function $v : \mathbb{R}^n \to \mathbb{R}$ and $\xi, \rho \in \mathbb{R}^n$ we define the *Dini derivative of v in the direction of ρ at ξ* to be

$$\overline{D^+} v(\xi, \rho) = \limsup_{t \to 0+} \; [v(\xi + t\rho) - v(\xi)]/t.$$

In what follows we will be considering "closed loop" systems of the form $f(\xi, u_*(\xi))$ where the feedback controller u_* is measurable and locally bounded but possibly discontinuous, hence the classical notion of solution need not apply. To deal with this situation we will use the generalized solution due to Filippov [5]. Set $f_{u_*}(\xi) = f(\xi, u_*(\xi))$ and associate to u_* the set valued map

$$\mathcal{F}_{u_*}(\xi) = \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} \overline{\text{co}}\{f_{u_*}(\mathcal{B}(\xi, \delta) \setminus N)\}, \tag{2}$$

where $\mathcal{B}(\xi, \delta)$ is a ball of radius δ centered at ξ, $\overline{\text{co}}$ denoted the closure of the convex hull, and μ is the usual Lebesque measure on $\mathbb{R}^n$. A Filippov solution on an interval $I \subset \mathbb{R}$ is function $x : I \to \mathbb{R}^n$ such that $x(\cdot)$ is absolutely continuous on any interval $[t_1, t_2] \subset I$ and

$$\dot{x}(t) \in \mathcal{F}_{u_*}(x(t)) \tag{3}$$

almost everywhere on I (cf. [2, 5]). Solutions $x(t)$ to (3) are thus state trajectories of system (1) under the feedback controller $u = u_*(x)$ which are differentiable almost everywhere with respect to t.

Since $f(\xi, u_*(\xi))$ is measurable and locally bounded the set valued map $\mathcal{F}_{u_*}(\xi)$ is upper semicontinuous, compact and convex valued, and locally bounded (cf. [2, 10]). In particular the differential inclusion (2) satisfies the *basic conditions* of [5, p. 76] and thus has a Filippov solution for each initial state. The solution $x(t) = 0$ of a differential inclusion $\dot{x}(t) = \mathcal{F}(x(t))$ is called:

- **stable** [5] if for each $\epsilon > 0$ there exists $\delta > 0$ with the following property: for each x_0 such that $\|x_0\| < \delta$, each solution $x(t)$ to $\dot{x}(t) \in \mathcal{F}(x(t))$ with initial data $x(0) = x_0$ exists for $0 \leq t < \infty$ and satisfies the inequality $\| x(t)\| < \epsilon$ $(0 \leq t < \infty)$.
- **asymptotically stable** if $x(t) = 0$ is stable and, in addition, $x(t) \to 0$ as $t \to 0$.

The possibly discontinuous feedback controller $x \mapsto u_*(x)$ *stabilizes* system (1) iff $x(t) = 0$ is a stable solution of the corresponding differential inclusion (3).

3. Lower bounded Lyapunov pairs. A *Lipschitz continuous Lyapunov pair* (v, w) consists of a locally Lipschitz continuous, positive definite, proper function $v : \mathbb{R}^n \to \mathbb{R}$ and a non-negative continuous function $w : \mathbb{R}^n \to \mathbb{R}$ such that, for $\xi \neq 0$, there exists $\rho \in f(\xi, U)$ with

$$\overline{D^+} v(\xi, \rho) \leq -w(\xi).$$

REMARK 3.1. In [13] Sontag and Sussmann examine the connection between Lyapunov pairs and null asymptotic controllability using a Lyapunov pair (v, w) where v only needs to be continuous. This necessitates the use of the more general directional subderivative of v in the direction ρ at ξ, $D^-_\rho v(\xi)$, which can be infinite. They require $D^-_\rho v(\xi) \leq -w(\xi)$ for some $\rho \in \overline{\text{co}} f(\xi, U)$, where

$$D^-_\rho v(\xi) = \liminf_{t \to 0+, \omega \to \rho} \; [v(\xi + t\omega) - v(\xi)]/t.$$

The existence of this more relaxed Lyapunov pair with w positive definite is necessary and sufficient for null controllability of system (1) [13]. Our interest is in the design of a stabilizing feedback law which necessitates using $\rho \in f(\xi, U)$ rather than $\rho \in \overline{\text{co}} f(\xi, U)$ and the lower-bounded control-Lyapunov function introduced below. □

For control-affine systems with $f = f_0 + \sum_{i=1}^m u_i f_i$ the existence of a Lipschitz continuous Lyapunov pair (v, w) with w positive definite and v continuously differentiable implies the existence of a feedback law $u = k(x)$ so that the origin is a global asymptotically stable equilibrium with u continuous on $\mathbb{R}^n \setminus \{0\}$ (cf. [1, 14]). In [14] Sontag presents a "universal construction" of Artstein's Theorem for control-affine systems. This gives rise to a Lyapunov pair (v, w) with somewhat special properties. Let $v : \mathbb{R}^n \to \mathbb{R}$ be a *smooth* control Lyapunov function for a single-input control affine system $\dot{x} = f_0(x) + f_1(x)u$ - that is $v : \mathbb{R}^n \to \mathbb{R}$ is positive definite, proper and

$$\inf_{u \in \mathbb{R}^n} [\nabla v(\xi) f_0(\xi) + u \nabla v(\xi) f_1(\xi)] < 0$$

for each $\xi \neq 0$. Sontag's "universal" feedback stabilizer is given by the formula

$$u = -\frac{a + \sqrt{a^2 + b^4}}{b},$$

where $a = \nabla v f_0$ and $b = \nabla v f_1$. This controller is *smooth* on $\mathbb{R}^n \setminus \{0\}$ and along the resulting state trajectory $dv/dt = -\sqrt{a^2 + b^4}$ (which implies stability). If v is a CLF with the *small control property* (for $\epsilon > 0$ there exists $\delta > 0$ such that, for $0 < ||\xi|| < \delta$ there exists some $|\ell| < \epsilon$ with $a(\xi) + \ell b(\xi) < 0$) then $u_* = -(a + \sqrt{a^2 + b^4})/b$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and *continuous* on all of $\mathbb{R}^n$. Set

$$w = \sqrt{a^2 + b^4}, \tag{4}$$

a positive definite function. Here $\overline{D^+} v(\xi, f(\xi, u)) = a(\xi) + b(\xi)u$ and Sontag's "universal" feedback stabilizer u_* is the *unique* solution u to

$$\overline{D^+} v(\xi, f(\xi, u)) = -w(\xi).$$

We note that the range (space) of the map $u \mapsto D^+ v(\xi, f(\xi, u))$ will be $\mathbb{R}$ in the case $b(\xi) \neq 0$ and $\{-w(\xi)\}$ if $b(\xi) = 0$. In particular, for $\xi \neq 0$, there exists $u_1, u_2 \in \mathbb{R}$ such that $\overline{D^+} v(\xi, f(\xi, u_1)) \leq -w(\xi) \leq \overline{D^+} v(\xi, f(\xi, u_2))$. We note that once we have selected the positive definite function w we can immediately solve for the corresponding stabilizing feedback controller.

We note that in the above case v is continuously differentiable. Thus the gradient of v is continuous hence bounded on compact neighbourhoods of the origin. This implies that "$\dot{v}$" is both lower and upper bounded. If v is not continuously differentiable this lower bound is not automatic. This motivates the following definition.

Definition 3.2. *A* **lower bounded Lyapunov pair** *for the system (1) is a Lipschitz continuous Lyapunov pair (v, w) such that*

1. *for $\xi \neq 0$ there exists $\rho_1, \rho_2 \in f(\xi, U)$ such that*

$$\overline{D^+} v(\xi, \rho_1) \leq -w(\xi) \leq \overline{D^+} v(\xi, \rho_2), \tag{5}$$

2. *for $\epsilon > 0$, there exists $\delta > 0$ such that, for $0 < ||\xi|| < \delta$, (5) holds with $\rho_i = f(\xi, u_i)$, $||u_i|| < \epsilon$, $i = 1, 2, \ldots, m$,*

Item (2) above is essentially the small controls property of [14], a natural requirement for practical implementation. That ρ_1, ρ_2 are in $f(\xi, U)$ rather than $\overline{\text{co}} f(\xi, U)$ reflects our desire to steer the state to the origin using a feedback controller as opposed to an open loop control.

Suppose that (v, w) is a lower bounded Lyapunov pair for system (1). Then, corresponding to each $\xi \in \mathbb{R}^n$, there is at least one solution $u \in U$ to $\overline{D^+}v(\xi, f(\xi, u)) = -w(\xi)$. We now address the question of how to choose a feedback control $u(\xi)$ to ensure that the resulting feedback controller remains bounded along the closed loop state trajectories. Set

$$g(\xi, u) = \overline{D^+}v(\xi, f(\xi, u)) + w(\xi), \tag{6}$$

and let

$$\mathcal{U}^{v,w}(\xi) = \{u \in U \mid g(\xi, u) = 0, \text{ and, if } g(\xi, \tilde{u}) = 0,\ \tilde{u} \in U, \text{ then } \|u\| \leq \|\tilde{u}\|\},$$

the set of minimum norm vectors $u \in U$ with the property that $g(\xi, u) = 0$. We associate with (v, w) the set valued map

$$\mathcal{F}^{v,w}(\xi) = \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} \overline{\text{co}}\{f^{v,w}(\mathcal{B}(\xi, \delta) \setminus N)\} \tag{7}$$

where $f^{v,w}(\xi) = f(\xi, \mathcal{U}^{v,w}(\xi))$.

Remark 3.3. The minimum norm solution to

$$\overline{D^+}v(\xi, f(\xi, u)) = -w(\xi) \tag{8}$$

often will be unique. In this common situation the set of minimum norm control values $\mathcal{U}^{v,w}(\xi)$ which enforce (8) consists of a single element $\{u(\xi)\}$. In this case $\mathcal{F}^{v,w}(\xi)$ contains a single vector ρ with

$$\overline{D^+}v(\xi, \rho) = -w(\xi). \tag{9}$$

In the case where the "Lyapunov function" v is differentiable at ξ, then $\rho \to \overline{D^+}v(\xi, \rho)$ is a linear map and (9) holds. Finally, if $\mathcal{F}^{v,w}(\xi)$ consists of a single element then (9) holds as well.

If v is not differentiable at ξ ($\rho \to \overline{D^+}v(\xi, \rho)$ is not a linear map) and $\mathcal{F}^{v,w}(\xi)$ contains more than a single vector we can no longer be assured that $\overline{D^+}v(\xi, \rho) = -w(\xi)$ for every $\rho \in \mathcal{F}^{v,w}(\xi)$. On the other hand, for every $\rho \in \mathcal{F}^{v,w}(\xi)$ the differential inclusion

$$\dot{\xi}(t) \in \mathcal{F}^{v,w}(\xi(t)) \tag{10}$$

need not have a solution $\xi(t)$ with $\xi(0) = \xi_0$ and $\dot{\xi}(0) = \rho$. Thus, as in [2], we introduce the (possibly empty) set $\mathcal{E}^{v,w}(\xi)$ of vectors $\rho \in \mathcal{F}^{v,w}(\xi)$ such that there exists a solution $\xi(t)$ of (10) defined for $t \geq 0$ with the property that $\xi(0) = \xi$ and $\dot{\xi}(0)$ exists and is equal to ρ. Clearly the differential inclusion

$$\dot{\xi}(t) \in \mathcal{E}^{v,w}(\xi(t))$$

has the same solutions as (10). For example suppose that $\mathcal{F}^{v,w}(\xi)$ arises from a sliding mode controller which steers states to a sliding surface S in finite time and has $x(t) \in S$ thereafter. We have smoothness away from the switching surface S hence $\mathcal{F}^{v,w}(\xi)$ consists of a single vector when $\xi \notin S$ and $\mathcal{E}^{v,w}(\xi) = \mathcal{F}^{v,w}(\xi)$. If $\xi_0 \in S$ then $\mathcal{F}^{v,w}(\xi_0)$ is a closed convex set of vectors containing a unique vector v_0 in the tangent space to S. The only solution $\xi(t)$ with $\xi(t_0) = \xi_0$ has $\dot{\xi}(t_0) = v_0$ hence $\mathcal{E}^{v,w}(\xi_0) = \{v_0\}$. □

As in [2] we use $\mathcal{E}^{v,w}$ to weaken the regularity conditions to deal with the degenerate case of points $\xi \in \mathbb{R}^n$ where the map $\rho \mapsto \overline{D^+}v(\xi,\rho)$ is not linear (which implies v is not differentiable at ξ).

Definition 3.4. *A lower bounded Lyapunov pair (v, w) said to be* **regular** *if*

1. *the set valued map $\mathcal{F}^{v,w}(\xi)$ is upper semicontinuous, compact and convex valued, and locally bounded,*
2. *there is a positive definite continuous function $\tilde{w}$ on $\mathbb{R}^n$ such that at points ξ where the map v is not differentiable (or, more generally, where $\rho \mapsto \overline{D^+}v(\xi,\rho)$ is not linear)*

$$\overline{D^+}v(\xi,\rho) \leq -\tilde{w}(\xi)\ \forall \rho \in \mathcal{E}^{v,w}(\xi). \tag{11}$$

Definition 3.5. *A locally Lipschitz continuous, positive definite and proper function v is called a* **lower bounded control-Lyapunov function (LB-CLF)** *if there exists a positive definite function w such that (v, w) is a regular lower bounded Lyapunov pair. The pair (v, w) will be called a* **regular Lyapunov pair for v**.

We note that if (v, w) is a regular Lyapunov pair for v then the differential inclusion (10) has a Filippov solution for each initial state. In [11] Smirnov considers the feedback stabilization problem for a class of linear systems with constrained controls. We now show that for the systems considered in Theorem 1 of [11] the existence of a stabilizing feedback is equivalent to the existence of a regular lower bounded control-Lyapunov function.

Proposition 3.6. *Consider the linear system with constrained controls*

$$\dot{x} = Ax + u, \tag{12}$$

where $x \in \mathbb{R}^n, u \in K$, a closed convex cone. Then there exists a feedback controller u_* such that the differential inclusion

$$\dot{x}(t) \in \mathcal{F}_{u_*}(x(t))$$

is asymptotically stable if and only if there exists a regular lower bounded control-Lyapunov function for the system (12).

Proof. In Theorem 1 of [11] Smirnov establishes that the existence of an asymptotically stabilizing feedback controller u_* for system (12) is equivalent to the existence of a convex positively homogeneous function $v(x)$ and $\theta > 0$ such that, for any $x \in \mathbb{R}^n$,

$$Dv(x)w \leq -\theta v(x)$$

for some vector $w \in Ax+K$. Here $v(x)$ is the Minkowski function of a set $M_\omega = M_I \times \omega M_J$ where M_J is an ellipsoid, M_I the convex hull of a finite set of points I in $\mathbb{R}^n$, and $\omega > 0$ [11]. The homogeneous function v need not be differentiable everywhere but the directional derivative $Dv(x)w$ is well defined (cf [12]). We note that by construction $Dv(x)$ is continuous almost everywhere and $||Dv(x)||$ is bounded on the compact set $\{v(x) = 1\}$. Thus some positive multiple w of v has the property that for $\xi \in \{v(x) = 1\}$ there exists $u_1, u_2 \in K$ such that $\overline{D^+}\hat{v}(\xi, f(\xi, u_1)) \leq -w(\xi) \leq \overline{D^+}\hat{v}(\xi, f(\xi, u_2))$ where $f(\xi, u) = A\xi + u$. This, combined with the homogeneity of v and w, means that (v, w) satisfies 1. of Definition 3.2. Since v is positively homogeneous and $Dv = \overline{D^+}v$ a.e. it is easy to show that 2. of Definition 3.2 holds as well and hence (v, w) is a lower-bounded Lyapunov pair. The proof that (v, w) is regular follows from the homogeneity of v. To see this we first note that, as a consequence of the construction of v we have $\mathcal{U}^{v,w}(\xi) = \{u_*(\xi)\}$ for some continuous function $u_* : M \to \mathbb{R}^m$ where $M \subset \mathbb{R}^n$ is open and $\mathbb{R}^n \setminus M$ is the union of hyperplanes $S_1, \ldots, S_k$ (cf [12]). It is straightforward to verify that on the compact set $\{v(x) = 1\}$ we can find a positive continuous function α such that $\overline{D^+}v(\xi, \rho) \leq -\alpha(\xi)w(\xi)$ $\forall \xi \in S_j \cap \{v(x) = 1\}$, $\rho \in \mathcal{F}^{v,w}(\xi) \cap T_\xi S_j$ for $j \in \{1, 2, \ldots, k\}$. Using the homogeneity of v and w we can extend the definition of α to a positive definite function defined on all of $\mathbb{R}^n$. The regularity of (v, w) follows directly from Proposition 3.8 below. □

REMARK 3.7. The linear system model with non-negative inputs modelled by

$$\begin{aligned} \dot{x}_1(t) =& x_2(t), \\ \dot{x}_2(t) =& -x_1(t) + u^2(t), \end{aligned} \tag{13}$$

mirrors the linear system with with non-negative inputs examined in Smirnov [11]. Define a continuously differentiable function $v_A(\xi_1, \xi_2) = \alpha\xi_1 + \beta\xi_2$ where $\alpha = e^{3\pi/2}(\sqrt{2}e^{-7\pi/4} - 1)$, $\beta = -e^{3\pi/2}$ and the continuously differentiable function expressed in polar coordinates as $v_B(r, \theta) = re^\theta$, and set

$$v(\xi) = \begin{cases} v_A(\xi) & \text{for } \xi_2 \leq -\xi_1 \leq 0, \\ v_B(\xi) & \text{otherwise.} \end{cases}$$

To motivate this choice of v we note that $\dot{x}_2$ can be assigned a large positive value by an appropriate choice for u but is only negative when x_1 is positive. The level

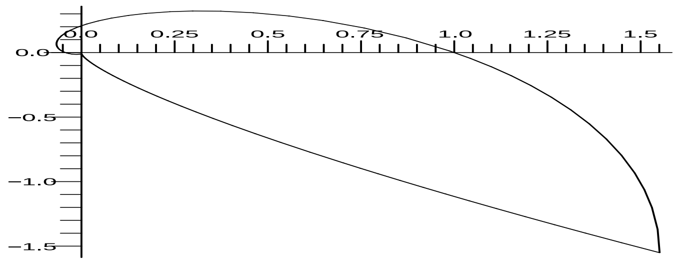

FIG. 1. *Level Set for CLF in Remark 3.7*

set for v must be designed to compensate for this lack of control over $\dot{x}_2$. A level set for v is shown in Figure 1. The choice of v_A and v_B was made to ensure that, for $x_2 > 0$ (where the control cannot be used to directly decrease x_2), the drift vector field points into the interior of region bounded by the level set.

One can show that v is a regular lower-bounded CLF and, setting

$$u_*(\xi) = \begin{cases} \sqrt{\xi_1 - \alpha\xi_2/\beta - v_A/\beta} & \text{for } \xi_2 \leq -\xi_1 \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that $\dot{v} = -v$ on M and the differential inclusion

$$\dot{\xi}(t) \in \mathcal{F}_{u_*}(\xi(t))$$

is asymptotically stable. It is fairly straightforward to find functions v_A and v_B which act as Lyapunov functions on disjoint regions of the state space - our task is made more challenging because of the compatibility needed on the closures of these disjoint sets. The results of a simulation of this case performed using SIMNON/PCW for Windows, Version 2.01 (SSPA Maritime Consulting AB, Sweden) is presented in Figure 2 where $x_0 = (-1, -1)$. We note that in this example solving "$\dot{v} = -w$" for u has two solutions. The minimum norm solution is u_* defined above. Other solutions can be unbounded along state trajectories!

$\square$

The following proposition shows that a lower bounded Lyapunov pair (v, w) is regular in many common cases. We note that if v is differentiable on $\mathbb{R}^n$ then $M = \emptyset$ and (ii) of Proposition 3.8 is automatically satisfied.

PROPOSITION 3.8. *Let (v, w) be a lower bounded Lyapunov pair for system (1) and $M \subset \mathbb{R}^n$ an open set on which v is differentiable (or, more generally, $\rho \mapsto \overline{D^+}v(\xi, \rho)$ is linear) and such that $\mathbb{R}^n \backslash M$ is the union of a finite number of hyperspaces $S_1, \ldots S_k$ (cf [5]). Then (v, w) will be regular if*

(i)

$$\mathcal{U}^{v,w}(\xi) = \{u_1(\xi), \ldots, u_\ell(\xi)\},\ \xi \in M$$

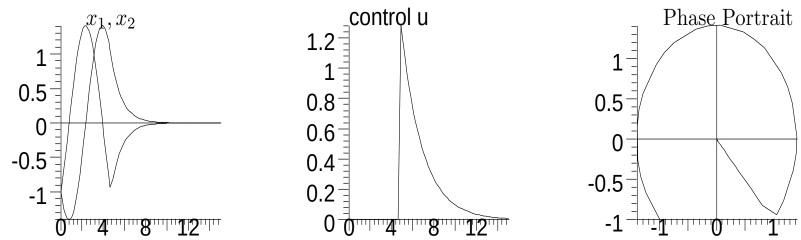

FIG. 2. *Asymptotic stabilization in using a USFC, Example 4.4*

for continuous functions u_j, $j = 1, \ldots, \ell$ *on* M *and*

(ii) there exists a continuous and positive definite function α *on* $\mathbb{R}^n$ *such that* $\overline{D^+}v(\xi, \rho) \le -\alpha(\xi)w(\xi)\ \forall \xi \in S_j, \rho \in \mathcal{F}^{v,w}(\xi) \cap T_\xi S_j$ *for* $j \in \{1, 2, \ldots, k\}$.

Proof. In the case where $\ell = 1$ it is straightforward to show that Definition 3.4 1. holds (cf [5] Chapter 2). The extension to the case were $\ell > 1$ is straightforward. To establish Definition 3.4 2. we first note that, without loss of generality, we can assume that $|\alpha(\xi)| \le 1$ (if not replace α by $\min\{1, \alpha\}$). Setting $\tilde{w} = \alpha w$ it follows that $\overline{D^+}v(\xi, \rho) \le -\tilde{w}(\xi)\ \forall \rho \in \mathcal{F}^{v,w}(\xi)$. Noting that $\mathcal{E}^{v,w}(\xi) \subset \mathcal{F}^{v,w}(\xi)$ establishes Definition 3.4 2. □

4. Universal stabilizing feedback controllers and asymptotic stabilization.

DEFINITION 4.1. *Suppose that* v *is a LB-CLF for system (1). A* **universal stabilizing feedback controller (USFC) for v** *is any measurable and locally bounded function* $u^{v,w} : \mathbb{R}^n \to U$ *with the property that* $f(\xi, u^{v,w}(\xi)) \in \mathcal{F}^{v,w}(\xi)$ *where* (v, w) *is any regular Lyapunov pair for* v.

REMARK 4.2. If v is a CLF for a single-input control-affine system with the small controls property the "universal stabilizing feedback controller" u_* is uniquely defined and continuous on $\mathbb{R}^n$ (cf [14]) and $\mathcal{F}^{v,w}(\xi) = \{f(\xi, u_*(\xi))\}$ is regular, with w given by (4) (Proposition 3.8 above). Furthermore $g(\xi, u) = a(\xi) + b(\xi)u + w(\xi)$ where $b(\xi) = \nabla v(\xi) f_1(\xi)$. Generically $\partial g/\partial u = b$ is nonzero on some open subset M_0 of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus M_0$ has measure zero. For multi-input control-affine systems $g(\xi, u) = a(\xi) + \sum_{j=1}^m b_j(\xi)u_j + w(\xi)$ but $g(\xi, u) = 0$ does not uniquely determine u (cf. [14]). Finding the minimum norm solution u_* to $g(\xi, u) = 0$ is a linear optimization problem with a unique solution $u_{*j} = -((w + a)/\sum_{i=1}^m b_i^2)b_j$ which is smooth on M_0. Thus Proposition 3.8 (i) is satisfied. Since v is differentiable on $\mathbb{R}^n$ we can take $M = \mathbb{R}^n$ and Proposition 3.8 (ii) is satisfied as well. In particular (v, w) is a regular lower-bounded Lyapunov pair. For systems which are not control-affine degenerate situations abound. For example the scalar system $\dot{x} = -xu^2$ with $v(x) = w(x) = |x|, x \in \mathbb{R}$ has $\dot{v}(x(t)) = -w(x(t))$ if and only if $u^2 = 1$ for all x, hence there are stabilizing feedback controllers which are not USFCs for v. Of course for this example one can also find stabilizing feedback controllers which are USFCs for v. □

THEOREM 4.3. *Let v be a LB-CLF for system (1) and u_* any USFC for v. Then $x(t) = 0$ is an asymptotically stable solution of (3). In particular $u = u_*$ is an asymptotic stabilizing feedback controller for system (1).*

Proof. Since u_* is a universal stabilizing feedback controller we have $f(\xi, u^{v,\bar{w}}(\xi)) \in \mathcal{F}^{v,\bar{w}}(\xi)$ for some regular Lyapunov pair $(v, \bar{w})$ for v. Since $(v, \bar{w})$ is a regular Lyapunov pair there exists a continuous positive definite function $\tilde{w}$ such that at points ξ where the map v is not differentiable

$$\overline{D^+}v(\xi, \rho) \leq -\tilde{w}(\xi) \; \forall \rho \in \mathcal{E}^{v,w}(\xi).$$

Define $w(\xi)$ by $w(\xi) = \min\{\bar{w}(\xi), \tilde{w}(\xi)\}$, so that w is a continuous positive definite function and (v, w) is a regular Lyapunov pair for v. If $\xi_*(t)$ denotes the state trajectory of system (1) under the feedback control law u_* then $\xi_*(t)$ will be differentiable for almost all t and

$$\overline{D^+}v(\xi_*(t), \dot{\xi}_*(t)) = \limsup_{h \to 0+} \; [v(\xi_*(t) + h\dot{\xi}_*(t)) - v(\xi_*(t))]/h.$$

This implies that $\dot{v}(\xi_*(t))$, the rate of change of v at time t along the state trajectory $\xi_*(t)$, satisfies $\dot{v}(\xi_*(t)) \leq -w(\xi_*(t)) \leq 0$. Then the extension of Theorem 1 in Chapter 3, section 15 of [5] to the case where v is only locally Lipschitz continuous (cf. page 159 of [5]) establishes that $\xi(t) = 0$ is an asymptotically stable solution of (3). □

5. Examples.

5.1. Example 1. Consider the non-affine system model

$$\begin{aligned} \dot{x}_1(t) &= u(t), \\ \dot{x}_2(t) &= -x_1^2(t) + u^2(t). \end{aligned} \tag{14}$$

Here $\dot{x} = f(x, u)$ with $f(x, u) = (u, -x_1^2 + u^2)$. This system does not satisfy the necessary conditions for the construction of a continuous stabilizing feedback controller [3, 8]. Here f is homogeneous with respect to the dilation $r = (1, 2)$ (cf. [4] and so it is natural to seek a CLF which is also homogeneous with respect to the dilation $r = (1, 2)$S. Consider the continuously differentiable CLF candidate

$$v(\xi) = \xi_1^4 - |\xi_2|^{3/2}\xi_1 + \xi_2^2.$$

To show that v is positive definite and radially unbounded (proper) we employ the Holder inequality $xy \leq x^p/p + y^q/q$ where $x, y > 0$ and $1/p + 1/q = 1$. Setting $x = |\xi_2|^{3/2}$ and $y = |\xi_1|$ we have

$$|\xi_2|^{3/2}|\xi_1| \leq \frac{3}{4}\xi_2^2 + \frac{1}{4}\xi_1^4.$$

Thus, for $\xi_1 \leq 0$, it follows that $v(\xi) \geq \xi_1^4 + \xi_2^2$ which implies v is proper and positive definite. In the case where $\xi_1 > 0$ we have

$$v(x) \geq \xi_1^4 - \frac{3}{4}\xi_2^2 - \frac{1}{4}\xi_1^4 + \xi_2^2.$$

This means $v(\xi) \geq \frac{1}{4}\xi_2^2 + \frac{3}{4}\xi_1^4$ and is proper and positive definite. Since v is continuously differentiable the Dini derivative

$$\begin{aligned} \overline{D^+}v(\xi, f(\xi,u)) =& dv_\xi f(\xi,u), \\ =& a(\xi)u^2 + b(\xi)u + c(\xi) \end{aligned} \tag{15}$$

where

$$\begin{aligned} a(\xi) =& 2\xi_2 - (3/2)\xi_1|\xi_2|^{1/2}\mathrm{sign}(\xi_2), \\ b(\xi) =& 4\xi_1^3 - |\xi_2|^{3/2}, \\ c(\xi) =& -\xi_1^2 a(\xi). \end{aligned}$$

Set $w_k(\xi) = v(\xi)/k$ where k is a positive integer. To enforce $\overline{D^+}v(\xi, f(\xi,u)) = -w_k(\xi)$ requires that u satisfy

$$a(\xi)u^2 + b(\xi)u + (w_k(\xi) - \xi_1^2 a(\xi)) = 0. \tag{16}$$

We first consider the case $a(\xi) \neq 0$ which means $u = [-b(\xi) \pm \sqrt{\alpha(\xi)}]/(2a(\xi))$ where $\alpha(\xi) = b^2(\xi) - 4a(\xi)(w_k(\xi) - \xi_1^2 a(\xi))$. Since u must be real $\alpha(\xi) \geq 0$. It is easy to show that, for ξ fixed and k sufficiently large, this will be the case. Since α is homogeneous of degree 6 with respect to the dilation $r = (1,2)$ and the compact $n-$sphere is compact, the standard arguments can be used to establish that, for some k, we have $\alpha(\xi) \geq 0$ for all ξ (cf. [4]). We set $w = w_k$.

Suppose $a(\xi) \neq 0$. We know that u satisfies (16) iff $u = -b(\xi) \pm \sqrt{\alpha(\xi)})/(2a(\xi))$ and the minimum norm solution $u = u_*(\xi)$ is

$$u_*(\xi) = \begin{cases} \frac{-b(\xi)+\sqrt{\alpha(\xi)}}{2a(\xi)} & \text{if } b(\xi) > 0, \\ \frac{-b(\xi)-\sqrt{\alpha(\xi)}}{2a(\xi)} & \text{if } b(\xi) < 0, \\ \pm\frac{\sqrt{\alpha(\xi)}}{2a(\xi)} & \text{if } b(\xi) = 0. \end{cases}$$

Suppose $a(\xi) = 0,\ b(\xi) \neq 0$. Then there is only one possible solution to (16), namely $u = u_*(\xi) = -(w(\xi) - \xi_1^2 a(\xi))/b(\xi)$. Setting $M = \{b(\xi) \neq 0\}$, an open set which is a union of hypersurfaces (in the sense of [5]) whose complement is a set of measure zero, we have unique solutions to (16) on M. In particular $\mathcal{U}_{(v,w)}(\xi)) = \{u_*(\xi)\}$ for $\xi \in M$ where u_* is defined above and is continuous on M. This means that (v,w) is regular (Proposition 3.8). We will define our controller to be the continuous function $u_* = -(w(\xi) - \xi_1^2 a(\xi))/b(\xi)$ on M and either $\sqrt{\alpha(\xi)}/2a(\xi)$ or $-\sqrt{\alpha(\xi)}/2a(\xi)$ on $\mathbb{R}^n \setminus M$. It follows [5] that

$$\mathcal{F}_{u_*}(\xi) = \overline{\mathrm{co}}\{\lim f(\xi_i, u_*(\xi_i)) \mid \xi_i \to \xi,\ \xi_i \notin M\}.$$

From our definition of u_* we have $\mathcal{F}_{u_*}(\xi) = f(\xi_i, u_*(\xi_i))$ if $b(\xi) \neq 0$ and $\mathcal{F}_{u_*}(\xi) = \overline{\text{co}}\{f(\xi_i, \sqrt{\alpha(\xi)}/2a(\xi)), f(\xi_i, -\sqrt{\alpha(\xi)}/2a(\xi))\}$ if $b(\xi) = 0$. Here the set $\{b(\xi) = 0\}$ divides M into two domains, M^+ where $b(\xi) > 0$ and M^- where $b(\xi) < 0$. As in [5] we set $f^+(\xi, u_*(\xi_i))$ and $f^-(\xi, u_*(\xi_i))$ to be the limiting values of $f(\xi, u_*(\xi_i))$ as ξ approaches $\{b(\xi) = 0\}$ from M^+ and M^- respectively. If N is a unit normal field to $\{b(\xi) = 0\}$ we set $f_N^+(\xi)$ and $f_N^-(\xi)$ to be the projections of f^+ and f^- onto N. It is straightforward to verify that $f_N^- > 0$ and $f_N^+ < 0$. This implies [Corollary 2 p 108 [5]] that the differential inclusion $\dot{x}(t) \in \mathcal{F}_{u_*}(x(t))$ has unique solutions $x(t)$ with the additional property that if $b(x(t_0)) = 0$ then $b(x(t)) = 0 \forall t > t_0$. That is $\{b(\xi) = 0\}$ is a "sliding surface" for $\dot{x}(t) \in \mathcal{F}_{u_*}(x(t))$ and u_* a sliding mode controller.

As noted above in the case where u_* is a sliding mode controller the set $\mathcal{E}^{v,w}(\xi) = \mathcal{F}^{v,w}(\xi)$ when $\xi \in M$ ($b(\xi) \neq 0$) and when $b(\xi) = 0$ then $\mathcal{E}_*^a(\xi)$ is the unique vector $q(\xi)$ in $\overline{\text{co}}\{f(\xi_i, \sqrt{\alpha(\xi)}/2a(\xi)), f(\xi_i, -\sqrt{\alpha(\xi)}/2a(\xi))\}$ which is tangent to $\{b(\xi) = 0\}$. Since $\overline{D^+}v(\xi, f(\xi, u_*(\xi))) = -w(\xi)$ when $b(\xi) \neq 0$ and $q(\xi) = sf(\xi_i, \sqrt{\alpha(\xi)}/2a(\xi)) + (1 - s)f(\xi_i, -\sqrt{\alpha(\xi)}/2a(\xi))$ for $0 \leq s \leq 1$ it follows that $\overline{D^+}q(\xi) = -w(\xi)$. In particular we have $\dot{v}^*(\xi) = -w(\xi) < 0$. Then Theorem 4.3 implies that the solution $x(t) = 0$ of (10) is asymptotically stable.

The results of a simulation of this case performed using SIMNON/PCW for Windows, Version 2.01 (SSPA Maritime Consulting AB, Sweden) is presented in Figure 3 where $x_0 = (-0.5, 1)$, $k = 4$.

5.2. Example 2. Consider the control-affine system model

$$\begin{aligned} \dot{x}_1(t) &= u_1(t), \\ \dot{x}_2(t) &= u_2(t), \\ \dot{x}_3(t) &= x_1^2 - x_2^2. \end{aligned} \tag{17}$$

Here $\dot{x} = f(x, u)$ with $f(x, u) = (u, -x_1^2 + u^2)$. This system does not satisfy the necessary conditions for the construction of a differentiable stabilizing feedback controller [3, 8].

We modify the CLF candidate from example 5.1 as follows: set

$$\begin{aligned} v^+(\xi) &= 1.2\xi_1^4 + \xi_2^4 - \text{sign}(x_3)|\xi_3|^{3/2}(\xi_1 + \xi_2) + 2\xi_3^2, \\ v^-(\xi) &= \xi_1^4 + 1.2\xi_2^4 - \text{sign}(x_3)|\xi_3|^{3/2}(\xi_1 + \xi_2) + 2\xi_3^2. \end{aligned} \tag{18}$$

Computing the rate of change of v^+, v^- along the state trajectory one finds that

$$\begin{aligned} \dot{v}^+(\xi(t)) &= a(\xi(t)) + b_1^+(\xi(t))u_1(t) + b_2^+(\xi(t))u_2(t), \\ \dot{v}^-(\xi(t)) &= a(\xi(t)) + b_1^-(\xi(t))u_1(t) + b_2^-(\xi(t))u_2(t), \end{aligned} \tag{19}$$

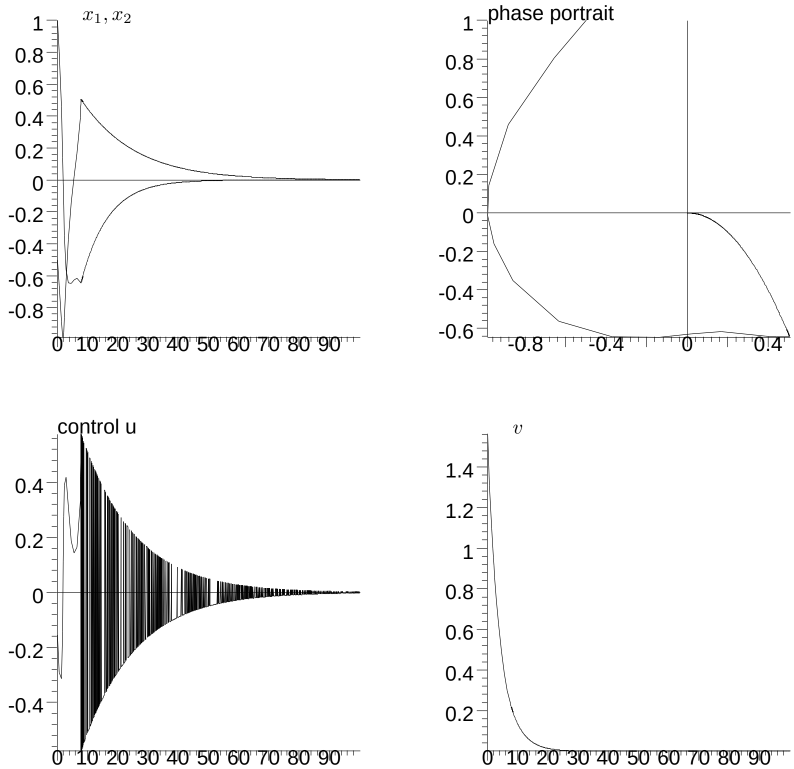

FIG. 3. *Asymptotic stabilization using a USFC, Example 4.5*

where

$$
\begin{aligned}
a(\xi) =& \{4\xi_3 - 3/2(\xi_1+\xi_2)|\xi_3|^{3/2}\}\{\xi_1^2 - \xi_2^2)\}, \\
b_1^+(\xi) =& 4.8\xi_1^3 - \mathrm{sign}(\xi_3)|\xi_3|^{3/2}, \\
b_2^+(\xi) =& 4\xi_2^3 - \mathrm{sign}(\xi_3)|\xi_3|^{3/2}, \\
b_1^-(\xi) =& 4\xi_1^3 - \mathrm{sign}(\xi_3)|\xi_3|^{3/2}, \\
b_2^-(\xi) =& 4.8\xi_1^3 - \mathrm{sign}(\xi_3)|\xi_3|^{3/2}.
\end{aligned}
\tag{20}
$$

It follows that $\dot{v}^+(\xi(t)) < 0$ when $b_1^+ = b_2^+ = 0$ and $x_3 > 0$ while $\dot{v}^-(\xi(t)) < 0$ when $b_1^- = b_2^- = 0$ and $x_3 < 0$. As in example 5.1 we set

$$
\begin{aligned}
w^+ =& |a| + k_1^+|b_1^+|^{\frac{4}{3}} + k_2^+|b_2^+|^{\frac{4}{3}}, \\
w^- =& |a| + k_1^-|b_1^-|^{\frac{4}{3}} + k_2^-|b_2^-|^{\frac{4}{3}}.
\end{aligned}
\tag{21}
$$

where $k_1^+, k_2^+, k_1^-, k_2^- > 0$ and it is straightforward to verify that the functions v^+, v^-, w^+, w^- are positive definite. We define a lower bounded Lyapunov pair (v, w) by setting $v = v^+,\ w = w^+$ where $\xi_3 > \sqrt{v^+}$ and $v = v^-,\ w = w^-$ where $\xi_3 < -\sqrt{v^-}$.

This leaves the case $\sqrt{v^+} \geq \xi_3 \geq -\sqrt{v^-}$ to consider. We first define the surface

$$S_0 = \{(1.1 + 0.1\xi_3)\xi_1^4 + (1.1 - 0.1\xi_3)\xi_2^4 - \xi_3(\xi_1 + \xi_2) + 2\xi_3^2 = 1\} \text{ where } -1 \leq \xi_3 \leq 1$$

and let v^0 be the unique continuous function homogeneous of degree 4 with respect to the dilation $(r = (1, 1, 2)$ which takes on the value 1 on S_0. We define w^0 similarly, and set $v = v^0,\ w = w^0$ in the case $\sqrt{v^+} \geq \xi_3 \geq -\sqrt{v^-}$. We note that the surface S_0 is transversal to $\mathbb{R}^2 \times \{0\}$, the set of possible control directions, one consequence of which is controls can be chosen to give v any desired rate of change in the region $M_0 = \{\sqrt{v^+} \geq \xi_3 \geq -\sqrt{v^-}\}$. Finally we let $b_i = b_i^+$ for $\xi_3 \geq 0$ and b_i^- otherwise, and define our controller as in the above example, namely

$$u_i^* = -\frac{w + a}{{b_1}^2 + {b^2}^2} b_i. \tag{22}$$

It follows directly that $\dot{v} = -w$ with u_i continuous everywhere. Then, for k_i sufficiently large, v is a LB-CLF and $u^* = (u_1^*, u_2^*)$ a USFC for v. Theorem 4.3 asserts that $u = u^*$ is an asymptotically stabilizing feedback controller for system (17).

REFERENCES

[1] Z. Artstein, *Stabilization with relaxed controls*, Nonlinear Analysis, Theory, Methods and Applications, 7(1983), pp. 1160-1173.

[2] A. Bacciotti and L. Rosier, *Lyapunov and Lagrange stability: Inverse theorems for discontinuous systems*, Math. Control Signals Systems, 11(1998), pp. 101-128.

[3] R. W. Brockett, *Pattern generation and the control of nonlinear systems*, IEEE Transactions on Automatic Control, 48(2003), pp. 1699-1711.

[4] S. Celikovsky and E. Aranda-Bricaire, *Constructive nonsmooth stabilization of triangular systems*, Syst. Contr. Lett., 36(1999), pp. 21–37.

[5] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic Publishers, Dordrecht, 1988.

[6] L. Grune, *Homogeneous state feedback stabilization of homogeneous systems*, SIAM Journal Control Optimization, 38(2000), pp. 1288-1308.

[7] V. Guillemin and A. Pollack, *Differential Topology*, Prentice Hall, Englewood Cliffs, 1974.

[8] E. P. Ryan, *On Brockett's condition for smooth stabilizability and its necessity in a contest of nonsmooth feedback*, SIAM Journal Control Optimization, 32(1994), pp. 1597-1604.

[9] I. W. Sandberg, *Global Inverse Function Theorems*, IEEE Transactions on Circuits and Systems, 27(1980), pp. 998-1004.

[10] D. Shevitz and B. Paden, *Lyapunov stability theory of nonsmooth systems*, IEEE Transactions on Automatic Control, 39(1994), pp. 1910-1914.

[11] G. Smirnov, *Stabilization by constrained controls*, SIAM Journal Control Optimization, 34(1996), pp. 1616-1649.

[12] G. Smirnov, *Introduction to the theory of differential inclusions*, Graduate Studies in Mathematics, 41(2002), American mathematics Society, Providence, Rhode Island.

[13] E. D. Sontag and H. J. Sussmann, *Nonsmooth Control-Lyapunov Functions*, Proc. IEEE Conf. Decision and Control, New Orleans, 1995, pp. 2799-2805.

[14] E. D. Sontag, *A Universal Construction of Artsein's Theorem on Nonlinear Stabilization*, Systems and Control Letters, 13(1989), pp. 117-123.