Communications in Information and Systems
Vol. 7, No. 2, pp. 111-132, 2007

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USING STOCHASTIC OPTIMIZATION METHODS FOR STOCK SELLING DECISION MAKING AND OPTION PRICING: NUMERICS AND BIAS AND VARIANCE DEPENDENT CONVERGENCE RATES

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Abstract. This paper is concerned with using stochastic approximation and optimization methods for stock liquidation decision making and option pricing. For stock liquidation problem, we present a class of stochastic recursive algorithms, and make comparisons of performances using stochastic approximation methods and that of certain commonly used heuristic methods, such as moving averaging method and moving maximum method. Stocks listed in NASDAQ are used for making the comparisons. For option pricing, we design stochastic optimization algorithms and present numerical experiments using data derived from Berkeley Options Data Base. An important problem in these studies concerns the rate of convergence taking into consideration of bias and noise variance. In an effort to ascertain the convergence rates incorporating the computational efforts, we use a Liapunov function approach to obtain the desired convergence rates. Variants of the algorithms are also suggested.

1. Introduction. The original motivation of stochastic approximation introduced by Robbins and Monro [13] is concerned with finding roots of a continuous function $f(\cdot)$, where either the precise form of the function is unknown, or it is too complicated to compute; the experimenter is able to take "noisy" measurements at desired values. A classical example is to find appropriate dosage level of a drug, provided only $f(x)$+noise is available, where x is the level of dosage and $f(x)$ is the probability of success (leading to the recovery of the patient) at dosage level x. The classical Kiefer–Wolfowitz (KW) algorithm introduced by Kiefer and Wolfowitz [6] concerns the minimization of a real-valued function using only noisy functional measurements. Tremendous progress has been made in the study of stochastic approximation methods for the past a half of century. The interesting theoretical issues in the analysis of iteratively defined stochastic processes and a wide variety of applications focus on the

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basic paradigm of stochastic difference equations. Simple statistical treatment has been substantially extended to accommodate complex stochastic dynamic systems. Emerging applications have arisen in queueing theory, manufacturing and production planning, adaptive control, signal processing, and wireless communications. A most up-to-date development on stochastic approximation can be found in the recent book [7] and references therein. This work continues our effort in using stochastic approximation and optimization methods to make decisions for financial engineering applications. Specifically, we focus our attention on stock selling decision making and option pricing.

Investing in a financial market requires constantly making decision on timing certain actions with respect to the market. Frequently, a question asked by a stock holder is: What is the "best" time to sell the stock? The action of selling is taken to realize one's return or to cut short one's loss. The term "best" here means either to take profit or to stop loss in order to maximize an expected return over time. Likewise, for portfolio manager involving options, a question of concern, for example, is: When should we exercise an American put option?

Since it is a decision making process involving randomness and uncertainty, stochastic control techniques naturally come into play. In [16], liquidation of a stock was formulated as an optimization problem. Assuming that the stock price is represented by a regime-switching geometric Brownian motion model, a diffusion process modulated by a continuous-time Markov chain with finite state space, optimal strategies are shown to be of threshold type. When the continuous-time Markov chain has only two states, the optimal selling decision can be made by solving a system of two-point boundary value problems. For Markov chains having more than two states, although it can still be demonstrated that the optimal strategy is of threshold type, an analytic solution may not be possible. Although the aforementioned reference provides insight into the structure of the optimal solution, in trading practice, a more systematic approach is much appreciated. Moreover, generally, one does not know the precise model; the return rate of the stock may be unknown, and the volatility is very likely to be stochastic. Calibration of the precise model may require sophisticated estimation and filtering techniques. Nevertheless, the stock price can be observed. In fact, to most of the investors, the stock price is the main or only available information to them. For example, plotting the daily closing prices of a stock, one traces out a curve. Figure 1 presents daily closes of Microsoft for a period of more than two years. The data were downloaded from Yahoo finance. One question of crucial importance is: Having observed stock closing prices, based on only such observations without knowing the exact model or precise parameter values, can we figure out the right time to liquidate a stock? Not only is it of practical value, but also it is an interesting

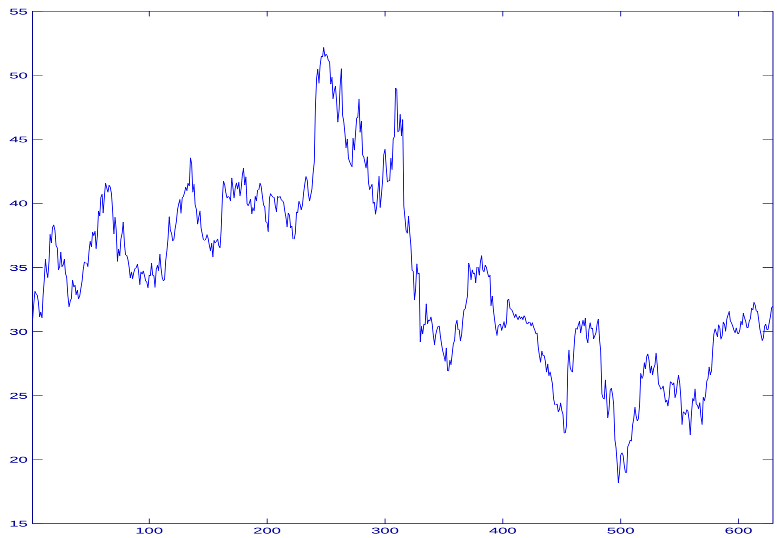


FIG. 1. *Daily closing prices of Microsoft (adjusted to stock splits and dividend payouts). Horizontal axis–days; Vertical axis–stock prices*

mathematical question.

Focusing on threshold-dependent decisions, a stochastic approximation method was developed in [14]. A class of easily implementable recursive algorithms was proposed; convergence of the algorithms was obtained via the limit ordinary differential equation of the interpolated sequence of the iterates; rates of convergence were ascertained using stationary covariance of associated diffusions together with scaling factors. Simulations and real data were also demonstrated. Subsequently, in [15], probability estimates of the iterates escaping from a neighborhood of the optimal threshold value were derived. Moreover, we used issues from NASDAQ 100 daily closes and intra-day data to demonstrate the effectiveness of our algorithms.

Pricing options has been a major research topic in financial engineering for years. There is a substantial research devoted to it; see for example, [1, 2, 5] and references therein. As is well known, pricing American put options can be formulated as an optimal stopping problem. A typical method of solution is a variational or quasi-variational inequality approach. Although such an approach provides theoretically interesting results, the computation required often deemed to be infeasible in real applications. Thus, taking such practical concern into consideration, Monte Carlo methods were suggested in [4, 10], and references therein. Nevertheless, the computation required is normally rather extensive; the number of simulation runs is often large (e.g., from 50000 to 100000 or even more). Designing more efficient procedures will be beneficial. Here, our approach is to use stochastic optimization methods to design recursive algorithms that can be easily implemented and that depends only on the closing pricing of the underlying stock up to current time. It requires neither calibration of system parameters nor estimation of states of the switching process. Although the result obtained via stochastic approximation may be only suboptimal, the method serves practical purpose well. These are distinct advantages of the SA methods. The rationale of our approach is to focus on the class of stopping rules depending on some threshold values. We do not attempt to solve the corresponding variational inequalities or partial differential equations; the underlying problem is treated parametrically. Stochastic recursive algorithms are developed to approximate the optimal parameter values. We demonstrate that the stochastic approximation and optimization approach provides an efficient and systematic computation scheme. In the proposed algorithm, since the noise varies much faster than that of the parameter, certain averaging takes place and the noise is averaged out resulting in a projected ordinary differential equation whose stationary point is the optimal parameter we are searching for. After establishing the convergence of the algorithm, we reveal how a suitably scaled and centered estimation error sequence evolve dynamically.

The rest of the paper is arranged as follows. Section 2 presents the stochastic approximation and optimization algorithms. Section 3 examines algorithms with constant stepsize and exploits the connections of bias, noise, and stepsizes. Section 4 is devoted to numerics of stock liquidation problems, whereas Section 5 is concerned with pricing options. we make comparisons of our stochastic approximation approach with heuristic approaches such as moving average and moving maximum are often used by practitioners. Section 5 presents numerical experiments using data derived from Berkeley Options Data Base by means of stochastic optimization methods. Section 6 is concerned with variants of the basic algorithm including soft constraints, and robust procedures. Finally, the paper is concluded with a few more remarks.

2. Recursive Algorithms.

This section is divided into two parts. The first part is concerned with stock liquidation, and the second part focuses on pricing American put options. Recursive algorithms of stochastic approximation type are provided.

2.1. Algorithms for Stock Liquidation.

We consider one stock at a time. Denote the stock price at time $t \in [0, \infty)$ by $S(t)$, and the observed stock prices at discrete time n by $S_n \in \mathbb{R}$. In what follows, n is used as the iteration number of the recursive algorithms. To avoid fast growth of the iterates of the stochastic approximation algorithm, we use the log price instead. That is, we use $X_n = \ln S_n$. We are concerned with threshold type of selling rule and use $\theta = (\theta^1, \theta^2)' \in \mathbb{R}^2$ to denote the threshold vector. Focusing on threshold-dependent stopping time, with the threshold value fixed at $\theta_n = (\theta_n^1, \theta_n^2)' \in \mathbb{R}^2$, we compute τ_n the first exit time of $X(t)$ from $I_{\theta_n} = (-\theta_n^1, \theta_n^2)$ (the interval with the lower and upper boundaries set at

$-\theta_n^1$ and θ_n^2, respectively) by

$$\tau_n = \inf\{t > 0 : X(t) \notin I_{\theta_n}\}. \tag{1}$$

We consider the expected profit

$$\phi(\theta) = E\Phi(\theta, \xi_n), \tag{2}$$

where Φ is a utility function and $\{\xi_n\}$ is a stationary sequence of random variables representing the observation noise and $\phi(\cdot)$ is a smooth function. Note that ξ_n is a combined process, which includes the random effects from $X(t)$ and the stopping time τ_n as

$$\xi_n = (X(\tau_n), \tau_n)'. \tag{3}$$

Our objective can be stated as: Choose θ so as to maximize the expected profit $\phi(\theta)$. [That is, the objective of choosing the right time τ is converted to choose the best θ so that $\phi(\theta)$ is maximized.] In [14], exponential type utility functions were used. Here, we do not specify utility functions. Thus, functions other than exponential utility may also be used if it is desired so.

We are facing a situation that the precise form of $\phi(\cdot)$ is unknown, only noise corrupted observations are available at any parameter value θ. To solve the problem, we use stochastic approximation/optimization methods to construct a sequence $\{\theta_n\}$ to approximate the optimum. Assuming $\phi(\cdot)$ to be a smooth function, the maximization problem is equivalent to finding the stationary points of $\phi(\cdot)$ (the points at which $\nabla\phi(\theta) = 0$). The precise value of $\phi(\cdot)$ is not available. Therefore, we replace the gradient of $\phi(\cdot)$ by its noisy finite difference estimate. Let $\{c_n\}$ be the finite difference interval, a sequence of positive real numbers satisfying $c_n \to 0$. Suppose that θ_n has been obtained. Define the finite difference approximation of the gradient as

$$\Delta\phi_n = \begin{pmatrix} \dfrac{\Phi(\theta_n + c_n e_1, \xi_n^+) - \Phi(\theta_n - c_n e_1, \xi_n^-)}{2c_n} \\ \dfrac{\Phi(\theta_n + c_n e_2, \xi_n^+) - \Phi(\theta_n - c_n e_2, \xi_n^-)}{2c_n} \end{pmatrix},$$

where $\{c_n\}$ is a sequence of positive real numbers tending to 0 as $n \to \infty$ representing the finite difference interval, e_1 and e_2 are two-dimensional standard unit vectors. In the above, central finite difference is used and $\{\xi_n^\pm\}$ are observation noises associated with the use of $\theta_n \pm c_n e_i$. The stochastic approximation/optimization algorithm is given by

$$\theta_{n+1} = \theta_n + \varepsilon_n \Delta\phi_n, \tag{4}$$

where $\{\varepsilon_n\}$ is a sequence of positive real numbers satisfying $\varepsilon_n \to 0$ as $n \to \infty$ and $\sum_n \varepsilon_n = \infty$. In order to avoid the iterates becoming unbounded, adjustments may be made through projection and/or truncations. In lieu of (4), we could consider

$$\theta_{n+1} = \Pi_H[\theta_n + \varepsilon_n \Delta\phi_n], \tag{5}$$

where Π_H is the projection onto the constraint set $H = \{\theta : \theta^{i,l} \le \theta^i \le \theta^{i,u}\}$, and $\theta^{i,l}$, $\theta^{i,u}$ are real numbers being the lower and upper bounds of θ^i. If the iterates are in the interior of H, we keep the values as they are. If the iterates ever escape from the bounded region, we project them back to the boundary of the respective intervals. That is, if $\theta_n^i > \theta^{i,u}$ (resp. $\theta_n^i < \theta^{i,l}$), we reset the iterate to $\theta^{i,u}$ (resp. $\theta^{i,l}$).

Using the stochastic recursive algorithm, we obtain the estimate of the optimal threshold vector $\theta_* = (\theta_*^1, \theta_*^2)$. This in turn, determines the threshold values

$$(L, U) = (S_0 \exp(-\theta_*^1), S_0 \exp(\theta_*^2)),$$

where S_0 is the purchase price of the stock. The selling strategy is: Sell the stock when the first time the price is at or below the lower bound L or the price is at or above the upper bound U. Concerning exponential type utility functions, convergence and rates of convergence of algorithms (4) and (5) were studied in [14]. Further large deviations type results were obtained in [15].

2.2. Algorithms for Pricing American Put Options. At any given time $t \ge 0$, the stock price $S(t)$ is available to the investors. Denote $X(t) = \ln S(t)$. We again focus on threshold-type solutions. Let ς be a stopping time depending on the threshold value defined by

$$\varsigma = \varsigma(\theta) = \inf\{t > 0 : X(t) \notin \Xi(\theta)\}, \tag{6}$$

where $\Xi(\theta) = (\theta, \infty)$. The reason to use the logarithm of the price instead of the price itself is to avoid the possible exponential growth and numerical errors. We aim at finding the optimal threshold level θ_* so that the expected return is maximized. The problem can be rewritten as:

$$\begin{cases} \text{Find argmax } \phi(\theta), \\ \text{subject to } \phi(\theta) = E \exp(-\mu\varsigma(\theta))(K - S(\varsigma(\theta)))^+. \end{cases} \tag{7}$$

An analytic solution is often difficult to obtain in general. Noting the dependence of the optimal solution on the threshold values in connection with option pricing, we focus on a class of stopping times that depends on a parameter θ and convert the problem to a stochastic optimization problem. The basic premise stems from a twist of the optimal stopping rules. The rational is to focus on the class of stopping times depending on threshold values in lieu of finding the optimal stopping time among all stopping rules. Another distinct feature is that our approach enables us to handle perpetual options as well as pricing options in a finite horizon.

Let $\mathcal{F}_t$ be the σ-algebra generated by $\{X(s) : s \le t\}$, the logarithm of the price, and let $\mathcal{A}_T$ be the class of $\mathcal{F}_t$-stopping times that are bounded by T, i.e., $\mathcal{A}_T = \{\varsigma : \varsigma$ is an $\mathcal{F}_t$-stopping time and $\varsigma \le T$ w.p.1$\}$, where T is the expiration date. [Note that the perpetual option corresponds to the case $T \to \infty$.] We propose a stochastic optimization procedure to find the optimal threshold value θ_* using the recursive algorithm

$$\theta_{n+1} = \theta_n + \{\text{stepsize}\} \cdot \{\text{noisy gradient estimate of } \phi(\theta_n)\},$$

where the stepsize is a decreasing sequence of real numbers. The stochastic approximation procedures to be developed depend on how the gradient estimates of $\phi(\theta)$ are constructed. We use finite-difference schemes. Let us now describe the procedure.

1. Initial estimate: Choose an arbitrary initial estimate θ_0, we then compute the exit time $\varsigma(\theta_0)$ of $X(t)$ defined by

$$\varsigma(\theta_0) = \inf\{t > 0 : X(t) \notin \Xi(\theta_0)\}.$$

Observe $\widetilde{\phi}(\theta_0) = \exp(-\mu\varsigma(\theta_0))\,(K - S(\varsigma(\theta_0)))^+$ (with observation noise $\chi(\theta_0, \zeta_0)$)

$$\widehat{\Psi}(\theta_0, \zeta_0) = \widetilde{\phi}(\theta_0) + \chi(\theta_0, \zeta_0). \tag{8}$$

Construct derivative estimate

$$\Delta\widehat{\phi}_0 = \frac{\widehat{\Psi}(\theta_0 + \delta_0, \zeta_0^+) - \widehat{\Psi}(\theta_0 - \delta_0, \zeta_0^-)}{2\delta_0}.$$

where $\zeta_0^{\pm}$ are two observation noises and the sequence of positive real numbers $\{\delta_n\}$ is the finite difference stepsize satisfying $\delta_n \to 0$.

2. Update: Next, we compute θ_1 by using the recursive algorithms

$$\theta_1 = \theta_0 + \varepsilon_0 \Delta\widehat{\phi}_0.$$

3. Induction: Suppose that θ_n has been obtained. We compute the exit time

$$\varsigma(\theta_n) = \inf\{t > 0 : X(t) \notin \Xi(\theta_n)\},$$

and observe $\widetilde{\phi}(\theta_n) = \exp(-\mu\varsigma(\theta_n))\,(K - S(\varsigma(\theta_n)))^+$ with the observation noise $\chi(\theta_n, \zeta_n)$. Let

$$\widehat{\Psi}(\theta_n, \zeta_n) = \widetilde{\phi}(\theta_n) + \chi(\theta_n, \zeta_n). \tag{9}$$

Construct derivative estimate

$$\Delta\widehat{\phi}_n = \frac{\widehat{\Psi}(\theta_n + \delta_n, \zeta_n^+) - \widehat{\Psi}(\theta_n - \delta_n, \zeta_n^-)}{2\delta_n}.$$

We obtain the next estimate θ_{n+1} by using a stochastic approximation algorithm

$$\theta_{n+1} = \theta_n + \varepsilon_n \Delta\widehat{\phi}_n. \tag{10}$$

To ensure the boundedness of the iterates, we use a projection algorithm:

$$\theta_{n+1} = \Pi_{[\theta_l,\theta_u]}[\theta_n + \varepsilon_n \Delta\widehat{\phi}_n], \tag{11}$$

where the projection operator Π is defined by

$$\Pi_{[\theta_l,\theta_u]}(\theta) = \begin{cases} \theta_l, & \text{if} \quad \theta < \theta_l; \\ \theta_u, & \text{if} \quad \theta > \theta_u; \\ \theta, & \text{otherwise}, \end{cases}$$

and θ_l and θ_u are the lower bound and upper bound, respectively. As in [7], we rewrite (11) as

$$\theta_{n+1} = \theta_n + \varepsilon_n \Delta\widehat{\phi}_n + \varepsilon_n Z_n,$$

where $\varepsilon_n Z_n = \theta_{n+1} - \theta_n - \varepsilon_n \Delta\widehat{\phi}_n$ is the quantity with the shortest Euclidean length needed to take $\theta_n + \varepsilon_n \Delta\widehat{\phi}_n$ back to the constraint set $[\theta_l, \theta_u]$ if it ever escapes from there. Under broad conditions, it can be shown that the above algorithm converges. We refer the reader to [9] for further details. We reiterate that unlike the Monte Carlo approach, the precise model need not be known. The recursion depends on the observed data only. More will be said in the numerical experiment section. Henceforth, for simplicity, we write $\Delta\phi_n$ in lieu of $\Delta\widehat{\phi}_n$.

3. Bias, Noise, and Convergence Rates. This section is devoted to rate of convergence study taking into consideration of bias, noise variance, and stepsize. Concerning stochastic optimization using sequential Monte Carlo methods, convergence rates were evaluated for a class of stochastic optimization algorithms with decreasing stepsizes in conjunction with computational budget. For simplicity of argument, we consider algorithms without projection. Projection and truncation algorithms can be handled in essentially the same way. Our attention is on classes of constant stepsize algorithms.

3.1. Rate of Convergence: Algorithms for Stock Liquidation. In (4), we replace ε_n by ε, where $\varepsilon > 0$ is a constant stepsize. It is known that constant stepsize algorithms have the ability to track slight variation of the true parameter, and are easily implementable. Therefore, in this section, we consider

$$\theta_{n+1} = \theta_n + \varepsilon \Delta\phi_n. \tag{12}$$

We pose some conditions and assumptions. They are not the most general ones available. Weaker conditions on the regularity are possible (see [15]). However, here our main objective is to present the functional dependence of the error estimates on bias, noise, and stepsize. Thus we choose to use simple setting without undue technical complexity.

To proceed, let $\mathcal{F}_n$ be the σ-algebra generated by past data up to n (i.e., $\mathcal{F}_n$ measures at least $\{\theta_0, \xi_j : j < n\}$). Define the bias and "variance" as

$$\begin{aligned} b_n &= E_n \Delta\phi_n - \nabla\phi(\theta_n), \\ \psi_n &= \Delta\phi_n - E_n\Delta\phi_n. \end{aligned} \tag{13}$$

In view of (12) and (13),

$$\theta_{n+1} = \theta_n + \varepsilon\nabla\phi(\theta_n) + \varepsilon b_n + \varepsilon\psi_n. \tag{14}$$

Throughout this section, we assume the following conditions. Henceforth, K denotes a generic positive constant, whose value may change for different appearances. However, note that in (A1), $\kappa = \kappa(\omega)$ depends on the underlying sample point ω.

(A1) There is a $\beta > 0$ such that $|b_n| \le \kappa\varepsilon^\beta$ w.p.1 and $E|b_n|^2 \le K\varepsilon^{2\beta}$.

(A2) There is a $\delta_\varepsilon > 0$ such that $E|\psi_n|^2 \le K\delta_\varepsilon^{-2}$ such that as $\varepsilon \to 0$, $\delta_\varepsilon \to 0$, but $\varepsilon/\delta_\varepsilon^2 \to 0$.

(A3) $\nabla\phi(\cdot)$ is continuous; the autonomous ordinary differential equation

$$\dot{\theta} = \nabla\phi(\theta) \tag{15}$$

has a unique solution for each initial condition; $\nabla\phi(\theta) = 0$ has a unique root θ_*; $\nabla\phi(\theta) = A(\theta - \theta_*) + O(|\theta - \theta_*|^2)$, where A is a 2×2 stable matrix (i.e., all of its eigenvalues belong to the left half of the complex plane).

Note that θ_* is the precise stationary point of the objective function $\phi(\cdot)$ we are searching for. The uniqueness of θ_* implies that the function $\phi(\cdot)$ has a unique stationary point. Assumption (A2) stems from the use of finite difference approximation of the gradient estimates. One can think of a finite difference approximation with stepsize δ_ε replaces δ_n. Then this condition says nothing more than that the noise variance will be proportional to δ_ε^{-2} To study the asymptotic behavior of the algorithm, we take a continuous-time interpolation defined by $\theta^\varepsilon(t) = \theta_n$ for $t \in [n\varepsilon, n\varepsilon + \varepsilon)$. Then $\theta^\varepsilon(\cdot)$ belongs to the space of functions that are right continuous, have left limits, endowed with the Skorohod topology.

THEOREM 3.1. *Under assumptions* (A1)–(A3), $\theta^\varepsilon(\cdot)$ *converges weakly to* $\theta(\cdot)$ *such that* $\theta(\cdot)$ *is the solution of* (15). *Furthermore, let* $t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$. *Then* $\theta^\varepsilon(\cdot + t_\varepsilon)$ *to* θ_* *weakly as* $\varepsilon \to 0$.

Proof. The proof is, in fact, simpler than that of [14] since the conditions there are weaker. We omit the details. □

Our main interest here is to establish the following results. It gives us a precise order estimate on the convergence rate with respect to the bias, noise, and stepsize.

THEOREM 3.2. *Assume* (A1)–(A3)*, and* $E|\theta_0|^2 < \infty$*. Then we have for sufficiently large* n*,*

$$E|\theta_n - \theta_*|^2 = O(\varepsilon) + O(\varepsilon^{1+2\beta}) + O(\varepsilon\delta_\varepsilon^{-2}). \tag{16}$$

Proof. Without loss of generality and for simplicity, assume $\theta_* = 0$. Define $V(\theta) = (1/2)|\theta|^2$. To proceed, we first show that

$$\sup_n EV(\theta_n) < \infty. \tag{17}$$

The w.p.1 convergence of $\theta_n \to 0$ and (17) enable us to conclude $EV(\theta_n) \to 0$ as $n \to \infty$. Next, we refine the order of magnitude estimate.

(i) Since θ_n is measurable with respect to the $\mathcal{F}_n$,

$$E_n\theta_n'\psi_n = \theta_n' E_n[\Delta\phi_n - E_n\Delta\phi_n] = 0. \tag{18}$$

For the remainder of the proof, we suppose that $n \geq n_0$ is large enough. Observe that

$$|\theta'(A\theta - \nabla\phi(\theta))| \leq |\theta|(|A||\theta| + K(1 + |\theta|)) \leq K(1 + V(\theta)),$$

for each θ and for some $K > 0$. In the above and henceforth, K is taken to be a generic positive constant, whose value may change for different usage. Using (14), (18), and (A1)–(A3), we obtain

$$\begin{aligned}
&E[V(\theta_{n+1}) - V(\theta_n)] \\
&\quad = E(E_nV(\theta_{n+1}) - V(\theta_n)) \\
&\quad = EE_nV_\theta'(\theta_n)(\theta_{n+1} - \theta_n) + \frac{1}{2}E|\theta_{n+1} - \theta_n|^2 \\
&\quad = \varepsilon E\theta_n'[A\theta_n + (\nabla\phi(\theta_n) - A\theta_n) + b_n + \psi_n] + \frac{1}{2}E|\theta_{n+1} - \theta_n|^2 \\
&\quad \leq -\varepsilon\lambda_0 EV(\theta_n) + K\varepsilon(1 + EV(\theta_n)) + \varepsilon E\theta_n' b_n \\
&\qquad + K\varepsilon^2 E\Big(|\nabla\phi(\theta_n)|^2 + |b_n|^2 + |\psi_n|^2\Big),
\end{aligned} \tag{19}$$

where λ_0 is a positive real number.

It is easily seen that

$$E|\nabla\phi(\theta_n)|^2 \leq K(1 + EV(\theta_n)). \tag{20}$$

We also have that by use of the familiar inequality $ab \leq (a^2 + b^2)/2$ for $a, b \in \mathbb{R}$,

$$\varepsilon E|\theta_n' b_n| \leq O(\varepsilon)EV(\theta_n) + O(\varepsilon^{1+2\beta}). \tag{21}$$

In addition,

$$\begin{aligned}
\varepsilon^2 E|b_n|^2 &\leq O(\varepsilon^{2+2\beta}), \\
\varepsilon^2 E|\psi_n|^2 &\leq O(\varepsilon^2\delta_\varepsilon^{-2}).
\end{aligned} \tag{22}$$

By virtue of (19)–(22), we obtain that

$$\begin{aligned} &EV(\theta_{n+1}) - EV(\theta_n) \\ &\leq -\lambda_0 \varepsilon EV(\theta_n) + O(\varepsilon)(1+EV(\theta_n)) + O(\varepsilon^{1+2\beta}) \\ &+O(\varepsilon^2)(1+EV(\theta_n)) + O(\varepsilon^{2+2\beta}) + O(\varepsilon^2)O(\delta_\varepsilon^{-2}). \end{aligned} \tag{23}$$

Moreover, there is a $\lambda_1 > 0$ such that $-\lambda_0 + O(\varepsilon) < -\lambda_1$. Then we obtain

$$\begin{aligned} E[V(\theta_{n+1}) - V(\theta_n)] &\leq -\lambda_1 \varepsilon EV(\theta_n) + O(\varepsilon)(1+EV(\theta_n)) \\ &\quad +O(\varepsilon^{1+2\beta}) + O(\varepsilon^2) + O(\varepsilon^{2+2\beta}) + O(\varepsilon^2)O(\delta_\varepsilon^{-2}). \end{aligned} \tag{24}$$

It follows that

$$\begin{aligned} EV(\theta_{n+1}) &\leq (1-\lambda_1\varepsilon)EV(\theta_n) + O(\varepsilon)(1+EV(\theta_n)) \\ &\quad +O(\varepsilon^{1+2\beta}) + O(\varepsilon^2) + O(\varepsilon^{2+2\beta}) + O(\varepsilon^2)O(\delta_\varepsilon^{-2}). \end{aligned}$$

Iterating on the above inequality leads to

$$\begin{aligned} EV(\theta_{n+1}) \leq (1-\lambda_1\varepsilon)^n EV(\theta_0) \; &+K\varepsilon \sum_{j=0}^{n} (1-\lambda_1\varepsilon)^{n-j} \\ &+K\varepsilon \sum_{j=0}^{n} (1-\lambda_1\varepsilon)^{n-j} EV(\theta_j) \\ &+K\varepsilon \sum_{j=0}^{n} (1-\lambda_1\varepsilon)^{n-j} O(\varepsilon^{2\beta}) \\ &+K\varepsilon^2 \sum_{j=0}^{n} (1-\lambda_1\varepsilon)^{n-j} \\ &+K\varepsilon^2 \sum_{j=0}^{n} (1-\lambda_1\varepsilon)^{n-j} O(\varepsilon^{2\beta}) \\ &+K\varepsilon^2 \sum_{j=0}^{n} (1-\lambda_1\varepsilon)^{n-j} O(\delta_\varepsilon^{-2}). \end{aligned} \tag{25}$$

An application of the Gronwall's inequality yields

$$EV(\theta_{n+1}) \leq K\varepsilon \sum_{j=0}^{n} (1-\lambda_1\varepsilon)^{n-j} = O(1).$$

Furthermore, the bound holds uniformly in n. That is, (17) holds. It then follows from the w.p.1 convergence of $\theta_n \to 0$ and (17), $EV(\theta_n) \to 0$ as $n \to \infty$.

(ii) Next, we refine the order estimates. In view of (A1)–(A3) and (i),

$$\begin{aligned} &E[\theta_n' \nabla\phi(\theta_n)] = E[\theta_n' A\theta_n] + o(EV(\theta_n)), \\ &E|\nabla\phi(\theta_n)|^2 \leq E[\theta_n' A' A\theta_n] + o(EV(\theta_n)). \end{aligned} \tag{26}$$

Then

$$|o(EV(\theta_n))| \leq \frac{1}{2}\lambda_0 EV(\theta_n).$$

Similar to (19),

$$
\begin{aligned}
&E[V(\theta_{n+1}) - V(\theta_n)] \\
&\quad = \varepsilon E\theta_n'[\nabla\phi(\theta_n) + b_n + \psi_n] + \frac{1}{2}E|\theta_{n+1} - \theta_n|^2 \\
&\quad \le -\varepsilon\lambda_1 EV(\theta_n) + \varepsilon E\theta_n' b_n + K\varepsilon^2 E\Big(|\nabla\phi(\theta_n)|^2 + |b_n|^2 + |\psi_n|^2\Big) \\
&\quad \le -\varepsilon\lambda_1 EV(\theta_n) + O(\varepsilon^2)(1 + EV(\theta_n)) + O(\varepsilon^{2+2\beta}) + O(\varepsilon^2\delta_\varepsilon^{-2}) \\
&\quad \le -\varepsilon\lambda_2 EV(\theta_n) + O(\varepsilon^2) + O(\varepsilon^{2+2\beta}) + O(\varepsilon^2\delta_\varepsilon^{-2}),
\end{aligned}
\tag{27}
$$

where $0 < \lambda_2 < \lambda_1 < \lambda_0$. Iterating on (27),

$$
\begin{aligned}
EV(\theta_{n+1}) &\le K\varepsilon \sum_{j=0}^{n} (1 - \lambda_2\varepsilon)^{n-j}[O(\varepsilon) + O(\varepsilon^{1+2\beta}) + O(\varepsilon\delta_\varepsilon^{-2})] \\
&\le O(\varepsilon) + O(\varepsilon^{1+2\beta}) + O(\varepsilon\delta_\varepsilon^{-2}).
\end{aligned}
$$

The desired result thus follows. □

3.2. Rate of Convergence: Algorithms for Pricing Options. Here all processes under consideration are real valued.

$$
\theta_{n+1} = \theta_n + \varepsilon\Delta\phi_n. \tag{28}
$$

Redefine $\mathcal{F}_n$ to be the σ-algebra generated by past data up to n (i.e., $\mathcal{F}_n$ measures at least $\{\theta_0, \zeta_j : j < n\}$). Redefine the bias and "variance" as

$$
\begin{aligned}
b_n &= E_n\Delta\phi_n - \nabla\phi(\theta_n), \\
\psi_n &= \Delta\phi_n - E_n\Delta\phi_n.
\end{aligned}
\tag{29}
$$

In view of (12) and (13),

$$
\theta_{n+1} = \theta_n + \varepsilon\nabla\phi(\theta_n) + \varepsilon b_n + \varepsilon\psi_n. \tag{30}
$$

Throughout this section, in lieu of conditions (A1)–(A3), we use the following conditions.

(A4) (A1)–(A3) hold with the following modifications: $\nabla\phi(\cdot)$ is continuous; the autonomous ordinary differential equation

$$
\dot\theta = \nabla\phi(\theta) \tag{31}
$$

has a unique solution for each initial condition; $\nabla\phi(\theta) = 0$ has a unique root θ_*; $\nabla\phi(\theta) = A(\theta - \theta_*) + O(|\theta - \theta_*|^2)$, where $A < 0$.

THEOREM 3.3. *Under* (A4)*, Theorems* 3.1 *and* 3.2 *continue to hold.*

The proof of Theorem 3.3 is similar to that of Theorems 3.1 and 3.2. We thus omit the verbatim argument.

4. Numerics.

4.1. Stock Liquidation: Comparisons of Stochastic Optimization and Moving Average Methods. Concerning liquidation of a stock, one of the approaches used by many investors in practice is a moving average method. One superimposes an m-day moving average on the same plot of the daily closes. The investor buys the stock as soon as its price rises above the m-day moving average, and sells it as soon as the stock price cross down the m-day moving average.

In our numerical study, we have chosen $m = 10$. That is, we examine the 10-day moving average to make a decision on if the stock should be sold. Then we compare the performance of the selling decisions based on the moving average method with the stochastic approximation method. Real data from NASDAQ 100 were used for the numerical comparison. We choose the period of comparison to be the beginning of 1999 to the end of 2000. Mainly, in this duration, the stock prices vary drastically. Many of the stocks soared in 1999 and took a downturn from the middle of 2000. Figure 2 shows the daily closing prices of Cisco from January 1999 to June 2001. The

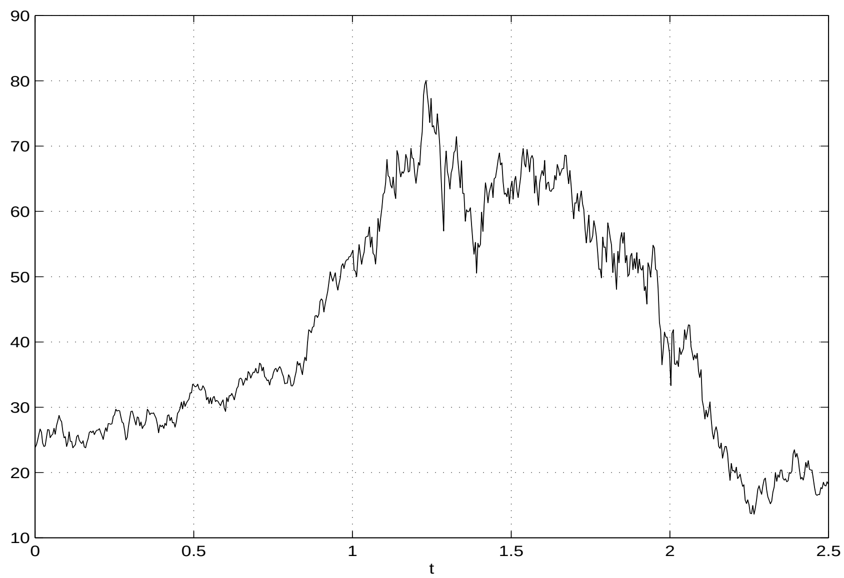


FIG. 2. *Closing Prices of Cisco: January 1999–June 2001. Horizontal axis–days; Vertical axis–stock prices*

up and down trends are clearly seen from the picture, and are rather pronounced.

To make comparisons of the stochastic approximation method with that of the moving averaging procedure, we use real data stocks from NASDAQ 100, downloaded from Yahoo finance web site. We used only 91 stocks from the list because a substantial portion of data are missing for the other 9 stocks. Adjusted daily closing prices are used taking into consideration of stock splits and dividend payments. The purchase date of each stock is determined by the moving averaging method. As soon as the stock price is higher than the past 10-day average, a buying decision is rendered. Then, to compare the liquidation rules, we first use the moving average method to determine the right time to sell a stock, and then we figure out the time to sell using stochastic approximation method for each of the 91 stocks individually. For comparison purposes, we only consider one transaction for each stock during three different periods (beginning of 1999, 2000 and middle of 2000). We will not consider short selling either. The percentage of return of each stock using each method is noted. Average percentage returns of the 91 stocks are computed, the total holding days are calculated. The computation and comparison results are depicted in Tables 1–3. In these tables, MA denotes the use of moving average method, whereas SA indicates stochastic approximation method. In these tables, average percentage return of the 91 stocks, the total number of holding days of 91 stocks, and average return per holding day are displayed.

Table 1

Comparisons of Stochastic Approximation and Moving Average Method for the Beginning of 1999 Period

Method	Average Return	Total # of Holding Days	Gain per Holding Day
SA	7.22%	4,575	0.0016%
MA	0.40%	747	0.0005%

Table 2

Comparisons of Stochastic Approximation and Moving Average Method for the Beginning of 2000 Period

Method	Average Return	Total # of Holding Days	Gain per Holding Day
SA	14.05%	4,323	0.0033%
MA	1.35%	989	0.0014%

Table 3

Comparisons of Stochastic Approximation and Moving Average Method for the Middle of 2000 Period

Method	Average Return	Total # of Holding Days	Gain per Holding Day
SA	−0.30%	3,994	−0.0001%
MA	−2.29%	778	−0.0029%

We make the following observations regarding the comparisons.

(a) The stochastic approximation method and the moving averaging method generate different selling dates.

(b) Using stochastic approximation method, the holding time appears to be much longer than that of the moving average counter part.

(c) Overall, stochastic approximation method performs much better. Even in the bear market case, the stochastic approximation method still perform better than the moving averaging method.

(d) Since the holding time for each method is different, a fair comparison involves figure out the gain or loss per holding day. So we computed this for each of the three period of testing. Again, the stochastic approximation method provides much better return.

4.2. Stock Liquidation: Comparisons of Stochastic Optimization and Moving Maximum Methods. This part is inspired by an idea of O'Neil [11]. Starting from a certain day, we check the price of each stock, and make a purchase decision if it is greater than its maximal price of the last 20 days. Once the stock is bought, we use both stochastic approximation methods and experiential methods to determine the optimal threshold values. Then comparisons are made for selling decision using stochastic approximation methods and that of heuristically selected upper and lower bounds for profit and stop-loss limits. Denote the closing price of stock i at the time of sale by S_i and its initial price by $S_i(0)$. Let h_i be the number of holding days for stock i. To make fair comparisons, we examine several criteria. They include average (a) profit or loss per stock (obtained by averaging the gain of loss over 82 participating stocks), (b) average rate of return $([\sum_{i=1}^{82}(S_i - S_i(0))]/[\sum_{i=1}^{82} S_i(0)])$, (c) average profit or loss per holding day $([\sum_{i=1}^{82}(S_i - S_i(0))]/\sum_{i=1}^{82} h_i)$, (d) average rate of return per holding day (average rate of return / average number of holding days). Using historical data, O'Neil argued that successful stocks after breaking out, tend to move up 20% to 25% and then build new base from there, and in some cases resume their advance. Thus, he suggested that one should sell a stock when it has already increased by 20% and avoid getting caught in the 20% to 40% corrections, which occur periodically in market leaders. On the other hand, if the stock decline below their purchase prices by, e.g. 8%, they should be sold and the loss is taken. These are mainly heuristic reasonings, but the strategies suggested have been used by investors. Here we suggest an alternative that is a systematic scheme.

Consider the period of January 1999 to June 2001. Taking averages of the 82 stocks selected, we obtain the following plot in Figure 3. During this period, the stock market experienced a huge and long up turn and then suffered a sharp down turn, which serves our purpose of testing well. Then, we picked out four periods for comparisons with starting date October 1, 1999, January 3, 2000, June 1, 2000, and February 1, 2001, respectively. For example, suppose that we started on October 1, 1999. by performing the moving maximum purchasing strategy, we would have

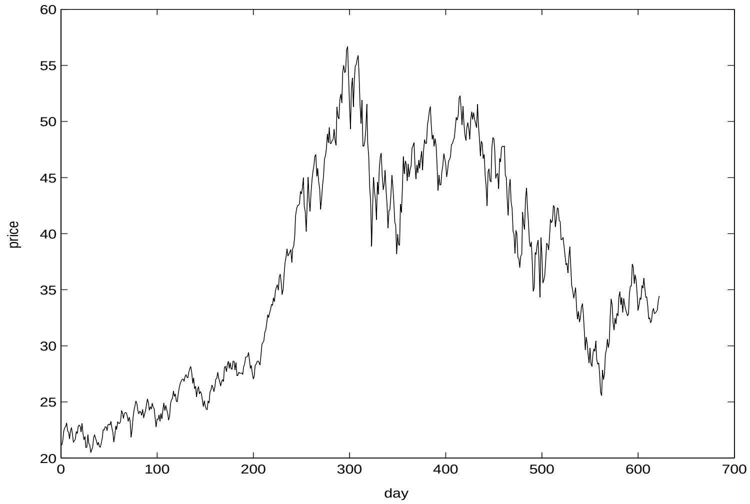


FIG. 3. *Average daily closing prices of 82 NASDAQ 100 stocks from Jan. 4 1999 to June 29, 2001*

bought the stock of Adobe Systems Inc. (ADBE) on October 4, 1999 with a price of \$29.22; see Figure 4 Using stochastic approximation methods, we obtain the cut-loss

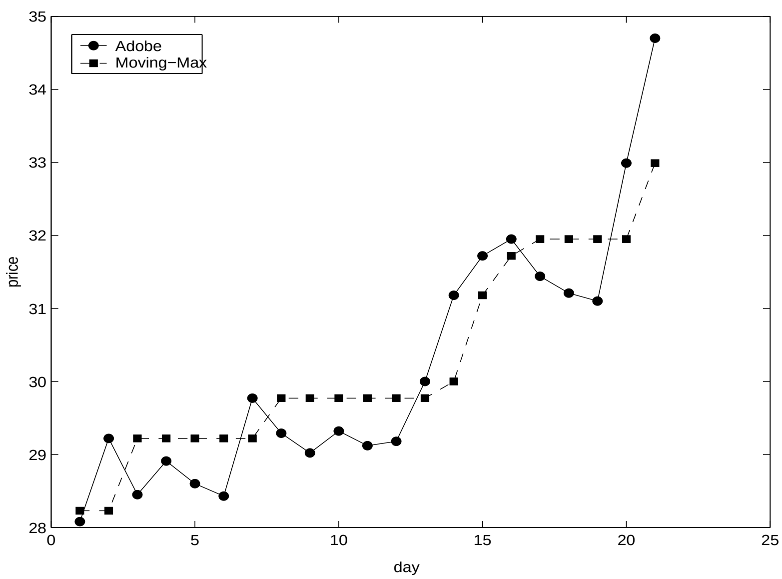


FIG. 4. *Stock prices of and moving-maxima of Adobe from October 1 1999 to October 29, 1999*

and profit thresholds should be $[23.92, 37.01]$. By observing the closing price, the first time that price of the stock is outside the above interval would be Nov. 12, 1999 with a closing price of \$37.27. The resulting profit would be \$8.05 with the total number of holding days being 29. These results are summarized in Tables 4–7. They show that the SA approach outperforms the moving average method.

TABLE 4

Comparisons of Stochastic Approximation and Moving Maximum Method for the Period Starting on October 1, 1999. Gain:=profit/loss $ amount, Rate:=average rate of return, Gain/Day:=average profit/loss per holding day, Rate/Day:=average daily return rate per holding day, # of Days:=average number of holding days

Strategies	Gain	Rate	Gain/Day	Rate/Day	# of Days
EM (−7%, 12%)	1.21	4.14%	0.10	0.36%	11.5
EM (−7%, 20%)	2.41	8.23%	0.13	0.45%	18.4
EM (−15%, 20%)	5.15	17.58%	0.18	0.62%	28.2
SA	5.74	19.61%	0.22	0.74%	26.4

TABLE 5

Comparisons of Stochastic Approximation and Moving Maximum Method for the Period Starting on January 3, 2000. Gain:=profit/loss $ amount, Rate:=average rate of return, Gain/Day:=average profit/loss per holding day, Rate/Day:=average daily return rate per holding day, # of Days:=average number of holding days

Strategies	Gain	Rate	Gain/Day	Rate/Day	# of Days
EM (−7%, 12%)	−1.08	−2.33%	−0.15	−0.32%	7.4
EM (−7%, 20%)	−0.60	−1.28%	−0.06	−0.12%	10.7
EM (−15%, 20%)	1.50	3.24%	0.07	0.14%	22.4
SA	3.58	7.71%	0.11	0.25%	31.4

TABLE 6

Comparisons of Stochastic Approximation and Moving Maximum Method for the Period Starting on June 1, 2000. Gain:=profit/loss $ amount, Rate:=average rate of return, Gain/Day:=average profit/loss per holding day, Rate/Day:=average daily return rate per holding day, # of Days:=average number of holding days

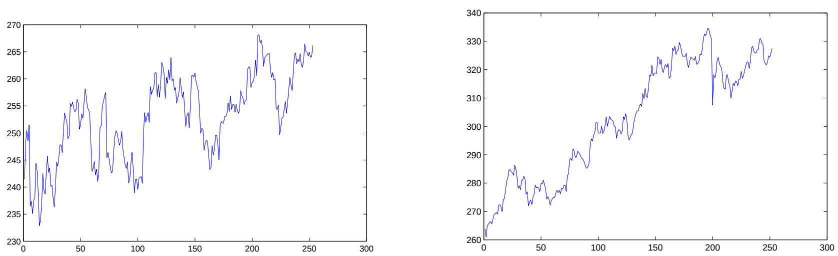
Strategies	Gain	Rate	Gain/Day	Rate/Day	# of Days
EM (−7%, 12%)	0.46	0.97%	0.04	0.09%	11.2
EM (−7%, 20%)	0.55	1.18%	0.03	0.07%	18.0
EM (−15%, 20%)	0.57	1.22%	0.02	0.04%	32.9
SA	1.08	2.30%	0.03	0.07%	34.3

4.3. Pricing American Put Options. Here our numerical experiments were done using the data derived from Berkeley Options Data Base. First, to get the feeling of the stock performance, let us plot the following two figures, which are the stock

TABLE 7

Comparisons of Stochastic Approximation and Moving Maximum Method for the Period Starting on February 1, 2001. Gain:=profit/loss $ amount, Rate:=average rate of return, Gain/Day:=average profit/loss per holding day, Rate/Day:=average daily return rate per holding day, # of Days:=average number of holding days

Strategies	Gain	Rate	Gain/Day	Rate/Day	# of Days
EM (−7%, 12%)	−0.94	−2.83%	−0.10	−0.30%	9.5
EM (−7%, 20%)	−0.38	−1.14%	−0.03	−0.09%	12.3
EM (−15%, 20%)	−0.28	−0.84%	−0.01	−0.03%	26.6
SA	1.94	5.87%	0.10	0.30%	19.3



(a) Stock price vs time: 1988 data

(b) Stock price vs time: 1989 data

FIG. 5. *Stock closing prices of 1988 and 1989*

prices in the years of 1988 and 1989 in Figure 5. As can be seen that 1988 was a more volatile year, whereas in 1989, although there were ups and downs, the stock price seems to have increasing tendency. From the data of the stock prices, we use stochastic approximation algorithms to price American put options.

Using the real data, we use both the stochastic approximation (SA) method and the well-known binomial (BIN) method. The programs are written using C++ to compare the performance of the algorithms. For SA, we use $\theta_0 = \log(250)$, and stepsizes $\varepsilon_n = 1/n$ and $\delta_n = 1/n^{\frac{1}{6}}$, respectively. For BIN, the time to maturity is divided into 2000 periods. The results are displayed in Table 8.

It is easily seen that the results obtained by the two methods are comparable. A desk top computer with a Pentium 4 processor (2793 Mhz) was used for the computations. The CPU time for BIN is of the order $O(10^{-2})$ seconds, whereas for SA, is of the order $O(10^{-3})$ seconds. Although only two year's data have been examined, the results are promising. It indicates that the stochastic approximation algorithm is easily implementable. It gives us a good insight on how we would proceed in general.

We remark that the data we examined tend out to be in the "early" years of option trading. In addition, the strike prices in the data were mostly higher than the

TABLE 8

Comparisons of SA and BIN for American put options

Date	Initial Price	Expiration Date	Strike Price	Volatility	Interest Rate	SA	BIN
01/05/1988	253.51	02/20/1988	290.00	0.2238	0.0581	36.49	36.49
03/02/1988	256.56	04/16/1988	255.00	0.2805	0.0570	8.52	8.55
04/22/1988	246.57	05/21/1988	270.00	0.2128	0.0591	23.43	23.45
05/05/1988	245.00	05/21/1988	260.00	0.1801	0.0626	15.00	15.00
07/22/1988	251.09	08/20/1988	275.00	0.2088	0.0673	23.91	23.91
04/21/1989	288.87	04/22/1989	290.00	0.0639	0.0865	1.13	1.15
05/04/1989	288.26	05/20/1989	300.00	0.1475	0.0843	11.74	11.77
11/06/1989	311.04	11/18/1989	330.00	0.2018	0.0769	18.96	18.97
12/15/1989	329.06	12/16/1989	330.00	0.0488	0.0763	0.94	0.96

initial stock prices, and there was a tendency of stock price increase in the two year period. It would be nice if we could get more numerical experiments on different real data sets in order to gain further insight. Furthermore, it would certainly be desirable to examine option data of more recent years.

5. Further Remarks. In addition to the projection algorithms studied, we may consider two variants of the algorithms. The first one is the so-called soft constraint algorithm, and the second one is a robust algorithm. We describe them below.

5.1. Soft Constraints. The idea of soft constraints is that these constraints may be violated but cannot be violated too much. Take for instance, the soft constraint to be the circle in two-dimensional Euclidean space centered at the origin with radius r_0. $B_0 = \{\theta : |\theta| \le r_0\}$. Define $c(\theta) = d(\theta, B_0)$, the distance from θ to the sphere B_0. Thus

$$c(\theta) = \begin{cases} (|\theta| - r_0)^2 & |\theta| \ge r_0, \\ 0, & \text{otherwise.} \end{cases}$$

The soft constraint algorithm can be written as

$$\theta_{n+1} = \theta_n + \varepsilon_n D\phi_n - \varepsilon_n K_0 \nabla c(\theta_n), \tag{32}$$

where K_0 is a sufficiently large positive number. Note that in this case, the mean limit ordinary differential equation becomes

$$\dot{\theta} = \nabla\phi(\theta) - K_0 \nabla c(\theta).$$

5.2. Robust Algorithm. The motivation comes from a work of [12]. Since the actual dynamics might be hardly known, one may use the idea in robust statistical analysis. Let $\widetilde{B}_i(\cdot)$, $i = 1, 2$ be bounded real-valued functions. In lieu of (4), consider

$$\theta_{n+1} = \theta_n + \varepsilon_n \widetilde{B}(D\phi_n), \tag{33}$$

where $\widetilde{B}(\theta) = (\widetilde{B}_1(\theta^1), \widetilde{B}_2(\theta^2))'$. The function $\widetilde{B}_i(\cdot)$ are monotonically nondecreasing satisfying $\widetilde{B}_i(0) = 0$, $\widetilde{B}_i(u) = -\widetilde{B}_i(u)$, and $\widetilde{B}_i(u)/u \to 0$ as $u \to \infty$. A commonly used function is $\widetilde{B}_i(u) = \min(K_i, u)$ for $u \geq 0$, where K_i are given constants.

5.3. Concluding Remarks. In this paper, stochastic approximation and optimization algorithms are considered for selling decision making of a stock. Comparisons of stochastic approximation methods with heuristic moving average methods are made. Then convergence rates are studied by taking into consideration of bias, noise variance, and stepsize of the algorithm. So far, the liquidation of a stock is done in the following fashion. When one decides to sell it, the entire collection of the stock will be sold. A worthwhile undertaking is to consider situations that only part of the stock shares is sold and develop the corresponding stochastic approximation algorithms. Recent advances in fractional Brownian motion [3] open up new research avenues; designing the associated stochastic optimization algorithms deserves further study and investigation.

Acknowledgement. We are very grateful to Dr. Reza Kamaly and Necessity and Chance LLC for providing us with data derived from Berkeley Options Data Base. This research was supported in part by the National Science Foundation under grant DMS-0603287, and in part by the National Security Agency under grant MSPF-068-029.

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