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MULTIDIMENSIONAL SCHUR COEFFICIENTS AND BIBO STABILITY*

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Abstract. In the framework of BIBO stability tests for one-dimensional (1-D) linear systems, the Schur-Cohn stability test has the appealing property of being a recursive algorithm. This is a consequence of the simultaneously algebric and analytic aspect of the Schur coefficients, which can be also regarded as *reflection coefficients.* In the multidimensional setting, this dual aspect gives rise to two extension of the Schur coefficients that are no longer equivalent. This paper presents the two extensions of the Schur-Cohn stability test that derive from these extended Schur coefficients. The reflection-coefficient approach was recently proposed in the 2-D case as a necessary but non sufficient condition of stability. The Schur-type multidimensional approach provides a stronger condition of stability, which is necessary and sufficient condition of stability for multidimensional linear system. This extension is based on so-called slice function associated to n-variable analytic functions. Several examples are given to illustrate this approach.

1. Introduction. This paper presents some recent advances in the problem of the BIBO stability of multidimensional linear systems. The extension to the multidimensional case of the one-dimensional BIBO stability theory encounters several major difficulties, of both algebraic and analytic nature. One such difficulty is the lack, in several dimensions, of a canonical analogous of the Schur coefficients, which are crucial in characterizing the one-dimensional BIBO stability.

The remarkable fact about the Schur coefficients in the one-dimensional setting is that they have simultaneously an algebraic and an analytic content. As explained in [11], the algebraic feature of the Schur coefficients appears through their occurrence in the Levinson algorithm as reflection coefficients, in solving the Yule-Walker equations for an AR model, while the analytic feature comes directly from the original Schur algorithm, which characterizes bounded analytic functions in the unit disk. Among many applications that take benefits from this double aspect of the Schur coefficients, the BIBO stability of a one-dimensional linear system with rational transfer function can be tested through a finite iterative –thus "computable" - algorithm, known as the Schur-Cohn stability test.

Unfortunately, when trying to extend the theory to the multidimensional case, it seems that the algebraic aspect and the analytic aspect are no longer sides of the same object. Two very natural extensions, on one side of the reflection coefficients, and on the other side of the Schur coefficients, become different notions, with different impact on BIBO stability.

The aim of the paper is to present two extensions of the Schur Cohn criterion that

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derive from these two viewpoints. On one side, a reflection coefficient-type extension, based on the prediction-theoretical setting for causal quarter-plane AR models, yields a necessary stability condition, which is "computable", but not sufficient. On the other side, a Schur coefficient-type extension, based on *slice functions* [16], produces a necessary and sufficient condition, but which is not algebraically. However, these latter coefficients were proved to have a continuity property that makes them computable by approximations.

An n-dimensional linear system with analytic transfer function $H : \mathbf{D}^n \to \mathbf{C}$, $H(z) = \sum_{\alpha \in \mathbf{N}^n} c_\alpha z^\alpha$ is called BIBO-stable if the Taylor coefficients of H belong to $l^1(\mathbf{N}^n)$, i.e. $\sum_{\alpha \in \mathbf{N}^n} |c_\alpha| < \infty$. If the transfer function H is rational, so $H(z) = \frac{B(z)}{A(z)}$ for two mutually prime polynomials, then the BIBO stability is implied by the fact that the denominator A has no zeroes in the closed unit polydisk $\bar{\mathbf{D}}^n$[8]. The converse is not true in general, since H might have removable singularities on $\mathbf{T}^n$ and still being stable, as shown in [8]. However, as long as H has no such removable singularities, the BIBO stability is equivalent to non-canceling of the denominator A in $\bar{\mathbf{D}}^n$. This happens in particular for autoregressive filters, for which the transfer function has the special form $H(z) = \frac{1}{A(z)}$. Therefore checking the BIBO stability is reduced for these systems to a zero localization problem that we will consider in the following.

2. BIBO stability conditions for 2-D AR models. This sections presents the first approach announced, i.e. the multidimensional Schur-Cohn type extension based on the lattice representation.

This work has been mainly inspired by the work performed with T. Kailath and his co-workers ([12],[13],[15],[20]). They raised the nice properties of the lattice representation and its impact in many areas from communications to speech analysis and processing.

In such a way one of the co-authors has extended various adaptive algorithms from 1 to 2-D case deriving lattice representation for 2-D random field and proposing recursive estimation of the 2-D reflection [4][14][21] [22]. Then arise naturally the idea to use these parameters to extend the Schur Cohn stability test to the 2-D case.

We consider a two-dimensional random field $\{y(m,n)\}$ satisfying a causal quarter-plane AR model of order (M, N). This model is described by the difference equation

$$y(i,j) = -\sum_{\substack{m=0 \\ (m,n)\neq(0,0)}}^{M} \sum_{n=0}^{N} a_{mn} y(i-m, j-n) + e(i,j) \tag{2.1}$$

where $\{a_{m,n}\}$ are the transversal AR parameter coefficients.

The considered AR process can be regarded as the output of a 2-D synthesis filter, with the transfer function

$$H(z_1,z_2) = \frac{1}{A(z_1,z_2)} \tag{2.2}$$

with

$$A(z_1,z_2) = 1 + \sum_{m=0}^{M}\sum_{n=0}^{N} a_{m,n} z_1^m z_2^m. \tag{2.3}$$

In this case, the BIBO stability conditions of the synthesis filter is that [19]

$$A(z_1,z_2) \neq 0,\ |z_1| \leq 1 \text{ and } |z_2| \leq 1. \tag{2.4}$$

The idea of the Levinson-type extension, that the authors recently proposed in [2], is based on a lexicographic image scanning and geometrical linear prediction approach. This method uses the two-dimensional lattice representation of the considered AR process which can be expressed as follows for $1 \leq m \leq M$ [14]

$$\begin{cases} E_m^f(s) = E_{m-1}^f(s-1) - E_{m-1}^b(s-1)\,K_m^f(s), \\ E_m^b(s) = E_{m-1}^b(s-1) - E_{m-1}^f(s)\,K_m^b(s) \end{cases} \tag{2.5}$$

where s is the lexicographic scanning index, $E_m^f(s)$, $E_{m-1}^b(s)$ are respectively the forward and backward prediction errors , and K_m^f and K_m^b are the reflection coefficient matrices, as the natural extension of the 1-D case. Based on this 2-D lattice representation, the following recursions relations for the errors correlations matrices are obtained:

$$\begin{cases} E_m^f = E_{m-1}^f - E_{m-1}^b K_m^f, \\ E_m^b = E_{m-1}^b - E_{m-1}^f K_m^b. \end{cases} \tag{2.6}$$

According to the recursion relations in (29), it is shown in [2] that the BIBO stability of 2-D synthesis filter associated to the 2-D QP AR model implies the two following conditions:

The matrices $K_m^f K_m^b, m = 1, \ldots, M$ have eigenvalues strictly less than 1.

The one variable polynomial $A^s(z)$ satisfies the condition $A^s(z) \neq 0$ for $|z| < 1$, where

$$A^s(z) = 1 - \sum_{\mu=1}^{N} k_{\mu,N+1}^{f,0}(s)\, z^{\mu} \tag{2.7}$$

and $k_{\mu,j}^{f,0}$, are the elements of the reflection coefficient matrix K_0^f.

The first condition expressed in terms of the reflection coefficients matrices can be interpreted as an extension of the 1-D Schur–Cohn test, .i.e. $K_m^b \equiv K_m^b \equiv k_m$, which is performed based on the magnitude of the reflection coefficients. In fact, the 1-D AR filters is stable if and only if all these reflection coefficients are of magnitudes strictly less than unity.

This approach provides some advantage for 2-D signal analysis. Indeed, checking the BIBO stability conditions becomes a one variable stability problem and classical numerical matrix computation can be used. However, this 2-D stability conditions is necessary but not sufficient. The reason is that the 1-D Schur (i.e. reflection) coefficients have a natural non-linear behavior that cannot be entirely reproduced, in several variables, by essentially linear objects such as the reflection coefficients matrices. In order to capture this non-linearity, one needs extensions of the Schur coefficients that carry more information. Such an extension is presented in the next section, yielding necessary and sufficient conditions of stability.

3. Functional Schur coefficients and stability. In this section we show how one can use slice functions in order to define the notion of Schur coefficient in several variables.

We denote by $\mathbf{D}$ the open unit disk and by $\mathbf{T} = \partial\bar{\mathbf{D}}$ the closed unit circle in the complex plane $\mathbf{C}$. The notation $H^\infty(\mathbf{D})$ stands for the Banach algebra of all bounded analytic functions in $\mathbf{D}$. We also denote by S the closed unit ball of $H^\infty(\mathbf{D})$.

We recall the main object of the (1-D) Schur algorithm, i.e. the mapping$\Phi : S \to S$ defined by

$$[\Phi(f)](z) = \begin{cases} \frac{f(z)-f(0)}{z(1-\overline{f(0)}f(z))} & z \neq 0 \\ f'(0)(1-|f(0)|^2)^{-1} & z = 0. \end{cases} \tag{3.1}$$

If f is not a unimodular constant function, and $\Phi(f) = 0$ otherwise (see [3]). For any function f in the Schur class we call *the Schur iterates* of f the sequence of functions

$$f_n \stackrel{d}{=} \Phi^{(n)}(f) = \underbrace{\Phi \circ \Phi \circ \cdots \circ \Phi}_{n}(f) \quad (n \geq 0), \tag{3.2}$$

and we call the numbers $\gamma_n \stackrel{d}{=} f_n(0)$ *the Schur parameters* of f.

For any polynomial $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ of degree n with complex coefficients, define the transpose polynomial $P^T(z) = \bar{a}_n + \bar{a}_{n-1} z + \ldots + \bar{a}_0 z^n$. The well known Schur-Cohn criterion [10] can be stated in the following equivalent form: *the polynomial P has no zeroes in the closed unit disk if and only if the Schur parameters $\gamma_k (k \geq 0)$ of the function $f = \frac{P^T}{P}$ satisfy $|\gamma_k| < 1$ for $k = 0, 1, \ldots, n-1$.*

The connection between the Schur parameters γ_k defined above and the Hermitian Schur-Cohn matrix of the polynomial P is the that the values $1 - |\gamma_k| \quad (0 \leq k \leq p-1)$ are exactly the determinants of the principal minors in the Schur-Cohn matrix of P.

Consider now the case of an n-variable polynomial P for which we want to check if it has zeroes in the closed unit polydisk $\bar{\mathbf{D}}^n$. Of course a "brutal" method would be to check all the points of $\bar{\mathbf{D}}^n$, which would take a very long time (and, fortunately, is not necessary). One would prefer instead to look for zeroes in subsets $\Sigma \subset \overline{\mathbf{D}}^n$ that are as small as possible. Then raises the following question: are there subsets $\Sigma \subset \overline{\mathbf{D}}^n$ which contain at least one zero for any polynomial P that has zeroes in $\bar{\mathbf{D}}^n$?

How small are they?

It follows from a general result on the homotopy structure of $\mathbf{T}^n$ (Rudin [16], Theorem 4.7.2.), that such a set is the "diagonal set" of $\bar{\mathbf{D}}^n$:

$$\Sigma = \{(z_1, \dots, z_n) \in \overline{\mathbf{D}}^n : |z_1| = |z_2| = \dots = |z_n|\}$$

which is shown [16] to contain at least one zero not only for any polynomial P that has zeroes in $\bar{\mathbf{D}}^n$, but also for any function f which is analytic in $\mathbf{D}^n$ and continuous on $\bar{\mathbf{D}}^n$ (called a disk algebra function) that has zeroes in $\bar{\mathbf{D}}^n$.

To have an idea about the "size" of Σ, observe that Σ can be parameterized via a set of n parameters $\lambda, w_1, \dots, w_{n-1}$ with $\lambda \in \overline{\mathbf{D}}$ and$w_1, \dots w_{n-1} \in \mathbf{T}$, by the transform

$$(z_1, \dots, z_n) = (\lambda w_1, \dots \lambda w_{n-1}, \lambda),$$

so Σ has (real) dimension $n{+}1$ compared to $\bar{\mathbf{D}}^n$ which has (real) dimension $2n$.

One can see that for any fixed vector $w = (w_1, \dots w_{n-1}) \in \mathbf{T}^{n-1}$, the set $D_w = \{(\lambda w_1, \dots \lambda w_{n-1}, \lambda) : \lambda \in \overline{\mathbf{D}}\}$ is a one (complex) dimensional disk in $\bar{\mathbf{D}}^n$ that "slices" $\bar{\mathbf{D}}^n$ through the origin and the point $(w_1, \dots w_{n-1}, 1) \in \mathbf{T}^n$. Two such disks have only the origin in common and the union of all these disks is Σ.

Now let $f : \mathbf{D}^n \to \mathbf{C}$ be an n-variable analytic function and fix $w = (w_1, \dots, w_{n-1}) \in \mathbf{T}^{n-1}$. The restriction of f to the disk D_w can be regarded as the one complex variable analytic function $f_w : \mathbf{D} \to \mathbf{C}$ defined by:

$$f_w(\lambda) = f(\lambda w_1, \dots, \lambda w_{n-1}, \lambda) \quad (\lambda \in \mathbf{D}), \tag{3.3}$$

called [16] *the (diagonal) slice of f through w.*

Consequently an algebra disk function f has no zeroes in $\bar{\mathbf{D}}^n$ if and only if none of the slices f_w has zeroes in $\overline{\mathbf{D}}$. Since these slices belong to the Schur class provided that f is in the closed unit ball of $H^\infty(\mathbf{D}^n)$, it is natural to consider the sequences of their Schur coefficients

$$\gamma_k(w) = \Phi^{(k)}(f_w)(0) \quad (k \geq 0, w \in \mathbf{T}^{n-1})$$

as functions of $w = (w_1, \dots, w_{n-1})$ defined on $\mathbf{T}^{n-1}$. We called these functions *the functional Schur coefficients* of the function f.

It is shown in [18] that the properties of these coefficients can be used to derive a Schur algorithm in several variables. More precisely, it is shown that a rational inner function belongs to the polydisk algebra if and only if its functional Schur coefficients are all continuous, if and only if its denominator has no zeroes in the closed unit polydisk.

The proof relies upon two facts. The first one is the weak continuity of the map that carries a point w on the polytorus into the slice f_w of $f$through w, and the continuity of the mapping Φ that defines the Schur coefficients. The second is a homotopic-type relationship between slice functions and the so called *determinig sets* [16], i.e. subsets of the polytorus that are "large enough" for not being zero sets for any non-zero polynomial.

One consequence is the following natural extension of the Schur-Cohn criterion:

THEOREM 1. *Multivariable Schur-Cohn test.*

Let P be a polynomial in n variables of degree p. For each $w = (w_1, \dots w_{n-1}) \in \mathbf{T}^{n-1}$, let $\gamma_k(w)$ the functional Schur coefficients of $\frac{(P_w)^T}{P_w}$. The following are equivalent:

a) The polynomial P has no zeroes in $\bar{\mathbf{D}}^n$;

b) $\sup\limits_{w \in \mathbf{T}^{n-1}} |\gamma_k(w)| < 1$for $k = 0, \dots, p-1$.

The continuity of the functional Schur coefficients, which are rational functions in w, allows the numerical computation of the supremum (in fact a maximum) in b).

Here is a computed example to illustrate the test:

EXAMPLE 1. Consider the following test polynomial:

$$P(z_1, z_2) = 1 + 0.5z_1 + 0.5z_2 + 0.25z_1z_2 + 0.25z_1^2 + 0.25z_2^2$$

taken from [10]. The slices of P are:

$$P_w(\lambda) = P(\lambda w, \lambda) = 1 + 0.5(w+1)\lambda + 0.25(1 + w + w^2)\lambda^2, \quad (w \in \mathbf{T})$$

so

$$f_w(\lambda) = \frac{P_w^T(\lambda)}{P_w(\lambda)} = \frac{0.25(1 + \overline{w} + \overline{w}^2) + 0.5(\overline{w}+1)\lambda + \lambda^2}{1 + 0.5(w+1)\lambda + 0.25(1 + w + w^2)\lambda^2} \quad (w \in \mathbf{T}).$$

Since P has degree 2, we are interested by the first two functional Schur coefficients of f, which are, by (3.1) and (3.2)

$$\gamma_0(w) = 0.25(1 + \overline{w} + \overline{w}^2)$$

and

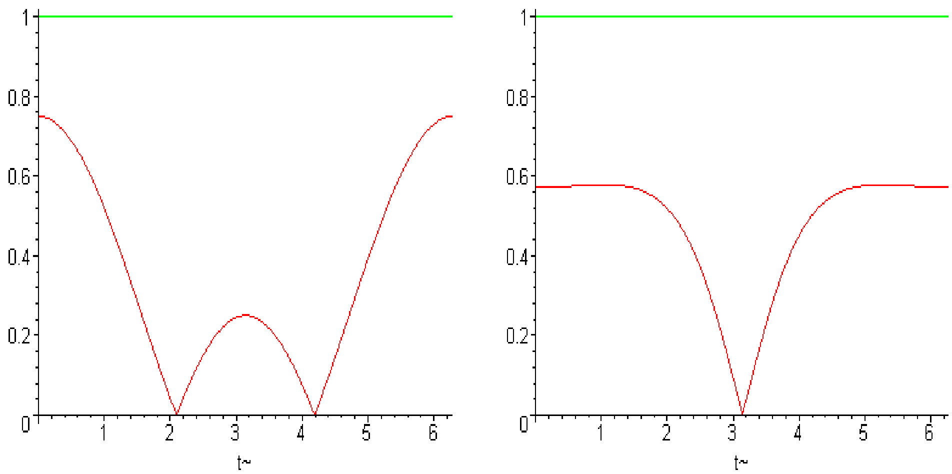
$$\gamma_1(w) = \frac{0.25(1 + \overline{w}) - 0.125(w + \overline{w}^2)}{0.8125 - 0.0625(w^2 + \overline{w}^2) - 0.125(w + \overline{w})}, (w \in \mathbf{T})$$

with absolute values

$$\left|\gamma_0(e^{it})\right|^2 = 0.625[(1 + \cos t + \cos 2t)^2 - (\sin t + \sin 2t)^2] \quad (t \in [0, 2\pi))$$

$$\left|\gamma_1(e^{it})\right|^2 = \frac{8(1 + \cos 2t)}{(\cos 2t + 0.5\cos t + 1)^2 + (\sin 2t + 5\sin t)^2} \quad (t \in [0, 2\pi)).$$

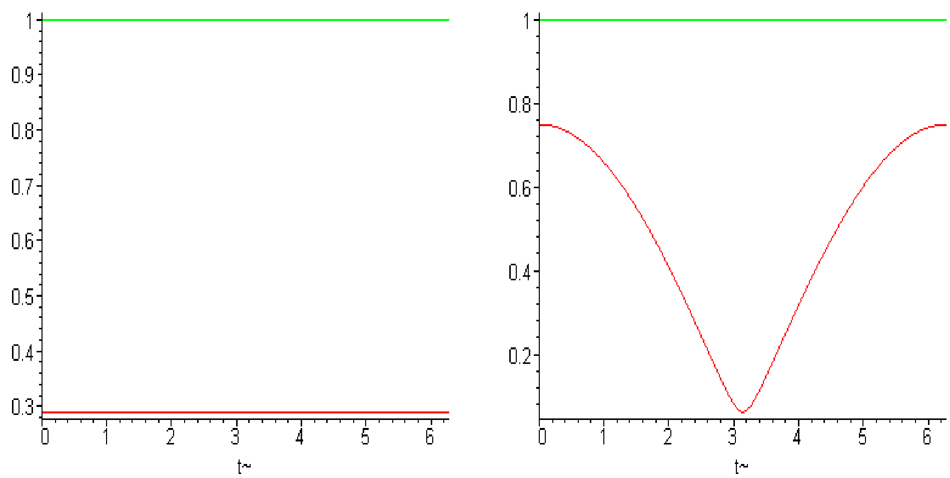
Therefore we find $\sup_{t\in[0,2\pi)} \left|\gamma_0(e^{it})\right| = 0.75 < 1$ and $\sup_{t\in[0,2\pi)} \left|\gamma_1(e^{it})\right| \approx 0.55735 < 1$, so P is stable. Here are the graphs of the absolute values of the functional Schur coefficients:



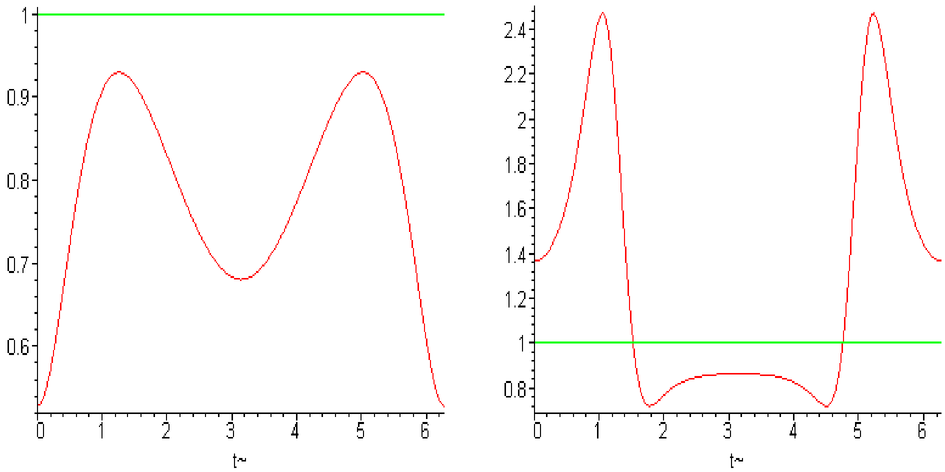
$\left|\gamma_0(e^{it_1}, e^{it_2})\right|$ $\left|\gamma_1(e^{it_1}, e^{it_2})\right|$

FIG. 1. *The absolute values of the functional Schur coefficients for Example 1.*

EXAMPLE 2. $P(z_1, z_2) = 1 - 1.5z_1 + 0.6z_1^2 - 1.2z_2 + 1.8z_1z_2 - 0.72z_1^2z_2 + 0.3z_2^2 - 0.75z_1z_2^2 + 0.29z_1^2z_2^2$ (in [10], ex. 3)



$\left|\gamma_0(e^{it_1}, e^{it_2})\right|$ $\left|\gamma_1(e^{it_1}, e^{it_2})\right|$



$\left|\gamma_2(e^{it_1}, e^{it_2})\right|$ $\left|\gamma_3(e^{it_1}, e^{it_2})\right|$

FIG. 2. *The absolute values of the functional Schur coefficients for Example 2.*

We see that $|\gamma_3|$ exceeds 1, so P is an unstable polynomial.

The next examples illustrate the test for 3 variable polynomials. The graphs represent the absolute values of the functional Schur coefficients $\gamma_k(e^{it_1}, e^{it_2})$, $t_1, t_2 \in [0, 2\pi]$.

EXAMPLE 3. $P(z_1, z_2, z_3) = 4.5 + z_1 z_3 - z_2^2 + 2z_2 z_3 + z_1 z_2^2$, of degree 3.

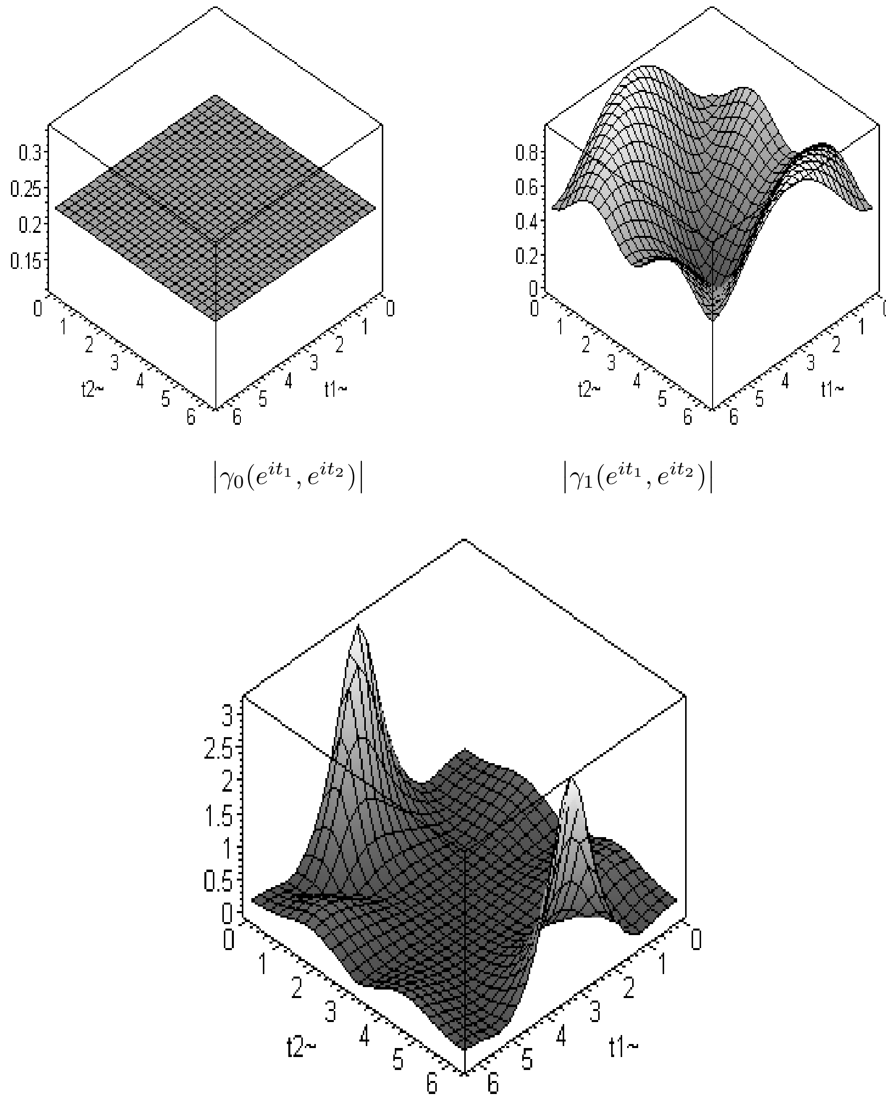
The polynomial is unstable because $|\gamma_3|$ exceeds 1 (see fig. 3).

EXAMPLE 4. $P(z_1, z_2, z_3) = 5 + z_2 + z_1 z_3^3 - z_1 z_2 z_3 + z_2 z_3$, also of degree 3.

Since $\left|\gamma_k(e^{it_1}, e^{it_2})\right| \leq 0.5 < 1$ for $k = 0, 1, 2$ it follows that P is stable (see fig. 4).

4. Conclusion. This paper focuses on the Bounded-Input Bounded-Output (BIBO) stability problem of multidimensional linear systems. The necessary and sufficient criterion is based on the concept of slice functions and yields a full characterization of BIBO stability, in which the role of the Schur coefficients from one variable is played by the functional Schur coefficients that we have introduced. We have showed that a multidimensional linear system is BIBO stable if and only if the absolute values of the functional Schur coefficients of the denominator of its transfer function are less than unity.

The fact that the two approaches are not equivalent, as they are in one dimension, shows how complex the zero-localization and the BIBO stability problems are in several variables.



$\left|\gamma_0(e^{it_1}, e^{it_2})\right|$ $\left|\gamma_1(e^{it_1}, e^{it_2})\right|$

$\left|\gamma_2(e^{it_1}, e^{it_2})\right|$

FIG. 3. *The absolute values of the functional Schur coefficients for Example 3.*

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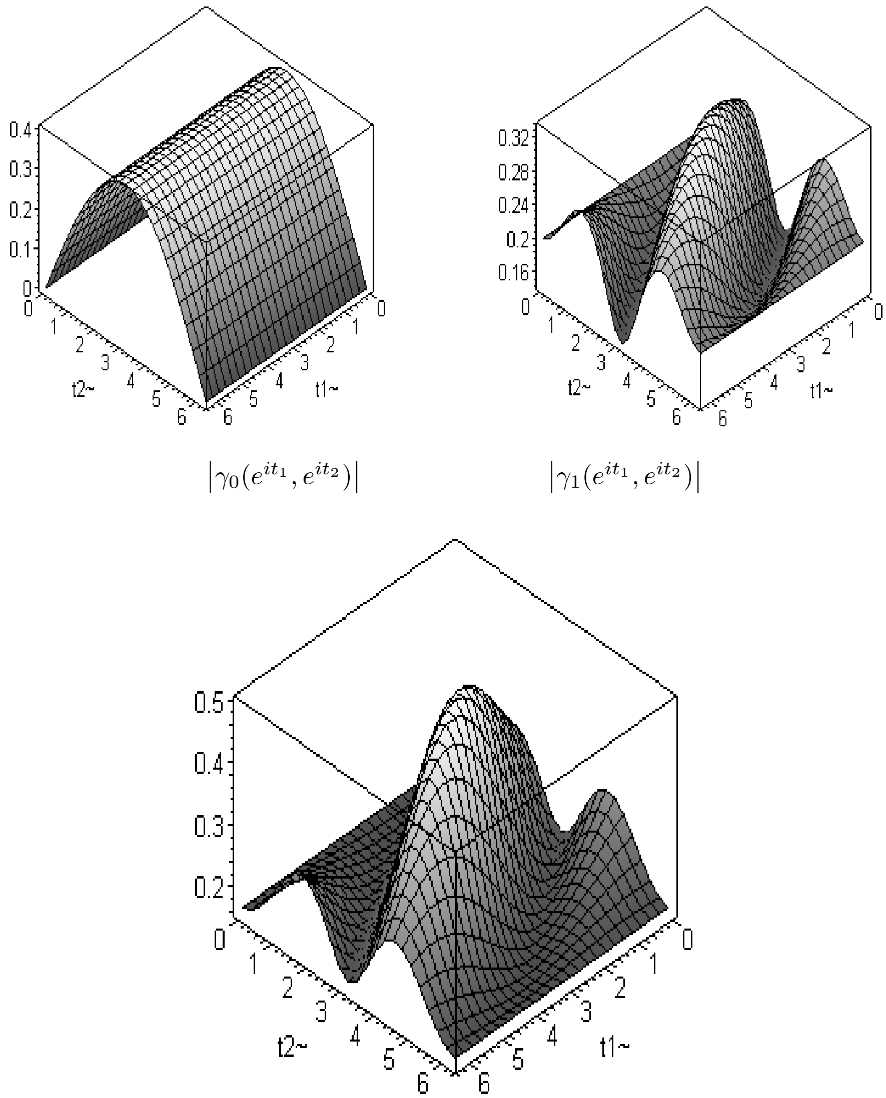

$\left|\gamma_0(e^{it_1}, e^{it_2})\right|$ $\left|\gamma_1(e^{it_1}, e^{it_2})\right|$

$\left|\gamma_2(e^{it_1}, e^{it_2})\right|$

FIG. 4. *The absolute values of the functional Schur coefficients for Example 4.*

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