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INDEFINITE STOCHASTIC LQ CONTROLS WITH MARKOVIAN JUMPS IN A FINITE TIME HORIZON*

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Abstract. This paper is concerned with a stochastic linear–quadratic (LQ) control problem over a finite time horizon with Markovian jumps in the problem parameters. The problem is indefinite in that the cost weighting matrices for the state and control are allowed to be indefinite. A system of coupled generalized (differential) Riccati equations (CGREs) is introduced to cope with the indefiniteness of the problem. Specifically, it is proved that the solvability of the CGREs is sufficient for the well-posedness of the stochastic LQ problem. Moreover, it is shown that the solvability of the CGREs is necessary for the well-posedness of the stochastic LQ problem and the existence of optimal (feedback/open-loop) controls via the dynamic programming approach. An example is presented to illustrate the results established.

Keywords. Indefinite stochastic LQ control, jump linear systems, coupled generalized Riccati equations, matrix pseudo-inverse, Hamilton-Jacobi-Bellman equations

1. Introduction.

Optimal LQ control is one of the fundamental problems in the engineering field. In most literature of the LQ theory, the cost function is assumed to have positive semidefinite state weighting matrix and positive definite control weighting matrix. However, recent studies of the stochastic LQ problem [5, 2, 3] show that when the diffusion term depends on the control the stochastic LQ problem may still be well-posed even if the cost weighting matrices are indefinite. This interesting phenomenon has to do with the deep nature of uncertainty/risk as well as its control, and has led to applications to the financial mean–variance portfolio selection problems. See [5] for a detailed discusion on the motivation and practical significance of the indefinite stochastic LQ control, and [13] for a tutorial paper on its applications to finance. The objective of this paper is to extend the indefinite stochastic LQ control to the so-called Markov-modulated systems, namely, ones where there are jumps in the problem parameters modeled by a continuous-time Markov chain.

The study of jump linear systems can be traced back at least to the work of Krasosvkii and Lidskii [8]. The LQ control with Markovian jumps has been very widely studied for the last decade; see, for example, Ait Rami and El Ghaoui [1], Mariton [9], Ji and Chizeck [6, 7], Zhang and Yin [12], among others. Most works

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focus on the following class of LQ control problems

$$
\begin{aligned}
&\text{Minimize} \quad J = E\Big\{ \int_0^T [x(t)'Q(t,r_t)x(t) + u(t)'R(t,r_t)u(t)]dt \\
&\qquad\qquad + x(T)'Hx(T) \;\Big|\; r_0 = i \Big\}, \qquad (1)\\
&\text{subject to} \quad \begin{cases} dx(t) = [A(t,r_t)x(t) + B(t,r_t)u(t)]dt + \sigma(t,r_t)dW(t), \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}
\end{aligned}
$$

where r_t is a Markov chain taking values in $\{1, \cdots, l\}$, $W(t)$ is a standard Brownian motion independent of r_t, and $A(t,r_t) = A_i(t)$, $B(t,r_t) = B_i(t)$, $\sigma(t,r_t) = \sigma_i(t)$, $Q(t,r_t) = Q_i(t)$ and $R(t,r_t) = R_i(t)$ when $r_t = i$ $(i = 1, \cdots, l)$. Here the matrix functions $A_i(\cdot)$, etc. are given with appropriate dimensions. The Markov chain r_t has the transition probabilities given by:

$$
\mathbf{P}\{r_{t+\Delta t} = j | r_t = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & \text{if } i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & \text{else,} \end{cases} \qquad (2)
$$

where $\pi_{ij} \geq 0$ for $i \neq j$ and $\pi_{ii} = -\sum_{j \neq i} \pi_{ij}$.

As with the traditional LQ control problems without jumps, in the literature where the above type of problems is tackled, it is usually required that the state weighting matrices, $Q_i(t)$, and the control weighting matrices, $R_i(t)$, be positive semidefinite and positive definite, respectively. Also, in the existing works the diffusion coefficients are usually set as independent of the state and control variables. To motivate our study here, let us look at an example.

Example 1.1. Consider the following problem

$$
\begin{aligned}
&\min \quad J = E\Big\{ \int_0^2 [x(t)^2 + R(t,r_t)u(t)^2]dt + x(2)^2 \;\Big|\; r_0 = i \Big\}, \\
&\text{s.t.} \quad \begin{cases} dx(t) = dW(t), \\ x(0) = 0, \end{cases}
\end{aligned} \qquad (3)
$$

where $r_t = 1$ and $R(t,r_t) = R_1(t) < 0$ when $t \in [0,1)$, and $r_t = 2$ and $R(t,r_t) = R_2(t) < 0$ when $t \in [1,2]$. This problem is ill-posed since $J = E\{\int_0^1 R_1(t)u(t)^2dt + \int_1^2 R_2(t)u(t)^2dt\} + 4 \to -\infty$ as $|u(t)| \to +\infty$. Now, let us modify the problem into the following

$$
\begin{aligned}
&\min \quad J = E\Big\{ \int_0^2 [x(t)^2 + R(t,r_t)u(t)^2]dt + x(2)^2 \;\Big|\; r_0 = i \Big\}, \\
&\text{s.t.} \quad \begin{cases} dx(t) = u(t)dW(t), \\ x(0) = 0, \end{cases}
\end{aligned} \qquad (4)
$$

where $r_t = 1$ and $R(t,r_t) = R_1(t)$ when $t \in [0,1)$, and $r_t = 2$ and $R(t,r_t) = R_2(t)$ when $t \in [1,2]$. Substituting $Ex(t)^2 = E\int_0^t u(s)^2ds$ into the cost function, we obtain via a simple calculation

$$J = E\left\{\int_0^1 [R_1(t) + (3-t)]u(t)^2dt + \int_1^2 [R_2(t) + (3-t)]u(t)^2dt\right\}.$$

Hence, the problem (4) is well-posed when $R_1(t) > t - 3$ and $R_2(t) > t - 3$. In this case, $R_1(t)$ and $R_2(t)$ could be negative. Of course, they cannot be too negative because the problem will be ill-posed when $R_1(t) < -3$ or $R_2(t) < -3$.

The above example shows an interesting feature of stochastic systems when the diffusion terms depend on the control. In general, we may consider problem (1) where the diffusion coefficient $\sigma(t, r_t)$ is replaced by $C(t, r_t)x(t) + D(t, r_t)u(t)$. As with the case without jumps [2], this problem is intimately related to the following system of coupled constrained Riccati equations (t is suppressed)

$$(5)\qquad \begin{cases} \dot{P}_i + P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \sum_{j=1}^{l}\pi_{ij}P_j \\ \qquad -(P_iB_i + C_i'P_iD_i)(R_i + D_i'P_iD_i)^{-1}(B_i'P_i + D_i'P_iC_i) = 0, \\ P_i(T) = H, \\ R_i + D_i'P_iD_i > 0, \qquad \text{a.e. } t \in [0,T], \qquad \text{for } i = 1, \cdots, l. \end{cases}$$

However, the third constraint of (5) is rather restrictive that will likely lead to the non-existence of its solution even when the corresponding LQ problem is well-posed. To handle more general indefinite stochastic LQ problems with jumps, we further relax the positive definiteness of the term $R_i(t) + D_i(t)'P_i(t)D_i(t)$ which enables us to deal with the possible singularity. The generalized form of the equations (5) will be shown to be correct for studying indefinite stochastic LQ control with Markovian jumps, in the sense that its solvability is equivalent to the well-posedness of the LQ problem. Moreover, we can construct all optimal controls via the solution of the generalized Riccati equations.

The rest of this paper is organized as follows. In Section 2 the indefinite stochastic LQ control problem is formulated, some preliminaries are given and the coupled generalized Riccati equations (CGREs) is introduced. It is shown in Section 3 that the solvability of the CGREs is sufficient for the well-posedness of the LQ problem and the existence of an optimal control. Moreover, we construct all the optimal controls via the solution of the CGREs. In Sections 4 and 5 we prove that the solvability of the CGREs is also necessary for the existence of optimal feedback controls and optimal open-loop controls via dynamic programming approach, respectively. An example is presented in Section 6 to illustrate the results established. Finally, Section 7 gives some conclusions.

2. Problem Formulation and Preliminaries.

2.1. Notation. We make use of the following notation in this paper:

M'	:	the transpose of any matrix or vector M;
$M^\dagger$	:	the Moore–Penrose pseudo-inverse of a matrix M;
$M > 0$	:	the symmetric matrix M is positive definite;
$M \geq 0$	:	the symmetric matrix M is positive semidefinite;
$\mathbb{R}^n$	:	the n-dimensional Euclidean space;
$\mathbb{R}^{n\times m}$	:	the set of all $n \times m$ matrices;
$\mathcal{S}^n$	:	the set of all $n\times n$ symmetric matrices;
$\mathcal{S}^n_+$	:	the subset of all non-negative definite matrices of $\mathcal{S}^n$;
$(\mathcal{S}^n)^l$	:	$= \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{l}$;
$(\mathcal{S}^n_+)^l$	:	$= \underbrace{\mathcal{S}^n_+ \times \cdots \times \mathcal{S}^n_+}_{l}$;
$C(0,T;\mathbb{R}^{n\times m})$	:	the set of continuous functions $\phi : [0,T] \to \mathbb{R}^{n\times m}$;
$L^p(0,T;\mathbb{R}^{n\times m})$	:	the set of functions $\phi : [0,T] \to \mathbb{R}^{n\times m}$ such that $\int_0^T |\phi(t)|^p dt < \infty$ $(p \in [1,\infty))$;
$L^\infty(0,T;\mathbb{R}^{n\times m})$	:	the set of essentially bounded measurable functions $\phi : [0,T] \to \mathbb{R}^{n\times m}$;
$C^1(0,T;(\mathcal{S}^n)^l)$	:	the set of continuously differential functions $\phi : [0,T] \to (\mathcal{S}^n)^l$.

2.2. Problem Formulation. Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ and a Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|_{\mathcal{H}}$, define the Hilbert space

$$L^2_{\mathcal{F}}(0,T;\mathcal{H}) = \left\{ \phi(\cdot) \;\middle|\; \begin{array}{l} \phi(\cdot) \text{ is an } \mathcal{F}_t\text{-adapted, } \mathcal{H}\text{-valued measurable process} \\ \text{on } [0,T] \text{ and } E\int_0^T \|\phi(t,\omega)\|^2_{\mathcal{H}} dt < +\infty, \end{array} \right\}$$

with the norm

$$\|\phi(\cdot)\|_{\mathcal{F},2} = \left(E\int_0^T \|\phi(t,\omega)\|^2_{\mathcal{H}} dt \right)^{\frac{1}{2}}.$$

Consider the following linear stochastic differential equation (SDE) subject to Markovian jumps defined by

$$(6)\begin{cases} dx(t) = [A(t,r_t)x(t) + B(t,r_t)u(t)]dt + [C(t,r_t)x(t) + D(t,r_t)u(t)]dW(t), \\ x(s) = y, \end{cases}$$

where $(s,y) \in [0,T) \times \mathbb{R}^n$ are the initial time and initial state, respectively, and an admissible control $u(\cdot)$ is an $\mathcal{F}_t$-adapted, $\mathbb{R}^{n_u}$-valued measurable process on $[0,T]$. The set of all admissible controls is denoted by $\mathcal{U}_{ad} \equiv L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_u})$. The solution $x(\cdot)$ of the equation (6) is called the response of the control $u(\cdot) \in \mathcal{U}_{ad}$, and $(x(\cdot), u(\cdot))$ is called an admissible pair. Here, $W(t)$ is a one-dimensional standard $\mathcal{F}_t$-Brownian motion on $[0,T]$ (with $W(0) = 0$). Note that all the results in the following sections can be generalized to the case with a multi-dimensional Brownian motion without any difficulty. On the other hand, r_t is a Markov chain adapted to $\mathcal{F}_t$, taking values in $\{1, \cdots, l\}$, with the transition probabilities specified by (2). In addition, we assume that the processes r_t and $W(t)$ are independent.

For each (s, y) and $u(\cdot) \in \mathcal{U}_{ad}$, the associated cost is

$$\begin{aligned} & J(s, y, i; u(\cdot)) \\ = \; & E\left\{ \int_s^T \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(t, r_t) & L(t, r_t) \\ L(t, r_t)' & R(t, r_t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \right. \\ & \left. + x(T)' H(r_T) x(T) \,\middle|\, r_s = i \right\}. \end{aligned} \tag{7}$$

In (6) and (7), $A(t, r_t) = A_i(t)$, etc. whenever $r_t = i$, and $H(r_T) = H_i$ whenever $r_T = i$, whereas $A_i(\cdot)$ etc. are given matrix-valued functions and H_i are given matrices, $i = 1, \cdots, l$. The objective of the optimal control problem is to minimize the cost function $J(s, y, i; u(\cdot))$, for a given $(s, y) \in [0, T) \times \mathbb{R}^n$, over all $u(\cdot) \in \mathcal{U}_{ad}$. The value function is defined as

$$V(s, y, i) = \inf_{u(\cdot) \in \mathcal{U}_{ad}} J(s, y, i; u(\cdot)). \tag{8}$$

Definition 2.1. The optimization problem $(6) - (8)$ is called well-posed if

$$V(s, y, i) > -\infty, \quad \forall (s, y) \in [0, T) \times \mathbb{R}^n, \quad \forall i = 1, \cdots, l.$$

An admissible pair $(x^*(\cdot), u^*(\cdot))$ is called *optimal* (with respect to the initial condition (s, y, i)) if $u^*(\cdot)$ achieves the infimum of $J(s, y, i; u(\cdot))$.

The following basic assumption will be in force throughout this paper.

Assumption 2.1. The data appearing in the LQ problem $(6) - (8)$ satisfy, for every i,

$$\left\{ \begin{array}{rcl} A_i(\cdot), C_i(\cdot) & \in & L^\infty(0, T; \mathbb{R}^{n \times n}), \\ B_i(\cdot), D_i(\cdot) & \in & L^\infty(0, T; \mathbb{R}^{n \times n_u}), \\ Q_i(\cdot) & \in & L^\infty(0, T; \mathcal{S}^n), \\ L_i(\cdot) & \in & L^\infty(0, T; \mathbb{R}^{n \times n_u}), \\ R_i(\cdot) & \in & L^\infty(0, T; \mathcal{S}^{n_u}), \\ H_i & \in & \mathcal{S}^n. \end{array} \right.$$

We emphasize again that we are dealing with an *indefinite* LQ problem, namely, $Q_i(t), R_i(t)$ and H_i are all possibly indefinite.

2.3. Coupled Generalized Differential Riccati Equations. Let us first recall the properties of a pseudo matrix inverse [10].

Proposition 2.1. *Let a matrix $M \in \mathbb{R}^{m \times n}$ be given. Then there exists a unique matrix $M^\dagger \in \mathbb{R}^{n \times m}$ such that*

$$\left\{ \begin{array}{ll} M M^\dagger M = M, & M^\dagger M M^\dagger = M^\dagger, \\ (M M^\dagger)' = M M^\dagger, & (M^\dagger M)' = M^\dagger M, \end{array} \right. \tag{9}$$

where the matrix $M^\dagger$ is called the Moore–Penrose pseudo inverse of M.

Now we introduce a new type of coupled differential Riccati equations associated with the LQ problem (6)–(8).

DEFINITION 2.2. The following system of constrained differential equations (with the time argument t suppressed)

$$\tag{10}\left\{\begin{array}{l} \dot P_i + P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \sum_{j=1}^l \pi_{ij}P_j \\ \qquad -(P_iB_i + C_i'P_iD_i + L_i)(R_i + D_i'P_iD_i)^\dagger(B_i'P_i + D_i'P_iC_i + L_i') = 0, \\ P_i(T) = H_i, \\ (R_i + D_i'P_iD_i)(R_i + D_i'P_iD_i)^\dagger(B_i'P_i + D_i'P_iC_i + L_i') \\ \qquad -(B_i'P_i + D_i'P_iC_i + L_i') = 0, \\ R_i + D_i'P_iD_i \geq 0, \qquad \text{a.e. } t \in [0,T], \qquad\qquad i = 1,\cdots,l \end{array}\right.$$

is called a system of coupled generalized (differential) Riccati equations (CGREs).

If the term $(R_i + D_i'P_iD_i)$, for every i, is further required to be non-singular, then the CGREs reduce to the equations

$$\tag{11}\left\{\begin{array}{l} \dot P_i + P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \sum_{j=1}^l \pi_{ij}P_j \\ \qquad -(P_iB_i + C_i'P_iD_i + L_i')(R_i + D_i'P_iD_i)^{-1}(B_i'P_i + D_i'P_iC_i + L_i') = 0, \\ P_i(T) = H_i, \\ R_i + D_i'P_iD_i > 0, \qquad \text{a.e. } t \in [0,T], \qquad\qquad i = 1,\cdots,l. \end{array}\right.$$

Another interesting special case is when $R_i + D_i'P_iD_i \equiv 0$ for every i, the CGREs reduce to the following linear differential matrix system:

$$\tag{12}\left\{\begin{array}{l} \dot P_i + P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \sum_{j=1}^l \pi_{ij}P_j = 0, \\ P_i(T) = H_i, \\ B_i'P_i + D_i'P_iC_i + L_i' = 0, \\ R_i + D_i'P_iD_i = 0, \qquad \text{a.e. } t \in [0,T], \qquad\qquad i = 1,\cdots,l. \end{array}\right.$$

2.4. Some useful lemmas. In this subsection we collect a number of technical lemmas that are useful in our subsequent analysis. The first one is the *generalized Itô's formula.*

LEMMA 2.1 ([11]). *Let $x(t)$ satisfy*

$$dx(t) = b(t,x(t),r_t)dt + \sigma(t,x(t),r_t)dW(t),$$

and $\varphi(\cdot,\cdot,i) \in C^2([0,\infty) \times \mathbb{R}^n)$, $i = 1,\cdots,l$, be given. Then,

$$\tag{13}\begin{aligned} & E\Big\{\varphi(T,x(T),r_T) - \varphi(s,x(s),r_s) \mid r_s = i\Big\} \\ = \ & E\left\{ \int_s^T \varphi_t(t,x(t),r_t) + \Gamma\varphi(t,x(t),r_t)dt \;\middle|\; r_s = i\right\}, \end{aligned}$$

where

$$\begin{aligned} \Gamma\varphi(t,x,i) \ = \ & \tfrac{1}{2}\mathrm{tr}[\sigma(t,x,i)'\varphi_{xx}(t,x,i)\sigma(t,x,i)] \\ & + b(t,x,i)'\varphi_x(t,x,i) + \textstyle\sum_{j=1}^l \pi_{ij}\varphi(t,x,j). \end{aligned}$$

LEMMA 2.2 ([2, 3]). *For a symmetric matrix S, we have*

(i) $S^\dagger = (S^\dagger)'$;

(ii) $SS^\dagger = S^\dagger S$;

(iii) $S \geq 0$ *if and only if* $S^\dagger \geq 0$.

LEMMA 2.3 (Extended Schur's lemma [4]). *Let matrices $M = M'$, N and $R = R'$ be given with appropriate dimensions. Then the following conditions are equivalent:*

(i) $M - NR^\dagger N' \geq 0$ *and* $N(I - RR^\dagger) = 0$, $R \geq 0$;

(ii) $\begin{bmatrix} M & N \\ N' & R \end{bmatrix} \geq 0$;

(iii) $\begin{bmatrix} R & N' \\ N & M \end{bmatrix} \geq 0$.

LEMMA 2.4 ([2]). *Let matrices L, M and N be given with appropriate sizes. Then the following matrix equation*

$$LXM = N \tag{14}$$

has a solution X if and only if

$$LL^\dagger N M^\dagger M = N. \tag{15}$$

Moreover, any solution to (14) *is represented by*

$$X = L^\dagger N M^\dagger + S - L^\dagger L S M M^\dagger, \tag{16}$$

where S is a matrix with an appropriate size.

3. Sufficiency of the CGREs. In this section, we will show that the solvability of the CGREs is sufficient for the well-posedness of the LQ problem and the existence of an optimal feedback control. In addition, all optimal controls can be obtained via the solution to the CGREs (10).

THEOREM 3.1. *If the CGREs* (10) *admit a solution $(P_1(\cdot), \cdots, P_l(\cdot)) \in C^1(0,T; (\mathcal{S}^n)^l)$, then the stochastic LQ problem* (6) – (8) *is well-posed. Moreover, the set of all optimal controls with respect to the initial $(s, y) \in [0,T) \times \mathbb{R}^n$ is determined by the following (parameterized by (Y_i, z_i)):*

$$\begin{aligned} u(t) \;=\; & -\textstyle\sum_{i=1}^{l} \Big\{ [(R_i(t) + D_i(t)'P_i(t)D_i(t))^\dagger (B_i(t)'P_i(t) \\ & + D_i(t)'P_i(t)C_i(t) + L_i(t)') + Y_i(t) \\ & - (R_i(t) + D_i(t)'P_i(t)D_i(t))^\dagger (R_i(t) + D_i(t)'P_i(t)D_i(t))Y_i(t)]x(t) \\ & + z_i(t) - (R_i(t) + D_i(t)'P_i(t)D_i(t))^\dagger (R_i(t) \\ & + D_i(t)'P_i(t)D_i(t))z_i(t) \Big\} \chi_{\{r_t = i\}}(t), \end{aligned} \tag{17}$$

where $Y_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u \times n})$ and $z_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u})$. Furthermore, the value function is uniquely determined by $(P_1(\cdot), \cdots, P_l(\cdot)) \in C^1(0,T;(\mathcal{S}^n)^l)$:

$$V(s,y,i) \equiv \inf_{u(\cdot) \in \mathcal{U}_{ad}} J(s,y,i;u(\cdot)) = y'P_i(s)y, \quad i = 1, \cdots, l. \tag{18}$$

Proof. Let $(P_1(\cdot), \cdots, P_l(\cdot)) \in C^1(0,T;(\mathcal{S}^n)^l)$ be a solution of the CGREs (10). Setting $\varphi(t,x,i) = x'P_i(t)x$ and applying the generalized Itô's formula (Lemma 2.1) to the linear system (6), we have

$$\begin{aligned}
& E[x(T)'H_{r_T}x(T)] - y'P_i(s)y, \\
= \ & E[x(T)'P_{r_T}(T)x(T) - x(s)'P(r_s)x(s) \mid r_s = i] \\
= \ & E[\varphi(T,x(T),r_T) - \varphi(s,x(s),r_s) \mid r_s = i] \\
= \ & E\left\{ \int_s^T \Gamma\varphi(t,x(t),r_t)dt \;\Big|\; r_s = i \right\},
\end{aligned}$$

where

$$\begin{aligned}
\Gamma\varphi(t,x,i) \ = \ & \varphi_t(t,x,i) + b(t,x,u,i)'\varphi_x(t,x,i) \\
& + \tfrac{1}{2}\mathrm{tr}[\sigma(t,x,u,i)'\varphi_{xx}(t,x,i)\sigma(t,x,u,i)] + \textstyle\sum_{j=1}^l \pi_{ij}\varphi(t,x,j) \\
= \ & x'[\dot{P}_i(t) + A_i(t)'P_i(t) + P_i(t)A_i(t) + C_i(t)'P_i(t)C_i(t) + \textstyle\sum_{j=1}^l \pi_{ij}P_j(t)]x \\
& + 2u'[B_i(t)'P_i(t) + D_i(t)'P_i(t)C_i(t)]x + u'D_i(t)'P_i(t)D_i(t)u.
\end{aligned}$$

Hence, we can express the cost function as follows

$$\begin{aligned}
& J(s,y,i;u(\cdot)) \\
= \ & y'P_i(s)y + E\left\{ \int_s^T [\Gamma\varphi(t,x(t),r_t) + x(t)'Q(t,r_t)x(t) \right. \\
& \left. + 2u(t)'L(t,r_t)'x(t) + u(t)'R(t,r_t)u(t)]dt \;\Big|\; r_s = i \right\}.
\end{aligned} \tag{19}$$

From the definition of the CGREs, we have

$$\begin{aligned}
& \Gamma\varphi(t,x,i) + x'Q_i(t)x + 2u'L_i(t)'x + u'R_i(t)u \\
= \ & x'[\dot{P}_i(t) + A_i(t)'P_i(t) + P_i(t)A_i(t) + C_i(t)'P_i(t)C_i(t) + Q_i(t) + \textstyle\sum_{j=1}^l \pi_{ij}P_j(t)]x \\
& + 2u'[B_i(t)'P_i(t) + D_i(t)'P_i(t)C_i(t) + L_i(t)']x + u'[R_i(t) + D_i(t)'P_i(t)D_i(t)]u \\
= \ & x'\{[P_i(t)B_i(t) + C_i(t)'P_i(t)D_i(t) + L_i(t)][R_i(t) + D_i(t)'P_i(t)D_i(t)]^\dagger \\
& \cdot[B_i(t)'P_i(t) + D_i(t)'P_i(t)C_i(t) + L_i(t)']\}x \\
& + 2u'[B_i(t)'P_i(t) + D_i(t)'P_i(t)C_i(t) + L_i(t)']x + u'[R_i(t) + D_i(t)'P_i(t)D_i(t)]u.
\end{aligned}$$

Now, let $Y_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u \times n})$ and $z_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u})$ be given for every i. Set

$$G_i^1(t) = Y_i(t) - [R_i(t) + D_i(t)'P_i(t)D_i(t)]^\dagger[R_i(t) + D_i(t)'P_i(t)D_i(t)]Y_i(t),$$

$$G_i^2(t) = z_i(t) - [R_i(t) + D_i(t)'P_i(t)D_i(t)]^\dagger[R_i(t) + D_i(t)'P_i(t)D_i(t)]z_i(t).$$

Applying Proposition 2.1 and Lemma 2.2-(ii), we have for $k = 1, 2$,

$$[R_i(t) + D_i(t)'P_i(t)D_i(t)]G_i^k(t) = [R_i(t) + D_i(t)'P_i(t)D_i(t)]^\dagger G_i^k(t) = 0, \tag{20}$$

and

$$[P_i(t)B_i(t) + C_i(t)P_i(t)D_i(t) + L_i(t)]G_i^k(t) = 0.$$

Hence,

$$\begin{aligned} &\Gamma\varphi(t,x,i) + x'Q_i(t)x + 2u'L_i(t)'x + u'R_i(t)u \\ = \ &[u + (G_i^1(t) - K_i(t))x + G_i^2(t)]'[R_i(t) + D_i(t)'P_i(t)D_i(t)] \\ &\cdot[u + (G_i^1(t) - K_i(t))x + G_i^2(t)], \end{aligned}$$

where $K_i(t) = -[R_i(t) + D_i(t)'P_i(t)D_i(t)]^\dagger[B_i(t)'P_i(t) + D_i(t)'P_i(t)C_i(t) + L_i(t)']$. Then the equation (19) can be expressed as

$$\begin{aligned} &J(s,y,i;u(\cdot)) \\ = \ &y'P_i(s)y + \left\{ E\int_s^T [u(t) + (G^1(t,r_t) - K(t,r_t))x(t) + G^2(t,r_t)]' \right. \\ &\cdot[R(t,r_t) + D(t,r_t)'P(t,r_t)D(t,r_t)][u(t) \\ &\left. +(G^1(t,r_t) - K(t,r_t))x(t) + G^2(t,r_t)]dt \;\middle|\; r_s = i \right\}, \end{aligned} \tag{21}$$

where and $P(t,r_t) = P_i(t)$, $K(t,r_t) = K_i(t)$ and $G^k(t,r_t) = G_i^k(t)$ whenever $r_t = i$, $k = 1,2$. Thus, $J(s,y,i;u(\cdot))$ is minimized by the control given by (17) with the optimal value being $y'P_i(s)y$. □

COROLLARY 3.1. *The optimal controls are obtained in the following special cases:*

(i) *If $R_i(t) + D_i(t)'P_i(t)D_i(t) \equiv 0$, a.e. $t \in [s,T]$ for every i, then any admissible control is optimal.*

(ii) *If $R_i(t) + D_i(t)'P_i(t)D_i(t) > 0$, a.e. $t \in [s,T]$ for every i, then there is a unique optimal control that is given by the following linear feedback law:*

$$u(t) = \textstyle\sum_{i=1}^l K_i(t)x(t)\chi_{\{r_t=i\}}(t),$$

where $K_i(t) = -[R_i(t) + D_i(t)'P_i(t)D_i(t)]^{-1}[B_i(t)'P_i(t) + D_i(t)'P_i(t)C_i(t) + L_i(t)']$.

Proof. These are straightforward from Theorem 3.1. □

As an immediate consequence of Theorem 3.1, we have the uniqueness of the solution to the CGREs (10).

COROLLARY 3.2. *If there is a solution to the CGREs* (10)*, then it must be the unique solution.*

Proof. Let $(\tilde P_1(\cdot), \cdots, \tilde P_l(\cdot))$ and $(\hat P_1(\cdot), \cdots, \hat P_l(\cdot)) \in C^1(0,T;(\mathcal{S}^n)^l)$ be two solutions of the CGREs (10). Then Theorem 3.1 implies that

$$y'\tilde P_i(s)y = y'\hat P_i(s)y, \quad \forall y \in \mathbb{R}^n, \quad \forall s \in [0,T], \quad i = 1, \cdots, l.$$

Hence, $(\tilde P_1(t), \cdots, \tilde P_l(t)) \equiv (\hat P_1(t), \cdots, \hat P_l(t))$. □

It is interesting to see the specialization of our results in the deterministic case (i.e., $C_i(\cdot) = D_i(\cdot) \equiv 0$ for $i = 1, \cdots, l$). Assume that $R_i(\cdot) \in L^\infty(0,T;\mathcal{S}_+^{n_u})$, namely, the control weight is possibly singular. The corresponding CGREs are (t is suppressed)

$$(22)\quad \begin{cases} \dot{P}_i + P_iA_i + A_i'P_i - (P_iB_i + L_i)R_i^{\dagger}(B_i'P_i + L_i') + Q_i + \sum_{i=1}^{l} \pi_{ij}P_j = 0, \\ P_i(T) = H_i, \\ R_iR_i^{\dagger}(B_i'P_i + L_i') - (B_i'P_i + L_i') = 0, \quad \forall t \in [0,T], \qquad i = 1, \cdots, l. \end{cases}$$

According to Theorem 3.1, if the above equations admit a solution $(P_1(\cdot), \cdots, P_l(\cdot)) \in C^1(0,T;(\mathcal{S}^n)^l)$, then there may be infinitely many optimal controls, and any optimal control has the following form

$$\begin{aligned} u(t) \;=\; -\sum_{i=1}^{l} \Big\{ & [R_i(t)^{\dagger}B_i(t)'P_i(t) + Y_i(t) - R_i(t)^{\dagger}R_i(t)Y_i(t)]x(t) \\ & + z_i(t) - R_i(t)^{\dagger}R_i(t)z_i(t) \Big\} \chi_{\{r_t = i\}}(t), \end{aligned}$$

where $Y_i(\cdot) \in L^2(s,T;\mathbb{R}^{n_u \times n})$ and $z_i(\cdot) \in L^2(s,T;\mathbb{R}^{n_u})$ for every i.

4. Necessity of the CGREs. In the previous section, we proved that the solvability of the CGREs (10) is *sufficient* for the well-posedness of the LQ problem (6)–(8) and optimal feedback control laws can be constructed based on the solution to the CGREs (10). In this section, we shall derive the associated Hamilton–Jacobi–Bellman (HJB) equation by using the dynamic programming approach, and show that the solvability of the CGREs (10) is also *necessary* for the LQ problem to have an optimal *feedback* control. Furthermore, we will show that any optimal feedback control law has the form (17) with $z(t) \equiv 0$.

First we give the following lemma.

LEMMA 4.1. *Assume that the LQ problem* (6) − (8) *is well-posed. For any* $s \in [0,T)$*, if there exists* $(P_1(\cdot), \cdots, P_l(\cdot)) \in C^1(0,T;(\mathcal{S}^n)^l)$ *such that*

$$(23)\quad \begin{cases} \dot{P}_i + P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \sum_{i=1}^{l} \pi_{ij}P_j \\ \quad -(P_iB_i + C_i'P_iD_i + L_i)(R_i + D_i'P_iD_i)^{\dagger}(B_i'P_i + D_i'P_iC_i + L_i') = 0, \\ P_i(T) = H_i, \\ (R_i + D_i'P_iD_i)(R_i + D_i'P_iD_i)^{\dagger}(B_i'P_i + D_i'P_iC_i + L_i') \\ \quad -(B_i'P_i + D_i'P_iC_i + L_i') = 0, \qquad a.e.\ t \in [s,T], \quad i = 1, \cdots, l. \end{cases}$$

Then $(P_1(\cdot), \cdots, P_l(\cdot))$ *must satisfy*

$$R_i + D_i'P_iD_i \geq 0, \quad a.e.\ t \in [s,T], \quad i = 1, \cdots, l.$$

Proof. Let $\lambda_k(t)$ be a negative eigenvalue of the kth matrix $R_k(t) + D_k(t)'P_k(t)D_k(t)$, $t \in [s,T]$. We will show that $\mathbf{mes}(\{t \in [s,T] \mid \lambda_k(t) < 0\}) = 0$, where $\mathbf{mes}$ denotes the Lebesgue measure. Denote the unitary eigenvector with respect to $\lambda_k(t)$ as $v_{\lambda_k}(t)$ (i.e., $v_{\lambda_k}(t)'v_{\lambda_k}(t) = 1$). Define $I_n^k(\cdot)$ as the indicator function of the set $\{t \in [s,T] \mid \lambda_k(t) < -\frac{1}{n}\}$, $n = 1, 2, \cdots$. Let $\delta \neq 0$ be an arbitrary scalar and consider the state trajectory $x(\cdot)$ of the system (6) under the feedback control

$$u(t,x,i)=\begin{cases} K_i(t)x, & \text{if } i\neq k,\\ \delta|\lambda_k(t)|^{-\frac{1}{2}}I_n^k(t)v_{\lambda_k}(t)+K_i(t)x, & \text{if } i=k,\end{cases} \tag{24}$$

where $K_i(t)=-[R_i(t)+D_i(t)'P_i(t)D_i(t)]^{\dagger}[B_i(t)'P_i(t)+D_i(t)'P_i(t)C_i(t)+L_i(t)']$. Let $x(\cdot)$ be the corresponding state trajectory of (6) under the above feedback control and let $u(t)=u(t,x(t),r_t)$. Clearly, $u(\cdot)\in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u})$. Now,

$$\begin{aligned} J(s,y,i;u(\cdot)) \;=\; & y'P_i(s)y+E\Big\{\int_s^T[u(t)-K(t,r_t)x(t)]'\\ & \cdot[R(t,r_t)+D(t,r_t)'P(t,r_t)D(t,r_t)][u(t)-K(t,r_t)x(t)]dt\;\Big|\;r_s=i\Big\}.\end{aligned}$$

It follows from $\lambda_k(t)<0$ that

$$|\lambda_k(t)|^{-1}I_n^k(t)[R_k(t)+D_k(t)'P_k(t)D_k(t)]v_{\lambda_k}(t)=-I_n^k(t)v_{\lambda_k}(t).$$

Hence, we have

$$\begin{aligned} J(s,y,i;u(\cdot)) &= y'P_i(s)y-\delta^2\textstyle\int_s^T I_n^k(t)dt\\ &= y'P_i(s)y-\delta^2\mathbf{mes}\Big(\{t\in[s,T]\mid\lambda_k(t)<-\tfrac{1}{n}\}\Big).\end{aligned}$$

If $\mathbf{mes}(\{t\in[s,T]\mid\lambda_k(t)<-\frac{1}{n}\})>0$, then by letting $\delta\to\infty$ we have $J(s,y,i;u(\cdot))\to-\infty$ which contradicts to the well-posedness of the LQ problem. Then $\mathbf{mes}(\{t\in[s,T]\mid\lambda_k(t)<-\frac{1}{n}\})=0$. Since

$$\Big\{t\in[s,T]\mid\lambda_k(t)<0\Big\}=\textstyle\bigcup_{n\in\mathbf{N}}\Big\{t\in[s,T]\;\Big|\;\lambda_k(t)<-\frac{1}{n}\Big\}, \tag{25}$$

we conclude that $\mathbf{mes}(\{t\in[s,T]\mid\lambda_k(t)<0\})=0$, completing the proof. □

THEOREM 4.1. *Assume that $Q_i(t)$ and $R_i(t)$ are continuous in t for every i. In addition, assume that the LQ problem (6) − (8) is well-posed and a given feedback control $\bar{u}(t)=\sum_{i=1}^{l}\overline{K}_i(t)x(t)\chi_{\{r_t=i\}}(t)$ is optimal for (6) − (8) with respect to any initial $(s,y)\in[0,T]\times\mathbb{R}^n$. Then the CGREs (10) must have a solution $(P_1(\cdot),\cdots,P_l(\cdot))\in C^1(0,T;(\mathcal{S}^n)^l)$. Moreover, the optimal feedback control $\bar{u}(t)=\sum_{i=1}^{l}\overline{K}_i(t)x(t)\chi_{\{r_t=i\}}(t)$ can be represented via (17) with $z(t)\equiv0$.*

Proof. By the dynamic programming approach, the value functions $V(s,y,i)$ satisfy the following HJB equations for $i=1,\cdots,l$

$$\begin{aligned} &V_s(s,y,i)+\min_u\Big\{y'Q_iy+2y'L_iu+u'R_iu+[A_iy+B_iu]'V_y(s,y,i)\\ &\quad+\tfrac{1}{2}[C_iy+D_iu]'V_{yy}(s,y,i)[C_iy+D_iu]+\textstyle\sum_{j=1}^{l}\pi_{ij}V(s,y,j)\Big\}=0,\end{aligned} \tag{26}$$

with the boundary condition

$$V(T,y,i)=y'H_iy. \tag{27}$$

In view of the assumption of the theorem, a simple adaption to the proof of [2, Lemma 5.1] yields that the value function can be represented as

$$V(s,y,i) = y'P_i(s)y, \quad i = 1,\cdots,l \tag{28}$$

for a symmetric $m \times m$ matrix $P_i(\cdot)$. Moreover, $P_i(t)$ is differentiable at any $t \in [0,T]$. Substituting (28) into (26), we have the equations (s is suppressed)

$$\begin{cases} y'\Big(\dot{P}_i + P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \sum_{j=1}^{l} \pi_{ij}P_j\Big)y \\ \quad + \min\limits_{u} \Big\{u'(R_i + D_i'P_iD_i)u + 2y'(P_iB_i + C_i'P_iD_i + L_i)u\Big\} = 0, \\ P_i(T) = H_i, \qquad\qquad i = 1,\cdots,l. \end{cases} \tag{29}$$

By assumption, a minimizer u in (29) is given by $u(s,y,i) = K_i(s)y$ for i, and hence (29) are reduced to the following equations,

$$\begin{cases} y'\Big(\dot{P}_i + P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \sum_{j=1}^{l} \pi_{ij}P_j\Big)y \\ \quad + \min\limits_{K_i} \Big\{y'\Big[K_i'(R_i + D_i'P_iD_i)K_i + 2(P_iB_i + C_i'P_iD_i + L_i)K_i\Big]y\Big\} = 0, \\ P_i(T) = H_i, \qquad\qquad i = 1,\cdots,l. \end{cases} \tag{30}$$

The second term of the left-hand side of the first equation above reaches the minimum if and only if

$$\frac{\partial}{\partial K_i}\Big[K_i'(R_i + D_i'P_iD_i)K_i + 2(P_iB_i + C_i'P_iD_i + L_i)K_i\Big]\Big|_{K_i=\overline{K}_i} = 0, \quad i = 1,\cdots,l,$$

i.e.,

$$(R_i + D_i'P_iD_i)\overline{K}_i + (B_i'P_i + D_i'P_iC_i + L_i') = 0, \quad i = 1,\cdots,l. \tag{31}$$

Now, we apply Lemma 2.4 to the equations (31) with

$$L = R_i + D_i'P_iD_i, \; M = I, \; N = -(B_i'P_i + D_i'P_iC_i + L_i'), \quad i = 1,\cdots,l.$$

First of all, by virtue of the assumption we know *a priori* that the equations (31) *do* have a solution $\overline{K}_i$. Hence (15) in this case is equivalent to

$$(R_i + D_i'P_iD_i)(R_i + D_i'P_iD_i)^{\dagger}(B_i'P_i + D_i'P_iC_i + L_i') = B_i'P_i + D_i'P_iC_i + L_i'.$$

Moreover, by (16), $\overline{K}_i$ has the following form

$$\begin{aligned} \overline{K}_i \;=\; & -\Big[(R_i + D_i'P_iD_i)^{\dagger}(B_i'P_i + D_i'P_iC_i + L_i') \\ & + Y_i - (R_i + D_i'P_iD_i)^{\dagger}(R_i + D_i'P_iD_i)Y_i\Big], \qquad i = 1,\cdots,l. \end{aligned} \tag{32}$$

Replacing $(\overline{K}_1(\cdot),\cdots,\overline{K}_l(\cdot))$ into the first l equations of (30), we can see by a simple calculation that $(P_1(\cdot),\cdots,P_l(\cdot)) \in C^1(0,T;(\mathcal{S}^n)^l)$ satisfies the following equations

$$\begin{aligned} & \dot{P}_i + P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \textstyle\sum_{j=1}^{l} \pi_{ij}P_j \\ & \quad -(P_iB_i + C_i'P_iD_i + L_i)(R_i + D_i'P_iD_i)^{\dagger} \\ & \quad \cdot(B_i'P_i + D_i'P_iC_i + L_i') = 0, \quad i = 1,\cdots,l. \end{aligned} \tag{33}$$

Noting that Lemma 4.1 implies that $R_i + D_i' P_i D_i \geq 0$ for every i, we easily conclude that $(P_1(\cdot), \cdots, P_l(\cdot)) \in C^1(0,T;(\mathcal{S}^n)^l)$ solves (10). The representation of $(\overline{K}_1(\cdot), \cdots, \overline{K}_l(\cdot))$ is given by (32). This completes the proof. $\square$

5. Open-Loop Optimal Controls. In the previous analysis we have shown that the solvability of the CGREs (10) is equivalent to that the LQ problem is solvable by feedback controls. In this section we further prove that the solvability of the CGREs (10) is also equivalent to that the LQ problem is solvable by continuous *open-loop* controls.

Set

$$\begin{cases} \mathcal{M}_i(t) &= \dot{P}_i(t) + A_i'(t)P_i(t) + P_i(t)A_i(t) \\ & \quad + C_i(t)'P_i(t)C_i(t) + Q_i(t) + \sum_{j=1}^l \pi_{ij}P_j(t), \\ \mathcal{N}_i(t) &= R_i(t) + D_i(t)'P_i(t)D_i(t), \\ \mathcal{L}_i(t) &= P_i(t)B_i(t) + C_i(t)'P_i(t)D_i(t) + L_i(t). \end{cases}$$

Consider the following convex set of linear matrix inequalities (LMIs) on $[0,T]$:

$$\mathcal{P} \triangleq \left\{ \begin{array}{c} (P_1(\cdot), \cdots, P_l(\cdot)) \\ \in C^1(0,T;(\mathcal{S}^n)^l) \end{array} \;\middle|\; \begin{array}{l} \left[\begin{array}{c|c} \mathcal{M}_i(t) & \mathcal{L}_i(t) \\ \hline \mathcal{L}_i(t)' & \mathcal{N}_i(t) \end{array}\right] \geq 0, \text{ a.e. } t \in [0,T], \\ P_i(T) \leq H_i, \quad i = 1, \cdots, l \end{array} \right\}. \tag{34}$$

Let us show the following theorem which provides a sufficient condition for the well-posedness of the LQ problem.

THEOREM 5.1. *The LQ problem* (6) – (8) *is well-posed if the set* $\mathcal{P}$ *is nonempty.*

Proof. Let $(P_1(\cdot), \cdots, P_l(\cdot)) \in \mathcal{P}$. Setting $\varphi(t,x,i) = x'P_i(t)x$ and applying the generalized Itô formula, we have, for any admissible (open-loop) control $u(\cdot)$ and any initial $(s,y) \in [0,T) \times \mathbb{R}^n$,

$$\begin{aligned} J(s,y,i;u(\cdot)) &= y'P_i(s)y + E\Big[x(T)'(H_{r_T} - P_{r_T}(T))x(T) \;\Big|\; r_s = i\Big] \\ &\quad + E\Big\{\int_s^T [\Gamma\varphi(t,x(t),r_t) + x(t)'Q(t,r_t)x(t) \\ &\quad + 2u(t)'L(t,r_t)'x(t) + u(t)'R(t,r_t)u(t)]dt \;\Big|\; r_s = i\Big\}, \end{aligned} \tag{35}$$

where

$$\begin{aligned} & \Gamma\varphi(t,x,i) + x'Q_i(t)x + 2u'L_i(t)'x + u'R_i(t)u \\ = {} & x'[\dot{P}_i(t) + A_i(t)'P_i(t) + P_i(t)A_i(t) + C_i(t)'P_i(t)C_i(t) + Q_i(t) + \textstyle\sum_{j=1}^l \pi_{ij}P_j(t)]x \\ & + 2u'[B_i(t)'P_i(t) + D_i(t)'P_i(t)C_i(t) + L_i(t)']x + u'[R_i(t) + D_i(t)'P_i(t)D_i(t)]u \\ = {} & \begin{bmatrix} x \\ u \end{bmatrix}' \left[\begin{array}{c|c} \mathcal{M}_i(t) & \mathcal{L}_i(t) \\ \hline \mathcal{L}_i(t)' & \mathcal{N}_i(t) \end{array}\right] \begin{bmatrix} x \\ u \end{bmatrix}. \end{aligned}$$

Thus $J(s,y,i;u(\cdot)) \geq y'P_i(s)y$, implying $V(s,y,i) > -\infty, \forall (s,y) \in [0,T) \times \mathbb{R}^n$. $\square$

The following is the main result of this section.

THEOREM 5.2. *Assume that $B_i(t), C_i(t), D_i(t), Q_i(t), R_i(t)$ and $L_i(t)$ are continuous in t. Then the LQ problem (6) – (8) has a continuous optimal open-loop control for any initial $(s, y) \in [0, T] \times \mathbb{R}^n$ if and only if the CGREs (10) have a solution $(P_1(\cdot), \cdots, P_l(\cdot)) \in \mathcal{P}$.*

Proof. The "if" part follows from Theorem 3.1. Let us now show the "only if" part. Similar to Theorem 4.1, the dynamic programming approach yields that the value function $V(s, y, i)$ satisfy the HJB equation (26) and the boundary condition (27). Taking $u(\cdot) \equiv \bar{u} \in \mathbb{R}^{n_u}$, we obtain from (29)

$$\begin{cases} y'\Big(\dot{P}_i + P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \sum_{j=1}^{l} \pi_{ij}P_j\Big)y \\ \quad +\Big\{\bar{u}'(R_i + D_i'P_iD_i)\bar{u} + 2y'(P_iB_i + C_i'P_iD_i + L_i)\bar{u}\Big\} \geq 0, \\ P_i(T) = H_i, \qquad\qquad i = 1, \cdots, l. \end{cases}$$

This is equivalent to

$$\begin{cases} \begin{bmatrix} y \\ \bar{u} \end{bmatrix}' \left[\begin{array}{c|c} \mathcal{M}_i(s) & \mathcal{L}_i(s) \\ \hline \mathcal{L}_i(s)' & \mathcal{N}_i(s) \end{array}\right] \begin{bmatrix} y \\ \bar{u} \end{bmatrix} \geq 0, \quad \text{a.e. } s \in [0, T], \\ P_i(T) = H_i, \qquad\qquad i = 1, \cdots, l. \end{cases}$$

Since $y \in \mathbb{R}^n$ and $\bar{u} \in \mathbb{R}^{n_u}$ are arbitrary, we obtain

$$\begin{cases} \left[\begin{array}{c|c} \mathcal{M}_i(s) & \mathcal{L}_i(s) \\ \hline \mathcal{L}_i(s)' & \mathcal{N}_i(s) \end{array}\right] \geq 0, \quad \text{a.e. } s \in [0, T], \\ P_i(T) = H_i, \qquad\qquad i = 1, \cdots, l. \end{cases} \tag{36}$$

Applying Lemma 2.3 to (36) and noting Lemma 2.2-(ii), we have

$$\begin{cases} \mathcal{M}_i(t) - \mathcal{L}_i(t)\mathcal{N}_i(t)^\dagger\mathcal{L}_i(t)' \geq 0, \\ \mathcal{N}_i(t)\mathcal{N}_i(t)^\dagger\mathcal{L}_i(t)' - \mathcal{L}_i(t)' = 0, \\ \mathcal{N}_i(t) \geq 0, \qquad \text{a.e. } t \in [0, T], \\ P_i(T) = H_i, \qquad i = 1, \cdots, l. \end{cases} \tag{37}$$

Now, Let $(x^*(\cdot), u^*(\cdot))$ be an optimal open-loop control for (6)–(8) with respect to the initial condition $x^*(s) = y$. Setting $\varphi(t, x, i) = x'P_i(t)x$ and applying the generalized Itô formula, we have, as with (35),

$$\begin{aligned} &V(s, y, i) \\ = \;& y'P_i(s)y + E\Big\{\int_s^T [\Gamma\varphi(t, x^*(t), r_t) + x^*(t)'Q(t, r_t)x^*(t) \\ &+2u^*(t)'L(t, r_t)'x^*(t) + u^*(t)'R(t, r_t)u^*(t)]dt \;\Big|\; r_s = i\Big\}, \end{aligned} \tag{38}$$

where

$$\begin{aligned}
&\Gamma\varphi(t,x,i) + x'Q_i(t)x + 2u'L_i(t)'x + u'R_i(t)u \\
=& \begin{bmatrix} x \\ u \end{bmatrix}' \left[\begin{array}{c|c} \mathcal{M}_i(t) & \mathcal{L}_i(t) \\ \hline \mathcal{L}_i(t)' & \mathcal{N}_i(t) \end{array}\right] \begin{bmatrix} x \\ u \end{bmatrix} \\
=& \; x'[\mathcal{M}_i(t) - \mathcal{L}_i(t)\mathcal{N}_i(t)^\dagger \mathcal{L}_i(t)']x + [u + \mathcal{N}_i(t)^\dagger \mathcal{L}_i(t)'x]'\mathcal{N}_i(t)[u + \mathcal{N}_i(t)^\dagger \mathcal{L}_i(t)'x].
\end{aligned}$$

By virtue of the relation $V(s,y,i) = y'P_i(s)y$ and (38), we obtain

$$\mathcal{M}_i(t) - \mathcal{L}_i(t)\mathcal{N}_i(t)^\dagger \mathcal{L}_i(t)' = 0, \quad i = 1, \cdots, l,$$

i.e., (t is suppressed)

$$\begin{aligned}
&\dot{P}_i + P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \textstyle\sum_{j=1}^l \pi_{ij}P_j \\
&\quad -(P_iB_i + C_i'P_iD_i + L_i')(R_i + D_i'P_iD_i)^\dagger(B_i'P_i + D_i'P_iC_i + L_i') = 0, \\
&\qquad\qquad i = 1, \cdots, l,
\end{aligned} \tag{39}$$

which, along with (37), implies that $(P_1(\cdot), \cdots, P_l(\cdot))$ is a solution to the CGREs (10).

What remains to show is that *any* optimal control $u^*(\cdot)$ can be represented by (17) for some $Y_i(\cdot)$ and $z_i(\cdot)$. Since $u^*(\cdot)$ is optimal, by (38), the integrand in the right hand side of (38) must be zero almost everywhere in t. This implies, for every i,

$$\mathcal{N}_i(t)^{\frac{1}{2}}[u^*(t) + \mathcal{N}_i(t)^\dagger \mathcal{L}_i(t)'x^*(t)] = 0,$$

which leads to

$$\mathcal{N}_i(t)[u^*(t) + \mathcal{N}_i(t)^\dagger \mathcal{L}_i(t)'x^*(t)] = 0,$$

or equivalently,

$$\mathcal{N}_i(t)u^*(t) + \mathcal{L}_i(t)'x^*(t) = 0,$$

a.e. $t \in [s,T]$. To solve the above equation in $u^*(t)$, we apply Lemma 2.4 with

$$L = \mathcal{N}_i(t), \;\; M = I, \;\; N = -\mathcal{L}_i(t)'x^*(t).$$

Note that the condition (15) in the present case is implied by the third constraint in the CGREs (10), hence the general solution (16) with $z_i(t) = S$ and $Y_i(t) = 0$ yields that $u^*(\cdot)$ can be represented by (17). This completes the proof. $\square$

Theorem 5.1 indicates that the non-emptiness of the set $\mathcal{P}$ is sufficient for the well-posedness of the original LQ problem. The following result states that the non-emptiness of the set $\mathcal{P}$ is *necessary* for the attainability of the LQ problem.

THEOREM 5.3. *Under the same assumption of Theorem* 5.2, *the LQ problem* $(6)-(8)$ *has a continuous optimal open-loop control for any initial* $(s,y) \in [0,T] \times \mathbb{R}^n$ *only if the set* $\mathcal{P}$ *is nonempty.*

Proof. This is seen from (36). $\square$

6. An Example. In this section we give an example where the Markov chain has two states and where the singularity of $R_i(t) + D_i(t)'P_i(t)D_i(t)$ $(i = 1, 2)$ occurs, but the LQ problem is well-posed and attainable. Moreover, the example shows that the stochastic LQ problem can be well-posed even if $R_i(t)$, $i = 1, 2$, are negative.

Consider the following one-dimensional LQ problem

$$\begin{aligned}
\min \quad & J = E\Big\{\int_0^T [Q(t,r_t)x(t)^2 + 2L(t,r_t)x(t)u(t) + R(t,r_t)u(t)^2]dt \\
& \qquad + Hx(T)^2 \,\Big|\, r_0 = i\Big\}, \\
\text{s.t.} \quad & \begin{cases} dx(t) = [A(t,r_t)x(t) + B(t,r_t)u(t)]dt + [C(t,r_t)x(t) + D(t,r_t)u(t)]dW(t), \\ x(0) = x_0, \end{cases}
\end{aligned}$$

where $A(t,r_t) = A_i$, $B(t,r_t) = B_i$, $C(t,r_t) = C_i$, $D(t,r_t) = D_i$, $Q(t,r_t) = Q_i$ and $L(t,r_t) = L_i$ are constants, and $R(t,r_t) = R_i(t)$ when $r_t = i$. The coefficients are chosen such that $D_i \neq 0$, $B_i + D_iC_i = 0$, $L_i = 0$, $Q_i = 0$, $\pi_{ii} < 0$ for $i = 1, 2$, and $\pi_{11} \neq \pi_{22}$. Moreover, $R_i(t) = -D_i^2 P_i(t)$ $(i = 1, 2)$, where $P_i(\cdot)$ satisfies

$$\begin{cases} \dot{P}_1(t) = -[2A_1 + C_1^2 + \pi_{11}]P_1(t) + \pi_{11}P_2(t), \\ \dot{P}_2(t) = \pi_{22}P_1(t) - [2A_2 + C_2^2 + \pi_{22}]P_2(t), \\ P_1(T) = H, \\ P_2(T) = H. \end{cases} \tag{40}$$

It is easy to see that (40) is nothing but the system of CGREs (10) in the present case.

Set

$$a = -(2A_1 + C_1^2 + \pi_{11}) \quad \text{and} \quad b = -(2A_2 + C_2^2 + \pi_{22}).$$

Then (40) reduces to

$$\begin{cases} \dot{P}_1(t) = aP_1(t) + \pi_{11}P_2(t), \\ \dot{P}_2(t) = \pi_{22}P_1(t) + bP_2(t), \\ P_1(T) = H, \\ P_2(T) = H. \end{cases}$$

The above equation is solvable by

$$\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = He^{\zeta_1(t-T)} \cdot \frac{\zeta_2 - (a+\pi_{11})}{\sqrt{\Delta}} \cdot \begin{bmatrix} 1 \\ \frac{\zeta_1 - a}{\pi_{11}} \end{bmatrix} + He^{\zeta_2(t-T)} \cdot \frac{(a+\pi_{11}) - \zeta_1}{\sqrt{\Delta}} \cdot \begin{bmatrix} 1 \\ \frac{\zeta_2 - a}{\pi_{11}} \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = He^{\zeta_1(t-T)} \cdot \frac{\zeta_2 - (\pi_{22}+b)}{\sqrt{\Delta}} \cdot \begin{bmatrix} \frac{\zeta_1 - b}{\pi_{22}} \\ 1 \end{bmatrix} + He^{\zeta_2(t-T)} \cdot \frac{(\pi_{22}+b) - \zeta_1}{\sqrt{\Delta}} \cdot \begin{bmatrix} \frac{\zeta_2 - b}{\pi_{22}} \\ 1 \end{bmatrix},$$

where

$$\begin{cases} \zeta_1 = \frac{1}{2}[(a+b) - \sqrt{\Delta}], \\ \zeta_2 = \frac{1}{2}[(a+b) + \sqrt{\Delta}], \\ \Delta = (a-b)^2 + 4\pi_{11}\pi_{22}. \end{cases}$$

In fact, ζ_1 and ζ_2 are two different real roots of the following quadratic algebraic equation

$$\zeta^2 - (a+b)\zeta + ab - \pi_{11}\pi_{22} = 0. \tag{41}$$

Moreover, the following equalities hold

$$\begin{cases} \zeta_2 - (a+\pi_{11}) \cdot \frac{\zeta_1 - a}{\pi_{11}} = \zeta_2 - (\pi_{22} + b), \quad (a+\pi_{11}) - \zeta_1 \cdot \frac{\zeta_2 - a}{\pi_{11}} = (\pi_{22} + b) - \zeta_1, \\ \zeta_2 - (\pi_{22} + b) \cdot \frac{\zeta_1 - b}{\pi_{22}} = \zeta_2 - (a+\pi_{11}), \quad (\pi_{22} + b) - \zeta_1 \cdot \frac{\zeta_2 - b}{\pi_{22}} = (a+\pi_{11}) - \zeta_1. \end{cases}$$

By Theorem 3.1 and Corollary 3.1-(i), we see that the LQ problem is well-posed, and *any* admissible control is optimal with the optimal cost $P_i(0)x_0^2$ if $r_0 = i$. Furthermore, taking π_{11} and π_{22} so that $-\sqrt{\Delta} \leq (a-b) + 2\pi_{11} \leq \sqrt{\Delta}$ and $-\sqrt{\Delta} \leq (b-a) + 2\pi_{22} \leq \sqrt{\Delta}$, we have

$$\begin{cases} \zeta_2 - (a+\pi_{11}) \geq 0, \quad (a+\pi_{11}) - \zeta_1 \geq 0, \\ \zeta_2 - (\pi_{22} + b) \geq 0, \quad (\pi_{22} + b) - \zeta_1 \geq 0. \end{cases} \tag{42}$$

Hence, if H is chosen to be nonnegative, then $P_i(t)$ $(i = 1, 2)$ will be *nonnegative*; otherwise if H is negative then $P_i(t)$ $(i = 1, 2)$ will be *negative*. Finally, we note that the LQ problem is well-posed even though $R_i(t) = -D_i^2 P_i(t) \leq 0$ when choosing $P_i(t) \geq 0$.

7. Conclusion. This paper investigated the indefinite stochastic LQ control with Markovian jumps. A new type of Riccati equations was introduced to identify optimal controls and calculate the optimal cost value. Certain equivalent relations were established between the solvability of the Riccati equations and the well-posedness/solvability of the LQ problem.

The research on the indefinite stochastic LQ control has so far been limited to the so-called full information case, namely, the state and the Markov chain are both completely observable. In reality, however, most of the interesting cases are those of the partial information. Partially observed indefinite stochastic LQ control remains an important and challenging open problem.

REFERENCES

[1] M. Ait Rami and L. El Ghaoui, *LMI optimization for nonstandard Riccati equations arising in stochastic control*, IEEE Transactions on Automatic Control, 41(1996), pp. 1666–1671.

[2] M. Ait Rami, J. B. Moore, and X. Y. Zhou, *Indefinite stochastic linear quadratic control and generalized differential Riccati equation*, SIAM Journal on Control and Optimization, 40(2001), pp. 1296–1311.

[3] M. Ait Rami and X. Y. Zhou, *Linear matrix inequalities, Riccati equations, indefinite stochastic quadratic controls*, IEEE Transactions on Automatic Control, 45(2000), pp. 1131–1143.

[4] A. Albert, *Conditions for positive and nonnegative definiteness in terms of pseudo-inverse*, SIAM Journal on Applied Mathematics, 17(1969), pp. 434–440.

[5] S. Chen, X. Li and X. Y. Zhou, *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM Journal on Control and Optimization, 36(1998), pp. 1685–1702.

[6] Y. Ji and H. J. Chizeck, *Controllability, stabilizability, and continuous-time Markovian jump linear quadratic*, IEEE Transactions on Automatic Control, 35(1990), pp. 777–788.

[7] Y. Ji and H. J. Chizeck, *Jump linear quadratic Gaussian control in continuous-time*, IEEE Transactions on Automatic Control, 37(1992), pp. 1884–1892.

[8] N. N. Krasosvkii and E. A. Lidskii, *Analytical design of controllers in systems with random attributes I, II, III*, Automation and Remote Control, 22(1961), pp. 1021–1025, 1141–1146, 1289–1294.

[9] M. Mariton, *Jump Linear Systems in Automatic Control*, Marcel Dekker, New York, 1990.

[10] R. Penrose, *A generalized inverse of matrices*, Proceedings of the Cambridge Philosophical Society, 51(1955), pp. 406–413.

[11] W. M. Wonham, *Random differential equations in control theory*, Probabilistic Methods in Applied Mathematics, Academic Press, New York, 2(1970), pp. 131–212.

[12] Q. Zhang and G. Yin, *On nearly optimal controls of hybrid LQG problems*, IEEE Transactions on Automatic Control, 44(1999), pp. 2271–2282.

[13] X. Y. Zhou, *Markowitz's world in continuous time, and beyond*, in: Stochastic Models and Optimization, D.D. Yao, H. Zhang and X.Y. Zhou (eds.), Springer, New York, 2003.