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AN APPROACH TO H_∞ CONTROLLER SYNTHESIS OF PIECEWISE LINEAR SYSTEMS*

GANG FENG†, G. P. LU†, and S. S. ZHOU†

Abstract. This paper presents an H_∞ controller synthesis method for piecewise linear systems based on a piecewise smooth Lyapunov function. It is shown that the closed loop system is globally stable with guaranteed disturbance attenuation performance and the control law can be obtained by solving a set of Linear Matrix Inequalities (LMI) that is numerically feasible with commercially available software. A simulation example is presented to demonstrate the performance of the proposed method.

Keywords. Controller synthesis, Linear matrix inequality, Lyapunov functions, Piecewise linear systems

1. Introduction. Piecewise linear systems have been a subject of research in the systems and control community for some time, see for example [1-12]. In fact, the piecewise linear systems constitute a special class of hybrid systems [8] and arise often in practical control systems when piecewise linear components are encountered. These components include dead-zone, saturation, relays, and hysteresis. In addition, many other classes of nonlinear systems can also be approximated by the piecewise linear systems. Thus the piecewise linear systems provide a powerful means of analysis and design for nonlinear control systems.

A number of significant results have been obtained on analysis and controller design of such piecewise linear systems during the last few years. For example, the authors in [1] studied a basic issue, that is, the well-posedness of piecewise linear systems. Necessary and sufficient conditions for bimodal systems to be well-posed have been derived, and the extension to the multimodal case has also been discussed. The authors in [2-3] presented results on stability and optimal performance analysis for piecewise linear systems based on a piecewise continuous Lyapunov function. It has been shown that lower bounds, as well as upper bounds, on the optimal control cost can be obtained by semidefinite programming, and the framework of piecewise linear systems can be used to analyze smooth nonlinear systems with arbitrary accuracy. The authors in [4] discussed stability analysis and controller design of piecewise linear systems which may involve multiple equilibrium points based on a common quadratic Lyapunov function and a piecewise quadratic Lyapunov function. It has been shown that stability and performance analysis can be cast as convex optimization problems.

*Received on June 14, 2002; accepted for publication on September 20, 2002.
†Dept. of Manufacturing Engineering & Engineering Management, City University of Hong Kong, Tat Chee Ave., Kowloon, Hong Kong, Tel: (852) 2788 9032, Fax: (852) 2788 8423. E-mail: megfeng@cityu.edu.hk

A controller design method based on a common quadratic Lyapunov function and a linear matrix inequality has been proposed. However, it has been pointed out [4] that the controller synthesis based on the piecewise quadratic Lyapunov function cannot be easily cast as a convex optimization problem. In fact, there are very few results, to our best knowledge, on effective controller synthesis methods of piecewise linear systems based on piecewise Lyapunov functions.

Motivated from the results of piecewise continuous Lyapunov functions in [2-5], we will first present a new stability result based on a novel piecewise Lyapunov function and then develop a new constructive H_∞ controller synthesis method. It will be shown that global stability of the resulting closed loop system can be established, and moreover, the controller design procedure can be cast as solving a set of LMIs that is numerically feasible with commercially available software.

The rest of the paper is organised as follows. Section 2 introduces the piecewise linear system model and its piecewise quadratic stability. Section 3 presents an H_∞ controller synthesis method for the piecewise linear systems based on the piecewise Lyapunov function, which is followed by a simulation example in section 4. Finally, conclusions are given in section 5.

2. Piecewise Linear System Model and Its Piecewise Quadratic Stability. Consider the piecewise linear systems of the form

$$\dot{x}(t) = A_l x(t) + B_l u(t) + D_l v(t), \tag{2.1}$$

$$z(t) = H_l x(t) + G_l u(t),$$

$$\text{for } x \in \bar{S}_l, l = 1, 2, ..., m,$$

where $\{\bar{S}_l\}_{l \in L} \subseteq \Re^n$ denotes a partition of the state space into a number of closed polyhedral subspaces, L is the index set of subspaces, $x(t) \in \Re^n$ the system state variables, $u(t) \in \Re^p$ the system input variable, $v(t) \in \Re^q$ the external disturbance, and $(A_l, B_l, D_l, H_l, G_l)$ the *l-th* nominal local model of the system. For the definition of state trajectory and solution to the piecewise linear system in (2.1) please refer to [1-3] for details. Here we assume that given any initial condition $x(0) = x_0$, input signal u, and disturbance v, the differential equation (2.1) has a unique solution for all $t > 0$.

Our goal is to design a control law such that stability and H_∞ performance of the closed loop control system is guaranteed. We assume that each local model is controllable.

Recently, the authors in [2] introduced a kind of piecewise Lyapunov functions and developed a stability result based on this piecewise Lyapunov function for the piecewise linear systems. The key idea is to make the piecewise Lyapunov function continuous across the subspace boundaries.

As shown in [2], in order to find the piecewise Lyapunov function continuous across subspace boundaries, the matrices $F_l = \; l \in L$ need to be constructed, which is used to characterize the boundary among the subspaces,

$$F_l x = F_j x, \quad x \in \bar{S}_l \cap \bar{S}_j, \quad l, j \in L. \tag{2.2}$$

Then the piecewise Lyapunov function candidates that are continuous across the subspace boundaries can be parameterized as,

$$V(x) = x^T P_l x, \quad x \in \bar{S}_l, \; l \in L \tag{2.3}$$

with

$$P_l = F_l^T T F_l, \quad l \in L, \tag{2.4}$$

where the free parameters of the Lyapunov function candidate are characterized by the symmetric matrix T.

Remark 2.1. *A systematic procedure for constructing these matrices* F_l, $l \in L$ *for a given piecewise linear system can be found in [2-3]. It is also noted that* F_l, $l \in L$ *can be always constructed such that they are of full column rank. The interested readers please refer to [2-3] for details.*

The stability result Theorem 1 based on the above piecewise Lyapunov function presented in [2] can be easily used for stability checking. But unfortunately it seems difficult for the stability result to be directly used for numerically tractable controller synthesis discussed in the next section. In particular, with this stability result, it seems that it is difficult to formulate the controller synthesis problem as a numerically tractable LMI. In order to solve this problem, we develop the following alternative stability result.

Theorem 2.1. *Consider the piecewise linear system (2.1) with* $u \equiv v \equiv 0$. *If there exists a non-singular symmetric matrix* T *such that with*

$$P_l = (F_l^T F_l)^{-1} F_l^T T F_l (F_l^T F_l)^{-1}, \quad l \in L, \tag{2.5}$$

the following LMIs are satisfied,

$$0 < P_l, \quad l \in L \tag{2.6}$$

$$P_l A_l^T + A_l P_l < 0, \quad l \in L \tag{2.7}$$

then the piecewise linear system is globally exponentially stable, that is, $x(t)$ *tends to the origin exponentially for every continuous piecewise trajectory in the state space.*

Proof. Consider the following Lyapunov function candidate $V(t)$,

$$V(x) = x^T P_l^{-1} x, \quad x \in \bar{S}_l,\ l \in L. \tag{2.8}$$

It follows from (2.5) that

$$P_l^{-1} = F_l^T T^{-1} F_l, \quad l \in L, \tag{2.9}$$

and thus the function $V(t)$ is continuous across the subspace boundaries as the function in (2.3). It is obvious from (2.8) and (2.6) there exists constants $\alpha > 0$ and $\beta > 0$such that

$$\alpha \, ||x||^2 \leq V(t) \leq \beta \, ||x||^2. \tag{2.10}$$

In addition, (2.7) imply that there exists a constant $\rho > 0$such that

$$A_l^T P_l^{-1} + P_l^{-1} A_l + \rho I < 0. \tag{2.11}$$

Then along trajectories of the system, we have

$$\begin{aligned} \tfrac{d}{dt} V(t) &= x^T [A_l^T P_l^{-1} + P_l^{-1} A_l] x \\ &\leq x^T (-\rho I) x \\ &= -\rho ||x||^2. \end{aligned} \tag{2.12}$$

Therefore, the desired result follows directly from (2.10) and (2.12) based on the standard Lyapunov theory (see the Lemma 1 in [2]). □

The conditions in the theorem 2.1 are linear matrix inequalities in the variable T. A solution to those inequalities ensures $V(x)$ defined in (2.3) to be a Lyapunov function for the system.

REMARK 2.2. *The stability test of the piecewise linear system in eqn. (2.6)-(2.7) can be easily facilitated by a commercially available software package Matlab LMI toolbox [13,14].*

The objective of this paper is to design a suitable controller for the system (2.1) with a guaranteed performance in the H_∞ sense, that is, given a prescribed level of disturbance attenuation $\gamma > 0$, find a controller such that the induced L_2-norm of the operator from $v(t)$ to the controlled output $z(t)$ is less than γ under zero initial conditions,

$$\|z(t)\|_2 < \gamma \, \|v(t)\|_2$$

for all nonzero $v(t) \in l_2$. In this case, the closed loop control system is said to be globally stable with disturbance attenuation γ.

3. $\mathbf{H_\infty}$ Controller Design of Piecewise Linear Systems. In this section, we will address the H_∞ state feedback controller design problem for the piecewise linear systems introduced in the last section. The proposed controller design approach is based on the local linear model defined in each subspace. Consider the piecewise linear system model (2.1) on every subspace,

$$\dot{x}(t) = A_l x(t) + B_l u(t) + D_l v(t), \tag{3.1}$$

$$z(t) = H_l x(t) + G_l u(t),$$

$$\text{for } x(t) \in \bar{S}_l.$$

For the stabilization of the piecewise linear system (2.1) or equivalently (3.1), we consider the following piecewise continuous controller as

$$u(t) = K_l x(t) \quad x(t) \in \bar{S}_l, \quad l \in L. \tag{3.2}$$

With the control law (3.2), the global closed loop system is obtained by combining the piecewise linear system (3.1) and the controller (3.2), and can be described by the following equation in each local subspace as,

$$\dot{x}(t) = A_{cl} x(t) + D_{cl} v(t), \quad x(t) \in \bar{S}_l \tag{3.3}$$

$$z(t) = H_{cl} x(t).$$

where

$$A_{cl} = A_l + B_l K_l, \quad D_{cl} = D_l, \quad H_{cl} = H_l + G_l K_l.$$

Then we are ready to present the following lemma.

LEMMA 3.1. *Given a constant $\gamma > 0$, the piecewise linear system (3.3) is globally stable with disturbance attenuation γ, if there exists a non-singular symmetric matrix Tsuch that with*

$$P_l = (F_l^T F_l)^{-1} F_l^T T F_l (F_l^T F_l)^{-1}, \quad l \in L, \tag{3.4}$$

the following matrix inequalities are satisfied,

$$0 < P_l, \tag{3.5}$$

$$0 > P_l A_{cl}^T + A_{cl} P_l + \gamma^{-2} D_{cl} D_{cl}^T + P_l H_{cl}^T H_{cl} P_l, \tag{3.6}$$

$$\text{for } \; l \in L.$$

Proof. It is easily seen that eqn. (3.6) implies the following inequality,

$$0 > P_l A_{cl}^T + A_{cl} P_l, \quad l = 1, 2, ..., m$$

and thus with this inequality and (3.5), it follows from Theorem 2.1 that the closed loop system is globally stable.

Now we show the disturbance attenuation performance. Consider the Lyapunov function,

$$V(x) = x^T P_l^{-1} x, \quad x \in \bar{S}_l,\ l \in L \tag{3.7}$$

which is continuous across the subspace boundaries. Differentiating $x^T P_l^{-1} x,\ x \in \bar{S}_l,\ l \in L$ and then integrating from zero to infinity, we obtain, for $l \in L$,

$$\begin{aligned}
&\int_0^\infty \frac{d}{dt}(x^T P_l^{-1} x)\, dt \\
&= \int_0^\infty [x^T (P_l^{-1} A_{cl}^T + A_{cl} P_l^{-1}) x + v^T D_{cl}^T P_l^{-1} x + x^T P_l^{-1} D_{cl} v]\, dt \\
&< \int_0^\infty [x^T (-\gamma^{-2} P_l^{-1} D_{cl} D_{cl}^T P_l^{-1} - H_{cl}^T H_{cl}) x + v^T D_{cl}^T P_l^{-1} x + x^T P_l^{-1} D_{cl} v]\, dt \\
&= \int_0^\infty [-z^T z + \gamma^2 v^T v - (v^T - \gamma^{-2} x^T P_l^{-1} D_{cl}) \gamma^2 (v - \gamma^{-2} D_{cl}^T P_l^{-1} x)]\, dt \\
&\le \int_0^\infty [-z^T z + \gamma^2 v^T v]\, dt
\end{aligned} \tag{3.8}$$

that is,

$$V(x(\infty)) - V(x(0)) \le \int_0^\infty [-z^T z + \gamma^2 v^T v]\, dt$$

which implies that with $x(0) = 0$,

$$||z||_2 \le \gamma ||v||_2$$

and thus the proof is completed. □

REMARK 3.1. *It is noted that the state trajectory of the system may pass a number of subspaces and thus the subscript l in the integration of eqn.(3.8) may change. However, this does not alter the result obtained in eqn.(3.8).*

Then we have the following result.

THEOREM 3.1. *Given a constant $\gamma > 0$, the piecewise linear system (3.3) is globally stable with disturbance attenuation γ, if there exist a non-singular symmetric matrix T, and matrices Q_l, such that with*

$$P_l = (F_l^T F_l)^{-1} F_l^T T F_l (F_l^T F_l)^{-1}, \quad l \in L \tag{3.9}$$

the following LMIs are satisfied,

$$0 < P_l, \quad l = 1, 2, ..., m \tag{3.10}$$

$$0 > \begin{bmatrix} \Omega_l & P_l H_l^T + Q_l^T G_l^T \\ H_l P_l + G_l Q_l & -I \end{bmatrix}, \quad l = 1, 2, ..., m \tag{3.11}$$

where

$$\Omega_l := P_l A_l^{\mathrm{T}} + A_l P_l + Q_l^T B_l^{\mathrm{T}} + B_l Q_l + \gamma^{-2} D_l D_l^T.$$

Moreover, the controller gain for each local subsystem is given by

$$K_l = Q_l P_l^{-1}, \quad l \in L. \tag{3.12}$$

Proof. Based on the Lemma 3.1, we learn that the system (3.3) is globally stable with disturbance attenuation γ, if there exists a non-singular symmetric matrix T such that P_l defined in (3.9) is positive definite and satisfies the following inequality

$$0 > P_l A_{cl}^T + A_{cl} P_l + \gamma^{-2} D_{cl} D_{cl}^T + P_l H_{cl}^T H_{cl} P_l, \quad l = 1, 2, ..., m. \tag{3.13}$$

It is noted that the right hand side of inequality (3.13) can be expressed as,

$$\begin{aligned} RH &:= P_l A_{cl}^T + A_{cl} P_l + \gamma^{-2} D_{cl} D_{cl}^T + P_l H_{cl}^T H_{cl} P_l \\ &= P_l (A_l + B_l K_l)^T + (A_l + B_l K_l) P_l \\ &+\gamma^{-2} D_l D_l^T + P_l (H_l + G_l K_l)^T (H_l + G_l K_l) P_l \\ &= P_l A_l^{\mathrm{T}} + A_l P_l + Q_l^T B_l^{\mathrm{T}} + B_l Q_l \\ &+\gamma^{-2} D_l D_l^T + (P_l H_l^T + Q_l^T G_l^T)(H_l P_l + G_l Q_l) \end{aligned}$$

where $Q_l = K_l P_l$. Then it follows that the following inequality

$$\begin{aligned} 0 > {} & P_l A_l^{\mathrm{T}} + A_l P_l + Q_l^T B_l^{\mathrm{T}} + B_l Q_l \\ & +\gamma^{-2} D_l D_l^T + (P_l H_l^T + Q_l^T G_l^T)(H_l P_l + G_l Q_l) \end{aligned} \tag{3.14}$$

is equivalent to (3.13). Using the Schur complements, it then can be easily shown that (3.11) is equivalent to (3.14), and thus equivalent to (3.13). Therefore, it can be concluded that the closed loop control system is globally stable with disturbance attenuation γ, and the controller gains are given by (3.12), and thus the proof is completed. □

REMARK 3.2. *It should be noted that the conditions expressed in the theorem are only sufficient and thus the closed loop control system may still be stable even if the piecewise Lyapunov function cannot be identified from the above controller synthesis method.*

4. Simulation Example. In this section, we will use a numerical example to demonstrate the advantage of the proposed approach over the approach based on the common Lyapunov function and the performance of the proposed approach.

Consider a piecewise linear system

$$\dot{x}(t) = A_l x(t) + B_l u(t) + D_l v(t)$$

$$z_l(t) = H_l x(t) \tag{4.1}$$

$$l = 1, \cdots, 4,$$

with four region partitions shown in Fig. 1. The system matrices are given by

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.1 \\ -0.5 & -1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & -1 \end{bmatrix},$$

$$B_1 = B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D_1 = D_2 = D_3 = D_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$H_1 = H_2 = H_3 = H_4 = [1 \quad 0],$$

and $v(t) = 2\sin(2\pi t)$.

The matrices characterising the regions are given as follows,

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} E_1 \\ I \end{bmatrix}, \quad F_2 = \begin{bmatrix} E_2 \\ I \end{bmatrix}, \quad F_3 = \begin{bmatrix} E_3 \\ I \end{bmatrix}, \quad F_4 = \begin{bmatrix} E_4 \\ I \end{bmatrix}.$$

It is noted that the open loop system is unstable and that there is no solution to the common quadratic Lyapunov function based approach. However, if using the piecewise Lyapunov function approach proposed in this paper, then with $\gamma = 0.2$ the following solutions have been found,

$$T = \begin{bmatrix} 17.41 & -5.55 & 0 & 0 \\ -5.55 & 34.78 & 0 & 0 \\ 0 & 0 & -40.83 & -22.88 \\ 0 & 0 & -22.88 & 121.99 \end{bmatrix},$$

$$P_1 = P_3 = \begin{bmatrix} 0.0279 & -0.6130 \\ -0.6130 & 20.5864 \end{bmatrix}, \quad P_2 = P_4 = \begin{bmatrix} 2.4949 & -0.6130 \\ -0.6130 & 18.1194 \end{bmatrix},$$

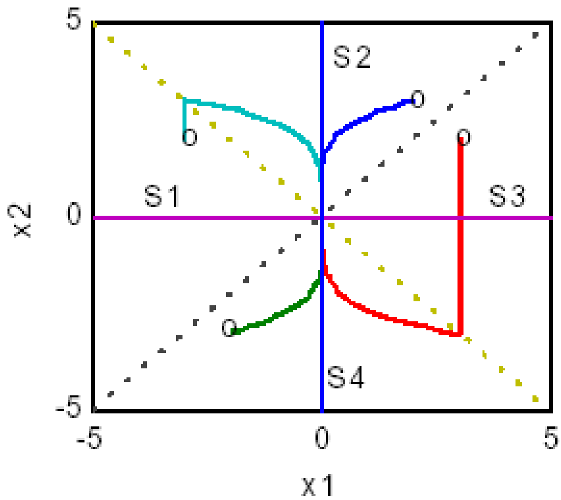

FIG. 1. *Responses of the closed loop control system*

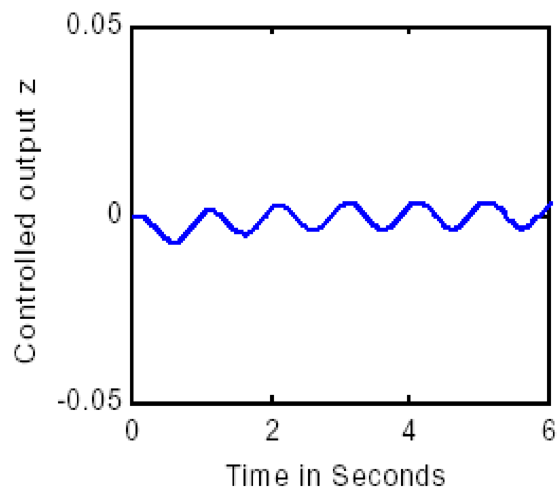

FIG. 2. *Response of the closed loop control system with disturbance*

$$K_1 = K_3 = \left[\begin{array}{cc} -217.68 & -6.54 \end{array}\right], \quad K_2 = K_4 = \left[\begin{array}{cc} -5.74 & -0.59 \end{array}\right].$$

It thus follows from the Theorem 3.1 that the stability and the disturbance performance of the closed loop control system are guranteed. Simulation results of four different initial conditions in the absence of the disturbance are reported in Fig. 1, which illustrate the stability of the closed loop control systems.

The simulation results in the presence of the disturbance and zero intial conditions are also reported in Fig. 2, which clearly demonstrate the disturbance attenuation performance of the proposed controller.

5. Conclusions. In this paper, a new method is developed to design an H_∞ controller for piecewise linear systems based on a piecewise Lyapunov function. It is shown that the controller can be obtained by solving a set of LMIs. It is believed that the idea can be extended to the controller synthesis of piecewise linear systems based on the discontinuous piecewise Lyapunov function proposed in [5].

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