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ON LYAPUNOV MAPPING AND ITS APPLICATIONS*

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Abstract. In this paper we investigate the Lyapunov mapping. Some results about the nature of the Lyapunov mapping are revealed. We first consider the properties of the spectrum and the norm of a Lyapunov mapping with those of its restrictions to the subspaces of symmetric and skew symmetric matrices. Secondly, some sufficient conditions are given for the existence of the common quadratic Lyapunov functions for a set of stable matrices. A norm estimation inequality is presented. Next effort is devoted to an algorithm for constructing a common quadratic Lyapunov function of a pair of matrices. The algorithm is based on norm estimation and several properties of Lyapunov mapping which are related to the numerical computations.

Key Words. Lyapunov mapping, system with switching models, spectrum, norm, common quadratic Lyapunov function.

1. Introduction. In recent years there has been much interest in the problem of finding a common Lyapunov function of a set of systems [1-3]. It is closely related to the stability and stabilization of the systems with switching or uncertain models[4-7], H_∞ control [8], and hybrid systems[9-10].

A particular attention has been paid to the common quadratic Lyapunov function for a set of linear systems because it has some special properties such as

1. The set of systems which share a common quadratic Lyapunov function is convex. Hence it assures that in a linear or near-linear switching the switching process is also stable.

2. It can be applied to nonlinear systems for local stability near equilibrium states.

The main tool for the approach in this paper is the Lyapunov mapping. We give the definition first. Through the paper we use M_n for the vector space of $n \times n$ matrices, use S_n and K_n for its symmetric and skew-symmetric subspaces respectively.

DEFINITION 1.1. [1-2] *1. Given an $n \times n$ matrix A. The mapping, $L_A : M_n \to M_n$, defined as*

$$L_A(X) := AX + XA^T \tag{1.1}$$

is called the Lyapunov mapping of A.

2. A is anti-stable if $\Re\sigma(A) > 0$.

In fact $-A$ is stable.

A positive definite matrix P is called the common quadratic Lyapunov function of two anti-stable matrices A and B if $L_A(P) > 0$ and $L_B(P) > 0$. In fact, it means

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the quadratic form $x^T Px$ is a common Lyapunov function of the two linear systems

$$\dot{x} = -Ax, \quad \dot{x} = -Bx.$$

The *matrix expression* of L_A is a matrix M_A such that $V(L_A(X)) = M_A V(X)$. Then (L_A is used for M_A) [1]

$$L_A = A \otimes I + I \otimes A. \tag{1.2}$$

That is

$$V(L_A(X)) = (A \otimes I + I \otimes A)V(X).$$

Let $X \in M_n$. $V : M_n \to R^{n^2}$ arranges the elements of X row by row as $V(X) = (x_{11}\ x_{12}\ \cdots\ x_{nn})^T$. V^{-1} stands for the inverse mapping of V.

We define an $n^2 \times n^2$ square matrix, $W_{[n]}$, in the following way: index the rows by: $(1,1)\ (1,2)\ \cdots\ (1,n)\ \cdots\ (n,1)\ \cdots\ (n,n)$ and the columns by: $(1,1)\ (2,1)\ \cdots\ (n,1)\ \cdots\ (1,n)\ \cdots\ (n,n)$. The $W_{[n]} = (w_{(i,j)(I,J)})$ is defined as

$$w_{(i,j)(I,J)} = \begin{cases} 1, & (i,j) = (I,J) \\ 0, & otherwise. \end{cases} \tag{1.3}$$

We call W_n the swap matrix because for a given $n \times n$ matrix A

$$V(A^T) = W_{[n]}V(A), \quad \text{and} \quad V(A) = W_{[n]}V(A^T).$$

2. The Spectrum of L_A. Recall that S_n and K_n are the symmetric and skew-symmetric subspaces of $M_{n\times n}$. It is easy to see that both S_n and K_n are invariant subspaces of L_A.

In the following we use L_A^S and L_A^K for the restrictions of L_A on S_n and K_n respectively. The spectrum of L_A^S and L_A^K will be determined in this section. To begin with, we consider the spectrum of L_A.

The following Lemma and Theorem are well known or easily verifiable.

LEMMA 2.1. *Assume two sets of vectors $X_i,\ i = 1,\cdots,s \in R^n$ and $Y_j,\ j = 1,\cdots,t \in R^n$ are both linearly independent. Then $X_i \otimes Y_j,\ i = 1,\cdots,s,\ q = 1,\cdots t$ are linearly independent.*

THEOREM 2.2. *Let $\{\lambda_1,\cdots,\lambda_n\}$ be eigenvalues of A. Then the eigenvalues of L_A^S are $\{\lambda_i+\lambda_j \mid 1 \le i \le j \le n\}$ and the eigenvalues of L_A^K are $\{\lambda_i+\lambda_j \mid 1 \le i < j \le n\}$.*

As an application, we consider the problem of structure invariance. Let N be an n dimensional manifold and $\omega \in T^2(N)$ be a quadratic tensor field. Given a dynamic system

$$\dot{x} = f(x), \quad x \in N.$$

ω is said to be f-invariant if along the integral curve, $\phi_f^t(x)$, of f the ω is invariant. That is, $(\phi_f^t) * (\omega) = \omega$(Remark: "*"is a supscription). (Refer to [11] for the concepts.)

This definition is a generalization of the structure invariance of Hamiltonian systems [12], where ω is required to be a non-singular closed two forms.

Consider a linear dynamic system

$$\dot{x} = Ax, \quad x \in R^n, \tag{2.1}$$

and looking for a quadratic tensor field, ω with a constant structure matrix, M_ω. That is

$$\omega(x,y) = x^T M_\omega y, \quad \forall x, y \in R^n.$$

The above result can be used to test whether (2.1) has an invariant structure.

COROLLARY 2.3. *System (2.1) has a symmetric invariant structure, iff there exist eigenvalues, λ_i, $\lambda_j \in \sigma(A)$, such that $\lambda_i + \lambda_j = 0$. A has a skew-symmetric quadratic invariant structure, iff there exist eigenvalues, λ_i, $\lambda_j \in \sigma(A)$, $i \neq j$, such that $\lambda_i + \lambda_j = 0$.*

Proof. Since $\phi^t_{Ax}(x) = exp(At)x$. ω is Ax invariant, iff for any $x, y \in R^n$

$$\omega(exp(At)x, exp(At)y) = \omega(x,y), \quad \forall x, y \in R^n.$$

That is equivalent to

$$exp(At)^T M_\omega exp(At) = M_\omega. \tag{2.2}$$

Using Taylor expansion, we have, by simply denote $M = M_\omega$ and collecting terms,

$$\begin{cases} A^T M + MA = 0 \\ \frac{1}{2}A^2TM + A^T MA + \frac{1}{2}MA^2 = 0 \\ \cdots \\ \frac{1}{k!}\sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} A^{-(k-i)T} MA^i \\ \cdots . \end{cases} \tag{2.3}$$

We have only to find M, which satisfies the first equation of (2.3). Because if M satisfies it, then exchanging A^T with M, we have

$$\frac{1}{k!}\sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} A^{-(k-i)T} MA^i = (1-1)^k MA^k = 0.$$

Now the problem becomes find a solution of

$$L_{A^T}(M) = 0.$$

The exitance of nonzero M follows from Theorem 2.3. The only thing we would like to show is how to construct M. When A has a pair of opposite eigenvalues, then we have ξ_i and ξ_j as the corresponding eigenvectors. Set $\xi_i \otimes \xi_j + \xi_j \otimes \xi_i$. According to Lemma 2.1, it is non-zero and $M = V^{-1}(\xi_i \otimes \xi_j + \xi_j \otimes \xi_i)$ is symmetric. Similarly, $M = V^{-1}(\xi_i \otimes \xi_j - \xi_j \otimes \xi_i)$ is also non-zero and skew-symmetric. □

3. Norm of L_A^S. For estimating stability radius of a stable matrix, A, the norm of L_A^S plays an important role. This section will investigate $\|L_A^S\|$.

PROPOSITION 3.1. *The norm of L_A satisfies*

$$\|L_A\| = \max\{\|L_A^S\|, \|L_A^K\|\}. \tag{3.1}$$

Proof.

$$\begin{aligned}
\|L_A\|^2 &= \sup_{0\neq M\in M_n} \frac{< L_AM, L_AM >}{< M, M >} \\
&= \sup_{0\neq (S,K)\in (S_n,K_n)} \frac{< L_A(S+K), L_A(S+K) >}{< S+K, S+K >} \\
&= \sup_{0\neq (S,K)\in (S_n,K_n)} \frac{< L_A^S S, L_A^S S > + < L_A^K K, L_A^K K >}{< S, S > + < K, K >} \\
&\leq \max\{\|L_A^S\|^2, \|L_A^K\|^2\}
\end{aligned}$$

The other direction of the inequality is trivial. The conclusion follows. □

Recall the definition of the swap matrix $W = W_{[n]}$, the following expression is obvious.

$$L_A^S = L_A\left(\frac{I+W}{2}\right)|_{S_n}. \tag{3.2}$$

$$L_A^K = L_A\left(\frac{I-W}{2}\right)|_{K_n}. \tag{3.3}$$

In fact we can prove the following:

PROPOSITION 3.2.

$$\|L_A^S\| = \|L_A\left(\frac{I+W}{2}\right)\|, \tag{3.4}$$

and

$$\|L_A^K\| = \|L_A\left(\frac{I-W}{2}\right)\|. \tag{3.5}$$

Proof.

$$\begin{aligned}
\|L_A(\frac{I+W}{2})\| &= \sup_{0\neq X\in M_n} \frac{\|L_A(\frac{I+W}{2})X\|}{\|X\|} \\
&\geq \sup_{0\neq S\in S_n} \frac{\|L_A(\frac{I+W}{2})S\|}{\|S\|} = \sup_{0\neq S\in S_n} \frac{\|L_A^S\|}{\|S\|} = \|L_A^S\|.
\end{aligned}$$

On the other hand, decompose X into symmetric and skew symmetric parts as

$$X = X_S + X_K, \quad \text{where } X_S \in S_n, \quad X_K \in K_n.$$

Then

$$\begin{aligned}\|L_A(\frac{I+W}{2})\| &= \sup_{0\neq X\in M_n} \frac{\|L_A(\frac{I+W}{2})(X_S+X_K)\|}{\|X_S+X_K\|} \\ &= \sup_{0\neq X\in M_n} \frac{\|L_A X_S\|}{\|X_S+X_K\|} \leq \sup_{0\neq X\in M_n} \frac{\|L_A X_S\|}{\|X_S\|} \\ &= \sup_{0\neq X_S\in S_n} \frac{\|L_A X_S\|}{\|X_S\|} = \|L_A^S\|.\end{aligned}$$

Both inequalities imply (3.4). Equation (3.5) can be proved in a similar way. □

Next, we consider a special case when A is normal.

PROPOSITION 3.3. *Assume A is normal. Then*

1. *All L_A, $L_A(\frac{I+W}{2})$ and $L_A(\frac{I-W}{2})$ are normal.*
2. *The norm of L_A (or L_A^S) is $2\max\{|\lambda| \mid \lambda \in \sigma(A)\}$.*
3. *If in addition, $\Re\sigma(A) < 0$ (or $\Re\sigma(A) > 0$), then $\|(L_A)^{-1}\| = \|(L_A^S)^{-1}\|$.*

Proof. 1. Since $AA^T = A^TA$, then

$$\begin{aligned}&(A\otimes I + I\otimes A)^T(A\otimes I + I\otimes A) = (A^T\otimes I + I\otimes A^T)(A\otimes I + I\otimes A)\\ &\quad = A^TA\otimes I + A^T\otimes A + A\otimes A^T + I\otimes A^TA\\ &\quad = AA^T\otimes I + A^T\otimes A + A\otimes A^T + I\otimes AA^T\\ &\quad = (A\otimes I + I\times A)(A\otimes I + I\otimes A)^T,\end{aligned}$$

L_A is normal.

It is easy to show that for any A

$$L_A(\frac{I+W}{2}) = (\frac{I+W}{2})L_A.$$

Using this fact and (3.11), we have

$$\begin{aligned}&\tfrac{(I+W)^T}{2}(A\otimes I + I\otimes A)^T(A\otimes I + I\otimes A)\tfrac{I+W}{2}\\ &= \tfrac{I+W^{-1}}{2}(A^T\otimes I + I\otimes A^T)(A\otimes I + I\otimes A)\tfrac{I+W}{2}\\ &= \tfrac{I+W^{-1}}{2}(A^TA\otimes I + I\otimes A^TA + A\otimes A + A\otimes A^T)\tfrac{I+W}{2}\\ &= \tfrac{I+W^{-1}}{2}(A\otimes I + I\times A)(A^T\otimes I + I\times A^T)\tfrac{I+W}{2}\\ &= (A\otimes I + I\otimes A)\tfrac{I+W^{-1}}{2}\tfrac{I+W}{2}(A^T\otimes I + I\otimes A^T)\\ &= (A\otimes I + I\otimes A)\tfrac{I+W}{2}\tfrac{I+W^{-1}}{2}(A^T\otimes I + I\otimes A^T)\\ &= (A\otimes I + I\otimes A)\tfrac{I+W}{2}\tfrac{(I+W)^T}{2}(A^T\otimes I + I\otimes A^T).\end{aligned}$$

Hence $L_A(\frac{I+W}{2})$ is normal. Similarly, $L_A(\frac{I-W}{2})$ is normal.

2. Since L_A is normal, $\|L_A\| = max\{|r| \mid r\in\sigma(L_A)\} = max\{|\lambda_i+\lambda_j| \mid \lambda_i,\lambda_j \in \sigma(A)\}$. According to Theorem 2.2, $\lambda_i+\lambda_j \in \sigma(L_A^S)$. Hence $\|L_A^S\| \geq max\{|r| \mid r\in \sigma(L_A^S)\} = max\{|\lambda_i+\lambda_j| \mid \lambda_i,\lambda_j\in\sigma(A)\}$. But $\|L_A^S\| \leq \|L_A\|$. Hence $\|L_A^S\| = \|L_A\|$.

3. Note that the additional condition of stable (or anti-stable) assures the invertibility of L_A. Now since S_n is L_A invariant, it is easy to see that

$$(L_A^S)^{-1} = (L_A)^{-1}|_S.$$

Similar argument as in 2 shows that

$$\|(L_A)^{-1}\| = \|(L_A^S)^{-1}\| = max\{\frac{1}{|\lambda_i + \lambda_j|} \mid \lambda_i, \lambda_j \in \sigma(A)\}.$$

□

When $n = 2$, assume $\sigma(A) = \{\lambda_1, \lambda_2\}$. Using Theorem 2.2, since $\dim(K_2) = 1$, $\|K_2\| = |\lambda_1 + \lambda_2|$. But for any matrix M, $\|M\| \geq \max\{|\lambda_i| \mid \lambda_i \in \sigma(M)\}$. Theorem 2.2 implies that $\|L_A^S\| \geq \|L_A^K\|$. So, $\|L_A\| = \|L_A^S\|$.

Our conjecture is $\|L_A\| = \|L_A^S\|$.

4. Common Lyapunov Functions for Two Matrices. Based on the technique used in [1], this section will give conditions under which two matrices share a common quadratic Lyapunov function.

Denote by P_n the set of positive semi-definite matrices, and its interior, $int(P_n)$, the set of positive definite matrices. Let $A_i, \quad i = 1, \cdots, k$ be a set of finite square matrices. $A_i, \quad i = 1, \cdots, k$ share a common quadratic Lyapunov function if there exists a $P \in int(P_n)$ such that

$$A_i P + P A_i^T \in int(P_n), \quad i = 1, \cdots, k. \tag{4.1}$$

The dual equivalent statement [13] says that (4.1) is equivalent to:

$$\sum_{i=1}^{k} (A_i^T Y_i + Y_i A_i) \in -P_n \tag{4.2}$$

has no nonzero solution $(Y_1, \cdots, Y_k) \in (P_n)^k$.

For a notational convenience, we consider when a set of anti-stable matrices A_i satisfy (4.1). It is obvious that this statement is equivalent to a set of stable matrices $B_i = -A_i$ to share a common quadratic Lyapunov function. Precisely, let

$$\dot{x} = B_i x, \quad i = 1, \cdots, k.$$

Set a quadratic Lyapunov function $V = x^T P x$. Then (4.1) assures that

$$\dot{V}|_i = B_i P + P B_i^T < 0, \quad i = 1, \cdots, k.$$

So V is a common quadratic Lyapunov function for all k linear dynamic systems.

The following lemma and two convenient sufficient conditions for them to share a common quadratic Lyapunov function are presented in [1] as:

LEMMA 4.1. *Let L be a linear mapping on S_n. If*

$$L^T + L > 0, \tag{4.3}$$

then there exists a $Q \in int(P_n)$ such that $L(Q) \in int(P_n)$.

Then the following result is obtained as

THEOREM 4.2. [1] *Assume either A or B is anti-stable, and one of the following two conditions is satisfied, then A and B share a common quadratic Lyapunov function.*

$$(L_A)^T L_B + (L_B)^T L_A > 0. \tag{4.4}$$

$$(L_A)^T (L_B)^T L_A L_B + (L_B)^T (L_A)^T L_B L_A > 0. \tag{4.5}$$

Based on Lemma 4.1 , these two conditions in Theorem 4.2 can be generalized as

THEOREM 4.3. *Let A, B be as in Theorem 4.2. They share a common quadratic Lyapunov function if there exists a linear mapping $G : S_n \to S_n$, which is invertible and $G^{-1} : int(P_n) \to int(P_n)$, such that*

$$(L_A)^T G L_B + (L_B)^T G^T L_A > 0; \tag{4.6}$$

Particularly, a class of such G has the form $G = \prod_{i=1}^k L_{S_i}$, where S_i, $i = 1, \cdots, k$, are anti-stable matrices. Set $S_i = I_n$, (4.4) follows. Set $S_1 = B^T$, $S_2 = A$, (4.5) follows.

If we set $S_1 = (A^{-1})^T$, $S_2 = B^T$, we get a new condition as

$$(L_A)^T (L_{A^{-1}})^T (L_B)^T L_B + (L_B)^T L_B L_{A^{-1}} L_A > 0; \tag{4.7}$$

EXAMPLE 4.4. Consider two matrices

$$A = \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0.7 & -1 \\ 1 & -0.3 \end{pmatrix}.$$

It is easy to check that A and B satisfy neither (4.4) nor (4.5), but they do satisfy (4.7). Hence, they have common quadratic Lyapunov function. □

Next, we try to unify the test conditions in [1] with those in [2]. Let $\pi(P_n)$ be the set of linear mappings: $P_n \to P_n$, and $\pi(\text{int}(P_n))$ be the set of linear mappings: $\text{int}(P_n) \to \text{int}(P_n)$.

Two more conditions for anti-stable matrices A and B to share a common quadratic Lyapunov function are given as

THEOREM 4.5. [2] *If one of A and B is anti-stable, and there exists a mapping $G > 0$, $G^{-1} \in \pi(P_n)$ and one of the following holds:*

$$(L_A)^T G L_B + (L_B)^T G L_A > 0, \tag{4.8}$$

$$(L_A)^T (L_B)^T G L_A L_B + (L_B)^T (L_A)^T G (L_A L_B) > 0. \tag{4.9}$$

This result can be improved. We need the following lemma.

LEMMA 4.6. *Assume a linear mapping $G : S_n \to S_n$ is nonsingular. Then $G \in \pi(P_n)$, iff, $G \in \pi(\text{int}(P_n))$.*

Proof. Let $G \in \pi(\text{int}(P_n))$. Since P_n is a closed subset of M_n, which is a complete metric space with standard metric inherited from R^{n^2} and any linear operator over a finite dimensional normed space is continuous then by the continuity of G it is easy to see that $G(P_n) \subset (P_n)$, i.e., $G \in \pi(P_n)$.

Conversely, if $G \in \pi(P_n)$, we have only to show that for any $p \in \text{int}(P_n)$, $G(p) \in \text{int}(P_n)$. If not, then $G(p)$ should be on the boundary, B, of $\text{int}(P_n)$. Since $\text{int}(P_n)$ is an open set in S_n, there exists an open neighborhood, $p \in V \subset \text{int}(P_n)$. Since G is non-singular, it is a local diffeomorphism. That is, $G(V) \ni G(p)$ is an open set in S_n. But $G(p)$ is on the boundary. Note that under the subspace topology $S_n \subset R^{n\times n}$, the boundary is $B = P_n \backslash \text{int}(P_n)$. Hence $G(p) \in B$ implies that $G(V)$ contains points which are not in P_n. This is a contradiction. □

Using Lemma 4.6, and Theorem 4.5, the following is an immediate consequence.

THEOREM 4.7. *Assume the linear mapping $G : S_n \to S_n$ is non-singular and $G^{-1} \in \pi(P_n)$, or equivalently $G^{-1} \in \pi(\text{int}(P_n))$. Moreover*

$$(L_A)^T G L_B + (L_B)^T G^T L_A > 0, \tag{4.10}$$

where A or B is anti-stable. Then A and B share a common quadratic Lyapunov function.

According to Theorem 4.7, the condition $G > 0$ in Theorem 4.5 can be relaxed by "G is nonsingular". So Theorem 4.7 implies Theorems 4.2 and 4.5.

It is interesting that Theorem 4.7 has a dual result as

THEOREM 4.8. *Assume that $G \in \pi(P_n)$ is linear and that*

$$(L_A)^T G L_B + (L_B)^T G^T L_A \leq 0, \tag{4.11}$$

where either A or B is anti-stable. Then A and B share no common quadratic Lyapunov function.

Proof. Suppose a common quadratic Lyapunov function, $x^T P x$, exists. Then

$$\begin{aligned} &V^T(P)((L_A)^T G L_B + (L_B)^T G^T L_A) V(P) = \\ &\langle L_A(P), G(L_B(P))\rangle + \langle G(L_B(P)), L_A(P)\rangle > 0, \end{aligned} \tag{4.12}$$

a contradiction. □

Corresponding to the positive statements of equations (4.4), (4.5) and (4.7), we have the following:

COROLLARY 4.9. *Assume A and B are anti-stable. They share no common quadratic Lyapunov function if one of the following holds:*

$$(L_A)^T L_B + (L_B)^T L_A \leq 0; \tag{4.13}$$

$$(L_A)^T(L_B)^{-T}(L_A)^{-1}L_B + (L_B)^T(L_A)^{-T}(L_B)^{-1}L_A \leq 0; \tag{4.14}$$

$$(L_A)^T(L_A)^{-T})(L_B)^{-T}L_B + (L_B)^T(L_B)^{-1}(L_A)^{-T}L_A \leq 0. \tag{4.15}$$

Same argument as used in the proof of Theorem 4.8 can be used to prove the Corollary 4.9.

5. A Common Lyapunov Function for a Set of Matrices. Let $\{A_1 \cdots A_k\}$ be a set of anti-stable matrices. We give a necessary and sufficient condition for them to have a common quadratic Lyapunov function.

PROPOSITION 5.1. *The set of matrices* $\{A_1 \cdots A_k\}$ *have a common quadratic Lyapunov function if and only if the following matrix inequalities*

$$[I + L_{D_i}(L_C)^{-1}](S) > 0, \quad i = 1, \cdots k-1 \tag{5.1}$$

have a positive solution $S > 0$*. Where* $C = A_k$*, and* $D_i = A_i - C$*.*

Proof. (Sufficiency) Since $L_A(X)$ is linear with respect to A, (5.1) can be re-written as

$$[L_{C+D_i}(L_C)^{-1}](S) = [L_{A_i}(L_C)^{-1}](S) > 0.$$

It is easy to see that $(L_C)^{-1}(S)$ is the common quadratic Lyapunov function.

(Necessity) Assume the common quadratic Lyapunov function is X^tPX. Then $S = L_C(P)$ is a required solution. □

Note that C can be any matrix in the set. Using the Lyapunov mapping, the above result becomes simple. In fact, some early results, e.g., some sufficient conditions in [6-7], can be obtained by setting $S = I$.

Proposition 5.1 cannot provide a common quadratic Lyapunov function directly. There are two problems, which have to be solved. First of all, how to choose C. Secondly, how to find a suitable S? The second is more critical.

It is easy to see that the set of positive definite matrices is an open convex cone in S_n. Therefore, we hope that the norm of the shifting term, operator $L_{D_i}(L_C)^{-1}$, can be as "small" as possible. In such a way, the image of S can be as close to S as possible, and hence remain in the cone.

We give an estimation of the norm of L_C.

PROPOSITION 5.2.

$$\sqrt{2}\|C\|_2 \leq \|L_C\| \leq 2\|C\|_2, \tag{5.2}$$

where the norm $\|\cdot\|_2$ *stands for the norm of linear operators on* R^n*.*

Proof. The right half is obvious because

$$\|L_C X\| = \|CX + XC^T\| \leq 2\|C\|_2\|X\|.$$

As for the left half of (5.2) let $r = \|C\|_2$. Then there exists a ξ, with $\|\xi\| = 1$ such that $C^T C\xi = r^2\xi$. Since $V^{-1}(\xi \otimes \xi) = \xi\xi^T$ and $\|\xi\xi^T\| = 1$ we have

$$
\begin{aligned}
\|L_C\|^2 &\geq \|L_C(\xi\xi^T)\|^2 \\
&= (\xi\otimes\xi)^T(C^T\otimes I + I\otimes C^T)(C\otimes I + I\otimes C)(\xi\otimes\xi) \\
&= (\xi\otimes\xi)^T(C^TC\otimes I + I\otimes C^TC)(\xi\otimes\xi) \\
&\quad +(\xi\otimes\xi)^T(C^T\otimes C + C\otimes C^T)(\xi\otimes\xi).
\end{aligned}
\tag{5.3}
$$

By definition of ξ, the first term is $2r^2$. Now we have only to show that the second term is non-negative. In fact it is because

$$
\begin{aligned}
&(\xi\otimes\xi)^T(C^T\otimes C + C\otimes C^T)(\xi\otimes\xi) \\
&= (\xi^TC^T\xi)\otimes(\xi^TC\xi) + (\xi^TC\xi)\otimes(\xi^TC^T\xi) = (\xi^TC\xi)^2 \geq 0.
\end{aligned}
$$

□

In general, (5.2) is sharp. Consider

$$
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
$$

It is easy to verify that for this A the left half of (5.2) becomes an equality.

Motivated by (5.2), we propose the following way to select C: Choose $C \in \{A_i, \mid i = 1, \cdots, k\}$ with largest norm $\|C\|_2$. Then

$$
\begin{aligned}
\|(L_C)^{-1}\| &= \sup_{X\neq 0}\frac{\|(L_C)^{-1}(X)\|}{\|X\|} = \sup_{X\neq 0}\frac{1}{\frac{\|L_C(X)\|}{\|X\|}} \\
&= \frac{1}{\inf_{X\neq 0}\frac{\|L_C(X)\|}{\|X\|}} \leq \frac{1}{\sqrt{2}\|C\|}.
\end{aligned}
$$

In the next section an algorithm will be developed to search S.

6. An Algorithm. Now we consider the problem of searching S with a fixed C. First of all, we consider the case of two matrices. This is the most important case because in most cases the switchings occur between two models. Given two anti-stable matrices A and B, say, we choose $C = B$ and $D = A - C$. Then consider the mapping

$$
F(S) := S + L_D(L_C)^{-1}(S). \tag{6.1}
$$

Since F is a linear mapping, we denote it by a matrix F. The rows of F are indexed by a set of double indices

$$
\Lambda = \{(ij) \mid 1 \leq i, j \leq n\}
$$

in the order of $(11, 21, \ldots, n1, \ldots, 1n, \ldots, nn)$(Remark: the notation above is not defined.).

Let $s = V(S)$ be the row stacking form of S, and $T = FS$ with the row stacking form $t = V(T)$. Then we can express (6.1) as

$$
t = \begin{pmatrix} f_{11} \\ \cdots \\ f_{nn}. \end{pmatrix}
$$

Equivalently, we have

$$t_{ij} = f_{ij}s, \quad i,j = 1,\cdots,n. \tag{6.2}$$

Now F is a given matrix. We are looking for an $S > 0$ such that $T = F(S) > 0$. Denote by $T^k(S)$ the k-th principal minor of T. A necessary condition is the following.

LEMMA 6.1. *$T = F(S)$ cannot be positive definite for any S if for some $1 \le k \le n$*

$$\det(T^k(S)) = 0, \quad \text{for all } S. \tag{6.3}$$

Equivalently, the following equation holds:

$$\sum_{\sigma\in\mathbf{S_k}} det \begin{pmatrix} f_{11}^{\lambda_{\sigma(1)}} & f_{12}^{\lambda_{\sigma(2)}} & \cdots & f_{1k}^{\lambda_{\sigma(k)}} \\ f_{21}^{\lambda_{\sigma(1)}} & f_{22}^{\lambda_{\sigma(2)}} & \cdots & f_{2k}^{\lambda_{\sigma(k)}} \\ & & \cdots & \\ f_{k1}^{\lambda_{\sigma(1)}} & f_{k2}^{\lambda_{\sigma(2)}} & \cdots & f_{kk}^{\lambda_{\sigma(k)}} \end{pmatrix} = 0, \tag{6.4}$$

for all distinct $\lambda_1, \lambda_2, \cdots, \lambda_k \in \Lambda$, where $f_{ij}^{\lambda_i}$ is the λ_i-th component of f_{ij}, $\mathbf{S_k}$ is the k-th symmetric group, i.e., σ is a permutation for k indices $\lambda_1 \cdots, \lambda_k$.

Proof. (6.3) is obvious. Denote

$$c_k(s) = \det(T^k(S)), \quad k = 1,\cdots,n.$$

Then

$$c_k(s) = \det \begin{pmatrix} \sum_{i,j=1}^n f_{11}^{ij}s_{ij} & \sum_{i,j=1}^n f_{12}^{ij}s_{ij} & \cdots & \sum_{i,j=1}^n f_{1k}^{ij}s_{ij} \\ \sum_{i,j=1}^n f_{21}^{ij}s_{ij} & \sum_{i,j=1}^n f_{12}^{ij}s_{ij} & \cdots & \sum_{i,j=1}^n f_{2k}^{ij}s_{ij} \\ & & \cdots & \\ \sum_{i,j=1}^n f_{k1}^{ij}s_{ij} & \sum_{i,j=1}^n f_{12}^{ij}s_{ij} & \cdots & \sum_{i,j=1}^n f_{kk}^{ij}s_{ij} \end{pmatrix}.$$

It is clear that $c_k(S)$ is a k-th homogeneous polynomial of S. Hence $c_k(S) \equiv 0$ if and only if

$$\frac{\partial^k c_k}{\partial s_{\lambda_1} \cdots \partial s_{\lambda_k}} = 0, \quad \text{for all } \lambda_1, \lambda_2, \cdots, \lambda_k \in \Lambda.$$

Since each column of $T^k(S)$ is linear in s, for a chosen set of $\lambda_1, \lambda_2, \cdots, \lambda_k \in \Lambda$ to get the above differentiation, we can only differentiate each column with distinct λ_j, $j = 1, 2, \cdots, k$, which yields (6.4). □

In general, (6.3) is not verifiable. Unlike (6.3), (6.4) is easily verifiable by computer.

As a necessary condition, we can assume that F satisfies the following

A1: For any k, equation (6.4) is not identical to zero.

Denote by G^k_{ij} the co-factor of t_{ij} in $T^k(S)$. Observe the following two facts: 1. since $c_k(S)=\sum\limits_{j=1}^{k} t_{ij}G^k_{ij}$ and G^k_{ij}, $\forall j$ are independent of $t_{i,j}$, we have

$$\frac{\partial c_k}{\partial t_{ij}} = G^k_{ij}.$$

2. Since t_{ij} is a linear function of s,

$$\nabla_s t_{ij} = (f_{ij})^T.$$

These two facts yield that the gradient of $c_k(s)$ is

$$\nabla c_k(s) = \sum_{i,j=1}^{k} \frac{\partial c_k(s)}{\partial t_{ij}} \nabla t_{ij} = \sum_{i,j=1}^{k} G^k_{ij}(f_{ij})^T. \tag{6.5}$$

We propose an algorithm for searching S. Choose, say $S_0 = I$. Then at i-th step, if for s_i all $c_k(s_i) > 0$, $k = 1,\cdots,n$, we are done. Otherwise, we search a *feasible direction* X_i such that for all k with $c_k(s) \le 0$, they will increase in the direction of X_i (for s near s_i), and then set $s_{i+1} = s_i + LX_i$, where $L > 0$ is a suitable step-length.

Let's see how to find X at point S. Say, $c_k(s) \le 0$ for $k = k_1, k_2, \cdots, k_p$. Without loss of generality, we can assume

$$X = \sum_{i=1}^{p} x_i \nabla c_{k_i}(s). \tag{6.6}$$

We want to find X such that

$$\langle \nabla c_{k_i}(S), X\rangle = dc_{k_i}(S)X > 0, \quad i = 1,2,\cdots,p,$$

where $dc_{k_i}(s) = (\nabla c_{k_i}(s))^T$.

We construct a $p \times n^2$ matrix G in the following way:

Step 1: Index the columns of G by Λ in natural order.

Step 2: Set

$$G_{l,(ij)} = \begin{cases} \frac{1}{k_l} G^{k_l}_{ij}, & i,j \le k_l \\ 0, & \text{otherwise}; \quad l = 1,2,\cdots,p \end{cases}$$

Note that if $k = 1$ then t_{ii} has no cofactor in T^1. We define $G^1_{11} = 1$. (It is consistent with the meaning of cofactors when calculating the determinant.) Then we have the following:

LEMMA 6.2.

1. Under assumption A1, for any $1 \le k \le n$, $G^kFx \ne 0$ for almost all s.

2.

$$\begin{pmatrix} < \nabla c_{k_1}(s), X > \\ \cdots \\ < \nabla c_{k_p}(s), X > \end{pmatrix} = GFF^TG^T \begin{pmatrix} x_1 \\ \cdots \\ x_p \end{pmatrix}. \tag{6.7}$$

3. $GFF^TG^T(s)$ is invertible for almost all s.

Proof. 1. Using (6.1) and the structure of G, one sees that

$$\det(T^k)(S) = G^kFs, \quad k = 1, 2, \cdots, n. \tag{6.8}$$

By assumption A1, $\{s \mid G^k(s)F = 0\}$ is an algebraic zero set.

2. From (6.5)

$$\begin{pmatrix} dc_{k_1}(s) \\ \cdots \\ dc_{k_p}(s) \end{pmatrix} = GF,$$

(6.7) follows.

3. It suffices to show that when $p = n$ the rows of $G(s)F$ are linearly independent for almost all S. Assume

$$\sum_{i=1}^{n} \lambda_i G^i(s)F = 0.$$

Right-multiply both sides by s, we have

$$\sum_{i=1}^{n} \lambda_i c_i(s) = 0.$$

Note that $c_i(S)$ is a homogeneous polynomial of degree i, So the above non-zero polynomial can have only an algebraic zero set. □

The following lemma shows that the search remains in S_n.

LEMMA 6.3. *$V^{-1}(X) \in S_n$, where X, defined by (6.6), is the searching direction.*

Proof. According to (6.5)-(6.6), it is enough to show that

$$V^{-1}(F^T(G^k)^T) \in S_n, \quad \text{for all } k.$$

Since G^k consists of cofactors of a symmetric matrix, it is easy to see that $V^{-1}((G^k)^T)$ $\in S_n$. Next

$$F^T = I + (L_C)^{-T}(L_A)^T = I + (L_{C^T})^{-1}L_{A^T}.$$

According to this form, to see that F^T maps S_n to S_n, we have only to show that $(L_{C^T})^{-1}$ maps S_n to S_n. It is guaranteed by the fact that both the symmetric set, S_n, and the skew symmetric set, K_n, are invariant subspaces of L_M for any $n \times n$ matrix M. Hence

$$X = F^TC^T \begin{pmatrix} x_{k_1} \\ \cdots \\ x_{k_p} \end{pmatrix} \in V(S_n).$$

□

Summarizing Lemmas 6.1, 6.2 and 6.3, we have

THEOREM 6.4. *Assume a point S is such that $c_{k_i}(s) \leq 0$, $i = 1, \cdots, p$. Under assumption A1 a feasible direction X, which makes*

$$< \nabla c_{k_i}(s), X > \;\; > 0, \quad i = 1, 2, \cdots, p$$

exists everywhere except an algebraic zero set.

Based on the above discussion, we propose the following algorithm for searching a suitable S.

ALGORITHM 6.5.

Step 1: Set $S_0 = I$;

Step j ($j \geq 1$): Check if $T^j(S_j) = F(S_j) > 0$. If the answer is "yes" we are done. Otherwise, say $k = k_1, \cdots, k_p$ are bad set, find a feasible direction by solving the following

$$GFF^TG^T\xi = \begin{pmatrix} \frac{|c_{k_1}(s_j)|}{\sum_{i=1}^{p} |c_{k_i}(s_j)|} \\ \cdots \\ \frac{|c_{k_p}(s_j)|}{\sum_{i=1}^{p} |c_{k_i}(s_j)|} \end{pmatrix} := b. \tag{6.9}$$

For some k if $c_k(s_j) = 0$, we may use a small $\epsilon > 0$ to replace $c_k(s_j)$. If GFF^TG is not invertible at s_j a small perturbation is used.

Find the step length $L > 0$ such that $s_{j+1} = s_j + LF^TG^T\xi > 0$. Check if $T^{j+1} = F(s_{j+1}) > 0$.

Note that in the algorithm $\min_{k=1}^n \det(T^k(S))$ is monotonically increasing. If finally we reach a step N such that $\min_{k=1}^n \det(T^k(S_N)) > 0$, we are done.

The algorithm can also be used for more than two matrices case. Say we have $A_1, \cdots, A_m$ and C, then construct G_i from A_i, get b_i by (6.9), and then we just need to solve

$$\begin{pmatrix} G_1F_1 \\ \cdots \\ G_mF_m \end{pmatrix} (F_1^TG_1^T \;\cdots\; F_m^TG_m^T)\xi = \begin{pmatrix} b_1 \\ \cdots \\ b_m \end{pmatrix}.$$

Finally, we produce $s_{j+1} = s_j + L(F_1^TG_1^T \;\cdots\; F_m^TG_m^T)\xi$.

Now we cannot assure the genetic exitance of ξ. But we still can use the algorithm, as long as a solution is obtained.

We give a detailed iteration description for the algorithm in the following example.

EXAMPLE 6.6. Consider the following two matrices A and B

$$A = \begin{pmatrix} 1.48 & 0 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 0.7 \end{pmatrix}, \quad B = \begin{pmatrix} 1.5 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Choose $C = B$, set $D = A - B$ and step length $L = 0.05$, and denote $c_{\min} = \min_{1 \leq r \leq n} c_r(s)$. Then

Time 1: $c_{\min} = -0.089221$

$$S_1 = \begin{pmatrix} 1.011334 & 0.015526 & -0.013654 \\ 0.015526 & 0.986942 & -0.019236 \\ -0.013654 & -0.019236 & 0.991166 \end{pmatrix}.$$

Time 2: $c_{\min} = -0.044875$

$$S_2 = \begin{pmatrix} 1.021876 & 0.037655 & -0.030578 \\ 0.037655 & 0.9683 & -0.043399 \\ -0.030578 & -0.043399 & 0.988218 \end{pmatrix}.$$

Time 3: $c_{\min} = -0.001875$

$$S_3 = \begin{pmatrix} 1.026091 & 0.070601 & -0.050337 \\ 0.070601 & 0.941036 & -0.071702 \\ -0.050337 & -0.071702 & 1.003722 \end{pmatrix}.$$

Time 4: $c_{min} = 10^{-6} > 0$. Now we are done. The solution is

$$S_4 = \begin{pmatrix} 1.026091 & 0.070601 & -0.050337 \\ 0.070601 & 0.941036 & -0.071702 \\ -0.050337 & -0.071702 & 1.003722 \end{pmatrix}.$$

The common quadratic Lyapunov function is determined by

$$P = (L_C)^{-1}(S_4) = \begin{pmatrix} 0.839428 & -0.746097 & 0.19495 \\ -0.746097 & 1.545942 & -0.537712 \\ 0.19495 & -0.537712 & 0.501861 \end{pmatrix}.$$

□

EXAMPLE 6.7. Consider the following three matrices:

$$A_1 = \begin{pmatrix} 2.3 & 0 & 0.8 \\ 1 & 3 & 2 \\ 1 & 0 & 0.7 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1.7 & 1 & 1 \\ 0.8 & 2.6 & 2.3 \\ 1.9 & 1.4 & 1.4 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1.9 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We choose $C = A_3$ and use above algorithm with $S_0 = I$ and step length $L = 0.05$. The following is the minimum values, $c_{\min}$, of $\det(T^k)$ in seven iterations:

$c_1 = -0.241598, c_2 = -0.19726, c_3 = -0.153845, c_4 = -0.109215, c_5 = -0.06367$, $c_6 = -0.019166$, $c_7 = 10^{-6} > 0$.

After 7 iterations we are done. The solution is

$$S_7 = \begin{pmatrix} 0.563234 & 0.201341 & 0.0629 \\ 0.201341 & 0.805909 & -0.173564 \\ 0.0629 & -0.173564 & 1.088162 \end{pmatrix}.$$

Then a common quadratic Lyapunov function is determined by the following $P > 0$

$$P = (L_C)^{-1}(S_7) = \begin{pmatrix} 0.500633 & -0.669585 & 0.239229 \\ -0.66958 & 1.664681 & -0.630863 \\ 0.239229 & -0.630863 & 0.544081 \end{pmatrix}.$$

□

7. Conclusion. In this paper certain properties of the Lyapunov mapping were investigated. Particularly, the spectrum decomposition and the norm are investigated. Some sufficient conditions for two matrices to share a common quadratic Lyapunov function were obtained.

The main effort is put on developing an numerical method for finding a common quadratic Lyapunov function for a set of anti-stable matrices. We first express a necessary and sufficient condition for the existence of such Lyapunov function by using the Lyapunov mappings. An estimation of the norm of the Lyapunov mappings was obtained to support the choice of a base matrix from the set. Certain properties were obtained for formulating the algorithm. A computer software was created to realize the algorithm. Two numerical examples, produced by the software, were presented to support the algorithm.

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