

## NECESSARY AND SUFFICIENT CONDITIONS FOR ADAPTIVE STABILIZABILITY OF JUMP LINEAR SYSTEMS\*

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**Abstract.** The adaptive stabilization problem of linear time-varying (LTV) systems with unknown parameters modeled as a hidden Markov chain is studied in this paper. A necessary and sufficient condition characterizing the adaptive stabilizability of the system is found, which hinges on the existence of a set of algebraic coupled Riccati-like equations that are different from those known for the non-adaptive case. Several equivalent characterizations and a constructive method for designing stabilizing feedback laws are also provided in this paper. It is worth mentioning that our results also reveal the capability and limitations of the feedback mechanism, as we have not restricted ourselves to any special (e.g., linear) class of feedback laws in our study.

**Key Words:** Adaptation, stabilization, estimation, Markov chain, feedback, Riccati equation.

**1. Introduction.** The primary motivation and the main advantage of adaptive control are linked to dynamical systems with uncertain and changing structure/parameters, and a standard model for the investigation of adaptive systems is described by finite-dimensional linear time-varying (LTV) systems with parameters governed by a finite state Markov chain.

In the non-adaptive case where the Markovian jump parameters can be observed directly, much progress has been made on feedback stabilization and the related linear-quadratic control problems; see [1],[2], [3], [4], [5] and [6], among many others. The adaptive case, however, has received less attention in the literature, and a complete characterization of stabilizability by adaptive feedback is still lacking, see for example, [7],[8],[9] and [10].

By adaptive feedback we mean the (nonlinear) feedback that captures the uncertain information of the system by properly utilizing the measured on-line system data. Intuitively, adaptive feedback should at least be able to capture slowly time-varying structures of a system and, in particular, to stabilize linear systems with hidden but slowly-jumping Markovian parameters. In a recent work [11], it has been shown in a rigorous way that adaptive stabilization is possible whenever the rate of transition of the underlying Markov chain is small enough.

However, to get a comprehensive understanding of the capability of adaptive feedback, one may naturally be concerned about the following questions: How fast the rate of parameter changes can be captured by adaptation? Can we find a critical value of the rate of parameter changes in characterizing the adaptive stabilizability? What are the key factors on which the capability of adaptation depends? These are puzzling questions which are still lack of general theoretical understandings. As a starting

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point towards understanding the above questions, some initial effort has been made in [12], where a simple first-order linear system with Markovian jump parameters was considered. It was demonstrated in [12] that the key factor inherent in characterizing the adaptive stabilizability is the information uncertainty of the underlying Markov chain, coupled with the model complexity exhibited by the dispersion of the state values of the system. The rate of parameter changes, however, is found to be not a key factor in characterizing the capability of adaptation.

In the present paper, we shall study the adaptive stabilization problem of a general class of linear time-varying (LTV) systems with hidden Markov jump parameters, and provide a general necessary and sufficient condition to characterize the adaptive stabilizability. Such a condition depends on the existence of a set of algebraic coupled Riccati-like equations that are different from those already known for the non-adaptive case. Several equivalent characterizations are also provided in this paper, which may be helpful in either numerical computations or further theoretical investigations. It is worth mentioning that our results may also give a quantitative evaluation of the capability and limitations of the feedback mechanism, as we have not restricted ourselves to any special (e.g., linear) class of feedback laws in our study.

In the next section, we will present the main results and give some discussions. Section III will give some auxiliary results that will be used in the proof of our main theorems in Section IV. Section V will conclude the paper with some remarks.

**2. The Main Results.** Consider the following linear time-varying model:

$$(1) \quad x_{t+1} = A(\theta_t)x_t + B(\theta_t)u_t + w_{t+1}, \quad t \geq 1;$$

where  $x_t \in R^n$ ,  $u_t \in R^m$  and  $w_{t+1} \in R^n$  are the state, input and noise vectors respectively. We assume that:

A1).  $\{\theta_t\}$  is an unobservable Markov chain which is homogeneous, non-reducible and non-periodic, and which takes values in a finite set  $\{1, 2, \dots, N\}$  with transition matrix denoted by  $P = (p_{ij})_{N \times N}$ , where by definition  $p_{ij} = P\{\theta_t = j | \theta_{t-1} = i\}$ ;

A2). There exists some  $m \times n$  matrix  $L$  such that

$$(2) \quad \det[(A_i - A_j) - (B_i - B_j)L] \neq 0, \quad \forall i \neq j.$$

where  $1 \leq i, j \leq N$ , and  $A_i \triangleq A(i) \in R^{n \times n}$ ,  $B_i \triangleq B(i) \in R^{n \times m}$  are the system matrices.

A3).  $\{w_t\}$  is a martingale difference sequence which is independent of  $\{\theta_t\}$ , and satisfies

$$(3) \quad \sigma I \leq Ew_t w_t', \quad Ew_t' w_t \leq \sigma_w, \quad \forall t$$

where  $\sigma$  and  $\sigma_w$  are two positive constants.

We remark that Condition A1) implies that each state in  $\{1, 2, \dots, N\}$  can be visited by  $\{\theta_t\}$  with positive probability when  $t$  is suitably large, while Condition A2)

is a sort of identifiability condition to be used later in the construction of stabilizing feedback laws. Moreover, the lower bound to the noise covariance in Condition A3) is assumed for simplicity of derivations, and the case where  $w_t = 0, \forall t$  can be treated analogously.

For simplicity of presentation, we will denote  $S \triangleq \{1, 2, \dots, N\}$  throughout the paper.

**Definition 2.1.** An input sequence  $\{u_t\}$  is said to be an admissible feedback if  $u_t \in \sigma\{x_0, x_1, \dots, x_t\}$  and  $E\|u_t\|^2 < \infty, \forall t$ , where  $\sigma\{x_0, x_1, \dots, x_t\}$  is the  $\sigma$ -algebra generated by the observed state information  $\{x_0, x_1, \dots, x_t\}$ . Moreover, the system (1) is said to be stabilizable by (adaptive) feedback if there exists an admissible feedback law  $\{u_t\}$  such that  $\sup_{t \geq 0} E\|x_t\|^2 < \infty$ .

The main result of this paper is stated as follows:

**Theorem 2.1.** Let the above assumptions A1)-A3) hold for the control system (1). Then the system is stabilizable by feedback if and only if the following coupled algebraic Riccati-like equations have a solution consisting of  $N$  positive definite matrices  $\{M_i > 0, i \in S\}$ :

$$(4) \quad \sum_j A'_j p_{ij} M_j A_j - \left( \sum_j A'_j p_{ij} M_j B_j \right) \left( \sum_j B'_j p_{ij} M_j B_j \right)^+ \left( \sum_j B'_j p_{ij} M_j A_j \right) - M_i = -I,$$

where  $i \in S$  and  $(\cdot)^+$  denotes the Moore-Penrose generalized-inverse of the corresponding matrix.

**Remark 2.1.** In contrast to most of the previous publications in the literature, we have neither restricted ourselves to the class of linear feedback laws, nor imposed any conditions on the rate of parameter changes. Hence, Theorem 2.1 enables us to explore the full capability and limitations of the feedback mechanism.

To further understand the key equation (4), we denote

$$(5) \quad L_i^* \triangleq L^*(M) \triangleq \left( \sum_{j=1}^N B'_j p_{ij} M_j B_j \right)^+ \left( \sum_{j=1}^N B'_j p_{ij} M_j A_j \right), \quad i \in S.$$

where and hereafter  $M \triangleq [M_1, \dots, M_N]'$ . Then by properties of generalized-inverse, it is easy to see that (4) can be rewritten in the following form:

$$(6) \quad \sum_{j=1}^N (A_j - B_j L_i^*)' p_{ij} M_j (A_j - B_j L_i^*) - M_i = -I, \quad i \in S.$$

For the convenience of future discussion, we denote the first term on the left-hand-side of (6) as  $\psi_i(M)$ , i.e.,

$$(7) \quad \psi_i(M) \triangleq \sum_{j=1}^N (A_j - B_j L_i^*)' p_{ij} M_j (A_j - B_j L_i^*), \quad i \in S.$$

Obviously,  $\psi_i(\cdot)$  is a nonlinear mapping from  $\prod_1^N R^{n \times n}$  to  $R^{n \times n}$ . Moreover, we denote

$$(8) \quad \psi(M) \triangleq (\psi_1(M), \psi_2(M), \dots, \psi_N(M))'$$

which plays a key role in characterizing the properties of the Riccati-like equation (4) as will be shown shortly.

The following theorem presents several conditions equivalent to the existence of the solution of the Riccati-like equations (4) used in Theorem 2.1, which may provide alternative ways for checking the stabilizability of control systems.

**Theorem 2.2.** The following four facts are equivalent :

(i). There exists a set of  $n \times n$  positive definite matrices  $\{M_i > 0, i \in S\}$  such that for any  $i \in S$

$$(9) \quad \sum_j A_j' p_{ij} M_j A_j - \left( \sum_j A_j' p_{ij} M_j B_j \right) \left( \sum_j B_j' p_{ij} M_j B_j \right)^+ \left( \sum_j B_j' p_{ij} M_j A_j \right) - M_i = -I.$$

(ii). There exists a set of  $m \times n$  matrices  $\{L_i, i \in S\}$  such that the following coupled equations have a solution consisting of positive definite matrices  $\{M_i > 0, i \in S\}$ :

$$(10) \quad \sum_{j=1}^N (A_j - B_j L_i)' p_{ij} M_j (A_j - B_j L_i) - M_i = -I, \quad i \in S.$$

(iii).  $\lim_{t \rightarrow \infty} V_t = 0$ , where  $\{V_t\}$  is defined recursively by

$$(11) \quad V_{t+1} = \psi(V_t), \quad V_0 = I^*$$

where  $\psi(\cdot)$  is defined in (8), and  $I^* \triangleq (I_{n \times n}, \dots, I_{n \times n})$ .

(iv).  $\lim_{t \rightarrow \infty} M_t^* < \infty$ , where  $\{M_k^*\} \in R^{nN \times n}$  is defined recursively by:

$$(12) \quad M_k^* = \psi(M_{k-1}^* + I^*); \quad k \geq 1, \quad M_0^* = 0^*.$$

where  $\psi(\cdot)$  and  $I^*$  are defined as in (iii) above, and  $0^* = (0_{n \times n}, \dots, 0_{n \times n})$ .

**Remark 2.2.** It is interesting to compare equation (6) with the following equation:

$$(13) \quad (A_i - B_i L_i)' \left[ \sum_{j=1}^N p_{ij} M_j \right] (A_i - B_i L_i) - M_i = -I, \quad i \in S.$$

which was used in Ji and Chizeck [6] for the case where the Markov chain  $\{\theta_t\}$  is assumed to be observable. Obviously, the existence of the solution of equation (6) derived in the present (adaptive) case should imply the existence of the solution of (13). This is the content of the following proposition whose proof is given in Appendix A.

**Proposition 2.1.** If there exists a set of  $m \times n$  matrices  $\{L_i, i \in S\}$  such that (6) has a solution  $\{M_i > 0, i \in S\}$ , then so does (13). However, the converse assertion is not true in general.

This proposition provides a rigorous comparison to the non-adaptive case, showing that the uncertainty in the system parameters  $\{\theta_t\}$  does indeed degrade the capability of the feedback mechanism.

**3. Some Auxiliary Lemmas. Lemma 3.1.** Let  $M = (M_1, \dots, M_N)'$  be a matrix consisting of positive definite matrices  $\{M_i, i \in S\}$ , and let  $\{\psi_i(M), i \in S\}$  be defined as in (7). Then for any  $m \times n$  matrices  $\{L_i, i \in S\}$ , the following matrix inequalities hold :

$$(14) \quad \sum_{j=1}^N (A_j - B_j L_i)' p_{ij} M_j (A_j - B_j L_i) \geq \psi_i(M) \quad \forall i \in S.$$

**Proof.** First of all, by the properties of the generalized-inverse, it is not difficult to verify that the matrix  $L_i^*, i \in S$  defined by (5) satisfies the following matrix identity:

$$\sum_j B_j' p_{ij} M_j A_j - \sum_j B_j' p_{ij} M_j B_j \cdot L_i^* = 0, \quad i \in S.$$

Then for any matrices  $L_i \in R^{m \times n}, i \in S$ , we have with some simple manipulations,

$$\begin{aligned} & \sum_j (A_j - B_j L_i)' p_{ij} M_j (A_j - B_j L_i) \\ &= \sum_j (A_j - B_j L_i^* + B_j (L_i^* - L_i))' p_{ij} M_j (A_j - B_j L_i^* + B_j (L_i^* - L_i)) \\ &= \sum_j (A_j - B_j L_i^*)' p_{ij} M_j (A_j - B_j L_i^*) + \sum_j (L_i^* - L_i)' B_j' p_{ij} M_j B_j (L_i^* - L_i) \\ & \quad + \sum_j (L_i^* - L_i)' B_j' p_{ij} M_j (A_j - B_j L_i^*) + \sum_j (A_j - B_j L_i^*)' p_{ij} M_j B_j (L_i^* - L_i) \\ &= \sum_j (A_j - B_j L_i^*)' p_{ij} M_j (A_j - B_j L_i^*) + \sum_j (L_i^* - L_i)' B_j' p_{ij} M_j B_j (L_i^* - L_i) \\ & \quad + (L_i^* - L_i)' \sum_j B_j' p_{ij} M_j (A_j - B_j L_i^*) + \left( (L_i^* - L_i)' \sum_j B_j' p_{ij} M_j (A_j - B_j L_i^*) \right)' \\ &= \sum_j (A_j - B_j L_i^*)' p_{ij} M_j (A_j - B_j L_i^*) + \sum_j (L_i^* - L_i)' B_j' p_{ij} M_j B_j (L_i^* - L_i) + 0 + 0 \\ &\geq \sum_j (A_j - B_j L_i^*)' p_{ij} M_j (A_j - B_j L_i^*) = \psi(M_i). \end{aligned}$$

This completes the proof.  $\square$

**Lemma 3.2.** Let the Markov chain  $\{\theta_t\}$  be given as in the system (1), and let us define for any  $1 \leq h \leq t$

$$(15) \quad \prod_{t,h} \triangleq \prod_{k=1}^h [A(\theta_{t+1-k}) - B(\theta_{t+1-k}) L_k(\theta_{t-k})],$$

where  $L_k(\theta_{t-k})$  is a  $\sigma\{\theta_{t-k}\}$  measurable  $m \times n$  random matrix for  $1 \leq k \leq h$ . Then we have:

(i). There exists a sequence of  $m \times n$  matrices  $\{L_{k,j}^*, k \geq 1, j \in S\}$  such that for any  $t \geq 1, i \in S$ ,

$$(16) \quad E \left\{ \prod_{t,t}^{*'} \prod_{t,t}^* | \theta_0 = i \right\} \leq E \left\{ \prod_{t,t}' \prod_{t,t} | \theta_0 = i \right\},$$

where  $\prod_{t,h}^* \triangleq \prod_{k=1}^h (A(\theta_{t+1-k}) - B(\theta_{t+1-k})L_k^*(\theta_{t-k}))$  with  $L_k^*(\theta_{t-k}) \triangleq \sum_{j=1}^N L_{k,j}^* I_{[\theta_{t-k}=j]}$ .

(ii). If we denote  $V_{t,i} \triangleq E \left\{ \prod_{t,t}^{*'} \prod_{t,t}^* | \theta_0 = i \right\}$ , and set  $V_t \triangleq (V_{t,1}, \dots, V_{t,N})'$ , then for any  $t \geq 0$ ,

$$V_{t+1} = \psi(V_t),$$

where  $V_0 = I^*$ , and  $\psi(\cdot)$  is defined as in (8).

**Proof.** We first recursively define a sequence of matrices  $\{\hat{V}_k\}$  as follows:

$$\hat{V}_{k+1} \triangleq \psi(\hat{V}_k), \quad k \geq 0, \quad \hat{V}_0 = I^*.$$

By (7) and (8), we know that there exists a sequence of  $m \times n$  matrices  $\{L_{k,i}^*, i \in S\}$  such that

$$(17) \quad \hat{V}_{k+1,i} = \sum_j (A_j - B_j L_{k+1,i}^*)' p_{ij} \hat{V}_{k,j} (A_j - B_j L_{k+1,i}^*).$$

We now proceed to show that  $\{L_{k,j}^*, j \in S, k \geq 1\}$  satisfies the requirement in (i) and that  $\hat{V}_t = V_t, t \geq 0$ .

Obviously, the above assertion holds trivially for  $t = 0$ . For  $t = 1$ , we have

$$\begin{aligned} & E \left\{ \prod_{1,1}' \prod_{1,1} | \theta_0 = i \right\} \\ &= E \left\{ (A(\theta_1) - B(\theta_1)L_1(\theta_0))' (A(\theta_1) - B(\theta_1)L_1(\theta_0)) | \theta_0 = i \right\} \\ &= E \left\{ (A(\theta_1) - B(\theta_1)L_{1,i})' (A(\theta_1) - B(\theta_1)L_{1,i}) | \theta_0 = i \right\} \\ &= \sum_j (A_j - B_j L_{1,i})' p_{ij} (A_j - B_j L_{1,i}) \end{aligned}$$

Hence by lemma 3.1 and the definition for  $L_{1,i}^*$  in (17) we know that the matrices  $\{L_{1,j}^*, j \in S\}$  satisfy (16) for the case of  $t = 1$ , and  $V_1 = \psi(V_0) = \psi(\hat{V}_0) = \hat{V}_1$ .

Now, let us assume that for  $t = \tau$ ,  $\{L_{k,j}^*, k \geq 1, j \in S\}$  satisfies the following inequality hold for any  $0 \leq h \leq \tau$

$$E \left\{ \prod_{h,h}^{*'} \prod_{h,h}^* | \theta_0 = i \right\} \leq E \left\{ \prod_{h,h}' \prod_{h,h} | \theta_0 = i \right\},$$

and at the same time make  $V_h = \hat{V}_h$ .

Then for  $t = \tau + 1$

$$\begin{aligned}
& E \left\{ \prod'_{\tau+1, \tau+1} \prod_{\tau+1, \tau+1} | \theta_0 = i \right\} \\
&= E \left\{ E \left\{ (A(\theta_1) - B(\theta_1)L_{\tau+1, i})' \prod'_{\tau+1, \tau} \prod_{\tau+1, \tau} (A(\theta_1) - B(\theta_1)L_{\tau+1, i}) | \theta_1 \right\} | \theta_0 = i \right\} \\
&= E \left\{ [A(\theta_1) - B(\theta_1)L_{\tau+1, i}]' E \left\{ \prod'_{\tau+1, \tau} \prod_{\tau+1, \tau} | \theta_1 \right\} [A(\theta_1) - B(\theta_1)L_{\tau+1, i}] | \theta_0 = i \right\} \\
&= \sum_j E \left\{ [A_j - B_j L_{\tau+1, i}]' E \left\{ \prod'_{\tau+1, \tau} \prod_{\tau+1, \tau} | \theta_1 = j \right\} [A_j - B_j L_{\tau+1, i}] I_{[\theta_1=j]} | \theta_0 = i \right\} \\
&= \sum_j [A_j - B_j L_{\tau+1, i}]' E \left\{ \prod'_{\tau+1, \tau} \prod_{\tau+1, \tau} | \theta_1 = j \right\} [A_j - B_j L_{\tau+1, i}] p_{ij}.
\end{aligned}$$

By the induction assumption and the homogeneous property, it is easy to see that

$$\begin{aligned}
& E \left\{ \prod'_{\tau+1, \tau} \prod_{\tau+1, \tau} | \theta_1 = j \right\} = E \left\{ \prod'_{\tau, \tau} \prod_{\tau, \tau} | \theta_0 = j \right\} \\
&\geq E \left\{ \prod^{*'}_{\tau, \tau} \prod^*_{\tau, \tau} | \theta_0 = j \right\} = E \left\{ \prod^{*'}_{\tau+1, \tau} \prod^*_{\tau+1, \tau} | \theta_1 = j \right\} \\
&= V_{\tau, j} = \hat{V}_{\tau, j}
\end{aligned}$$

Hence, we have

$$E \left\{ \prod'_{\tau+1, \tau+1} \prod_{\tau+1, \tau+1} | \theta_0 = i \right\} \geq \sum_j [A_j - B_j L_{\tau+1, i}]' p_{ij} V_{\tau, j} [A_j - B_j L_{\tau+1, i}]$$

By Lemma 3.1 and the definition of  $\{L_{k, j}^*, k \geq 1, j \in S\}$ , it is evident that if we take  $L_{\tau+1, i} = L_{\tau+1, i}^*, i \in S$ , then

$$E \left\{ \prod'_{\tau+1, \tau+1} \prod_{\tau+1, \tau+1} | \theta_0 = i \right\} \geq E \left\{ \prod^{*'}_{\tau+1, \tau+1} \prod^*_{\tau+1, \tau+1} | \theta_0 = i \right\} = V_{\tau+1} = \hat{V}_{\tau+1}.$$

Hence, the desired result is true by induction.  $\square$

**Lemma 3.3.** Let  $k \geq 1$ , and  $Z$  be a  $n \times n$  nonnegative definite random matrix which is bounded *a.s.* and is measurable with respect to  $\sigma(\theta_k, \theta_{k+1}, \dots)$ . Then the solution to the following optimization problem

$$\min_{u_k \in \mathcal{F}_k} E [A(\theta_k)x_k + B(\theta_k)u_k]' Z [A(\theta_k)x_k + B(\theta_k)u_k]$$

can be expressed as  $u_k = -L(\theta_{k-1})x_k$ , where  $L(\theta_{k-1}) \in \sigma(\theta_{k-1})$  is an  $m \times n$  random matrix and where  $\mathcal{F}_k \triangleq \sigma(\theta_0, \theta_1, \dots, \theta_{k-1}; w_1, w_2, \dots, w_k)$ .

**Proof.** Let us denote  $Z_j \triangleq E(Z | \theta_k = j), j \in S$ , and  $L^*(\theta_{k-1}) \triangleq \sum_{i=1}^N L_i^* I(\theta_{k-1} = i)$ , where for  $i \in S$

$$L_i^* \triangleq \left( \sum_{j=1}^N B_j' p_{ij} Z_j B_j \right)^+ \left( \sum_{j=1}^N B_j' p_{ij} Z_j A_j \right).$$

By the properties of the generalized-inverse, it is easy to see that

$$\sum_j B_j' p_{ij} Z_j (A_j - B_j L_i^*) = 0.$$

Hence,

$$\begin{aligned} & E\{B'(\theta_k)Z[A(\theta_k) - B(\theta_k)L^*(\theta_{k-1})]|\theta_{k-1}\} \\ &= \sum_{i=1}^N E\{B'(\theta_k)Z[A(\theta_k) - B(\theta_k)L_i^*]|\theta_{k-1} = i\}I(\theta_{k-1} = i) \\ (18) \quad &= \sum_{i,j}^N B_j' E[ZI(\theta_k = j)|\theta_{k-1} = i](A_j - B_j L_i^*)I(\theta_{k-1} = i) \\ &= \sum_{i,j}^N B_j' Z_j p_{ij} (A_j - B_j L_i^*)I(\theta_{k-1} = i) \\ &= 0. \end{aligned}$$

Now, for any  $u_k \in \mathcal{F}_k$ , we denote  $\tilde{u}_k = u_k + L^*(\theta_{k-1})x_k$ , then

$$\begin{aligned} & [A(\theta_k)x_k + B(\theta_k)u_k]' Z [A(\theta_k)x_k + B(\theta_k)u_k] \\ &= [(A(\theta_k) - B(\theta_k)L^*(\theta_{k-1}))x_k + B(\theta_k)\tilde{u}_k]' Z [(A(\theta_k) \\ (19) \quad & - B(\theta_k)L^*(\theta_{k-1}))x_k + B(\theta_k)\tilde{u}_k] \\ &= x_k' [A(\theta_k) - B(\theta_k)L^*(\theta_{k-1})]' Z [A(\theta_k) - B(\theta_k)L^*(\theta_{k-1})] x_k \\ & \quad + \tilde{u}_k' B(\theta_k)' Z B(\theta_k) \tilde{u}_k + 2\tilde{u}_k' B(\theta_k)' Z [A(\theta_k) - B(\theta_k)L^*(\theta_{k-1})] x_k \end{aligned}$$

By noticing (18), we have

$$\begin{aligned} & E\tilde{u}_k' B(\theta_k)' Z [A(\theta_k) - B(\theta_k)L^*(\theta_{k-1})] x_k \\ &= E\{E[\tilde{u}_k' B(\theta_k)' Z (A(\theta_k) - B(\theta_k)L^*(\theta_{k-1}))x_k | \mathcal{F}_k]\} \\ &= E\{\tilde{u}_k' E[B(\theta_k)' Z (A(\theta_k) - B(\theta_k)L^*(\theta_{k-1})) | \theta_{k-1}] x_k\} \\ &= 0 \end{aligned}$$

Hence it follows from (19) that for any  $u_k \in \mathcal{F}_k$ ,

$$\begin{aligned} & E[A(\theta_k)x_k + B(\theta_k)u_k]' Z [A(\theta_k)x_k + B(\theta_k)u_k] \\ & \geq E x_k' [A(\theta_k) - B(\theta_k)L^*(\theta_{k-1})]' Z [A(\theta_k) - B(\theta_k)L^*(\theta_{k-1})] x_k. \end{aligned}$$

Therefore the lemma is true and  $u_k = -L^*(\theta_{k-1})x_k$  is the desired minima .  $\square$

#### 4. The Proof of the Theorems.



**4.1. The proof of Theorem 2.1. Necessity:** Let the system (1) be stabilizable by feedback, i.e., there exists an admissible feedback law  $\{u_t\}$  such that  $\limsup_{t \rightarrow \infty} E\|x_t\|^2 < \infty$ . Then by Theorem 2.2 we need only to show that  $\lim_{t \rightarrow \infty} V_t = 0$  holds true, where  $V_t$  is defined as in (11).

Throughout the sequel, we denote

$$(20) \quad \mathcal{F}_t \triangleq \sigma(\theta_0, \theta_1, \dots, \theta_{t-1}; w_1, w_2, \dots, w_t).$$

For the system (1), by Lemma 3.3 we have

$$\begin{aligned} Ex'_{t+1}x_{t+1} &= E[A(\theta_t)x_t + B(\theta_t)u_t]'[A(\theta_t)x_t + B(\theta_t)u_t] + Ew'_{t+1}w_{t+1} \\ &\geq \min_{u_t \in \mathcal{F}_t} E[A(\theta_t)x_t + B(\theta_t)u_t]'[A(\theta_t)x_t + B(\theta_t)u_t] + Ew'_{t+1}w_{t+1} \\ &= Ex'_t[A(\theta_t) - B(\theta_t)L_1(\theta_{t-1})]'[A(\theta_t) - B(\theta_t)L_1(\theta_{t-1})]x_t + Ew'_{t+1}w_{t+1} \\ &= E[A(\theta_{t-1})x_{t-1} + B(\theta_{t-1})u_{t-1}]'[A(\theta_t) - B(\theta_t)L_1(\theta_{t-1})]' \\ &\quad [A(\theta_t) - B(\theta_t)L_1(\theta_{t-1})][A(\theta_{t-1})x_{t-1} + B(\theta_{t-1})u_{t-1}] \\ &\quad + Ew'_t[A(\theta_t) - B(\theta_t)L_1(\theta_{t-1})]'[A(\theta_t) - B(\theta_t)L_1(\theta_{t-1})]w_t + Ew'_{t+1}w_{t+1} \\ &\geq Ex'_{t-1} \left( \prod_{k=1}^2 [A(\theta_{t+1-k}) - B(\theta_{t+1-k})L_k(\theta_{t-k})] \right)' \\ &\quad \left( \prod_{k=1}^2 [A(\theta_{t+1-k}) - B(\theta_{t+1-k})L_k(\theta_{t-k})] \right) x_{t-1} \\ &\quad + Ew'_t[A(\theta_t) - B(\theta_t)L_1(\theta_{t-1})]'[A(\theta_t) - B(\theta_t)L_1(\theta_{t-1})]w_t + Ew'_{t+1}w_{t+1} \end{aligned}$$

Continuing this argument, we will get

$$(21) \quad Ex'_{t+1}x_{t+1} \geq \sum_{h=1}^t Ew'_{t+1-h} \prod'_{t,h} \prod_{t,h} w_{t+1-h},$$

where  $\prod_{t,h} \triangleq \prod_{k=1}^h [A(\theta_{t+1-k}) - B(\theta_{t+1-k})L_k(\theta_{t-k})]$ .

Now, it follows from the assumption A1) that there exists some  $t_0$  and  $\epsilon > 0$ , such that whenever  $t \geq t_0$ ,  $P(\theta_t = i) \geq \epsilon > 0$ ,  $\forall i \in S$ . For the convenience of presentation,

we may take  $t_0 = 0$ . Hence by the Markovian property and Lemma 3.2 we have:

$$\begin{aligned}
Ew'_{t+1-h} \prod'_{t,h} \prod_{t,h} w_{t+1-h} &= Ew'_{t+1-h} \left( \sum_{i=1}^N E(\prod'_{t,h} \prod_{t,h} | \theta_{t-h} = i) \cdot I_{[\theta_{t-h}=i]} \right) w_{t+1-h} \\
&= Ew'_{t+1-h} \left( \sum_{i=1}^N E(\prod'_{h,h} \prod_{h,h} | \theta_0 = i) \cdot I_{[\theta_{t-h}=i]} \right) w_{t+1-h} \\
&\geq Ew'_{t+1-h} \left( \sum_{i=1}^N V_{h,i} \cdot I_{[\theta_{t-h}=i]} \right) w_{t+1-h} \\
&\geq \sum_{i=1}^N \epsilon Ew'_{t+1-h} V_{h,i} w_{t+1-h} \\
&= \epsilon \cdot \sum_{i=1}^N \text{tr} \{ E(V_{h,i} w_{t+1-h} w'_{t+1-h}) \} \\
&= \epsilon \cdot \sum_{i=1}^N \text{tr} \{ V_{h,i} \cdot (Ew_t w'_t) \} \geq \epsilon \sigma \sum_{i=1}^N \text{tr}(V_{h,i}).
\end{aligned}$$

where  $\sigma$  is given in Assumption A3). Hence, it follows from (21) that

$$E x'_{t+1} x_{t+1} \geq \epsilon \sigma \sum_{h=1}^t \sum_{i=1}^N \text{tr}(V_{h,i}).$$

Hence we have  $\lim_{h=1}^t \text{tr}(V_{h,i}) < \infty, \forall i \in S$ . This implies that  $\lim V_t = 0$ . Hence the proof of the necessity part of Theorem 2.1 is completed.  $\square$

**4.2. Sufficiency.** The proof is divided into several steps.

**Step 1.** We first construct the stabilizing adaptive feedback.

By (6) and Assumption A2), it is not difficult to convince oneself that we may find a set of  $n \times m$  matrices  $\{L_i^\delta, i \in S\}$  from a small perturbation of the matrices  $\{L_i^*, i \in S\}$ , such that

$$(22) \quad \sum_{j=1}^N (A_j - B_j L_i^\delta)' p_{ij} M_j (A_j - B_j L_i^\delta) < M_i, \quad i \in S,$$

and that

$$(23) \quad \det[(A_i - A_j) - (B_i - B_j)L_l^\delta] \neq 0, \quad \forall i \neq j, l \in S.$$

Now, the adaptive feedback law can be defined as

$$(24) \quad u_t = -L^\delta(\hat{\theta}_{t-1}) \cdot x_t.$$

where  $\hat{\theta}_{t-1}$  is the estimate of  $\theta_{t-1}$ , defined by

$$(25) \quad \hat{\theta}_{t-1} = \underset{1 \leq i \leq N}{\text{argmin}} \|x_t - (A_i x_{t-1} + B_i u_{t-1})\|,$$

**Step 2.** We then show that

$$(26) \quad G_{k-1} \triangleq \{\hat{\theta}_{k-1} \neq \theta_{k-1}\} \subseteq \bigcup_{i \neq j, l} \{x'_{k-1} \mathcal{D}'_{ijl} \mathcal{D}_{ijl} x_{k-1} + 2w'_k \mathcal{D}_{ijl} x_{k-1} \leq 0\}$$

where  $\hat{\theta}_{k-1}$  is defined by (25) and  $\mathcal{D}_{ijl}$  is defined by

$$(27) \quad \mathcal{D}_{ijl} \triangleq [A_i - A_j - (B_i - B_j)L_l^\delta],$$

By (25),  $\hat{\theta}_{k-1} \neq \theta_{k-1}$  implies that there exists  $i \in S, i \neq \theta_{k-1}$  such that

$$\begin{aligned} & [x_k - (A_i - B_i L^\delta(\hat{\theta}_{k-2}))x_{k-1}]' [x_k - (A_i - B_i L^\delta(\hat{\theta}_{k-2}))x_{k-1}] \\ & \leq [x_k - (A(\theta_{k-1}) - B(\theta_{k-1})L^\delta(\hat{\theta}_{k-2}))x_{k-1}]' [x_k - (A(\theta_{k-1}) - B(\theta_{k-1})L^\delta(\hat{\theta}_{k-2}))x_{k-1}] \\ & = w'_k w_k. \end{aligned}$$

Substituting (1) into the above inequality, we see that there exists  $i \in S, i \neq \theta_{k-1}$  such that

$$x'_{k-1} \mathcal{D}'_{\theta_{k-1}, i, \hat{\theta}_{k-2}} \mathcal{D}_{\theta_{k-1}, i, \hat{\theta}_{k-2}} x_{k-1} + 2w'_k \mathcal{D}_{\theta_{k-1}, i, \hat{\theta}_{k-2}} x_{k-1} \leq 0.$$

Hence, we get

$$\{\hat{\theta}_{k-1} \neq \theta_{k-1}\} \subseteq \bigcup_{i \neq j, l} \{x'_{k-1} \mathcal{D}'_{ijl} \mathcal{D}_{ijl} x_{k-1} + 2w'_k \mathcal{D}_{ijl} x_{k-1} \leq 0\},$$

which is (26).

**Step 3.** We further show that the set  $G_{k-1}$  defined by (26) can be estimated by

$$(28) \quad G_{k-1} \subseteq \{x'_{k+1} x_{k+1} \leq \mu \|w_k\|^2 + 2\|w_{k+1}\|^2\}.$$

Obviously, we need only to consider the last term in (26). Note that

$$0 \geq x'_{k-1} \mathcal{D}'_{ijl} \mathcal{D}_{ijl} x_{k-1} + 2w'_k \mathcal{D}_{ijl} x_{k-1} \geq \|\mathcal{D}_{ijl} x_{k-1}\|^2 - 2\|w_k\| \cdot \|\mathcal{D}_{ijl} x_{k-1}\|,$$

which implies that

$$\|\mathcal{D}_{ijl} x_{k-1}\| \leq 2\|w_k\|, \text{ or } \|\mathcal{D}_{ijl} x_{k-1}\|^2 \leq 4\|w_k\|^2.$$

From this, it follows that

$$(29) \quad \|x_{k-1}\|^2 \leq \frac{4}{\lambda_-} \|w_k\|^2.$$

where  $\lambda_-$  is defined by

$$(30) \quad \min_{i \neq j, l} \lambda_{\min}(\mathcal{D}'_{ijl} \mathcal{D}_{ijl}) = \lambda_- > 0.$$

and where  $\lambda_{\min}(\cdot)$  denotes the minimum eigenvalue of a matrix. The property (23) has been used to guarantee that  $\lambda_-$  is positive.

Now, by (1) and (24), we have  $x_k = [A(\theta_{k-1}) - B(\theta_{k-1})L^\delta(\hat{\theta}_{k-2})]x_{k-1} + w_k$ , and therefore,

$$x'_k x_k \leq 2\|A(\theta_{k-1}) - B(\theta_{k-1})L^\delta(\hat{\theta}_{k-2})\|^2 \cdot \|x_{k-1}\|^2 + 2\|w_k\|^2.$$

Consequently, there exists a constant  $\mu_1 > 0$ , such that

$$\|x_k\|^2 \leq 2\mu_1^2 \cdot \frac{4}{\lambda_-} \|w_k\|^2 + 2\|w_k\|^2 = \left(\frac{8\mu_1^2}{\lambda_-} + 2\right) \|w_k\|^2$$

Similarly, from this and  $x_{k+1} = [A(\theta_k) - B(\theta_k)L^\delta(\hat{\theta}_{k-1})]x_k + w_{k+1}$  it follows that

$$\begin{aligned} x'_{k+1}x_{k+1} &\leq 2\|A(\theta_k) - B(\theta_k)L^\delta(\hat{\theta}_{k-1})\|^2 \cdot \|x_k\|^2 + 2\|w_{k+1}\|^2 \\ &\leq 2\mu_1^2 \cdot \left(\frac{8\mu_1^2}{\lambda_-} + 2\right) \|w_k\|^2 + 2\|w_{k+1}\|^2. \end{aligned}$$

Hence, the assertion (28) is true.

**Step 4.** Finally we estimate the upper bound of  $Ex'_{t+1}x_{t+1}$ .

For any  $t \geq 4$  and for  $\hat{\Pi}_{t,k}$  defined by (39), we have

$$\begin{aligned} Ex'_{t+1}x_{t+1} &= Ex'_{t+1}x_{t+1}I_{G_{t-1}^c} + Ex'_{t+1}x_{t+1}I_{G_{t-1}} \\ &\leq Ex'_t[A(\theta_t) - B(\theta_t)L^\delta(\theta_{t-1})]'[A(\theta_t) - B(\theta_t)L^\delta(\theta_{t-1})]x_t + \\ &\quad Ew'_{t+1}w_{t+1} + Ex'_{t+1}x_{t+1}I_{G_{t-1}} \\ &\leq Ex'_t\hat{\Pi}'_{t,1}\hat{\Pi}_{t,1}x_t + \sigma_w + Ex'_{t+1}x_{t+1}I_{G_{t-1}} \\ &= Ex'_t\hat{\Pi}'_{t,1}\hat{\Pi}_{t,1}x_tI_{G_{t-2}^c} + Ex'_t\hat{\Pi}'_{t,1}\hat{\Pi}_{t,1}x_tI_{G_{t-2}} + \sigma_w + Ex'_{t+1}x_{t+1}I_{G_{t-1}} \\ &\leq Ex'_{t-1}\hat{\Pi}'_{t,2}\hat{\Pi}_{t,2}x_{t-1} + Ew'_t\hat{\Pi}'_{t,1}\hat{\Pi}_{t,1}w_t + Ex'_t\hat{\Pi}'_{t,1}\hat{\Pi}_{t,1}x_tI_{G_{t-2}} \\ &\quad + \sigma_w + Ex'_{t+1}x_{t+1}I_{G_{t-1}}. \end{aligned}$$

Iterating this argument, we finally have for any  $t \geq 4$ ,

$$(31) \quad \begin{aligned} Ex'_{t+1}x_{t+1} &\leq Ex'_2\hat{\Pi}'_{t,t-1}\hat{\Pi}_{t,t-1}x_2 + \sum_{k=3}^t Ex'_k\hat{\Pi}'_{t,t-k+1}\hat{\Pi}_{t,t-k+1}x_kI_{G_{k-2}} \\ &\quad + Ex'_{t+1}x_{t+1}I_{G_{t-1}} + \sum_{k=3}^t Ew'_k\hat{\Pi}'_{t,t-k+1}\hat{\Pi}_{t,t-k+1}w_k + \sigma_w. \end{aligned}$$

Now, denote  $\mathcal{F}_k \triangleq \sigma(\theta_0, \theta_1, \dots, \theta_{k-1}; w_1, \dots, w_k)$ ,  $k \geq 1$ . Then by  $x_k \in \mathcal{F}_k$ ,  $G_{k-2} \in \mathcal{F}_{k-1}$ , we have

$$\begin{aligned} &Ex'_k\hat{\Pi}'_{t,t-k+1}\hat{\Pi}_{t,t-k+1}x_kI_{G_{k-2}} \\ &= E\left\{E[x'_k\hat{\Pi}'_{t,t-k+1}\hat{\Pi}_{t,t-k+1}x_kI_{G_{k-2}}|\mathcal{F}_k]\right\} \\ &= Ex'_kE[\hat{\Pi}'_{t,t-k+1}\hat{\Pi}_{t,t-k+1}|\mathcal{F}_k]x_kI_{G_{k-2}} \\ &= Ex'_kE[\hat{\Pi}'_{t,t-k+1}\hat{\Pi}_{t,t-k+1}|\theta_{k-1}]x_kI_{G_{k-2}}. \end{aligned}$$

Hence by Lemma 3.6 ii) in Appendix B and (28) we get

$$\begin{aligned}
(32) \quad E x'_k \hat{\prod}_{t,t-k+1}' \hat{\prod}_{t,t-k+1} x_k I_{G_{k-2}} &\leq c_0 \rho^{t-k+1} \cdot E x'_k x_k I_{G_{k-2}} \\
&\leq c_0 \rho^{t-k+1} \cdot E x'_k x_k I_{[x'_k x_k \leq \mu \|w_{k-1}\|^2 + 2\|w_k\|^2]} \\
&\leq c_0 \rho^{t-k+1} \cdot E(\mu \|w_{k-1}\|^2 + 2\|w_k\|^2) \\
&= c_0(\mu + 2)\sigma_w \rho^{t-k+1}.
\end{aligned}$$

Moreover, by Lemma A1 in Appendix B again, we have

$$(33) \quad E w'_k \hat{\prod}_{t,t-k+1}' \hat{\prod}_{t,t-k+1} w_k = E w'_k (E \hat{\prod}_{t,t-k+1}' \hat{\prod}_{t,t-k+1}) w_k \leq c_0 \rho^{t-k+1} \sigma_w.$$

Combining (32) and (33), it follows from (31) that

$$\begin{aligned}
E x'_{t+1} x_{t+1} &\leq E x'_2 \hat{\prod}_{t,t-1}' \hat{\prod}_{t,t-1} x_2 + \sum_{k=3}^t c_0(\mu + 2)\sigma_w \rho^{t-k+1} \\
&\quad + E x'_{t+1} x_{t+1} I_{G_{t-1}} + \sum_{k=3}^t c_0 \rho^{t-k+1} \sigma_w + \sigma_w.
\end{aligned}$$

Finally, by the assertion in Step 2 and Lemma A.1 in Appendix B we have

$$\begin{aligned}
E x'_{k+1} x_{k+1} &\leq E \{E[x'_2 \hat{\prod}_{t,t-1}' \hat{\prod}_{t,t-1} x_2 | \mathcal{F}_2]\} + c_0(\mu + 2)\sigma_w \cdot \frac{1}{1-\rho} \\
&\quad + E(\mu \|w_t\|^2 + 2\|w_{t+1}\|^2) + c_0 \sigma_w \frac{1}{1-\rho} + \sigma_w \\
&= E \{x'_2 E[\hat{\prod}_{t,t-1}' \hat{\prod}_{t,t-1} | \mathcal{F}_2] x_2\} + \frac{3c_0 + (1+c_0)\mu + 2}{1-\rho} \sigma_w + \sigma_w \\
&\leq c_0 \rho^{t-1} \cdot E x'_2 x_2 + \frac{3c_0 + (1+c_0)\mu + 2}{1-\rho} \sigma_w + \sigma_w \\
&\leq c_0 E x'_2 x_2 + \frac{3c_0 + (1+c_0)\mu + 2}{1-\rho} \sigma_w + \sigma_w.
\end{aligned}$$

This completes the proof of Theorem 2.1.  $\square$

**4.3. The proof of Theorem 2.2.** (1) We first show that (i)  $\Rightarrow$  (ii). However, this is obvious by (6).

(2) Next we show that (ii)  $\Rightarrow$  (iii). By (10), it is obvious that  $M_i \geq I$ , and that there exists a constant  $0 < \lambda_1 < 1$  such that  $M_i - I \leq \lambda_1 M_i$ ,  $i \in S$ . Hence, by (10) and the definition of  $V_t$ , we have

$$\begin{aligned}
V_1 &= \psi(I^*) \leq \psi(M) \leq \lambda_1 M, \\
V_2 &= \psi(V_1) \leq \psi(\lambda_1 M) \leq (\lambda_1)^2 M,
\end{aligned}$$

where  $M \triangleq (M_1, \dots, M_N)'$ . Interating this argument, we can get

$$V_t \leq (\lambda_1)^t M, \quad t \geq 1.$$

which implies the assertion (iii).

(3). We now prove that (iii)  $\Rightarrow$  (iv).

In what follows, for two  $nN \times n$  matrices  $P = [P_1, \dots, P_N]'$  and  $Q = [Q_1, \dots, Q_N]'$ , we will simply denote  $P \leq Q$  if  $P_j \leq Q_j$  for all  $1 \leq j \leq N$ . Thus, by (iii) we may take  $K > 0$  large enough such that  $V_K \leq \frac{1}{2}I^*$ .

From the definitions of  $V_k$  and  $\psi(\cdot)$  it follows that there exists  $\{L_{k,i}^*, i \in S, 1 \leq k \leq K\}$  such that

$$\sum_j (A_j - B_j L_{k,i}^*)' p_{ij} V_{k-1,j} (A_j - B_j L_{k,i}^*) = V_{k,i} \quad i \in S, 1 \leq k \leq K.$$

Now, for any  $Q = (Q_1, \dots, Q_N)'$  with  $Q_i \geq 0$ , and for any  $1 \leq k \leq K$ , we define the following mapping:

$$\phi_k(Q) = [\phi_{k1}(Q), \dots, \phi_{kN}(Q)]'$$

where  $\phi_{ki}(Q)$  is defined by

$$\phi_{ki}(Q) \triangleq \sum_j (A_j - B_j L_{k,i}^*)' p_{ij} Q_j (A_j - B_j L_{k,i}^*), \quad i \in S,$$

For the case where  $k \geq K + 1$ , we define  $\phi_k(Q)$  as follows:

$$\phi_k(Q) \triangleq \begin{cases} \phi_l(Q), & \text{if } k = mK + l, 1 \leq l \leq K - 1; \\ \phi_K(Q), & \text{if } k = mK, \end{cases}$$

It is quite obvious that these mappings are linear in  $Q$  and that the following holds:

$$(34) \quad \phi_K \circ \phi_{K-1} \circ \dots \circ \phi_1(I^*) = V_K \leq \frac{1}{2}I^*.$$

Next, we prove the following inequality by induction:

$$(35) \quad \sum_{h=1}^t \phi_t \circ \phi_{t-1} \circ \dots \circ \phi_h(I^*) \geq M_t^*, \quad t \geq 1.$$

In fact, the case where  $t = 1$  is obvious. Now, let us assume that for some  $t = t_0$  the assertion is true. Then by linearity of  $\phi_k(\cdot)$ , we have

$$\begin{aligned} & \sum_{h=1}^{t_0+1} \phi_{t_0+1} \circ \phi_{t_0} \circ \dots \circ \phi_h(I^*) \\ &= \phi_{t_0+1}(I^*) + \phi_{t_0+1} \left( \sum_{h=1}^{t_0} \phi_{t_0} \circ \phi_{t_0-1} \circ \dots \circ \phi_h(I^*) \right) \\ &\geq \phi_{t_0+1}(I^* + M_{t_0}^*) \\ &\geq M_{t_0+1}^*. \end{aligned}$$

where for the last step we have used the optimality of  $L_{t_0+1}^*$  as established in Lemma 3.1. Hence (35) is true.

Next, it is easy to see that there is a constant  $c > 0$  such that:

$$\phi_{k_2} \circ \phi_{k_2-1} \circ \cdots \circ \phi_{k_1}(I^*) \leq c \cdot I^*, \quad \forall 1 \leq k_1 \leq k_2 \leq K.$$

Hence, by the definition of  $\phi_k$  and (34) we have

$$\phi_t \circ \phi_{t-1} \circ \cdots \circ \phi_h(I^*) \leq c^2 \cdot \left(\frac{1}{2}\right)^{\lfloor \frac{t-h}{K} \rfloor - 2} \cdot I^*$$

from which we have ,

$$\begin{aligned} & \sum_{h=1}^t \phi_t \circ \phi_{t-1} \circ \cdots \circ \phi_h(I^*) \\ & \leq 4c^2 \cdot \left( \sum_{h=1}^t \frac{1}{2^{\lfloor \frac{t-h}{K} \rfloor}} \right) \cdot I^* \\ & = 4c^2 \cdot \left( \sum_{h=1}^t \frac{1}{2^{\lfloor \frac{h}{K} \rfloor}} \right) \cdot I^* < \infty. \end{aligned}$$

Hence, it follow from (35) that  $\{M_k^*\}$  is a bounded sequence. However, by induction and Lemma 3.1 it is esay to see that  $M_k^*$  is nondecreasing, or  $M_{k+1}^* \geq M_k^*$ ,  $\forall k \geq 0$ . Hence the assertion (iv) is true.  $\square$

(4). Finally, we show that (iv)  $\Rightarrow$  (i). Let the limit of  $M_{t,i}^*$  be denoted by  $M_i^*$ , and let us set  $M^* \triangleq (M_1^*, \dots, M_N^*)$ . Then, it is obvious that  $\psi(M^* + I^*) = M^*$ . Therefore, if we take  $L_i^*$ ,  $i \in S$  as

$$(36) \quad L_i^* = \left( \sum_j B_j' p_{ij}(M_j^* + I^*) B_j \right)^+ \left( \sum_j B_j' p_{ij}(M_j^* + I^*) A_j \right),$$

Then it follows that

$$\sum_j B_j' p_{ij}(M_j^* + I^*) A_j - \sum_j B_j' p_{ij}(M_j^* + I^*) B_j \cdot L_i^* = 0.$$

Hence for any  $i \in S$ ,

$$\begin{aligned} \psi_i(M^* + I^*) &= \sum_j (A_j - B_j L_i^*)' p_{ij}(M_j^* + I^*) (A_j - B_j L_i^*) \\ &= \sum_j A_j' p_{ij}(M_j^* + I^*) A_j - \sum_j A_j' p_{ij}(M_j^* + I^*) B_j L_i^* \\ &\quad - L_i^{*'} \cdot \left( \sum_j B_j' p_{ij}(M_j^* + I^*) A_j - \sum_j B_j' p_{ij}(M_j^* + I^*) B_j L_i^* \right) \\ &= \sum_j A_j' p_{ij}(M_j^* + I^*) A_j - \sum_j A_j' p_{ij}(M_j^* + I^*) B_j L_i^* \\ &= \sum_j A_j' p_{ij}(M_j^* + I^*) A_j \\ &\quad - \left( \sum_j A_j' p_{ij}(M_j^* + I^*) B_j \right) \left( \sum_j B_j' p_{ij}(M_j^* + I^*) B_j \right)^+ \left( \sum_j B_j' p_{ij}(M_j^* + I^*) A_j \right), \end{aligned}$$

Thus we have

$$\sum_j A'_j p_{ij}(M_j^* + I^*)A_j - \left(\sum_j A'_j p_{ij}(M_j^* + I^*)B_j\right)\left(\sum_j B'_j p_{ij}(M_j^* + I^*)B_j\right)^+ \left(\sum_j B'_j p_{ij}(M_j^* + I^*)A_j\right) = M_i^*.$$

Hence the first assertion (i) follows by setting  $M_i = M_i^* + I$ . This completes the proof of Theorem 2.2.

**5. Concluding Remarks.** We have in this paper presented a necessary and sufficient condition in characterizing the adaptive stabilizability of finite-dimensional linear systems with complete state information but with unknown time-varying parameters modeled as a finite state Markov chain. Such a characterization provides a quantitative assessment of the capability and limitations of adaptive feedback, which depend, in a complicated way, on both the information uncertainty described by the transition probability  $\{p_{ij}\}$  of the Markov chain and the model complexity described by the (dispersion of) system matrices  $\{A_j, B_j, 1 \leq j \leq N\}$ . Obviously, there are many directions in which the results of the present paper can be generalized, and which require further investigation.

**Appendix A: The proof of Proposition 2.1.** (i). We first show that if the Riccati-like equation (4) has a solution  $\{M_i > 0, i \in S\}$ , then so does the equation (13).

Let  $\{M_t^*, t \geq 0\}$  be the nondecreasing sequence on nonnegative matrices defined recursively as in (12). Then, by Theorem 2.2 we know that  $M_t^*$  has a finite limit  $M^* \triangleq (M_{\infty,1}, \dots, M_{\infty,N})$ .

Now, let us introduce another sequence of matrices  $\{U_t, t \geq 0\}$ :

$$(37) \quad \begin{aligned} U_0 &= 0^*, \\ U_{t+1,i} &= \min_{L_i \in R^{m \times n}} \sum_j (A_i - B_i L_i)' p_{ij}(U_{t,j} + I)(A_i - B_i L_i)', \\ U_{t+1} &= (U_{t+1,1}, U_{t+1,2}, \dots, U_{t+1,N})'. \end{aligned}$$

Similar to the proof of Lemma 3.1, it is easy to see that  $U_t, t \geq 0$  exist. Moreover, it can also be shown by induction that  $U_t, t \geq 0$  is nondecreasing

We now proceed to show that for any  $i \in S$  and  $t \geq 0$ ,

$$(38) \quad \sum_j p_{ij} U_{t,j} \leq M_{t,i}^*.$$

The case where  $t = 0$  is obvious. Now let us assume that for  $t = \tau$  we have



$\sum_j p_{ij} U_{\tau,j} \leq M_{\tau,i}^*, \forall i \in S$ . Then for  $t = \tau + 1$  we have

$$\begin{aligned} \sum_j p_{ij} U_{\tau+1,j} &= \sum_j p_{ij} \cdot \min_{L_j} \sum_l (A_j - B_j L_j)' p_{jl} (U_{\tau,l} + I) (A_j - B_j L_j) \\ &= \sum_j p_{ij} \cdot \min_{L_j} (A_j - B_j L_j)' \left( \sum_l p_{jl} U_{\tau,l} + I \right) (A_j - B_j L_j) \\ &\leq \sum_j p_{ij} \cdot \min_{L_j} (A_j - B_j L_j)' (M_{\tau,j}^* + I) (A_j - B_j L_j) \\ &\leq \min_{L_i} \sum_j p_{ij} (A_j - B_j L_j)' (M_{\tau,j}^* + I) (A_j - B_j L_j) \\ &= M_{\tau+1,i}^*. \end{aligned}$$

Hence (38) is true.

For further derivation, we assume (without loss of generality) that  $\sum_j p_{ij} > 0, \forall j \in S$ . Then, by letting  $t \rightarrow \infty$  in (38), we get  $\sum_j p_{ij} U_{\infty,j} \leq M_{\infty,i}^*, i \in S$ . Hence we must have  $U_{\infty,j} < \infty, j \in S$ . Furthermore, let  $t \rightarrow \infty$  in (37) we get

$$U_{\infty,i} = \min_{L_i} \sum_j (A_i - B_i L_i)' p_{ij} (U_{\infty,j} + I) (A_i - B_i L_i)'.$$

Therefore  $\{U_{\infty,i} + I, i \in S\}$  is the solution for (13).

(ii). We now give an example showing that the converse assertion of (i) is not true in general. Consider the scalar case where the Markov chain has two states only:  $a_1 = -1$  and  $a_2 = 1$  with  $p_{11} = p_{12} = p_{21} = p_{22} = \frac{1}{2}$ .

In this case the equation (13) takes the following form:

$$\begin{cases} (-1 - l_1)^2 \left( \frac{1}{2} M_1 + \frac{1}{2} M_2 \right) - M_1 = -1, \\ (1 - l_2)^2 \left( \frac{1}{2} M_1 + \frac{1}{2} M_2 \right) - M_2 = -1. \end{cases}$$

Obviously,  $l_1 = -1, l_2 = 1, M_1 = M_2 = 1$  satisfy this equation. However, we will shall that the Riccati-like equation (4) which takes the following form

$$\begin{cases} (-1 - l_1)^2 \frac{1}{2} M_1 + (1 - l_1)^2 \frac{1}{2} M_2 - M_1 = -1, \\ (-1 - l_2)^2 \frac{1}{2} M_1 + (1 - l_2)^2 \frac{1}{2} M_2 - M_2 = -1. \end{cases}$$

has no solution.

In fact, if  $\{l_1, l_2\}$  together with  $\{M_1 > 0, M_2 > 0\}$  were a solution to the above equation, then

$$\begin{aligned}
 0 &> (-1 - l_1)^2 \frac{1}{2} M_1 + (1 - l_1)^2 \frac{1}{2} M_2 - M_1 \\
 &\geq \frac{1}{2} ((-1 - l_1)^2 + (1 - l_1)^2) \min\{M_1, M_2\} - M_1 \\
 &\geq \frac{1}{2} \cdot 2 \cdot \min\{M_1, M_2\} - M_1 \\
 &= \min\{M_1, M_2\} - M_1.
 \end{aligned}$$

Hence we have  $M_1 > \min\{M_1, M_2\}$ . Similarly, we can get  $M_2 > \min\{M_1, M_2\}$ . This is certainly impossible, and the proof is completed.  $\square$

**Appendix B: Lemma A.1 and its Proof** **Lemma A.1.** Let  $\{L_i^\delta, i \in S\}$  be defined as in (22) and (23), and let

$$\begin{aligned}
 \hat{\Pi}_{t,0} &\triangleq I, \\
 \hat{\Pi}_{t,k} &\triangleq [A(\theta_t) - B(\theta_t)L^\delta(\theta_{t-1})] \cdots [A(\theta_{t-k+1}) - B(\theta_{t-k+1})L^\delta(\theta_{t-k})], \\
 &\quad 1 \leq k \leq t; \\
 \hat{V}_{t,i} &\triangleq E[\hat{\Pi}'_{t,t} \hat{\Pi}_{t,t} | \theta_0 = i], \quad i \in S
 \end{aligned}
 \tag{39}$$

Then, we have the following two assertions:

i). For any  $t \geq 0$  and  $i \in S$ , we have  $\hat{V}_{0,i} = I$  and

$$\hat{V}_{t+1,i} = \sum_{j=1}^N (A_j - B_j L_i^\delta)' p_{ij} \hat{V}_{t,j} (A_j - B_j L_i^\delta).$$

(40)

ii). There exist two constants  $c_0 > 0$  and  $0 < \rho < 1$  such that for any  $1 \leq k \leq t$

$$E[\hat{\Pi}'_{t,k} \hat{\Pi}_{t,k} | \theta_{t-k} = i] \leq c_0 \rho^k I, \quad i \in S.$$

(41)

**Proof.**

i). Note that

$$\begin{aligned}
 \hat{V}_{t+1,i} &= E\{\hat{\Pi}'_{t+1,t+1} \hat{\Pi}_{t+1,t+1} | \theta_0 = i\} \\
 &= E\{[A(\theta_1) - B(\theta_1)L^\delta(\theta_0)]' \hat{\Pi}'_{t+1,t} \hat{\Pi}_{t+1,t} [A(\theta_1) - B(\theta_1)L^\delta(\theta_0)] | \theta_0 = i\} \\
 &= E\{E\{[A(\theta_1) - B(\theta_1)L^\delta(\theta_0)]' \hat{\Pi}'_{t+1,t} \hat{\Pi}_{t+1,t} [A(\theta_1) - B(\theta_1)L^\delta(\theta_0)] | \theta_0, \theta_1\} | \theta_0 = i\},
 \end{aligned}$$

Then by the homogeneous and the strong Markovian properties, we get

$$\begin{aligned}
\hat{V}_{t+1,i} &= E \left\{ [A(\theta_1) - B(\theta_1)L^\delta(\theta_0)]' E[\hat{\Pi}'_{t+1,t} \hat{\Pi}_{t+1,t} | \theta_0, \theta_1] [A(\theta_1) - B(\theta_1)L^\delta(\theta_0)] | \theta_0 = i \right\} \\
&= E \left\{ [A(\theta_1) - B(\theta_1)L^\delta(\theta_0)]' E[\hat{\Pi}'_{t+1,t} \hat{\Pi}_{t+1,t} | \theta_1] [A(\theta_1) - B(\theta_1)L^\delta(\theta_0)] | \theta_0 = i \right\} \\
&= \sum_{j=1}^N E \left\{ [A_j - B_j L_i^\delta]' E[\hat{\Pi}'_{t+1,t} \hat{\Pi}_{t+1,t} | \theta_1 = j] [A_j - B_j L_i^\delta] I_{[\theta_1=j]} | \theta_0 = i \right\} \\
&= \sum_j [A_j - B_j L_i^\delta]' \hat{V}_{t,j} [A_j - B_j L_i^\delta] \cdot E[I_{[\theta_1=j]} | \theta_0 = i] \\
&= \sum_j [A_j - B_j L_i^\delta]' p_{ij} \hat{V}_{t,j} [A_j - B_j L_i^\delta].
\end{aligned}$$

ii). By the strong Markovian property of  $\{\theta_k\}$ , we have

$$E[\hat{\Pi}'_{t,k} \hat{\Pi}_{t,k} | \theta_{t-k}] = E[\hat{\Pi}'_{k,k} \hat{\Pi}_{k,k} | \theta_0],$$

Hence we need only to show that:  $E[\hat{\Pi}'_{t,t} \hat{\Pi}_{t,t} | \theta_0 = i] \leq c_0 \rho^t I$ , or  $\hat{V}_{t,i} \leq c_0 \rho^t I$ .

Since  $\{L_i^\delta, i \in S\}$  satisfies (22), there exists a constant  $0 < \rho < 1$  such that

$$(42) \quad \sum_{j=1}^N (A_j - B_j L_i^\delta)' p_{ij} M_j (A_j - B_j L_i^\delta) < \rho M_i, \quad i \in S.$$

Moreover, there exist constants  $c_1 > 0$  and  $c_2 > 0$  such that  $c_1 I \leq M_i \leq c_2 I, \forall i \in S$ .

Hence, by (i) we have

$$\begin{aligned}
\hat{V}_{1,i} &= \sum_j (A_j - B_j L_i^\delta)' p_{ij} I (A_j - B_j L_i^\delta) \leq \frac{1}{c_1} \sum_j (A_j - B_j L_i^\delta)' p_{ij} M_j (A_j - B_j L_i^\delta) \\
&\leq \frac{1}{c_1} \rho M_i; \\
\hat{V}_{2,i} &= \sum_j (A_j - B_j L_i^\delta)' p_{ij} \hat{V}_{1,j} (A_j - B_j L_i^\delta) \leq \frac{1}{c_1} \rho \sum_j (A_j - B_j L_i^\delta)' p_{ij} M_j (A_j - B_j L_i^\delta) \\
&\leq \frac{1}{c_1} \rho^2 M_i, \quad i \in S.
\end{aligned}$$

Similarly, we have

$$\hat{V}_{t,i} \leq \frac{1}{c_1} \rho^t M_i, \quad i \in S.$$

Hence, we have

$$\hat{V}_{t,i} \leq \frac{c_2}{c_1} \rho^t I, \quad i \in S.$$

and the proof of Lemma A.1 is completed.  $\square$

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