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Reflection Groups, Braid Groups, Hecke Algebras, Finite Reductive Groups

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ABSTRACT. Finite subgroups of $\mathrm{GL}_n(\mathbb{Q})$ generated by reflections, known as Weyl groups, classify simple complex Lie Groups as well as simple algebraic groups. They are also building stones for many other significant mathematical objects like braid groups and Hecke algebras.

Through recent work on representations of reductive groups over finite fields based upon George Lusztig's fundamental work, and motivated by conjectures about modular representations of general finite groups, it has become clearer and clearer that finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ generated by pseudo–reflections ("complex reflection groups") behave very much like Weyl groups, and might even be as important.

We present here a concatenation of some recent work (mainly by D. Bessis, G. Malle, J. Michel, R. Rouquier and the author) on complex reflection groups, their braid groups and Hecke algebras, emphasizing the general properties which generalize basic properties of Weyl groups.

By many aspects, the family of finite groups $\mathbf{G}(q)$ over finite fields with q elements behave as if they were the specialisations at $x = q$ of an object depending on an indeterminate x. Convincing indices tend to show that, although complex reflection groups which are not Weyl groups do not define finite groups over finite fields , they might be associated to similar mysterious objects. We present here some aspects of the machinery allowing to emphasize this point of view. We use this machinery to state the general conjectures about representations of finite reductive groups over ℓ–adic rings which, ten years ago, originated this work .

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Contents

INTRODUCTION

Weyl groups.

Weyl groups are finite groups acting as a reflection group on a rational vector space. These rational reflection groups appear as the "skeleton" of many important mathematical objects, like algebraic groups, Hecke algebras, braid groups, etc.

A particularly fascinating area is the study of algebraic reductive groups over finite fields ("finite reductive groups"). In the study of subgroup structure, ordinary and modular representation theory of finite reductive groups, it turns out that many properties behave in a generic manner, *i.e.,* the results can be phrased in terms independent of the order q of the field of definition. Following in particular [**BrMa1**], [**BMM1**], [**BMM2**], we shall present here (chap. 4) a framework in which this generic behaviour can be conveniently formulated (and partly also proved).

More precisely, assume we are given a family of groups of Lie type, like the groups $\mathrm{GL}_n(q)$ for fixed n but varying prime power q. Then a crucial role in the description of this family of groups is played by the Weyl group, which is the same for all members in the family, together with the action induced by the Frobenius morphism (the twisting). The Weyl group occurs in a natural way as a reflection group on the vector space generated by the coroots, and the Frobenius morphism induces an automorphism of finite order. This leads to the concept of reflection datum $\mathbb{G}$. The orders of the groups attached to a reflection datum $\mathbb{G}$ can be obtained as values of one single polynomial, the order polynomial $|\mathbb{G}|$ of $\mathbb{G}$. In analogy to the concept of ℓ-subgroup of a finite group, for any cyclotomic polynomial Φ_d dividing the order polynomial $|\mathbb{G}|$ one can develop a theory of "Φ_d–subdata" of $\mathbb{G}$. These satisfy a complete analogue of the Sylow theorems for finite groups [**BrMa1**].

Complex reflection groups.

By extension of base field, Weyl groups may be viewed as particular finite *complex reflection groups.* Such groups have been characterized by Shephard–Todd and Chevalley as those linear finite groups whose ring of invariants in the corresponding symmetric algebra is still a regular ring. The irreducible finite complex reflection groups have been classified by Shephard–Todd.

It has been recently discovered that complex reflection groups (and not only Weyl groups) play a key role in the structure as well as in representation theory of finite reductive groups.

In the representation theory of finite reductive groups, the right generic objects are the unipotent characters. It follows from results of Lusztig that they can be parametrized in terms only depending on the reflection datum $\mathbb{G}$. Moreover, the functor of Deligne–Lusztig induction can be shown to be generic. It turns out that for any d such that Φ_d divides $|\mathbb{G}|$ there holds a d–Harish–Chandra theory for unipotent characters. This gives rise to a whole family of generalized Harish–Chandra theories which contains the usual Harish–Chandra theory for complex characters as the case $d = 1$. The resulting generalized Harish–Chandra series allow a description of the subdivision into ℓ-blocks of the unipotent characters of a finite reductive group, at least for large primes ℓ, thus giving an application of the generic formalism to the ℓ-modular representation theory of groups of Lie type. The generalized Harish–Chandra series can be indexed by irreducible characters of suitable relative Weyl groups such that the decomposition of the functor of Deligne–Lusztig induction can be compared to ordinary induction in these relative Weyl groups [**BMM1**]. The most surprising fact is that these relative Weyl groups turn out to be (complex) reflection groups.

These discoveries led naturally to more work on complex reflection groups, which in turn unvailed the fact that complex reflection groups (as well as the associated braid groups, Hecke algebras, etc.) have many unexpected properties in common with Weyl groups.

In the course of his classification of unipotent characters of finite reductive groups (which are ultimately determined by Weyl groups), Lusztig [**Lu2**] observed that similar sets can also be attached to those finite real reflection groups which are not Weyl groups (namely, the finite Coxeter groups), just as if there existed a "fake algebraic group" whose Weyl group is non–crystallographic.

It was at the conference held on the little Greek island named *Spetses*, during the summer of 1993, that Gunter Malle, Jean Michel and the author realized that these constructions might also exist for *non real* reflection groups. Since then, this was shown to be true ([**Ma1**], [**BMM3**]) for certain types of non–real irreducible reflection groups (called "spetsial") by constructing sets of "unipotent characters", Fourier transform matrices, and "eigenvalues of Frobenius", which satisfy suitable generalizations of combinatorial properties of actual finite reductive groups. These were announced in the report [**Ma3**] and will be developed in [**BMM3**]. The mysterious objects which ought to be there, behind the scene, have be named "'spetses" (singular : "spets").

ℓ–blocks of finite reductive groups.

With what precedes in mind, we present the form taken by the *abelian defect group conjecture* (a conjecture about ℓ–adic representations of abstract finite groups, see [**Bro1**]) for the particular case of finite reductive groups. The whole machinery of complex reflection groups, their braid groups and Hecke algebras, is relevant here. Moreover, here again, the phenomenon which we predict seems to be "generic", and it will probably generalized, some day, to the mysterious spetses.

Notation

The complex conjugate of a complex number λ is denoted by λ^*.

If $P(t_1, t_2, \ldots, t_m)$ is a Laurent polynomial in the indeterminates $t_1, t_2, \ldots, t_m$ with complex coefficients, $P(t_1, t_2, \ldots, t_m)^*$ denotes the Laurent polynomial whose coefficients are the complex conjugate of the coefficients of $P(t_1, t_2, \ldots, t_m)$.

We denote by $\boldsymbol{\mu}_\infty$ the group of all roots of unity in $\mathbb{C}$. For d an integer ($d \geq 1$), we denote by $\boldsymbol{\mu}_d$ the subgroup of $\boldsymbol{\mu}_\infty$ consisting of all d-th roots of unity, and we set $\zeta_d := \exp(2\pi i/d)$.

The letter K will denote a subfield of the field of complex numbers $\mathbb{C}$. We denote by $\boldsymbol{\mu}(K)$ the group of roots of unity in K.

For V a K–vector space, we denote by $V^\vee$ the dual space $V^\vee = \mathrm{Hom}(V, K)$.

CHAPTER I

COMPLEX REFLECTION GROUPS

For most of the results quoted here, we refer the reader to the classical literature on complex reflection groups, such as [**Bou1**], [**Ch**], [**OrSo***j*] ($j = 1, 2, 3$), [**OrTe**], [**ShTo**], [**Sp**].

We have made constant use of the papers [**BMM2**] and [**BMR**].

Let V be a K–vector space of dimension r.

Let W be a finite subgroup of $\mathrm{GL}(V)$ We denote by S the symmetric algebra of V, by $R = S^W$ the algebra of invariants of W, by R_+ the ideal of R consisting of elements of positive degree. We set $S_W := S/(R_+.S)$, and call it the *coinvariant algebra* of (V, W).

A *pseudo–reflection* (or simply *reflection*) of $\mathrm{GL}(V)$ is a non trivial element s of $\mathrm{GL}(V)$ which acts trivially on a hyperplane, called the *reflecting hyperplane* of s.

The pair (V, W) is called a *K–reflection group* (or simply a reflection group if we omit to mention the ground field) of rank r whenever W is a finite subgroup of $\mathrm{GL}(V)$ generated by pseudo–reflections.

Characterization

After Shephard and Todd discovered the following theorem as a consequence of their classification theorem (see 1.5 below), Chevalley [**Ch**] gave an *a priori* (*i.e.*, classification free) proof of it.

1.1. THEOREM. *Let V be an r–dimensional complex vector space and let W be a finite subgroup of* $\mathrm{GL}(V)$. *The following assertions are equivalent :*

(i) *(V, W) is a (complex) reflection group.*

(ii) *The (graded) algebra $R = S^W$ is regular (i.e., it is a polynomial algebra over r algebraically independant homogeneous elements).*

(iii) *S is free as an R–module.*

Suppose that (V, W) is a complex reflection group. Let $f_1, f_2, \ldots, f_r$ be a family of algebraically independant homogeneous elements of S, with degrees respectively $(d_1, d_2, \ldots, d_r)$, such that $R = \mathbb{C}[f_1, f_2, \ldots, f_r]$. The family $(d_1, d_2, \ldots, d_r)$ depends only on (V, W). It is called the family of *degrees* of (V, W).

The Poincaré series $\mathrm{grdim} R(x)$ of R (an element of $\mathbb{Z}[[x]]$, also called the "graded dimension of R") satifies the following identity :

$$\mathrm{grdim} R(x) = \frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1 - xw)} = \frac{1}{\prod_{j=1}^{j=r} (1 - x^{d_j})}\,.$$

A consequence of the preceding identity is the following formula :

$$|W| = d_1 d_2 \cdots d_r\,.$$

Reflection groups inside reflection groups.

The next two theorems, which provide a way of constructing subgroups of a complex reflection group which are again complex reflection groups, both rely on the above characterization 1.1.

Parabolic subgroups and Steinberg theorem.

For X a subset of V, let us denote by W_X the pointwise stabilizer of X in W.

DEFINITION. The parabolic subgroups of W are the subgroups W_X where X runs over the set of subsets of V.

The next result is due to Steinberg ([**St1**], Thm. 1.5) — cf. also exercises 7 and 8 in [**Bou**], Ch. v, §6.

1.2. THEOREM. *Let X be a subset of V. Then the parabolic subgroup W_X, consisting of all elements of W which fix X pointwise, is generated by the pseudo-reflections in W whose reflecting hyperplane contains X. In particular (V, W_X) is a reflection group.*

Let $\mathcal{A}$ be the set of all reflecting hyperplanes of reflections in W. Let us denote by $I(\mathcal{A})$ the set of all intersections of elements of $\mathcal{A}$. The following result is a consequence of 1.2.

1.3. COROLLARY. *The map $X \mapsto W_X$ is a (non–increasing) bijection from the set $I(\mathcal{A})$ onto the set of all parabolic subgroups of W.*

Regular elements and Springer theorem.

An element of V is said to be *regular* if it does not belong to any reflection hyperplane of W. We introduce the *regular variety*

$$\mathcal{M} := V - \bigcup_{H \in \mathcal{A}} H\,.$$

Thus $\mathcal{M}$ is the set of regular elements.

DEFINITION. Let d a be positive integer and let $\zeta \in \boldsymbol{\mu}_d$.

- An element w of W is said to be ζ–regular if it has a regular ζ–eigenvector, *i.e.*, if $\ker(w - \zeta\mathrm{Id}) \cap \mathcal{M} \neq \emptyset$. If $\zeta = \zeta_d$, we also say that w is d–regular.
- If there exists a ζ–regular element in W, we say that d is a regular number for W.

Notice (see [**Bes2**]) that d is a regular number for W if and only if the set $(\mathcal{M}/W)^{\boldsymbol{\mu}_d}$ (set of fixed points of $\mathcal{M}/W$ under multiplications by d–the roots of unity) is not empty.

The next result is essentially due to Springer [**Sp**] (see [**Le**], 5.8, or [**DeLo**] for the third assertion). It has been generalized in [**LeSp2**] and in [**BrMa2**] (see below 5.7).

1.4. THEOREM. *Let d be a positive integer and let $\zeta \in \boldsymbol{\mu}_d$. Let w be a ζ–regular element of W. Let $W(w)$ denote the centralizer of w in W, and let* $V(w) := \ker(w - \zeta\mathrm{Id})$.

(1) *The pair $(V(w), W(w))$ is a complex reflection group.*
(2) *Its family of degrees consists of the degrees of W which are divisible by d.*

(3) *Let $\mathcal{A}(w)$ be the set of reflecting hyperplanes of $(V(w), W(w))$ and let $\mathcal{M}(w)$ be the corresponding regular variety. Then the elements of $\mathcal{A}(w)$ are the intersections with $V(w)$ of the elements of $\mathcal{A}$ and we have*

$$\mathcal{M}(w) = \mathcal{M} \cap V(w)\,.$$

Classification

For more details about the classification and the exceptional groups, one may refer for example to [**ShTo**] and to [**OrTe**].

The general infinite family $G(de, e, r)$.

Let d, e and r be three positive integers.

- Let $D_r(de)$ be the set of diagonal complex matrices with diagonal entries in the group $\boldsymbol{\mu}_{de}$ of all de–th roots of unity.
- The d–th power of the determinant defines a surjective morphism

$$\det{}^d\colon D_r(de) \twoheadrightarrow \boldsymbol{\mu}_e\,.$$

  Let $A(de, e, r)$ be the kernel of the above morphism. In particular we have $|A(de, e, r)| = (de)^r/e$.
- Identifying the symmetric group $\mathfrak{S}_r$ with the usual $r \times r$ permutation matrices, we define

$$G(de, e, r) := A(de, e, r) \rtimes \mathfrak{S}_r\,.$$

We have $|G(de, e, r)| = (de)^r r!/e$, and $G(de, e, r)$ is the group of all monomial $r \times r$ matrices, with entries in $\boldsymbol{\mu}_{de}$, and product of all non-zero entries in $\boldsymbol{\mu}_d$.

Let $(x_1, x_2, \ldots, x_r)$ be a basis of V. Let us denote by $(\Sigma_j(x_1, x_2, \ldots, x_r))_{1 \leq j \leq r}$ the family of fundamental symmetric polynomials. Let us set

$$\begin{cases} f_j := \Sigma_j(x_1^{de}, x_2^{de}, \ldots, x_r^{de}) & \text{for } 1 \leq j \leq r-1\,, \\ f_r := (x_1 x_2 \cdots x_r)^d\,. \end{cases}$$

Then we have

$$\mathbb{C}[x_1, x_2, \ldots, x_r]^{G(de,e,r)} = \mathbb{C}[f_1, f_2, \ldots, f_r]\,.$$

EXAMPLES.

- $G(e, e, 2)$ is the dihedral group of order $2e$.
- $G(d, 1, r)$ is isomorphic to the wreath product $\boldsymbol{\mu}_d \wr \mathfrak{S}_r$. For $d = 2$, it is isomorphic to the Weyl group of type B_r (or C_r).
- $G(2, 2, r)$ is isomorphic to the Weyl group of type D_r.

About the exceptional groups.

There are 34 exceptional irreducible complex reflection groups, of ranks from 2 to 8, denoted $G_4, G_5, \ldots, G_{37}$.

The rank 2 groups are connected with the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ (the *binary polyhedral groups*).

1.5. THEOREM. (Shephard–Todd) *Let (V, W) be an irreducible complex reflection group. Then one of the following assertions is true :*

- *There exist integers d, e, r, with $de \geq 2$, $r \geq 1$ such that $(V, W) \simeq G(de, e, r)$.*
- *There exists an integer $r \geq 1$ such that $(V, W) \simeq (\mathbb{C}^{r-1}, \mathfrak{S}_r)$.*
- *(V, W) is isomorphic to one of the 34 exceptional groups G_n $(n = 4, \ldots, 37)$.*

Field of definition

The following theorem has been proved (using a case by case analysis) by Bessis [**Bes1**] (see also [**Ben**]), and generalizes a well known result on Weyl groups.

1.6. THEOREM–DEFINITION.
Let (V, W) be a reflection group. Let K be the field generated by the traces on V of all elements of W. Then all irreducible KW–representations are absolutely irreducible.
The field K is called the field of definition of the reflection group W.

- If $K \subseteq \mathbb{R}$, the group W is a (finite) *Coxeter group*.
- If $K = \mathbb{Q}$, the group W is a Weyl group.

Differential forms and derivations

Here we follow closely Orlik and Solomon [**OrSo2**].

Generalities.

Let us denote by Δ_1 the S–module of derivations of the K–algebra S, and by Ω^1 the S–dual of Δ_1 ("module of 1–forms"). We have

$$\Delta_1 = S \otimes V^\vee \quad \text{and} \quad \Omega^1 = S \otimes V \,,$$

and there is an obvious duality

$$\langle\ ,\ \rangle \colon \Omega^1 \times \Delta_1 \longrightarrow S \,.$$

Let $(x_1, x_2, \ldots, x_r)$ be a basis of V. Note that the family $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_r})$ of elements of Δ_1 is the dual basis. Denote by $d \colon \Omega^1 \longrightarrow \Omega^1$ the derivation of the S–module $\Omega^1 = S \otimes V$ defined by $d(x \otimes 1) := 1 \otimes x$ for all $x \in V$. Then

$$\begin{cases} (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_r}) \text{ is a basis of the } S\text{–module } \Delta_1 \,. \\ (dx_1 \,,\, dx_2 \,,\, \ldots \,,\, dx_r) \text{ is a basis of the } S\text{–module } \Omega^1 \,. \end{cases}$$

Let us endow Δ_1 and Ω^1 respectively with the graduations defined by

$$\begin{cases} \Delta_1 = \displaystyle\bigoplus_{n=-1}^{\infty} \Delta_1^{(n)} \text{ where } \Delta_1^{(n)} := S^{n+1} \otimes V^\vee \\ \Omega^1 = \displaystyle\bigoplus_{n=1}^{\infty} \Omega^{1,(n)} \text{ where } \Omega^{1,(n)} := S^{n-1} \otimes V \end{cases}$$

In other words, we have

$$\begin{cases} (\delta \in \Delta_1^{(n)}) \iff (\delta(S^m) \subseteq S^{n+m}) \,, \\ (\omega \in \Omega^{1,(n)}) \iff (\langle \omega, \Delta_1^{(m)} \rangle \subseteq S^{n+m}) \,, \end{cases}$$

and so (with previous notation)

$$\begin{cases} \text{the elements } (\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dots, \dfrac{\partial}{\partial x_r}) \text{ have degree } -1, \\ \text{while the elements } (dx_1\,, dx_2\,, \dots\,, dx_r) \text{ have degree } 1. \end{cases}$$

We denote by Δ and Ω the exterior algebras of respectively the S–modules Δ_1 and Ω^1. We have

$$\Delta = S \otimes \Lambda(V^\vee) \quad \text{and} \quad \Omega = S \otimes \Lambda(V)\,.$$

We endow Δ and Ω respectively with the bi–graduations extending the graduations of Δ_1 and Ω^1 :

$$\begin{cases} \Delta^{(m,n)} := S^m \otimes \Lambda^{-n}(V^\vee) \\ \Omega^{(m,n)} := S^m \otimes \Lambda^{n}(V) \end{cases}$$

The Poincaré series of the preceding bigraded modules are defined as follows :

$$\begin{cases} \operatorname{grdim} \Delta(t,u) := \displaystyle\sum_{m=0}^{\infty} \sum_{n=0}^{n=r} \dim \Delta^{(m,-n)} t^m (-u)^{-n} \\ \operatorname{grdim} \Omega(t,u) := \displaystyle\sum_{m=0}^{\infty} \sum_{n=0}^{n=r} \dim \Omega^{(m,-n)} t^m (-u)^{n}\,. \end{cases}$$

Fixed points under W.

Since S is a free graded R–module, the fixed points modules $(\Omega^1)^W$ and $(\Delta_1)^W$ are free graded R–modules.

- *Degrees again* : If $f_1, \dots, f_r$ is a family of algebraically independant homogeneous elements of S such that $R = \mathbb{C}[f_1, f_2, \dots, f_r]$, then $df_1, \dots, df_r$ is a basis of $(\Omega^1)^W$ over R (thus consisting in a family of homogeneous elements with degrees respectively $(d_1, d_2, \dots, d_r)$).
- *Codegrees* : If $\delta_1, \delta_2, \dots, \delta_r$ is a basis of $(\Delta_1)^W$ over R consisting of homogeneous elements of degrees $(d_1^\vee, d_2^\vee, \dots, d_r^\vee)$, the family $(d_1^\vee, d_2^\vee, \dots, d_r^\vee)$ is called the family of codegrees of (V, W).

The Poincaré series of $(\Omega^1)^W$ and $(\Delta_1)^W$ are respectively

$$\begin{cases} \operatorname{grdim} (\Omega^1)^W(q) := \dfrac{q^{-1} \sum_{j=1}^{j=r} q^{d_j}}{\prod_{j=1}^{j=r}(1-q^{d_j})} \\ \operatorname{grdim} (\Delta_1)^W(q) := \dfrac{q \sum_{j=1}^{j=r} q^{d_j^\vee}}{\prod_{j=1}^{j=r}(1-q^{d_j})}\,. \end{cases}$$

Let us denote by $\Delta^W := (S \otimes \Lambda(V^\vee))^W$ and $\Omega^W := (S \otimes \Lambda(V))^W$ respectively the subspaces of fixed points under the action of W.

1.7. Theorem. (Orlik–Solomon [**OrSo2**]) *The R–modules Δ^W and Ω^W are the exterior algebras of respectively the R–modules $(\Delta_1)^W$ and $(\Omega^1)^W$:*

$$\begin{cases} \Delta^W = \Lambda_R((\Delta_1)^W) \\ \Omega^W = \Lambda_R((\Omega^1)^W) \end{cases}$$

Let us denote by $\operatorname{grdim}\Delta(t,u)$ and $\operatorname{grdim}\Omega(t,u)$ respectively the generalized Poincaré series of these modules defined by

$$\begin{cases} \operatorname{grdim}\Delta^W(t,u) := \sum_{m=0}^{\infty}\sum_{n=0}^{n=r} \dim\,(\Delta^{(m,-n)})^W t^m(-u)^{-n} \\ \operatorname{grdim}\Omega^W(t,u) := \sum_{m=0}^{\infty}\sum_{n=0}^{n=r} \dim\,(\Omega^{(m,-n)})^W t^m(-u)^{n}\,. \end{cases}$$

1.8. Corollary. *We have*

$$\begin{cases} \operatorname{grdim}\Delta^W(t,u) = \dfrac{1}{|W|}\displaystyle\sum_{w\in W}\frac{\det(1-w^{-1}u^{-1})}{\det(1-wt)} = \prod_{j=1}^{j=r}\frac{1-u^{-1}t^{d_j^\vee+1}}{1-t^{d_j}} \\ \operatorname{grdim}\Omega^W(t,u) = \dfrac{1}{|W|}\displaystyle\sum_{w\in W}\frac{\det(1-wu)}{\det(1-wt)} = \prod_{j=1}^{j=r}\frac{1-ut^{d_j-1}}{1-t^{d_j}} \end{cases}$$

Complement. The previous corollary implies a multiplicative identity for the Poincaré series which has been conjectured by Jean Michel from computations.

1.9. Lemma. *We have*

$$\left(\prod_{w\in W}\det_V(1-tw)\right)^{1/|W|} = \prod_{j=1}^{j=r}(1-t^{d_j})^{1/d_j}\,.$$

Proof of 1.9. By 1.8 we have

$$\frac{1}{|W|}\sum_{w\in W}\frac{\det(1-wu)}{\det(1-wt)} = \prod_{j=1}^{j=r}\frac{1-ut^{d_j-1}}{1-t^{d_j}}\,.$$

Let us differentiate with respect to u both sides of the preceding equality. We get

$$\frac{1}{|W|}\sum_{w\in W}\frac{\frac{d}{du}\det_V(1-wu)}{\det_V(1-tw)} = \sum_{j=1}^{j=r}-t^{d_j-1}\left(\frac{\prod_{k\neq j}(1-ut^{d_k-1})}{\prod_{k=1}^{k=r}(1-t^{d_k})}\right)\,.$$

Now specialize the preceding equality at $u=t$. We get

$$\frac{d}{dt}\operatorname{Log}\left(\prod_{w\in W}\det_V(1-tw)\right)^{1/|W|} = \frac{d}{dt}\operatorname{Log}\left(\prod_{j=1}^{j=r}(1-t^{d_j})^{1/d_j}\right)\,,$$

thus proving the identity announced in 1.9. □

We recall that $\mathcal{A}$ denotes the set of reflecting hyperplanes of (V,W), and we set $N := |\mathcal{A}|$. We denote by $N^\vee$ the number of pseudo-reflections in W (note that for real reflection groups we have $N = N^\vee$).

For $H\in\mathcal{A}$, we denote by $e_H := |W_H|$ the order of the pointwise stabilizer of H. The group W_H is a minimal non trivial parabolic subgroup of W. All its non trivial elements are pseudo-reflections. The group W_H is cyclic : if s_H denotes the element of W_H with determinant ζ_{e_H}, we have $W_H = \langle s_H\rangle$, the group generated by s_H. Such a pseudo–reflection as s_H is called a *distinguished* pseudo–reflection in W.

The centralizer $C_W(W_H)$ of W_H in W is also its normalizer, as well as the normalizer (set-wise stabilizer) of H.

For $\mathcal{C} \in \mathcal{A}/W$ an orbit of hyperplanes, we denote by $N_{\mathcal{C}}$ its cardinality. We have $N_{\mathcal{C}} = |W : C_W(W_H)|$ for $H \in \mathcal{C}$. We also set $e_{\mathcal{C}} := e_H$ for $H \in \mathcal{C}$.

The following identities are all consequences of 1.8.

- We have
$$(x + d_1 - 1)(x + d_2 - 1) \cdots (x + d_r - 1) = \sum_{w \in W} x^{\dim V^{\langle w \rangle}}$$

(where $V^{\langle w \rangle}$ denotes the space of fixed points of w). It follows that

$$\sum_{j=1}^{j=r} (d_j - 1) = \sum_{H \in \mathcal{A}} (e_H - 1) = \sum_{\mathcal{C} \in \mathcal{A}/W} N_{\mathcal{C}}(e_{\mathcal{C}} - 1) = N^\vee . \tag{1.10}$$

- We have
$$(x - d_1^\vee - 1)(x - d_2^\vee - 1) \cdots (x - d_r^\vee - 1) = \sum_{w \in W} \det_V(w) x^{\dim V^{\langle w \rangle}} .$$

It follows that

$$\sum_{j=1}^{j=r} (d_j^\vee + 1) = \sum_{H \in \mathcal{A}} 1 = \sum_{\mathcal{C} \in \mathcal{A}/W} N_{\mathcal{C}} = N , \tag{1.11}$$

and so

$$N + N^\vee = \sum_{j=1}^{j=r} (d_j + d_j^\vee) = \sum_{\mathcal{C} \in \mathcal{A}/W} N_{\mathcal{C}} e_{\mathcal{C}} . \tag{1.12}$$

Regular numbers, degrees and codegrees.

Lehrer and Springer ([**LeSp2**], 5.1) have noticed the following characterization of regular numbers.

1.13. Proposition. *An integer d is regular for W if and only if it divides as many degrees as codegrees.*

It should be noticed that the "if" implication is only known at the moment by case–by–case inspection.

Cohomology of the hyperplanes complement

For $H \in \mathcal{A}$, let us denote by α_H a linear form on V with kernel H, and let us define the holomorphic differential form ω_H on $\mathcal{M}$ by the formula

$$\omega_H := \frac{1}{2i\pi} \frac{d\alpha_H}{\alpha_H} ,$$

which we also write $\omega_H = \dfrac{1}{2i\pi} d\mathrm{Log}(\alpha_H)$. We denote by $[\omega_H]$ the corresponding de Rham cohomology class.

Brieskorn (cf. [**Br2**], Lemma 5) has proved the following result.

1.14. *Let* $\mathbb{C}[(\omega_H)_{H\in\mathcal{A}}]$ *(resp.* $\mathbb{Z}[(\omega_H)_{H\in\mathcal{A}}]$*) be the* $\mathbb{C}$*-subalgebra (resp. the* $\mathbb{Z}$*-subalgebra) of the* $\mathbb{C}$*-algebra of holomorphic differential forms on* $\mathcal{M}$ *which is generated by* $\{\omega_H\}_{H\in\mathcal{A}}$*. Then the map* $\omega_H \mapsto [\omega_H]$ *induces an isomorphism between* $\mathbb{C}[(\omega_H)_{H\in\mathcal{A}}]$ *and the cohomology algebra* $\mathrm{H}^*(\mathcal{M},\mathbb{C})$ *(resp. an isomorphism between* $\mathbb{Z}[(\omega_H)_{H\in\mathcal{A}}]$ *and the singular cohomology algebra* $\mathrm{H}^*(\mathcal{M},\mathbb{Z})$*).*

From now on, we write ω_H instead of $[\omega_H]$.

Orlik and Solomon (cf. [**OrSo1**]) give a description of the algebra $\mathrm{H}^*(\mathcal{M},\mathbb{C})$. Before stating their result, we need to introduce more notation.

- Let $\mathbb{C}\mathcal{A} := \bigoplus_{H\in\mathcal{A}} \mathbf{e}_H$ be the vector space with basis indexed by $\mathcal{A}$, and let $\Lambda\mathcal{A}$ be its exterior algebra, endowed with the usual Koszul differential map $\delta\colon \Lambda\mathcal{A} \to \Lambda\mathcal{A}$ of degree -1.
- For $\mathcal{B} = \{H_1, H_2, \dots, H_k\} \subset \mathcal{A}$, we denote by $D_\mathcal{B}$ the line generated by $\mathbf{e}_{H_1} \wedge \mathbf{e}_{H_2} \wedge \cdots \wedge \mathbf{e}_{H_k}$.
- We say that $\mathcal{B}$ is *dependent* if $\operatorname{codim}(\bigcap_{H\in\mathcal{B}} H) < |\mathcal{B}|$.
- We denote by $\mathrm{I}\Lambda\mathcal{A}$ the (graded) ideal of $\Lambda\mathcal{A}$ generated by the $\delta(D_\mathcal{B})$ where $\mathcal{B}$ runs overs the set of all dependent subsets of $\mathcal{A}$.

1.15. Theorem. (Orlik and Solomon) *The map* $\mathbf{e}_H \mapsto \omega_H$ *induces an isomorphism of graded algebras between* $\Lambda\mathcal{A}/\mathrm{I}\Lambda\mathcal{A}$ *and* $\mathrm{H}^*(\mathcal{M},\mathbb{C})$*.*

We recall that $I(\mathcal{A})$ denotes the set of intersections of elements of $\mathcal{A}$. For $X \in I(\mathcal{A})$, we set $\mathrm{H}^{(X)}(\mathcal{M},\mathbb{C}) := \sum D_\mathcal{B}$ where the summation is taken over all $\mathcal{B} \subset \mathcal{A}$, $|\mathcal{B}| = \operatorname{codim}(X)$, $\bigcap_{H\in\mathcal{B}} H = X$, and where $D_\mathcal{B}$ is the complex line generated by $\omega_{H_1} \wedge \omega_{H_2} \wedge \cdots \wedge \omega_{H_k}$ if $\mathcal{B} = (H_1, H_2, \dots, H_k)$.

Then it follows from Theorem 1.15 that

1.16. Corollary. *for any integer* n*, we have*

$$\mathrm{H}^n(\mathcal{M},\mathbb{C}) = \bigoplus_{\substack{(X\in I(\mathcal{A}))\\ (\operatorname{codim}(X)=n)}} \mathrm{H}^{(X)}(\mathcal{M},\mathbb{C})\,.$$

Moreover, we see that

1.17. Corollary.

(1) *the family* $(\omega_H)_{H\in\mathcal{A}}$ *is a basis of* $\mathrm{H}^1(\mathcal{M},\mathbb{C})$*,*

(2) *for* X *an element of* $I(\mathcal{A})$ *with codimension 2, if* H_X *denotes a fixed element of* $\mathcal{A}$ *which contains* X*,*
   - *whenever* H *and* H' *are two elements of* $\mathcal{A}$ *which contain* X*, we have* $\omega_H \wedge \omega_{H'} = \omega_{H_X} \wedge \omega_{H'} - \omega_{H_X} \wedge \omega_H$ *,*
   - *the family* $(\omega_{H_X} \wedge \omega_H)_{(H\supset X)(H\neq H_X)}$ *is a basis of* $\mathrm{H}^{(X)}(\mathcal{M},\mathbb{C})$*.*

The codegrees are determined by the arrangement $\mathcal{A}$, by the following consequence of Theorem 1.15.

1.18. Corollary. *The Poincaré polynomial*

$$P_\mathcal{M}(q) := \sum_n q^n \dim(\mathrm{H}^n(\mathcal{M},\mathbb{C}))$$

of the cohomology algebra $\mathrm{H}^*(\mathcal{M},\mathbb{C})$ *is given by the following formulae :*

$$P_\mathcal{M}(q) = (1 + (1 + d_1^\vee)q) \cdots (1 + (1 + d_r^\vee)q) = \sum_{w\in W} \det_V(w)(-q)^{\operatorname{codim}(V^{\langle w \rangle})}\,.$$

Coinvariant algebra and fake degrees

The coinvariant algebra $S_W = S/(S.(S_+^W)$, viewed as a KW–module, is isomorphic to the regular representation of W (see [**Bou**], chap V, 5.2, th. 2, (ii), or [**Ch**], p.779). Moreover we have an isomorphism of graded KW–modules (cf. for example [**Bou**], chap. V, §5, th. 2) :

$$S \simeq S_W \otimes_K R\,.$$

Thus the graded dimension $\sum_n \dim S_W^n x^n$ of S_W is

$$\mathrm{grdim} S_W(x) = \prod_{j=1}^{j=r} (1 + x + \cdots + x^{d_j - 1})\,.$$

Since $\sum_{j=1}^{j=r}(d_j - 1) = N^\vee$, it follows that

$$S_W = \bigoplus_{n=0}^{n=N^\vee} S_W^n \quad \text{with} \quad \dim S_W^0 = \dim S_W^{N^\vee} = 1\,. \tag{1.19}$$

Let χ be a character of W. *The fake degree of* χ is the element of $\mathbb{Z}[x]$ defined as follows:

$$\mathrm{Feg}_\chi(x) := \sum_n \langle \mathrm{tr}(\cdot, S_W^n), \chi\rangle\, x^n\,. \tag{1.20}$$

In particular, if χ is (absolutely) irreducible, its fake degree is the "graded" multiplicity of an irreducible representation with character χ in the graded module S_W.

1.21. Example. Assume that W is a cyclic group of order e. The set of irreducible characters consists of the characters $\det_V^j$ for $0 \leq j < e$. Then the fake degree of $\det_V^j$ is x^j.

The following formulae are well known and they may be found, for example, in [**Sp**] (see also chap. 4 below).

(1) We have

$$\mathrm{Feg}_\chi(x) = \frac{1}{|W|} \sum_{w \in W} \frac{\chi(w)}{\det(1 - xw)^*} \left(\prod_{j=1}^{j=r} (1 - x^{d_j}) \right)\,.$$

(2) Since the coinvariant algebra S_W, viewed as a KW–module, is isomorphic to the regular representation of W, we have

$$\mathrm{Feg}_\chi(1) = \chi(1)\,. \tag{1.22}$$

The integer $N(\chi)$ is defined as

$$N(\chi) := \frac{d}{dx} \mathrm{Feg}_\chi(x)_{|_{x=1}}\,.$$

One writes sometimes

$$\mathrm{Feg}_\chi(x) = x^{e_1(\chi)} + \cdots + x^{e_{\chi(1)}(\chi)}\,,$$

where the integers $e_j(\chi)$ are called the *exponents of* χ. Then we have

$$N(\chi) = e_1(\chi) + \cdots + e_{\chi(1)}(\chi)\,.$$

The following result is a reformulation of a theorem of Gutkin (see [**Gu**]).

1.23. PROPOSITION. *We have*

$$N(\chi) = \sum_{H\in\mathcal{A}} N(\mathrm{Res}^W_{W_H}\chi)\,,$$

where in the right hand side N is interpreted with respect to the reflection group (V, W_H).

PROOF OF 1.23. Let us consider the element $S_\chi(x) \in K(x)$ defined by

$$S_\chi(x) := \frac{1}{|W|}\sum_{w\in W}\frac{\chi(w)}{\det(1-xw)^*} = \left(\prod_{j=1}^{j=r}(1-x^{d_j})\right)^{-1}\mathrm{Feg}_\chi(x)\,.$$

We define $\psi(\chi)$ by writing the first terms of the expansion of $S_\chi(x)$ as a Laurent series in $(1-x)$ as follows:

$$S_\chi(x) = \frac{\chi(1)}{|W|}\frac{1}{(1-x)^r} + \frac{\psi(\chi)}{|W|}\frac{1}{(1-x)^{r-1}} + \cdots\,.$$

Let us compute $\psi(\chi)$ in two different ways.

- We have $\psi(\chi) = |W|\dfrac{d}{dx}((1-x)^r S_\chi(x))_{|x=1}$. Since

$$\prod_{j=1}^{j=r} d_j = |W| \quad\text{and}\quad \sum_{j=1}^{j=r}(d_j-1) = N^\vee\,,$$

an easy computation gives

$$\psi(\chi) = \frac{N^\vee}{2} - N(\chi)\,.$$

- On the other hand, if $\mathrm{Ref}(W)$ denotes the set of all reflections in W, we have

$$\psi(\chi) = \sum_{\rho\in\mathrm{Ref}(W)}\frac{\chi(\rho)}{1-\det(\rho)^*}\,.$$

Thus we get

$$N(\chi) = \frac{N^\vee}{2} - \sum_{\rho\in\mathrm{Ref}(W)}\frac{\chi(\rho)}{1-\det(\rho)^*} = \sum_{\rho\in\mathrm{Ref}(W)}\frac{1-\det(\rho)^*+2\chi(\rho)}{2(1-\det(\rho)^*)}\,. \tag{1.24}$$

Since $\mathrm{Ref}(W)$ is the disjoint union of the sets $\mathrm{Ref}(W_H)$ for $H\in\mathcal{A}$, 1.23 results now from the preceding formula. □

Let us denote by $m^\chi_{\mathcal{C},j}$ the multiplicity of $\det^j_V$ as a constituent of $\mathrm{Res}^W_{W_H}\chi$ (for $H\in\mathcal{C}$). In other words, we have, for all integers k,

$$\chi(s^k_H) = \sum_{j=0}^{j=e_\mathcal{C}-1} m^\chi_{\mathcal{C},j}\det{}^j_V(s^k_H)\,.$$

We denote by χ^* the complex conjugate of the character χ, which is then the character of the contragredient representation of a representation defining χ.

The following properties may be found in [**BrMi2**], 4.1 and 4.2.

1.25. Proposition. *Whenever $\chi \in \mathrm{Irr}(W)$, we have*

(1) $N(\chi) = \sum_{C\in\mathcal{A}/W} \sum_{j=0}^{j=e_C-1} j\, N_C\, m^{\chi}_{C,j}$,

(2) $N(\chi) + N(\chi^*) = \sum_{C\in\mathcal{A}/W} \sum_{j=1}^{j=e_C-1} N_C e_C m^{\chi}_{C,j}$,

(3) $\chi(1)(N + N^\vee) - (N(\chi) + N(\chi^*)) = \sum_{C\in\mathcal{A}/W} N_C e_C m^{\chi}_{C,0}$.

Coxeter–like presentations

Coxeter–like presentations and minimal number of "generating reflections".

One says that W has a Coxeter–like presentation (cf. [**Op2**], 5.2) if it has a presentation of the form

$$\langle s \in S \mid \{v_i = w_i\}_{i\in I}, \{s^{e_s} = 1\}_{s\in S}\rangle$$

where S is a finite set of distinguished reflections, and I is a finite set of relations which are multi–homogeneous, *i.e.*, such that (for each i) v_i and w_i are positive words with the same length in elements of S.

Bessis ([**Bes3**], Thm. 0.1) has recently proven *a priori* (without using the Shephard–Todd classification) a general result about presentations of braid groups (see below 2.27) which implies that any complex reflection group has a Coxeter–like presentation.

The following property follows partially from the main theorem of [**Bes3**], and partially from a case–by–case analysis (see *loc.cit.*, 4.2).

The first two assertions are by–products of analogous results about associated braid groups (see below 2.28). The third assertion is proved in [**BMR**] through a case–by–case analysis.

1.26. Theorem. *Let (V, W) be a complex reflection group. We set $r := \dim(V)$. Let $(d_1, d_2, \ldots, d_r)$ be the family of its invariant degrees, ordered to that $d_1 \le d_2 \le \cdots \le d_r$. Let $g(W)$ denote the minimal number of reflections needed to generate W.*

(1) *We have $g(W) = \lceil (N + N^\vee)/d_r \rceil$.*

(2) *We have either $g_W = r$ or $g_W = r + 1$.*

(3) *The group W has a Coxeter–like presentation by g_W reflections.*

The tables in Appendix 1 provide a complete list of the irreducible finite pseudo–reflection groups, as classified by Shephard and Todd, together with Coxeter–like presentations of these groups symbolized by diagrams "à la Coxeter", as well as some of the data attached to these groups.

Many of these presentations were previously known. This is the case of the rank r groups which are generated by r reflections, studied by Coxeter [**Cx**]. Some others (the ones corresponding to the infinite series) occurred in [**BrMa**] or were inspired by [**Ari**].

The reader may refer to Appendix 1 to understand what follows.

Isomorphisms between diagrams.

We may notice that the only isomorphisms between the diagrams of our tables are between the diagrams of $G(2,1,2)$ and $G(4,4,2)$, between the diagrams of $\mathfrak{S}_4$ and $G(2,2,3)$, between the diagrams of $\mathfrak{S}_3$ and $G(3,3,2)$, and between the diagrams of $\mathfrak{S}_2$ and $G(2,1,1)$.

Coxeter groups.

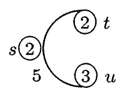
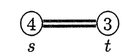
- $G(e,e,2)$ $(e \geq 3)$ is the dihedral group of order $2e$,
- G_{28} is the Coxeter group of type F_4,
- G_{35} is the Coxeter group of type E_6,
- G_{36} is the Coxeter group of type E_7,
- G_{37} is the Coxeter group of type E_8,
- G_{23} is the Coxeter group of type H_3,
- G_{30} is the Coxeter group of type H_4.

Admissible subdiagrams and parabolic subgroups.

Let $\mathcal{D}$ be one of the diagrams. Let us define an equivalence relation between nodes by $s \sim s$ and, for $s \neq t$,

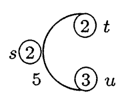
$$s \sim t \iff s \text{ and } t \text{ are not in a homogeneous relation with support } \{s,t\}.$$

Then we see that the equivalence classes have 1 or 3 elements, and that there is at most one class with 3 elements.

If there is no class with 3 elements, the rank r of the group is the number of nodes of the diagram, while it is this number minus 1 in case there is a class with 3 elements. Thus  has rank 2, as well as  .

REMARK. One must point out that, in the first of the preceding two diagrams, s, t and u must be considered as linked by a line (so t and u *do not* commute).

An **admissible subdiagram** is a full subdiagram of the same type, namely a diagram with 1 or 3 elements per class.

Thus, the diagram  has five admissible subdiagrams, namely the empty diagram, the three diagrams consisting of one node, and the whole diagram.

1.27. FACT. *Let $\mathcal{D}$ be the diagram of W as given in tables 1 to 4 in Appendix 2 below.*

(1) *If $\mathcal{D}'$ is an admissible subdiagram of $\mathcal{D}$, it gives a presentation of the corresponding subgroup $W(\mathcal{D}')$ of W. This subgroup is a parabolic subgroup.*
(2) *Assume W is neither G_{27}, G_{29}, G_{33} nor G_{34}. If $P_1 \subseteq P_2 \subseteq \cdots P_n$ is a chain of parabolic subgroups of W, there exist $g \in W$ and a chain $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots \mathcal{D}_n$ of admissible subdiagrams of $\mathcal{D}$ such that*

$$(P_1, P_2, \ldots, P_n) = {}^g(W(\mathcal{D}_1), W(\mathcal{D}_2), \ldots, W(\mathcal{D}_n)).$$

Remark.

For groups G_{27} and G_{29}, all isomorphism classes of parabolic subgroups are represented by admissible subdiagrams of our diagrams, but not all *conjugacy classes* of parabolics subgroups are represented by admissible subdiagrams, as noticed by Orlik.

For groups G_{33} and G_{34}, not all isomorphism classes of parabolic subgroups are represented by admissible subdiagrams of our diagrams.

CHAPTER II

ASSOCIATED BRAID GROUPS

Notation

For X a topological space, we denote by $\mathcal{P}(X)$ its fundamental groupoid, where the composition of (classes of) paths is defined so that, if γ_1 is a path going from x_0 to x_1 and γ_2 is a path going from x_1 to x_2, then the composite map going from x_0 to x_2 is denoted by $\gamma_2 \cdot \gamma_1$.

Given a point $x_0 \in X$, we denote by $\pi_1(X, x_0)$ (or $\pi_1(X)$ if the choice of x_0 is clear) the fundamental group with base point x_0. So we have $\pi_1(X, x_0) = \mathrm{End}_{\mathcal{P}(X)}(x_0)$. If $f\colon X \to Y$ is a continuous map, we denote by $\mathcal{P}(f)$ the corresponding functor from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$. We also denote by $\pi_1(f, x_0)$ (or $\pi_1(f)$) the group homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, f(x_0))$ induced by $\mathcal{P}(f)$.

We choose, once for all, a square root of (-1) in $\mathbb{C}$, which is denoted by i. Moreover, for every $z \in \mathbb{C}^\times$, we identify $\pi_1(\mathbb{C}^\times, z)$ with $\mathbb{Z}$ by sending onto 1 the loop $\lambda_z\colon [0,1] \to \mathbb{C}^\times$ defined by $\lambda_z(t) := z \exp(2i\pi t)$.

Generalities about hyperplane complements

What follows is probably well known to specialists of hyperplane complements and topologists. We include it for the convenience of the reader, and because of the lack of convenient references.

Distinguished braid reflections around an irreducible divisor.

We define here what we mean by a "*distinguished braid reflection around an irreducible divisor*", usually called "generator of the monodromy" (around this divisor), and we recall some well known properties.

Let Y be a smooth connected complex algebraic variety, I a finite family of irreducible codimension 1 closed subvarieties (irreducible divisors) and $Z := \cup_{D \in I} D$. Let $X := Y - Z$ and $x_0 \in X$.

For $D \in I$, let D_s be the smooth part of D and let us define $\tilde{D} := D_s - \left(D_s \cap \bigcup_{D' \in I, D' \neq D} D'\right)$.

A "*path from x_0 to D in X*" is by definition a path γ in Y such that $\gamma(0) = x_0$, $\gamma(1) \in \tilde{D}$ and $\gamma(t) \in X$ for $t \neq 1$.

Let γ' be another path from x_0 to D in X. We say that γ and γ' are *D-homotopic* if there is a continuous map $T : [0,1] \times [0,1] \to Y$ such that $T(t,0) = \gamma(t)$ and $T(t,1) = \gamma'(t)$ for $t \in [0,1]$, $T(0,u) = x_0$ and $T(1,u) \in \tilde{D}$ for all $u \in [0,1]$ and $T(t,u) \in X$ for $t \in [0,1[$ and $u \in [0,1]$. We denote by $[\gamma]$ the D-homotopy class of γ.

Given a path γ from x_0 to D in X, let B be a connected open neighbourhood of $\gamma(1)$ in $X \cup \tilde{D}$ such that $B \cap X$ has a fundamental group free abelian of rank 1. Let $u \in [0,1[$ such that $\gamma(t) \in B$ for $t \geq u$. Put $x_1 := \gamma(u)$. The orientation of $B \cap X$ coming from the orientation of X gives an isomorphism $f\colon \pi_1(B \cap X, x_1) \xrightarrow{\sim} \mathbb{Z}$. Let λ be a loop in $B \cap X$ from x_1 such that $f([\lambda]) = 1$.

Let γ_u be the "restriction" of γ to $[0,u]$, defined by $\gamma_u(t) := \gamma(ut)$ for all $t \in [0,1]$. Define $\rho_{\gamma,\lambda} := {\gamma_u}^{-1} \cdot \lambda \cdot \gamma_u$. Then, the homotopy class of $\rho_{\gamma,\lambda}$ in $\pi_1(\mathcal{M}, x_0)$ depends only on the D-homotopy class of γ and is denoted by $\rho_{[\gamma]}$. We call it *the distinguished braid reflection around D associated to* $[\gamma]$.

Given two paths γ and γ' from x_0 to D, the distinguished braid reflections $\rho_{[\gamma]}$ and $\rho_{[\gamma']}$ are conjugate.

2.1. PROPOSITION. *Let i be the injection of an irreducible divisor D in a smooth connected complex variety Y and $x_0 \in Y - D$. Then, the kernel of the morphism $\pi_1(i)\colon \pi_1(Y - D, x_0) \to \pi_1(Y, x_0)$ is generated by all the distinguished braid reflections around D.*

Indeed, note that the singular points of D form a closed subvariety D_{sing} of D, distinct from D, hence of (complex) codimension at least 2 in Y. Therefore the natural morphism $\pi_1(Y - D - D_{\text{sing}}, x_0) \to \pi_1(Y - D, x_0)$ is an isomorphism, and in order to prove 2.1 we may assume D is smooth, which we do now.

The lemma then follows from the fact that given a locally constant sheaf $\mathcal{F}$ over $Y - D$, its extension $i_*\mathcal{F}$ to Y is locally constant if and only if every distinguished braid reflection around D acts trivially on $\mathcal{F}$.

2.2. PROPOSITION. *Suppose that Y is simply connected. The fundamental group $\pi_1(X, x_0)$ is generated by all the distinguished braid reflections around the divisors $D \in I$.*

Indeed, this follows immediately from Proposition 2.1 by induction on $|I|$.

Lifting distinguished braid reflections.

Let $p\colon Y \to \overline{Y}$ be a finite covering between two smooth connected complex varieties. Let D be the branch locus of p and $\overline{D} = p(D)$. We assume $\overline{D}$ is an irreducible divisor. We set $\overline{X} := \overline{Y} - \overline{D}$ and $X := Y - D$.

We shall see that a distinguished braid reflection around $\overline{D}$ (associated to a path $\overline{\gamma}$ from $\overline{x}_0$ to $\overline{D}$ in $\overline{Y}$) may be naturally lifted to an element of $\mathcal{P}(X)$ (which depends only on the $\overline{D}$-homotopy class of $\overline{\gamma}$).

Indeed, let γ be the path from x_0 to an irreducible component, say D_γ, of D, which lifts $\overline{\gamma}$. Let $\overline{B}$ be an open neighbourhood of $\overline{x}_0$ in $\overline{Y}$ such that the fundamental group of $\overline{B} \cap \overline{X}$ is free abelian of rank 1 and $B \cap (X \cup \tilde{D}_\gamma) \to \overline{B}$ is unramified outside D_γ. Let $u \in [0,1[$ such that $\overline{\gamma}(t) \in \overline{B}$ for $t \geq u$. Let $\overline{\lambda}$ be a loop in $\overline{B} \cap \overline{X}$ with origin $\overline{\gamma}(u)$ which is a positive generator of $\pi_1(\overline{B} \cap \overline{X}, \overline{\gamma}(u))$.

Let λ be the path from $\gamma(u)$ which lifts $\overline{\lambda}$. Let γ_u be the restriction of γ to $[0,u]$. Let $\gamma_u^\vee$ be the path from $\lambda(1)$ which lifts $(\overline{\gamma}_u)^{-1}$, where $\overline{\gamma}_u$ is the "restriction" of $\overline{\gamma}$ to $[0,u]$.

The proof of the following proposition is left to the reader.

2.3. PROPOSITION. *We define $\rho_\gamma := \gamma_u^\vee \cdot \lambda \cdot \gamma_u$.*

(1) *The homotopy class of ρ_γ in $\mathcal{P}(X)$ depends only on the $\overline{D}$-homotopy class of $\overline{\gamma}$.*
(2) *Let e_D denote the ramification index of p on $\tilde{D}$. Then $\rho_\gamma^{e_D}$ is the distinguished braid reflection around D_γ associated to γ.*

Hyperplane complements.

Let $\mathcal{A}$ be a finite set of affine hyperplanes (*i.e.*, affine subspaces of codimension one) in a finite dimensional complex vector space V. We set $\mathcal{M} := V - \bigcup_{H \in \mathcal{A}} H$.

Let $x_0 \in \mathcal{M}$. We shall give now some properties of the fundamental group $\pi_1(\mathcal{M}, x_0)$.

Distinguished braid reflections around the hyperplanes.

For $H \in \mathcal{A}$, let α_H be an affine map $V \to \mathbb{C}$ such that $H = \{x \in V \mid \alpha_H(x) = 0\}$. Its restriction to $\mathcal{M} \to \mathbb{C}^\times$ induces a functor $\mathcal{P}(\alpha_H) \colon \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathbb{C}^\times)$, and in particular a group homomorphism $\pi_1(\alpha_H, x_0) \colon \pi_1(\mathcal{M}, x_0) \to \mathbb{Z}$

2.4. LEMMA. *For $H, H' \in \mathcal{A}$ and γ a path from x_0 to H, we have*

$$\pi_1(\alpha_{H'})(\rho_{[\gamma]}) = \delta_{H,H'} .$$

Indeed, let us set $\mathcal{M}_H := H - \bigcup_{\substack{H' \in \mathcal{A} \\ H' \neq H}} H'$. Let $x_\gamma := \gamma(1)$ and let B be an open ball with center x_γ contained in $\mathcal{M} \cup \mathcal{M}_H$. Let $u \in [0, 1[$ such that $\gamma(t) \in B$ for $t \geq u$. We set $x_1 := \gamma(u)$. Then, the restriction of α_H to $B \cap \mathcal{M}$ induces an isomorphism $\pi_1(\alpha_H) \colon \pi_1(B \cap \mathcal{M}, x_1) \to \mathbb{Z}$. Let λ be a loop in $B \cap \mathcal{M}$, with origin x_1, whose image under $\pi_1(\alpha_H)$ is 1. Let γ_u be the "restriction" of γ to $[0, u]$, defined by $\gamma_u(t) := \gamma(ut)$ for all $t \in [0, 1]$. Define $\rho_{\gamma,\lambda} := {\gamma_u}^{-1} \cdot \lambda \cdot \gamma_u$. Then the loop $\rho_{\gamma,\lambda}$ induces the distinguished braid reflection $\rho_{[\gamma]}$, and

$$\pi_1(\alpha_{H'})(\rho_{\gamma,\lambda}) = \pi_1(\alpha_{H'})(\lambda) = \delta_{H,H'} .$$

The following proposition is immediate.

2.5. PROPOSITION.

(1) *The fundamental group $\pi_1(\mathcal{M}, x_0)$ is generated by all the distinguished braid reflections around the affine hyperplanes $H \in \mathcal{A}$.*

(2) *Let $\pi_1(\mathcal{M}, x_0)^{\mathrm{ab}}$ denote the largest abelian quotient of $\pi_1(\mathcal{M}, x_0)$. For $H \in \mathcal{A}$, we denote by ρ_H^{ab} the image of $\rho_{H,\gamma}$ in $\pi_1(\mathcal{M}, x_0)^{\mathrm{ab}}$. Then*

$$\pi_1(\mathcal{M}, x_0)^{\mathrm{ab}} = \prod_{H \in \mathcal{A}} \langle \rho_H^{\mathrm{ab}} \rangle ,$$

where each $\langle \rho_H^{\mathrm{ab}} \rangle$ is infinite cyclic. Dually, we have

$$\mathrm{Hom}(\pi_1(\mathcal{M}, x_0), \mathbb{Z}) = \prod_{H \in \mathcal{A}} \langle \pi_1(\alpha_H) \rangle .$$

REMARK. Let us recall that we have natural isomorphisms

$$\pi_1(\mathcal{M}, x_0)^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{H}_1(\mathcal{M}, \mathbb{Z}) \quad \text{and} \quad \mathrm{Hom}(\pi_1(\mathcal{M}, x_0), \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(\mathcal{M}, \mathbb{Z}) .$$

Moreover, the duality between $\pi_1(\mathcal{M}, x_0)^{\mathrm{ab}}$ and $\mathrm{H}^1(\mathcal{M}, \mathbb{Z})$ may be seen as follows. For γ a loop in $\mathcal{M}$ with origin x_0 and for ω a holomorphic differential 1-form on $\mathcal{M}$, we set $\langle \gamma, \omega \rangle := \int_\gamma \omega$. It is then clear that, under the isomorphism

$$\mathrm{Hom}(\pi_1(\mathcal{M}, x_0), \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(\mathcal{M}, \mathbb{Z}) ,$$

the element $\pi_1(\alpha_H)$ is sent onto the class of the 1-form $\omega_H = \dfrac{1}{2i\pi} \dfrac{d\alpha_H}{\alpha_H}$ (see 1.14 for more details).

About the center of the fundamental group.

In this part, we assume the hyperplanes in $\mathcal{A}$ to be *linear*.

2.6. NOTATION. *We denote by* $\boldsymbol{\pi}$ *the loop* $[0,1] \to \mathcal{M}$ *defined by*

$$\boldsymbol{\pi} : t \mapsto x_0 \exp(2i\pi t)\,.$$

2.7. LEMMA.

(1) $\boldsymbol{\pi}$ *belongs to the center* $Z(\pi_1(\mathcal{M}, x_0))$ *of the fundamental group* $\pi_1(\mathcal{M}, x_0)$.

(2) *For all* $H \in \mathcal{A}$, *we have* $\pi_1(\alpha_H)(\boldsymbol{\pi}) = 1$.

Generalities about the braid groups

More notation.

We go back to notation introduced in chap. 1. In particular, $\mathcal{A}$ is now the set of reflecting hyperplanes of a finite subgroup W of $\mathrm{GL}(V)$ generated by pseudo–reflections. We denote by $p\colon \mathcal{M} \to \mathcal{M}/W$ the canonical surjection.

Let $x_0 \in \mathcal{M}$. We introduce the following notation for the fundamental groups:

$$P := \pi_1(\mathcal{M}, x_0) \quad \text{and} \quad B := \pi_1(\mathcal{M}/W, p(x_0))\,,$$

and we call B and P respectively the *braid group* (at x_0) and the *pure braid group* (at x_0) associated to W. We shall often write $\pi_1(\mathcal{M}/W, x_0)$ for $\pi_1(\mathcal{M}/W, p(x_0))$.

The covering $\mathcal{M} \to \mathcal{M}/W$ is Galois by Steinberg's theorem (see Theorem 1.2 above), hence the projection p induces a surjective map $B \twoheadrightarrow W$, $\sigma \mapsto \overline{\sigma}$, as follows :

Let $\tilde{\sigma}\colon [0,1] \to \mathcal{M}$ be a path in $\mathcal{M}$, such that $\tilde{\sigma}(0) = x_0$, which lifts σ. Then $\overline{\sigma}$ is defined by the equality $\overline{\sigma}(x_0) = \tilde{\sigma}(1)$.

Note that the map $\sigma \mapsto \overline{\sigma}$ is an *anti–morphism*.

Denoting by W^{op} the group opposite to W, we have the following short exact sequence :

$$1 \to P \to B \to W^{\mathrm{op}} \to 1\,, \tag{2.8}$$

where the map $B \to W^{\mathrm{op}}$ is defined by $\sigma \mapsto \overline{\sigma}$.

The spaces $\mathcal{M}$ and $\mathcal{M}/W$ are conjectured to be $K(\pi, 1)$-spaces.

The following result is due to Fox and Neuwirth [**FoNe**] for the type A_n, to Brieskorn [**Br2**] for Coxeter groups of type different from H_3, H_4, E_6, E_7, E_8, to Deligne [**De2**] for general Coxeter groups. The case of the infinite series of complex reflection groups $G(de, e, r)$ has been solved by Nakamura [**Na**]. For the non-real Shephard groups (non-real groups with Coxeter braid diagrams), this has been proven by Orlik and Solomon [**OrSo3**]. Note that the rank 2 case is trivial.

2.9. THEOREM. *Assume* W *has no irreducible component of type* G_{24}, G_{27}, G_{29}, G_{31}, G_{33} *or* G_{34}. *Then,* $\mathcal{M}$ *and* $\mathcal{M}/W$ *are* $K(\pi, 1)$*-spaces.*

Distinguished braid reflections around the hyperplanes.

For $H \in \mathcal{A}$, we set $\zeta_H := \zeta_{e_H}$, We recall that we denote by s_H and call *distinguished reflection* the pseudo-reflection in W with reflecting hyperplane H and determinant ζ_H. We set

$$L_H := \mathrm{im}(s_H - \mathrm{Id}_V)\,.$$

For $x \in V$, we set $x = \mathrm{pr}_H(x) + \mathrm{pr}_H^{\perp}(x)$ with $\mathrm{pr}_H(x) \in H$ and $\mathrm{pr}_H^{\perp}(x) \in L_H$.

Thus, we have $s_H(x) = \zeta_H \mathrm{pr}_H^{\perp}(x) + \mathrm{pr}_H(x)$.

If $t \in \mathbb{R}$, we set $\zeta_H^t := \exp(2i\pi t/e_H)$, and we denote by s_H^t the element of $\mathrm{GL}(V)$ (a pseudo–reflection if $t \neq 0$) defined by :

(2.10)
$$s_H^t(x) = \zeta_H^t \mathrm{pr}_H^{\perp}(x) + \mathrm{pr}_H(x) .$$

For $x \in V$, we denote by $\sigma_{H,x}$ the path in V from x to $s_H(x)$, defined by :

$$\sigma_{H,x} \colon [0,1] \to V \ , \ t \mapsto s_H^t(x) .$$

For any path γ in $\mathcal{M}$, with initial point x_0 and terminal point x_H, the path defined by $s_H(\gamma^{-1}) \colon t \mapsto s_H(\gamma^{-1}(t))$ is a path in $\mathcal{M}$ going from $s_H(x_H)$ to $s_H(x_0)$.

Whenever γ is a path in $\mathcal{M}$, with initial point x_0 and terminal point x_H, we define the path $\sigma_{H,\gamma}$ from x_0 to $s_H(x_0)$ as follows :

(2.11)
$$\sigma_{H,\gamma} := s_H(\gamma^{-1}) \cdot \sigma_{H,x_H} \cdot \gamma .$$

It is not difficult to see that, provided x_H is chosen "close to H and far from the other reflecting hyperplanes", the path $\sigma_{H,\gamma}$ is in $\mathcal{M}$, and its homotopy class does not depend on the choice of x_H, and the element it induces in the braid group B is actually a distinguished braid reflection around the image of H in $\mathcal{M}/W$.

The following properties are immediate.

2.12. Lemma.

(1) *The image of* $\mathbf{s}_{H,\gamma}$ *in* W *is* s_H.

(2) *Whenever* γ' *is a path in* $\mathcal{M}$, *with initial point* x_0 *and terminal point* x_H, *if* τ *denotes the loop in* $\mathcal{M}$ *defined by* $\tau := {\gamma'}^{-1}\gamma$, *one has*

$$\sigma_{H,\gamma'} = \tau \cdot \sigma_{H,\gamma} \cdot \tau^{-1}$$

and in particular $\mathbf{s}_{H,\gamma}$ *and* $\mathbf{s}_{H,\gamma'}$ *are conjugate in* P.

(3) *The path* $\prod_{j=e_H-1}^{j=0} \sigma_{H,s_H^j(\gamma)}$, *a loop in* $\mathcal{M}$, *induces the element* $\mathbf{s}_{H,\gamma}^{e_H}$ *in the braid group* B, *and belongs to the pure braid group* P. *It is homotopy equivalent, as a loop in* $\mathcal{M}$, *to the distinguished braid reflection* $\rho_{[\gamma]}$ *around* H *in* P.

2.13. Definition. *Let* s *be a distinguished pseudo–reflection in* W, *with reflecting hyperplane* H. *An* s*–distinguished braid reflection is a distinguished braid reflection* $\mathbf{s}$ *around the image of* H *in* $\mathcal{M}/W$ *such that* $\overline{\mathbf{s}} = s$.

Discriminants and length

Let $\mathcal{C}$ be an orbit of W on $\mathcal{A}$. Recall that we denote by $e_{\mathcal{C}}$ the (common) order of the pointwise stabilizer W_H for $H \in \mathcal{C}$. We call *discriminant at* $\mathcal{C}$ and we denote by $\delta_{\mathcal{C}}$ the element of the symmetric algebra of $V^\vee$ defined (up to a non zero scalar multiplication) by

$$\delta_{\mathcal{C}} := (\prod_{H \in \mathcal{C}} \alpha_H)^{e_{\mathcal{C}}} .$$

Since (see for example [**Co**], 1.8) $\delta_{\mathcal{C}}$ is W-invariant, it induces a continuous function $\delta_{\mathcal{C}} \colon \mathcal{M}/W \to \mathbb{C}^\times$, hence induces a functor $\mathcal{P}(\delta_{\mathcal{C}}) \colon \mathcal{P}(\mathcal{M}/W) \to \mathcal{P}(\mathbb{C}^\times)$, and in particular it induces a group homomorphism $\pi_1(\delta_{\mathcal{C}}) \colon B \to \mathbb{Z}$.

2.14. Proposition. *For any $H \in \mathcal{A}$, we have*

$$\pi_1(\delta_{\mathcal{C}})(\mathbf{s}_{H,\gamma}) = \begin{cases} 1 & \text{if } H \in \mathcal{C}\,, \\ 0 & \text{if } H \notin \mathcal{C}\,. \end{cases}$$

What precedes allows us to define length functions on B.

• There is a unique length function $\ell\colon B \to \mathbb{Z}$ defined as follows (see [**BMR**], Prop. 2.19): if $b = \mathbf{s}_1^{n_1} \cdot \mathbf{s}_2^{n_2} \cdots \mathbf{s}_m^{n_m}$ where (for all j) $n_j \in \mathbb{Z}$ and $\mathbf{s}_j$ is a distinguished braid reflection around an element of $\mathcal{A}$ in B, then

$$\ell(b) = n_1 + n_2 + \cdots + n_m\,.$$

Indeed, we set $\ell := \pi_1(\delta)$. Let $b \in B$. By Theorem 2.15 above, there exists an integer k and for $1 \leq j \leq k$, $H_j \in \mathcal{A}$, a path γ_j from x_0 to H_j and an integer n_j such that

$$b = \mathbf{s}_{H_1,\gamma_1}^{n_1}\mathbf{s}_{H_2,\gamma_2}^{n_2}\cdots\mathbf{s}_{H_k,\gamma_k}^{n_k}\,.$$

From Proposition 2.14 above, it then results that we have $\ell(b) = \sum_{j=1}^{j=k} n_j$.

If $\{\mathbf{s}\}$ is a set of distinguished braid reflections around hyperplanes which generates B, let us denote by B^+ the sub–monoid of B generated by $\{\mathbf{s}\}$. Then for $b \in B^+$, its length $\ell(b)$ coincide with its length on the distinguished set of generators $\{\mathbf{s}\}$ of the monoid B^+.

• More generally, given $\mathcal{C} \in \mathcal{A}/W$, there is a unique length function $\ell_{\mathcal{C}}\colon B \to \mathbb{Z}$ (this is the function denoted by $\pi_1(\delta_{\mathcal{C}})$ in [**BMR**], see Prop. 2.16 in *loc.cit.*) defined as follows: if $b = \mathbf{s}_1^{n_1} \cdot \mathbf{s}_2^{n_2} \cdots \mathbf{s}_m^{n_m}$ where (for all j) $n_j \in \mathbb{Z}$ and $\mathbf{s}_j$ is a distinguished braid reflection around an element of $\mathcal{C}_j$, then

$$\ell_{\mathcal{C}}(b) = \sum_{\{j \mid (\mathcal{C}_j = \mathcal{C})\}} n_j\,.$$

Thus we have, for all $b \in B$,

$$\ell(b) = \sum_{\mathcal{C} \in \mathcal{A}/W} \ell_{\mathcal{C}}(b)\,.$$

Generators and abelianization of B

2.15. Theorem.

(1) *The group B is generated by the generators $\{\mathbf{s}_{H,\gamma}\}$ (for all hyperplanes $H \in \mathcal{A}$ and all paths γ from x_0 to H in $\mathcal{M}$) of the monodromy (in B) around the elements of $\mathcal{A}$.*

(2) *We denote by B^{ab} the largest abelian quotient of B. For $\mathcal{C} \in \mathcal{A}/W$, we denote by $\mathbf{s}_{\mathcal{C}}^{\mathrm{ab}}$ the image of $\mathbf{s}_{H,\gamma}$ in B^{ab} for $H \in \mathcal{C}$. Then*

$$B^{\mathrm{ab}} = \prod_{\mathcal{C} \in \mathcal{A}/W} \langle \mathbf{s}_{\mathcal{C}}^{\mathrm{ab}} \rangle\,,$$

where each $\langle \mathbf{s}_{\mathcal{C}}^{\mathrm{ab}} \rangle$ is infinite cyclic. Dually, we have

$$\mathrm{Hom}(B, \mathbb{Z}) = \prod_{\mathcal{C} \in \mathcal{A}/W} \langle \pi_1(\delta_{\mathcal{C}}) \rangle\,.$$

Indeed, the second assertion is immediate by the first one and by Proposition 2.14. Let us sketch a proof of (1).

Since W is generated by the set $\{s_H\}_{(H\in\mathcal{A})}$ and since we have the exact sequence (2.8), it is enough to prove that the pure braid group P is generated by all the conjugates in P of the elements $\mathbf{s}_{H,\gamma}^{e_H}$. This is a consequence of Proposition 2.5, (1).

REMARK. We have natural isomorphisms

$$B^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{H}_1(\mathcal{M}/W,\mathbb{Z}) \quad \text{and} \quad \mathrm{Hom}(B,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(\mathcal{M}/W,\mathbb{Z}),$$

and, under the second isomorphism, we have

$$\pi_1(\delta_{\mathcal{C}}) \mapsto e_{\mathcal{C}} \sum_{H\in\mathcal{C}} \frac{1}{2i\pi}\frac{d\alpha_H}{\alpha_H} = \frac{1}{2i\pi} d\mathrm{Log}(\delta_{\mathcal{C}}).$$

Let us denote by $\mathrm{Gen}(B)$ the set of all distinguished braid reflections in B (see Definition 2.13 above). For $\mathbf{s} \in \mathrm{Gen}(B)$, we denote by $e_{\mathbf{s}}$ the order of $\overline{\mathbf{s}}$.

In other words, if $\mathbf{s}$ is a distinguished braid reflection around the reflecting hyperplane $H \in \mathcal{A}$, we set now (using the notation of Definition 2.13) : $e_{\mathbf{s}} := e_H$.

The following property is a consequence of general results recalled at the beginning of this section.

2.16. PROPOSITION.

(1) *The pure braid group P is generated by $\{\mathbf{s}^{e_{\mathbf{s}}}\}_{\mathbf{s}\in\mathrm{Gen}(B)}$.*

(2) *We have*

$$W \simeq B/\langle \mathbf{s}^{e_{\mathbf{s}}}\rangle_{\mathbf{s}\in\mathrm{Gen}(B)}.$$

About the center of B

2.17. NOTATION. *We denote by $\boldsymbol{\beta}$ the path $[0,1] \to \mathcal{M}$ defined by*

$$\boldsymbol{\beta}\colon t \mapsto x_0 \exp(2i\pi t/|Z(W)|).$$

From now on, we assume that W *acts irreducibly on* V. Note that, since W is irreducible on V, it results from Schur's lemma that

$$Z(W) = \{\zeta_{|Z(W)|}^k \mid (k \in \mathbb{Z})\},$$

and so in particular $\boldsymbol{\beta}$ defines an element of B, which we will still denote by $\boldsymbol{\beta}$.

2.18. LEMMA.

(1) *The image $\overline{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ in W is the scalar multiplication by $\zeta_{|Z(W)|}$. It is a generator of the center $Z(W)$ of W.*

(2) *We have $\boldsymbol{\beta} \in Z(B)$, $\boldsymbol{\pi} \in Z(P)$, and $\boldsymbol{\pi} = \boldsymbol{\beta}^{|Z(W)|}$.*

The following proposition (see **[BMR]** 2.23) is now easy.

2.19. PROPOSITION. *Let $\overline{\mathcal{M}}$ be the image of $\mathcal{M}$ in $(V-\{0\})/\mathbb{C}^\times$. Then, we have a commutative diagram, where all short sequences are exact :*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \langle \boldsymbol{\pi} \rangle \arrow[r] \arrow[d] & \langle \boldsymbol{\beta} \rangle \arrow[r] \arrow[d] & Z(W) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(\mathcal{M}, x_0) \arrow[r] \arrow[d] & \pi_1(\mathcal{M}/W, x_0) \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(\overline{\mathcal{M}}, \overline{x}_0) \arrow[r] \arrow[d] & \pi_1(\overline{\mathcal{M}}/W, \overline{x}_0) \arrow[r] \arrow[d] & W/Z(W) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

The following statement is known for Coxeter groups (see [**Del**] or [**BrSa**]). The result holds as well for G_{25}, G_{26}, G_{32}, since the corresponding braid groups are the same as braid groups of Coxeter groups. The proof for the particular case of groups in dimension 2 is easy (see [**BMR**]. A proof for all the infinite series is given in [**BMR**], §3. We conjecture it is still true in the case of G_{31}, as well as for G_{24}, G_{27}, G_{29}, G_{33}, G_{34}.

As for all results proved for all but a finite number of irreducible complex reflection groups and conjectured to be true in general, we call it a "*Theorem–Assumption*".

2.20. THEOREM–ASSUMPTION. *The center $Z(B)$ of B is infinite cyclic and generated by $\boldsymbol{\beta}$, the center $Z(P)$ of P is infinite cyclic and generated by $\boldsymbol{\pi}$, and the short exact sequence (2.8) induces a short exact sequence*

$$1 \to Z(P) \to Z(B) \to Z(W) \to 1\,.$$

2.21. COROLLARY. *Let $\boldsymbol{\beta}^{\mathrm{ab}}$ be the image in B^{ab} of the central element $\boldsymbol{\beta}$ of B. Then we have*

$$\boldsymbol{\beta}^{\mathrm{ab}} = \prod_{\mathcal{C}\in\mathcal{A}/W} (\mathbf{s}_{\mathcal{C}}^{\mathrm{ab}})^{e_{\mathcal{C}} N_{\mathcal{C}}/|Z(W)|}\,.$$

Indeed, it suffices to prove that, if $\mathcal{C}\in\mathcal{A}/W$, then $\pi_1(\delta_{\mathcal{C}})(\boldsymbol{\beta}^{\mathrm{ab}}) = e_{\mathcal{C}}N_{\mathcal{C}}/|Z(W)|$. This is immediate :

$$\begin{aligned}
\pi_1(\delta_{\mathcal{C}})(\boldsymbol{\beta}^{\mathrm{ab}}) &= \sum_{H\in\mathcal{C}} \frac{e_{\mathcal{C}}}{2i\pi}\int_0^1 \frac{d\alpha_H(x_0\exp(2i\pi t/|Z(W)|))}{\alpha_H(x_0\exp(2i\pi t/|Z(W)|))} \\
&= \frac{e_{\mathcal{C}}}{2i\pi}\sum_{H\in\mathcal{C}} \frac{2i\pi}{|Z(W)|}\int_0^1 dt = e_{\mathcal{C}}N_{\mathcal{C}}/|Z(W)|\,.
\end{aligned}$$

□

The following result is a consequence of Corollary 2.21. Notice that it generalizes a result of Deligne [**Del**], (4.21) (see also [**BrSa**]), from which it follows that if W is a Coxeter group, then $\ell(\boldsymbol{\pi}) = 2N$. It was noticed "experimentally" in [**BrMi2**], (4.8).

2.22. COROLLARY. *We have* $\ell_{\mathcal{C}}(\boldsymbol{\beta}) = N_{\mathcal{C}} e_{\mathcal{C}}/|Z(W)|$ *and* $\ell_{\mathcal{C}}(\boldsymbol{\pi}) = N_{\mathcal{C}} e_{\mathcal{C}}$. *In particular, we have* $\ell(\boldsymbol{\beta}) = (N + N^\vee)/|Z(W)|$ *and* $\ell(\boldsymbol{\pi}) = N + N^\vee$.

Roots of $\boldsymbol{\pi}$ and Bessis theorem

Here we follow [**BrMi**] and [**Bes2**]. The results of this paragraph should be viewed as the "lifting" up to braid groups level of Springer theory of regular elements.

Let d be a positive integer, let $\zeta \in \boldsymbol{\mu}_d$ and let w a ζ-regular element of W. Recall (see chap. 1 above) that we denote by $W(w)$ the centralizer of w in W, we set $V(w) := \ker(w - \zeta \mathrm{Id})$, $\mathcal{M}(w) := \mathcal{P} \cap V(w)$. By 1.4 we know that $(V(w), W(w))$ is a complex reflection group with regular variety $\mathcal{M}(w)$.

Let us choose a base point x_0 in $\mathcal{M}(w)$, and let us denote by $P(w)$ and $B(w)$ respectively the corresponding pure braid group and braid group.

Let us denote by $\mathbf{w}$ the element of B defined by the path

$$\mathbf{w}(t) \colon t \mapsto x_0 \exp(2\pi i t/d) .$$

The following proposition tells that a regular element can be lifted up to a root of $\boldsymbol{\pi}$ in B.

2.23. PROPOSITION.
(1) *The image of* $\mathbf{w}$ *through the natural morphism* $B \twoheadrightarrow W$ *is* w.
(2) *We have* $\mathbf{w}^d = \boldsymbol{\pi}$.

Roots of $\boldsymbol{\pi}$ (at least, for the case where W is a Weyl group) seem to play a key role in representation theory of the corresponding finite reductive groups (see below chap. VI).

Let us notice the following consequence of 2.23 and 2.22.

2.24. COROLLARY. *Let d be a regular number for W. Whenever* $\mathcal{C} \in \mathcal{A}/W$, *then d divides* $N_{\mathcal{C}} e_{\mathcal{C}}$. *In particular, d divides* $N + N^\vee$.

[Indeed, we have $\ell_{\mathcal{C}}(\mathbf{w}) = N_{\mathcal{C}} e_{\mathcal{C}}/d$ and $\ell(\mathbf{w}) = (N + N^\vee)/d$.]

The natural maps

$$\mathcal{M}(w) \hookrightarrow \mathcal{M} \quad \text{and} \quad \mathcal{M}(w)/W(w) \hookrightarrow \mathcal{M}/W(w) \twoheadrightarrow \mathcal{M}/W$$

induce the following commutative diagram, where the lines are exact :

$$\begin{tikzcd}
1 \arrow[r] & P(w) \arrow[r] \arrow[d] & B(w) \arrow[r] \arrow[d] & W(w)^{\mathrm{op}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P \arrow[r] & B \arrow[r] & W^{\mathrm{op}} \arrow[r] & 1
\end{tikzcd}$$

Answering positively (for almost all cases of complex reflection groups) a question raised in [**BrMi2**] (3.5), Bessis [**Bes2**] has proven the following theorem for the case where W is any irreducible complex reflection group different from the exceptional groups G_n for $n \in \{28, 30, 31, 33, 35, 36, 37\}$. Since we conjecture that the results holds in all cases, it is again a "theorem–assumption".

2.25. THEOREM–ASSUMPTION. *The vertical arrows in the preceding diagram are all injective. In particular, the map $B(w) \longrightarrow B$ identifies $B(w)$ with a subgroup of the centralizer $C_B(\mathbf{w})$ of $\mathbf{w}$ in B.*

REMARK. In the course of his proof, Bessis notices that the image of $\mathcal{M}(w)$ in $\mathcal{M}/W$ through any path of the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}(w) \arrow[r, hook] \arrow[d] & \mathcal{M} \arrow[d] \\
\mathcal{M}(w)/W(w) \arrow[r, hook] & \mathcal{M}/W(w) \arrow[d] \\
 & \mathcal{M}/W
\end{tikzcd}$$

is actually contained in the variety $(\mathcal{M}/W)^{\boldsymbol{\mu}_d}$ of fixed points of the group $\boldsymbol{\mu}_d$ acting on $\mathcal{M}/W$, and that the natural map $\mathcal{M}(w)/W(w) \longrightarrow (\mathcal{M}/W)^{\boldsymbol{\mu}_d}$ is an homeomorphism. Then his result may be reformulated as follows : the image under π_1 of the natural injection $(\mathcal{M}/W)^{\boldsymbol{\mu}_d} \hookrightarrow \mathcal{M}/W$ is injective

A natural question (see [**BDM**], 0.1) consists then in asking whether $B(w)$ is isomorphic to $C_B(\mathbf{w})$. A partial positive answer to this question has been given in [**BDM**], 0.2.

2.26. THEOREM. *Let (V, W) be an irreducible complex reflection group of the following types :*

$$\mathfrak{S}_r\,,\ G(d,1,r)\ (d>1)\,,\ G_n \textit{ for } n \in \{4,5,8,10,16,18,25,26,32\}\,.$$

Then the natural injection $B(w) \hookrightarrow C_B(\mathbf{w})$ is an isomorphism

Notice that the braid diagrams (see below) of the groups listed in theorem 2.26 above are all braid diagrams of Coxeter groups.

Parabolic braid subgroups

Let X be an intersection of reflecting hyperplanes: $X = \bigcap_{\substack{H\in\mathcal{A}\\ X\subseteq H}} H$. We set

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid (X \subseteq H)\} \quad \text{and} \quad \mathcal{M}_X := V - \bigcup_{H\in\mathcal{A}_X} H\,.$$

We recall (see 1.2 above) that the parabolic subgroup W_X, the pointwise stabilizer of X, is generated by all the cyclic reflection groups W_H for $H \in \mathcal{A}_X$.

For $x_0 \in \mathcal{M}$, we set

$$P_X := \pi_1(\mathcal{M}_X, x_0) \quad \text{and} \quad B_X := \pi_1(\mathcal{M}_X/W_X, p(x_0))\,.$$

We recall (see [**BMR**], Proposition 2.29) that there is a morphism of short exact sequences (see 2.8 above)

$$\begin{tikzcd} 1 \arrow[r] & P_X \arrow[r] \arrow[d, hook] & B_X \arrow[r] \arrow[d, hook] & W_X^{\mathrm{op}} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & P \arrow[r] & B \arrow[r] & W^{\mathrm{op}} \arrow[r] & 1 \end{tikzcd}$$

where all vertical arrows are injective, and where the corresponding injection $B_X \hookrightarrow B$ is well–defined up to P–conjugation.

Any subgroup of B which is the image of B_X by one of the preceding injections is called a *parabolic braid subgroup of* B *associated with* X. For a given X, such parabolic braid subgroups constitute a conjugacy class of subgroups under P.

Artin–like presentations and the braid diagrams

General results.

We say that B has an *Artin–like* presentation (cf. [**Op2**], 5.2) if it has a presentation of the form

$$\langle \mathbf{s} \in \mathbf{S} \mid \{\mathbf{v}_i = \mathbf{w}_i\}_{i \in I} \rangle$$

where $\mathbf{S}$ is a finite set of distinguished braid reflections, and I is a finite set of relations which are multi–homogeneous, *i.e.*, such that (for each i) $\mathbf{v}_i$ and $\mathbf{w}_i$ are positive words in elements of $\mathbf{S}$ (and hence, for each $\mathcal{C} \in \mathcal{A}/W$, we have $\ell_{\mathcal{C}}(\mathbf{v}_i) = \ell_{\mathcal{C}}(\mathbf{w}_i)$).

The following result, which has recently been proved by Bessis ([**Bes3**], Thm. 0.1), shows in particular that any braid group has an Artin–like presentation.

2.27. Theorem. *Let W be a complex reflection group of rank r, with associated braid group B. Let d be one of the degrees of W. Assume that d is a regular number for W. Let $n := (N + N^\vee)/d$. Then there exists a subset $\mathbf{S} = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$ of B such that*

(1) *The elements $\mathbf{s}_1, \dots, \mathbf{s}_n$ are distinguished braid reflections, and therefore their images $s_1, \dots, s_n$ in W are distinguished reflections.*
(2) *The set $\mathbf{S}$ generates B, and therefore $S := \{s_1, \dots, s_n\}$ generates W.*
(3) *The product $(\mathbf{s}_1 \cdots \mathbf{s}_n)^d$ is central in B and belongs to the pure braid group P.*
(4) *The product $c := s_1, \dots, s_n$ is a ζ_d–regular element in W.*
(5) *There exists a set $\mathcal{R}$ of relations of the form $\mathbf{w}_1 = \mathbf{w}_2$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are positive words of equal length in the elements of $\mathbf{S}$, such that $\langle S \mid \mathcal{R} \rangle$ is a presentation of B.*
(6) *Viewing now $\mathcal{R}$ as a set of relations in S, the group W is presented by*

$$\langle S \mid \mathcal{R};\ (\forall s \in S)(s^{e_s} = 1) \rangle$$

where e_s denotes the order of s in W.

The following result is mainly due to Bessis, who has shown that theorem 1.26 has a generalization in terms of braid groups (cf. [**Bes3**], 4.2).

The first assertion is due to Bessis. The second assertion is partially a consequence of *loc.cit.* and are partially proved in [**BMR**] through a case–by–case analysis.

2.28. THEOREM. *Let (V, W) be a complex reflection group. We set $r := \dim(V)$. Let $(d_1, d_2, \ldots, d_r)$ be the family of its invariant degrees, ordered to that $d_1 \leq d_2 \leq \cdots \leq d_r$.*

(1) *The following integers are equal.*
  (a) *The minimal number of reflections needed to generate W.*
  (b) *The minimal number of distinguished braid reflections around an hyperplane needed to generate B.*
  (c) $\lceil (N + N^\vee)/d_r \rceil$.

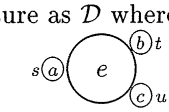
(2) *If $g(W)$ denotes the integer defined by properties (a) to (c) above, we have either $g_W = r$ or $g_W = r+1$, and the group B has an Artin–like presentation by g_W reflections.*

The braid diagrams.

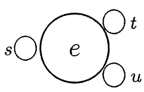
Let us first introduce some more notation.

Let (V, W) be a finite irreducible complex reflection group. As previously, we set $\mathcal{M} := V - \bigcup_{H \in \mathcal{A}} H$, $B := \pi_1(\mathcal{M}/W, x_0)$, and we denote by $\sigma \mapsto \overline{\sigma}$ the antimorphism $B \twoheadrightarrow W$ defined by the Galois covering $\mathcal{M} \twoheadrightarrow \mathcal{M}/W$.

Let $\mathcal{D}$ be one of the diagrams given in tables 1, 2, 3 (Appendix 1 below) symbolizing a set of relations as described in Appendix 1.

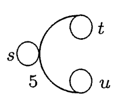
• We denote by $\mathcal{D}_{\mathrm{br}}$ and we call *braid diagram associated to $\mathcal{D}$* the set of nodes of $\mathcal{D}$ subject to all relations of $\mathcal{D}$ but the orders of the nodes, and we represent the braid diagram $\mathcal{D}_{\mathrm{br}}$ by the same picture as $\mathcal{D}$ where numbers insides the nodes are omitted. Thus, if $\mathcal{D}$ is the diagram 

, then $\mathcal{D}_{\mathrm{br}}$ is the diagram

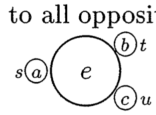


and represents the relations

$$\underbrace{stustu\cdots}_{e \text{ factors}} = \underbrace{tustus\cdots}_{e \text{ factors}} = \underbrace{ustust\cdots}_{e \text{ factors}}.$$

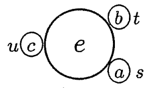
Note that this braid diagram for $e = 3$ is the braid diagram associated to $G(2d, 2, 2)$ $(d \geq 2)$, as well as G_7, G_{11}, G_{19}. Also, for $e = 4$, this is the braid diagram associated to G_{12} and for $e = 5$, the braid diagram associated to G_{22}. Similarly, the braid diagram 

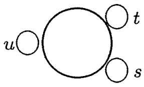
is associated to the diagrams of both G_{15} and $G(4d, 4, 2)$.

• We denote by $\mathcal{D}^{\mathrm{op}}$ and we call *opposite diagram associated to $\mathcal{D}$* the set of nodes of $\mathcal{D}$ subject to all opposite relations (words in reverse order) of $\mathcal{D}$. Thus, if $\mathcal{D}$ is the diagram 

, then $\mathcal{D}^{\mathrm{op}}$ represents the relations

$$s^a = t^b = u^c = 1 \text{ and } \underbrace{utsuts\cdots}_{e \text{ factors}} = \underbrace{sutsut\cdots}_{e \text{ factors}} = \underbrace{tsutsu\cdots}_{e \text{ factors}}.$$

Note that $\mathcal{D}^{\mathrm{op}}$ is the diagram 

. Finally, we denote by $\mathcal{D}_{\mathrm{br}}^{\mathrm{op}}$ the braid diagram associated with $\mathcal{D}^{\mathrm{op}}$. Thus, in the above case, $\mathcal{D}_{\mathrm{br}}^{\mathrm{op}}$ is the diagram 

. Note that if $\mathcal{D}_{\mathrm{br}}$ is a Coxeter type diagram, then it is equal to $\mathcal{D}_{\mathrm{br}}^{\mathrm{op}}$.

The following statement is well known for Coxeter groups (see for example [**Br1**] or [**De2**]). It has been noticed by Orlik and Solomon (see [**OrSo3**], 3.7) for the case of non real Shephard groups (*i.e.*, non real complex reflection groups whose braid diagram – see above – is a Coxeter diagram). It has been proved for all the infinite series, as well as checked case by case for all the exceptional groups but G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34} in [**BMM2**]. It is conjectured that it still holds for G_{31}.

2.29. THEOREM. *Let W be a finite irreducible complex reflection group, different from G_{24}, G_{27}, G_{29}, G_{33}, G_{34} – and also different from G_{31} for which the following assertions are still conjectural.*

Let $\mathcal{N}(\mathcal{D})$ be the set of nodes of the diagram $\mathcal{D}$ for W given in tables 1–3 below, identified with a set of pseudo-reflections in W. For each $s \in \mathcal{N}(\mathcal{D})$, there exists an s–distinguished braid reflection $\mathbf{s}$ in B such that the set $\{\mathbf{s}\}_{s\in\mathcal{N}(\mathcal{D})}$, together with the braid relations of $\mathcal{D}_{\mathrm{br}}^{\mathrm{op}}$, is a presentation of B.

Monoids.

Given a presentation of B as defined by a braid diagram $\mathcal{D}$, and since the relations symbolized by $\mathcal{D}$ only involve positive powers of the generators, one may consider the *monoid* $B_{\mathcal{D}}^{+}$ defined by "the same" presentation.

It is known (cf. for example [**De2**] or [**BrSa**]) that whenever B is the usual braid diagram associated with a Coxeter group, the follwoing two properties are satisfied.

1. The natural monoid morphism $B_{\mathcal{D}}^{+} \longrightarrow B$ is injective.
2. Identifying $B_{\mathcal{D}}^{+}$ with its image in B, we have $B = \{\boldsymbol{\pi}^n b \mid (n \in \mathbb{Z})(b \in B_{\mathcal{D}}^{+})\}$.

Ruth Corran ([**Cor**], chap. 8) has recently checked that

- the same properties hold for all the braid diagrams described in appendix 1 and associated with exceptional complex reflection groups but G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34},
- but the injectivity of the morphism $B_{\mathcal{D}}^{+} \longrightarrow B$ fails for all braid diagrams of the infinite series which are not Coxeter diagrams.

CHAPTER III

GENERIC HECKE ALGEBRAS

In this chapter we follow essentially [**BMR**] and [**BMM2**].

A monodromy representation of the braid group

We extend to the case of complex reflection groups the construction of generalized Knizhnik–Zamolodchikov connections for Weyl groups due to Cherednik ([**Ch1**], [**Ch2**], [**Ch3**]; see also the constructions of Dunkl [**Du**], Opdam [**Op**] and Kohno [**Ko1**]).[1] This allows us to construct explicit isomorphisms between the group algebra of a complex reflection group and its Hecke algebra.

Background from differential equations and monodromy.

What follows is well known, and is introduced here at an elementary level for the convenience of the unexperienced reader, since we only need this elementary approach. For a more general approach, see for example [**De1**].

We go back to the setting of chap. 1. Let A be a finite dimensional complex vector space. We denote by 1 a chosen non zero point of A — in the applications, A we will be an algebra. We set $E := \mathrm{End}(A)$. Let ω be a holomorphic differential form on $\mathcal{M}$ with values in E, *i.e.*, a holomorphic map $\mathcal{M} \to \mathrm{Hom}(V, E)$, where $\mathrm{Hom}(V, E)$ denotes the space of linear maps from V into E, such that (see 1.14 and 1.17, (1)) we have

$$\omega = \sum_{H \in \mathcal{A}} f_H \omega_H \,,$$

with $\omega_H = \dfrac{1}{2i\pi}\dfrac{d\alpha_H}{\alpha_H}$, and $f_H \in E$. For $x \in \mathcal{M}$ and $v \in V$, we have $\omega(x)(v) = \dfrac{1}{2i\pi}\sum_{H\in\mathcal{A}} \dfrac{\alpha_H(v)}{\alpha_H(x)} f_H$.

We consider the following linear differential equation

$$dF = \omega(F)\,, \qquad (\mathrm{Eq}(\omega))$$

where F is a holomorphic function defined on an open subset of $\mathcal{M}$ with values in A. In other words, for x in this open subset, we have $dF(x) \in \mathrm{Hom}(V, A)$, and we want F to satisfy, for all $v \in V$, $dF(x)(v) = \omega(x)(v)(F(x))$, or in other words $dF(x)(v) = \dfrac{1}{2i\pi}\sum_{H\in\mathcal{A}} \dfrac{\alpha_H(v)}{\alpha_H(x)} f_H(F(x))$.

For $y \in \mathcal{M}$, let us denote by $\mathcal{V}(y)$ the largest open ball with center y contained in $\mathcal{M}$. The existence and unicity theorem for linear differential equations shows that for each $y \in \mathcal{M}$, there exists a unique function

$$F_y \colon \mathcal{V}(y) \to A\,,\ x \mapsto F_y(x)\,,$$

[1] This construction has also been noticed independently by Opdam, who is able to deduce from it some important consequences concerning the "generalized fake degrees" of a complex reflection group.

solution of $\mathrm{Eq}(\omega)$ and such that $F_y(y) = 1$. From now on, we set

$$F(x,y) := F_y(x)\,.$$

Assume now that the finite group W acts linearly on A through a morphism $\varphi\colon W \to \mathrm{GL}(A)$. Then it induces an action of W on the space of differential forms on $\mathcal{M}$ with values in E, and an easy computation shows that ω is W-stable if and only if, for all $w \in W$,

$$\omega(w(x)) = \varphi(w)(\omega(x)\cdot w^{-1})\varphi(w^{-1})\,, \tag{3.1}$$

which can also be written, for all $x \in \mathcal{M}$ and $v \in V$:

$$\sum_{H\in\mathcal{A}} \omega_H(wx)(v) f_H = \sum_{H\in\mathcal{A}} \omega_H(x)(w^{-1}(v))\varphi(w) f_H \varphi(w^{-1})\,.$$

An easy computation shows that this is equivalent to

$$\sum_{H\in\mathcal{A}} f_{w(H)} \frac{d\alpha_{w(H)}}{\alpha_H} = \sum_{H\in\mathcal{A}} \varphi(w) f_H \varphi(w^{-1}) \frac{d\alpha_{w(H)}}{\alpha_H}\,. \tag{3.2}$$

In particular we see that

3.3. *If $f_{w(H)} = \varphi(w) f_H \varphi(w^{-1})$ for all $H \in \mathcal{A}$ and $w \in W$, then the form ω is W-stable.*

From (3.1) (and from the existence and unicity theorem), it follows that

3.4. *If ω is W-stable, then for all $y \in \mathcal{M}$, $x \in \mathcal{V}(y)$ and $w \in W$, the solution $x \mapsto F(x,y)$ satisfies*

$$\varphi(w)(F(x,y)) = F(w(x), w(y))\,.$$

The case of an interior W-algebra.

The following hypothesis and notation will be in force for the rest of this paragraph.

From now on, we assume that A is endowed with a structure of $\mathbb{C}$-algebra with unity, and that ω takes its values in the subalgebra of E consisting of the multiplications by the elements of A – which, by abuse of notation, we still denote by A. With this abuse of notation, we may assume that

$$\omega = \sum_{H\in\mathcal{A}} a_H \omega_H\,,$$

where $a_H \in A$, and the equation $\mathrm{Eq}(\omega)$ is written

$$dF = \omega F \;\text{ or }\; dF(x)(v) = \frac{1}{2i\pi} \sum_{H\in\mathcal{A}} \frac{\alpha_H(v)}{\alpha_H(x)} a_H F(x)\,.$$

Let γ be a path in $\mathcal{M}$. From the existence and unicity of local solutions of $\mathrm{Eq}(\omega)$, it results that the solution $x \mapsto F(x,\gamma(0))$ has an analytic continuation $t \mapsto (\gamma^*F)(t,\gamma(0))$ along γ, which satisfies the following properties.

Let us say that a sequence of real numbers $t_0 = 0 < t_1 < \ldots < t_{n-1} < t_n = 1$ is adapted to $(\gamma, \mathrm{Eq}(\omega))$ if for all $1 \le j \le n$, we have $\gamma([t_{j-1}, t_j]) \subset \mathcal{V}(\gamma(t_j))$.

Then :

(1) there exists $\varepsilon > 0$ such that $(\gamma^*F)(t,\gamma(0)) = F(\gamma(t),\gamma(0))$ for $0 \le t \le \varepsilon$,
(2) whenever $t_0 = 0 < t_1 < \ldots < t_{n-1} < t_n = 1$ is adapted to $(\gamma, \mathrm{Eq}(\omega))$, we have $(\gamma^*F)(t_j,\gamma(0)) = F(\gamma(t_j),\gamma(t_{j-1}))(\gamma^*F)(t_{j-1},\gamma(0))$ for all $j > 0$.

We see that

$$(\gamma^* F)(1,\gamma(0)) = \prod_{j=n}^{j=1} F(\gamma(t_j),\gamma(t_{j-1}))\,. \tag{3.5}$$

The case of an integrable form.

We recall that the form ω is said to be integrable if $d\omega + \omega\wedge\omega = 0$.

The following fact was noticed, for example, by Kohno (see [**Ko2**], 1.2). This is an immediate consequence of 1.17, (2).

3.6. LEMMA. *The form $\omega = \sum_{H\in\mathcal{A}} a_H\omega_H$ is integrable if and only if, for all subspaces X of V with codimension 2, and for all $H\in\mathcal{A}$ such that $X\subset H$, a_H commutes with $\sum_{\substack{(H'\in\mathcal{A})\\(H'\supset X)}} a_{H'}$.*

If ω is integrable, the value $(\gamma^*F)(1,\gamma(0))$ depends only on the homotopy class of γ. By (3.5), we see that we get a *covariant functor*

$$S\colon \begin{cases} \mathcal{P}(\mathcal{M}) \to A^\times \\ \gamma \mapsto (\gamma^*F)(1,\gamma(0)) \end{cases}$$

Action of W.

Assume now that A is an *interior* W-algebra, *i.e.*, that there is a group morphism $W \to A^\times$ (through which the image of $w\in W$ is still denoted by w), which defines a linear operation φ of W on A by composition with the injection $A^\times \hookrightarrow \mathrm{GL}(A)$. So, with our convention, for $w\in W$ and $a\in A$ we have $\varphi(w)(a) = wa$.

The form ω is then W-stable if and only if, for all $w\in W$ and $x\in\mathcal{M}$,

$$\omega(w(x)) = w(\omega(x)\cdot w^{-1})w^{-1}\,,$$

which can also be written, for all $x\in\mathcal{M}$ and $v\in V$:

$$\sum_{H\in\mathcal{A}} \omega_H(wx)(v)a_H = \sum_{H\in\mathcal{A}} \omega_H(x)(w^{-1}(v))wa_Hw^{-1}\,.$$

By 3.3, we have the following criterion.

3.7. *If, for all $H\in\mathcal{A}$ and $w\in W$, we have $a_{w(H)} = wa_Hw^{-1}$, then the form ω is W-stable.*

By 3.4, the solution F of $\mathrm{Eq}(\omega)$ then satisfies

$$wF(x,y)w^{-1} = F(w(x),w(y))\,.$$

3.8. DEFINITION–PROPOSITION. *Assuming that ω is W-stable, we define a group morphism*

$$T\colon \pi_1(\mathcal{M}/W,x_0) \to (A^\times)^{\mathrm{op}}$$

(or, in other words, a group anti-morphism $T\colon \pi_1(\mathcal{M}/W,x_0)\to A^\times$), called the monodromy morphism associated with ω, as follows.

For $\sigma\in B$, with image $\overline{\sigma}$ in W through the natural anti-morphism $B\to W$, we denote by $\tilde\sigma$ a path in $\mathcal{M}$ from x_0 to $\overline{\sigma}(x_0)$ which lifts σ. Then we set

$$T(\sigma) := S(\tilde\sigma^{-1})\overline{\sigma}\,.$$

Dependence of parameters.

Suppose the form ω depends holomorphically on m parameters $t_1, \dots, t_m$. Denoting by $\mathcal{O}$ the ring of holomorphic functions of the variables $t_1, \dots, t_m$, we have $\omega = \sum_{H\in\mathcal{A}} f_H \omega_H$ where $f_H \in \mathcal{O} \otimes_{\mathbb{C}} E$. Then, for $y \in \mathcal{M}$, the function F_y is a holomorphic function of $t_1, \dots, t_m$, *i.e.*, F_y has values in $\mathcal{O} \otimes_{\mathbb{C}} A$.

Then, given a path γ in $\mathcal{M}$, the analytic continuation $t \mapsto (\gamma^* F)(t, \gamma(0))$ depends holomorphically of $t_1, \dots, t_m$.

If ω is integrable and W-stable, then the monodromy morphism depends holomorphically on the parameters $t_1, \dots, t_m$. It follows that we have a monodromy morphism

$$T : \pi_1(\mathcal{M}/W, x_0)^{\mathrm{op}} \to (\mathcal{O} \otimes_{\mathbb{C}} A)^\times .$$

The main theorem.

From now on, we assume that $A = \mathbb{C}W$.

We denote by $\mathcal{O}$ the ring of holomorphic functions of a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$ variables

$$\mathbf{z} := (z_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)} \, .$$

For $H \in \mathcal{C}$, we set $z_{H,j} := z_{\mathcal{C},j}$.

We put

$$q_{\mathcal{C},j} = \exp(-2\pi i z_{\mathcal{C},j}/e_{\mathcal{C}}) \quad \text{for } \mathcal{C} \in \mathcal{A}/W \,,\, 0 \leq j \leq e_{\mathcal{C}} - 1 \, .$$

For $H \in \mathcal{C}$, we set $q_{H,j} := q_{\mathcal{C},j}$.

Let $\mathcal{C} \in \mathcal{A}/W$ and let $H \in \mathcal{C}$. For $0 \leq j \leq e_{\mathcal{C}} - 1$, we denote by $\varepsilon_j(H)$ the primitive idempotent of the group algebra $\mathbb{C}W_H$ associated with the character $\det_V^j$ of the group W_H. Thus we have

$$\varepsilon_j(H) = \frac{1}{e_{\mathcal{C}}} \sum_{k=0}^{k=e_{\mathcal{C}}-1} \zeta_{e_{\mathcal{C}}}^{jk} \, s_H^k \, .$$

We set

$$a_H := \sum_{j=0}^{j=e_H-1} z_{H,j} \varepsilon_j(H) \quad \text{and} \quad \omega := \sum_{H\in\mathcal{A}} a_H \frac{d\alpha_H}{\alpha_H} \, .$$

In other words, we have

$$\omega = \sum_{\mathcal{C}\in\mathcal{A}/W} \sum_{j=0}^{j=e_{\mathcal{C}}-1} \sum_{H\in\mathcal{C}} z_{\mathcal{C},j} \varepsilon_j(H) 2\pi i \omega_H \, .$$

The following lemma is clear.

3.9. Lemma. *The map $\mathcal{A} \to A$, $H \mapsto a_H$ has the following properties :*
(1) *it is W-stable, i.e., for all $w \in W$ and $H \in \mathcal{A}$, we have $a_{w(H)} = w a_H w^{-1}$,*
(2) *for all $H \in \mathcal{A}$, a_H belongs to the image of $\mathbb{C}W_H$ in A.*

The following property follows from 3.9.

3.10. Lemma. *The form ω is W-stable and integrable.*

The proof of the following result may be found in [**BMR**], §4.

3.11. THEOREM. *We denote by $T\colon B^{\mathrm{op}} \to (\mathbb{C}W)^\times$ the monodromy morphism associated with the differential form ω on $\mathcal{M}$. For all $H \in \mathcal{C}$, we have*

$$\prod_{j=0}^{j=e_{\mathcal{C}}-1} (T(\mathbf{s}_{H,\gamma}) - q_{H,j}\det_V(s_H)^j) = 0\,.$$

Furthermore, T depends holomorphically on the parameters $z_{\mathcal{C},j}$, i.e., arises by specialization from a morphism $\mathbf{T}\colon B^{\mathrm{op}} \to (\mathcal{O}W)^\times$.

Hecke algebras : first properties

We define a set

$$\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\le j\le e_{\mathcal{C}}-1)}$$

of $\sum_{\mathcal{C}\in\mathcal{A}/W}(e_{\mathcal{C}})$ indeterminates. We denote by $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ the ring of Laurent polynomials in the indeterminates $\mathbf{u}$.

Let $\mathfrak{I}$ be the ideal of the group algebra $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]B$ generated by the elements

$$(\mathbf{s}_{H,\gamma} - u_{\mathcal{C},0})(\mathbf{s}_{H,\gamma} - u_{\mathcal{C},1})\cdots(\mathbf{s}_{H,\gamma} - u_{\mathcal{C},e_{\mathcal{C}}-1})$$

where $\mathcal{C} \in \mathcal{A}/W$, $H \in \mathcal{C}$, $\mathbf{s}_{H,\gamma}$ is a distinguished braid reflection around H in B and s is the image of $\mathbf{s}_{H,\gamma}$ in W.

3.12. DEFINITION. *The Hecke algebra, denoted by $\mathcal{H}_{\mathbf{u}}(W)$, is the $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-algebra $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]B/\mathfrak{I}$.*

Notice that, since all the distinguished braid reflections around any hyperplane $H \in \mathcal{C}$ are conjugate under B, it suffices to take one of the $\mathbf{s}_H$ in the above relations to generate the ideal $\mathfrak{I}$.

From now on, whenever R is a ring endowed with a ring morphism $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}] \longrightarrow R$, we will write $R\mathcal{H}(W,\mathbf{u})$ for the implied tensor product $R\otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]}\mathcal{H}(W,\mathbf{u})$.

Now assume that W is a finite irreducible complex reflection group. Let $\mathcal{D}$ be the diagram of W, and let $\mathrm{s} \in \mathcal{N}(\mathcal{D})$ be a node of $\mathcal{D}$. We set $u_{\mathrm{s},j} := u_{\mathcal{C},j}$ for $j = 0,1,\dots,e_{\mathcal{C}}-1$, where $\mathcal{C}$ denotes the orbit under W of the reflecting hyperplane of s.

The following proposition is an immediate consequence of Theorem 2.29 :

3.13. PROPOSITION. *Assume W is different from G_{24}, G_{27}, G_{29}, G_{33}, G_{34} – and also different from G_{31} for which the following assertion is still conjectural. The Hecke algebra $\mathcal{H}_{\mathbf{u}}(W)$ is isomorphic to the $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-algebra generated by elements $(\mathrm{s})_{\mathrm{s}\in\mathcal{N}(\mathcal{D})}$ such that*

- *the elements* s *satisfy the braid relations defined by $\mathcal{D}_{\mathrm{br}}^{\mathrm{op}}$,*
- *we have $(\mathrm{s}-u_{\mathrm{s},0})(\mathrm{s}-u_{\mathrm{s},1})\cdots(\mathrm{s}-u_{\mathrm{s},e_{\mathrm{s}}-1}) = 0$.*

Note that the specialization $u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j$ takes the generic Hecke algebra $\mathcal{H}(W,\mathbf{u})$ to the group algebra of W^{op} over a cyclotomic extension of $\mathbb{Z}$. Any specialization of the generic Hecke algebra $\mathcal{H}(W,\mathbf{u})$ through which the previous one factorizes is called *admissible*.

We shall denote by $\mathbf{h}\colon \mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]B \to \mathcal{H}(W,\mathbf{u})$ the natural epimorphism. For a $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-linear map t defined on $\mathcal{H}(W,\mathbf{u})$, we shall often omit the letter $\mathbf{h}$ by writing $t(b)$ instead of $t(\mathbf{h}(b))$.

Hecke algebras and monodromy representations.

By Theorem 3.11, we see that the monodromy representation $\mathbf{T}$ factors through $\mathcal{H}_{\mathbf{u}}(W)$. Indeed, let us set

$$\begin{cases} u_{\mathcal{C},j} := \exp\left(\dfrac{2\pi i(j - z_{\mathcal{C},j})}{e_{\mathcal{C}}}\right) & \text{for all } (\mathcal{C},j)\,, \\ \mathbf{z} := (z_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)}\,, \end{cases}$$

and let us denote by $\mathcal{O}$ the ring of holomorphic functions of the set of variables $\mathbf{z}$. Then we have the following commutative diagram :

$$\begin{tikzcd} \mathcal{O}B \arrow[rr, "\mathbf{T}"] \arrow[dr] & & \mathcal{O}W^{\mathrm{op}} \\ & \mathcal{O}\otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]}\mathcal{H}_{\mathbf{u}}(W) \arrow[ur] & \end{tikzcd}$$

Let $\mathcal{K}$ be the field of fractions of $\mathcal{O}$.

The following lemma is a key point to understand the structure of $\mathcal{H}_{\mathbf{u}}(W)$. It is well-known to hold for Coxeter groups. For the infinite series of complex reflection groups, see [**ArKo**] for $G(d,1,r)$, [**BrMa**], (4.12) for $G(2d,2,r)$ and [**Ari**], Proposition 1.4 for the general case (it has been also checked for many of the remaining groups of small rank — see for example [**BrMa**], Satz 4.7). We conjecture it is true for all complex reflection groups.

3.14. Lemma–Assumption. *The $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-module $\mathcal{H}_{\mathbf{u}}(W)$ can be generated by $|W|$ elements.*

From this lemma, we can now deduce the following

3.15. Theorem–Assumption. *The monodromy representation $\mathbf{T}$ induces an isomorphism of $\mathcal{K}$-algebras*

$$\mathcal{K}\otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]}\mathcal{H}_{\mathbf{u}}(W) \xrightarrow{\sim} \mathcal{K}W^{\mathrm{op}}\,.$$

Furthermore, $\mathcal{H}_{\mathbf{u}}(W)$ is a free $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-module of rank $|W|$.

Indeed, by Lemma 3.14, there is a surjective morphism of $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-modules

$$\phi\colon \mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]^{|W|} \to \mathcal{H}_{\mathbf{u}}(W)\,.$$

Let $\mathfrak{m}$ be the ideal of $\mathcal{O}$ of the functions vanishing at the point $(t_{\mathcal{C},j}=1)$. The morphism $\mathcal{O}_{\mathfrak{m}}\otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]}\mathcal{H}_{\mathbf{u}} \to \mathcal{O}_{\mathfrak{m}}W$ induced by the monodromy is surjective by Nakayama's lemma, since it becomes an isomorphism after tensoring by $(\mathcal{O}_{\mathfrak{m}})/\mathfrak{m}$. Composing with $1_{\mathcal{O}_{\mathfrak{m}}}\otimes\phi$, we obtain an epimorphism $\mathcal{O}_{\mathfrak{m}}^{|W|}\to\mathcal{O}_{\mathfrak{m}}W$: this must be an isomorphism. Hence, $\ker\phi=0$, *i.e.*, ϕ is an isomorphism and $\mathcal{H}_{\mathbf{u}}$ is free of rank $|W|$ over $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$.

Since the morphism $K\otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]}\mathcal{H}_{\mathbf{u}}\to KW$ is a surjective morphism between two K-modules with same dimensions, it is an isomorphism and Theorem 3.15 follows. □

Grading the Hecke algebra.

The algebra $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ is multi-graded over $\mathbb{Z}$: for each $\mathcal{C}\in\mathcal{A}/W$, we define the grading

$$\deg_{\mathcal{C}}\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]\to\mathbb{Z}$$

by setting

$$\deg_{\mathcal{C}}(u_{\mathcal{C}',j}) := \begin{cases} 1 & \text{if } \mathcal{C}' = \mathcal{C}\,, \\ 0 & \text{if } \mathcal{C}' \neq \mathcal{C}\,. \end{cases}$$

If we set $\deg_{\mathcal{C}}(b) := \ell_{\mathcal{C}}(b)$ for $b \in B$ and $\mathcal{C} \in \mathcal{A}/W$, we see then that the group algebra $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B$ is endowed with a grading on $\mathbb{Z}^{\mathcal{A}/W}$. Since the ideal $\mathfrak{J}$ (see above 3.12) is homogeneous for this grading, it follows that the maps $\deg_{\mathcal{C}}$ endow the generic Hecke algebra $\mathcal{H}(W, \mathbf{u})$ with a structure of $\mathbb{Z}^{\mathcal{A}/W}$-graded algebra.

Parabolic Hecke sub–algebras.

Let X be an intersection of reflecting hyperplanes, and let W_X be the corresponding parabolic subgroup of W (see chap. 1 above). We recall that we denote by $\mathcal{A}_X$ the set of reflecting hyperplanes of W_X. Let $a : \mathcal{A}_X/W_X \to \mathcal{A}/W$ be the map sending a W_X-orbit of hyperplanes to the corresponding W-orbit.

Let $\mathbf{u}' = (u'_{\mathcal{C},j})_{(\mathcal{C} \in \mathcal{A}_X/W_X)(0 \le j \le e_{\mathcal{C}}-1)}$ be a set of $\sum_{\mathcal{C} \in \mathcal{A}_X/W_X} e_{\mathcal{C}}$ indeterminates. We have a morphism

$$\mathbb{Z}[\mathbf{u}', \mathbf{u}'^{-1}] \to \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]\,,\ u'_{\mathcal{C},j} \mapsto u_{a(\mathcal{C}),j}\,.$$

The injection $B_{W_X} \hookrightarrow B_W$, defined up to P–conjugation (see above), induces an inclusion, defined up to P–conjugation (see [**BMR**], §4)

$$\mathcal{H}(W_X, \mathbf{u}') \otimes_{\mathbb{Z}[\mathbf{u}', \mathbf{u}'^{-1}]} \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}] \to \mathcal{H}(W, \mathbf{u})\,,$$

whose image is called the generic *parabolic Hecke sub–algebra of* $\mathcal{H}(W, \mathbf{u})$ *associated with* X, and is denoted by $\mathcal{H}(W_X, W, \mathbf{u})$ (note that only the P–conjugacy class of this subalgebra is well defined).

Semi–simple Hecke algebras and absolutely irreducible characters.

The following assertion is known to hold

- for all infinite families of finite irreducible reflection groups by [**ArKo**], [**BrMa2**], [**Ar**],
- for all Coxeter groups (see for example [**Bou**], chap. IV, §2, ex. 23),
- for the groups G_4, G_5, G_8 and G_{25} which occur in [**BrMa2**].

We conjecture it to be true in all cases (see [**BMR**], §4).

It shows in particular that the algebra $\mathcal{H}(W, \mathbf{u})$ is a free graded $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$–module of rank $|W|$, since it has a basis over $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ consisting of homogeneous elements.

3.16. Theorem–Assumption. *There exists a family* $(b_w)_{w \in W}$ *of* $|W|$ *elements of the braid group* B *with*

- b_w *maps to* w *through the natural morphism* $B \twoheadrightarrow W^{\mathrm{op}}$,
- $b_1 = 1$,

such that the family $(\mathbf{h}(b_w))_{w \in W}$ *is a basis of* $\mathcal{H}(W, \mathbf{u})$ *over* $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$.

By Appendix 2, 8.4, (1) below, it follows that

3.17. Proposition. *For* W *satisfying Theorem-Assumption above, the* $\mathbb{Q}(\mathbf{u})$*–algebra* $\mathbb{Q}(\mathbf{u})\mathcal{H}(W, \mathbf{u})$ *is trace symmetric, hence separable and in particular semi–simple.*

Let $\mathbb{Q}(\boldsymbol{\mu}_\infty)$ be the subfield of $\mathbb{C}$ generated by all roots of unity. Assume that (V, W) is an irreducible K–reflection group, where K is a subfield of $\mathbb{Q}(\boldsymbol{\mu}_\infty)$ of finite degree over $\mathbb{Q}$. We denote by $K(\mathbf{u})$ the field of fractions of the Laurent polynomial ring $K[\mathbf{u},\mathbf{u}^{-1}]$, where as above $\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)}$.

The following theorem is proved for *any* reflection group in [**Ma4**] (see theorem 5.2. in *loc.cit.*), provided that the corresponding Hecke algebra is defined by generators and relations symbolized by the diagrams of Appendix 1 below. Since this last fact is unknown for the six exceptional groups G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}, we must call it a "theorem–assumption": it is proved for all irreducible groups but the six previous ones.

3.18. Theorem–Assumption. *Let W be a complex reflection group satisfying Theorem-Assumption 3.16. Let $\mathbf{t} = (t_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)}$ be a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$ indeterminates such that, for all $\mathcal{C}, j$, we have $t_{\mathcal{C},j}^{|\mu(K)|} = \zeta_{e_{\mathcal{C}}}^{-j} u_{\mathcal{C},j}$. The field $K(\mathbf{t})$ is a splitting field for the Hecke algebra $\mathbb{Q}(\mathbf{u})\mathcal{H}(W,\mathbf{u})$.*

In what follows the field $K(\mathbf{t})$ above is called a *good splitting field*.

Let us denote by $\mathbb{Z}_K$ the ring of integers of K. By Appendix 2, 8.2 below, the specialization

$$t_{\mathcal{C},j} \mapsto 1$$

induces a bijection $\chi \mapsto \chi_{\mathbf{t}}$ from the set $\mathrm{Irr}(W^{\mathrm{op}})$ of absolutely irreducible characters of W^{op} onto the set $\mathrm{Irr}(\mathcal{H}(W,\mathbf{u}))$ of absolutely irreducible characters of the Hecke algebra $\mathcal{H}(W,\mathbf{u})$, such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{H}(W,\mathbf{u}) & \xrightarrow{\chi_{\mathbf{t}}} & \mathbb{Z}_K[\mathbf{t},\mathbf{t}^{-1}] \\ \downarrow & & \downarrow \\ \mathbb{Z}_K W^{\mathrm{op}} & \xrightarrow{\chi} & \mathbb{Z}_K\,. \end{array}$$

3.19. Convention. As functions on the underlying set of W, the characters of W and of W^{op} coincide. We identify $\mathrm{Irr}(W)$ and $\mathrm{Irr}(W^{\mathrm{op}})$. Thus we get a bijection

$$\mathrm{Irr}(W) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}(W,\mathbf{u}))\ ,\ \chi \mapsto \chi_{\mathbf{t}}$$

such that, for $b \in B$ with image $\bar{b} \in W$, we have

$$(\chi_{\mathbf{t}}(b))_{|\mathbf{t}=1} = \chi(\bar{b})\,.$$

Linear characters.

Let $\mathrm{Hom}(B,\mathbb{Q}(\mathbf{u})^\times)$ denote the group of linear characters of B with values in $\mathbb{Q}(\mathbf{u})^\times$. We have an isomorphism (see for example [**BMR**], prop. 2.16)

$$\mathrm{Hom}(B,\mathbb{Q}(\mathbf{u})^\times) \simeq \underbrace{\mathbb{Q}(\mathbf{u})^\times \times \mathbb{Q}(\mathbf{u})^\times \times \cdots \times \mathbb{Q}(\mathbf{u})^\times}_{|\mathcal{A}/W| \text{ times}}\,,$$

given by the map

$$\theta \mapsto (\theta(\mathbf{s}_{\mathcal{C}}))_{\mathcal{C}\in\mathcal{A}/W}\,,$$

where $\mathbf{s}_{\mathcal{C}}$ denotes a distinguished braid reflection around a hyperplane belonging to $\mathcal{C}$.

It follows from what precedes that the linear characters of $\mathcal{H}(W,\mathbf{u})$ are induced by those linear characters of B described as follows: there exists a family of integers $\mathbf{j} := (j_{\mathcal{C}})_{\mathcal{C}\in\mathcal{A}/W}$ where $j_{\mathcal{C}} \in \{0,1,\ldots,e_{\mathcal{C}}-1\}$, such that $\mathbf{s} \mapsto u_{\mathcal{C},j_{\mathcal{C}}}$ if $\mathbf{s}$ is a distinguished braid reflection around an element of $\mathcal{C}$. We denote by $\theta_{\mathbf{j}}$ this linear character of $\mathcal{H}(W, \mathbf{u})$.

We denote by $\det_{\mathcal{C}}$ the linear character of W such that

$$\det_{\mathcal{C}}(s_H) = \begin{cases} \det_V(s_H) & \text{if } H \in \mathcal{C}\,, \\ 1 & \text{if } H \notin \mathcal{C}\,, \end{cases}$$

where s_H is any reflection with reflecting hyperplane H.

Then the character $(\det_{\mathcal{C},j})_{\mathbf{t}}$ of the Hecke algebra corresponding to $\det_{\mathcal{C}}^j$ is defined by

$$(\det_{\mathcal{C},j})_{\mathbf{t}}(\mathbf{s}_H) = \begin{cases} u_{\mathcal{C},j} & \text{if } H \in \mathcal{C}\,, \\ u_{\mathcal{C}',0} & \text{if } H \in \mathcal{C}'\,,\, \mathcal{C}' \neq \mathcal{C}\,, \end{cases}$$

where $\mathbf{s}_H$ is a distinguished braid reflection around H.

We see in particular that $(\det_{\mathcal{C},j})_{\mathbf{t}}$ is rational over $\mathbb{Q}(\mathbf{u})$: we have

$$(\det_{\mathcal{C},j})_{\mathbf{t}} \colon \mathcal{H}(W, \mathbf{u})^\times \to \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]^\times\,. \tag{3.20}$$

Notice that (with notation introduced above), for $\mathbf{j} = (j_{\mathcal{C}})_{\mathcal{C} \in \mathcal{A}/W}$ we have

$$\theta_{\mathbf{j}} = \prod_{\mathcal{C} \in \mathcal{A}/W} (\det_{\mathcal{C}, j_{\mathcal{C}}})_{\mathbf{t}}\,.$$

Central morphisms associated with irreducible characters.

More generally, let $\chi \in \mathrm{Irr}(W)$. Recall that, for $\mathcal{C} \in \mathcal{A}/W$, we denote by $m_{\mathcal{C},j}^{\chi}$ the multiplicity of $\det_V^j$ in the restriction of χ to the cyclic group W_H. It results from [**BrMi2**], 4.17, that

$$\frac{m_{\mathcal{C},j}^{\chi} N_{\mathcal{C}} e_{\mathcal{C}}}{\chi(1)} \in \mathbb{N}.$$

Since $K(\mathbf{t})$ is a splitting field for the algebra $K(\mathbf{t})\mathcal{H}(W, \mathbf{u})$, the irreducible character $\chi_{\mathbf{t}}$ defines an algebra morphism from the center of $K(\mathbf{t})\mathcal{H}(W, \mathbf{u})$ onto $K(\mathbf{t})$ which we denote by

$$\omega_{\chi_{\mathbf{t}}} \colon Z(K(\mathbf{t})\mathcal{H}(W, \mathbf{u})) \to K(\mathbf{t})\,.$$

By [**BrMi2**], prop. 4.16, its value on the image of the central element $\boldsymbol{\pi}$ equals

$$\omega_{\chi_{\mathbf{t}}}(\boldsymbol{\pi}) = \zeta_{\chi(1)}^{N(\chi)} \prod_{\mathcal{A}/W} \prod_{j=0}^{j=e_{\mathcal{C}}-1} u_{\mathcal{C},j}^{\frac{m_{\mathcal{C},j}^{\chi} N_{\mathcal{C}} e_{\mathcal{C}}}{\chi(1)}}\,.$$

Now using the equalities $t_{\mathcal{C},j}^{|\mu(K)|} = \zeta_{e_{\mathcal{C}}}^j u_{\mathcal{C},j}$ and 1.25 above, we get the equivalent formula

$$\omega_{\chi_{\mathbf{t}}}(\boldsymbol{\pi}) = \prod_{\mathcal{C} \in \mathcal{A}/W} \prod_{j=0}^{j=e_{\mathcal{C}}-1} t_{\mathcal{C},j}^{|\mu(K)| \frac{m_{\mathcal{C},j}^{\chi} N_{\mathcal{C}} e_{\mathcal{C}}}{\chi(1)}}\,. \tag{3.21}$$

A proof like in [**BrMi2**], 4.16, provides the following generalization of (3.21).

3.22. PROPOSITION. *Assume that* $\mathbf{w} \in B$ *is such that* $\mathbf{w}^d = \boldsymbol{\pi}$ *(for some positive integer d). Let* w *denote its image in* W. *Then we have*

$$\chi_{\mathbf{t}}(\mathbf{w}) = \chi(w) \prod_{C \in \mathcal{A}/W} \prod_{j=0}^{j=e_C-1} t_{C,j}^{\frac{|\mu(K)|}{d} \frac{m_{C,j}^{\chi} N_C e_C}{\chi(1)}} .$$

Characters as homogeneous functions.

The following proposition results from Appendix 2 below, 8.1, (2).

3.23. PROPOSITION. *Assume* W *satisfies Theorem-Assumption 3.16. Then for all* $\chi \in \mathrm{Irr}(W)$ *and* $b \in B$, *the value* $\chi_{\mathbf{t}}(b)$ *(as an element of* $\mathbb{Z}_K[\mathbf{t}, \mathbf{t}^{-1}]$*) is multi-homogeneous of degree* $|\boldsymbol{\mu}(K)|\ell_C(b)$ *in the indeterminates* $(t_{C,j})_{0 \leq j < e_C}$ *(for all* $C \in \mathcal{A}/W$*).*

Galois operations.

Both extensions $K(\mathbf{t})/K(\mathbf{u})$ and $K(\mathbf{t})/\mathbb{Q}(\mathbf{t})$ are Galois (although in general the extension $K(\mathbf{t})/\mathbb{Q}(\mathbf{u})$ is not Galois). Since $\mathcal{H}(W, \mathbf{u})$ is defined over $\mathbb{Q}(\mathbf{u})$, both groups $\mathrm{Gal}(K(\mathbf{t})/K(\mathbf{u}))$ and $\mathrm{Gal}(K(\mathbf{t})/\mathbb{Q}(\mathbf{t}))$ act on the set of irreducible characters of the algebra $K(\mathbf{t})\mathcal{H}(W, \mathbf{u})$, hence, through the preceding bijection, both groups act on $\mathrm{Irr}(W)$.

- For $g \in \mathrm{Gal}(K(\mathbf{t})/K(\mathbf{u}))$ and $\chi \in \mathrm{Irr}(W)$, we denote by $g(\chi)$ the irreducible character of W defined by the condition

$$g(\chi_{\mathbf{t}}) = (g(\chi))_{\mathbf{t}} .$$

- The group $\mathrm{Gal}(K(\mathbf{t})/\mathbb{Q}(\mathbf{t}))$ is isomorphic (through the restriction map from $K(\mathbf{t})$ to K) to the Galois group $\mathrm{Gal}(K/\mathbb{Q})$. Through this isomorphism we get an action of $\mathrm{Gal}(K/\mathbb{Q})$ on $\mathrm{Irr}(W)$. It is easy to see that this action coincides with the usual action of $\mathrm{Gal}(K/\mathbb{Q})$ on $\mathrm{Irr}(W)$. For $g \in \mathrm{Gal}(K/\mathbb{Q})$ and $\chi \in \mathrm{Irr}(W)$, we still denote by $g(\chi)$ the image of χ under g.

The following proposition is 4.1 and 4.2 in [**BrMi2**].

3.24. PROPOSITION.
(1) *The linear characters of* W *are fixed under* $\mathrm{Gal}(K(\mathbf{t})/K(\mathbf{u}))$.
(2) *Whenever* $\chi \in \mathrm{Irr}(W)$ *and* $g \in \mathrm{Gal}(K(\mathbf{t})/K(\mathbf{u}))$, *we have*
   (a) $\chi(1) = g(\chi)(1)$,
   (b) $\chi(s) = g(\chi)(s)$ *whenever* s *is a reflection in* W,
   (c) $N(\chi) = N(g(\chi))$.

Some anti-automorphisms of the generic Hecke algebra.

We denote by $\alpha \mapsto \alpha^{\vee}$ the automorphism of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ consisting of the simultaneous inversion of the indeterminates.

The following proposition is an immediate consequence of the definition of the Hecke algebra.

3.25. PROPOSITION. *There is a unique anti-automorphism of* $\mathcal{H}(W, \mathbf{u})$, *denoted by* $\mathbf{a}_1$, *such that:*
(a1) *For* $b \in B$, *we have*

$$\mathbf{a}_1(\mathbf{h}(b)) = \mathbf{h}(b^{-1}) .$$

(a2) *For* $\alpha \in \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$, *we have*

$$\mathbf{a}_1(\alpha) = \alpha^\vee .$$

The group

$$\mathfrak{S}_W := \prod_{\mathcal{C} \in \mathcal{A}/W} \mathfrak{S}_{e_\mathcal{C}}$$

permutes the set of indeterminates $\mathbf{u}$ in an obvious way, thus acts on the $\mathbb{Z}$–algebra $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$. For $\sigma \in \mathfrak{S}_W$, we still denote by σ the corresponding automorphism of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$, as well as the $\mathbb{Z}$–algebra automorphism of $\mathcal{H}(W, \mathbf{u})$ which extends the automorphism σ of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ by the condition $\sigma(\mathbf{h}(b)) = \mathbf{h}(b)$ for all $b \in B$. We define

$$\mathbf{a}_\sigma := \sigma\, \mathbf{a}_1 ; \tag{3.26}$$

then $\mathfrak{S}_W$ acts regularly on the set of all the anti-automorphisms $\mathbf{a}_\sigma$.

Anti–automorphisms and characters of the generic Hecke algebra.

A linear character θ is transformed into another linear character by an anti-automorphism $\mathbf{a}$ through the formula

$$^{\mathbf{a}}\theta := \mathbf{a}\theta\mathbf{a}^{-1} .$$

It is immediate to check that, for $\sigma \in \mathfrak{S}_W$, we have

$$^{\mathbf{a}_\sigma}\theta_{\mathbf{j}} = \theta_{\sigma(\mathbf{j})} .$$

Notation.

• We extend the automorphism $\alpha \mapsto \alpha^\vee$ of the ring $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ to the automorphism (still denoted by $\alpha \mapsto \alpha^\vee$) of the ring $\mathbb{Z}_K[\mathbf{t}, \mathbf{t}^{-1}]$ such that

$$t_{\mathcal{C},j}^\vee := t_{\mathcal{C},j}^{-1} \text{ (for all } \mathcal{C} \text{ and } j\text{), and } \lambda^\vee := \lambda^* \text{ (for all } \lambda \in \mathbb{Z}_K\text{)} .$$

• We extend the anti–automorphism $\mathbf{a}_1$ to the algebra $\mathbb{Z}_K[\mathbf{t}, \mathbf{t}^{-1}]\mathcal{H}(W, \mathbf{u})$ and to the algebra $K(\mathbf{t})\mathcal{H}(W, \mathbf{u})$ by stipulating that $\mathbf{a}_1$ induces the automorphism $\alpha \mapsto \alpha^\vee$ on the ring $\mathbb{Z}_K[\mathbf{t}, \mathbf{t}^{-1}]$. Thus, for $\lambda \in K$, $b \in B$, $\mathcal{C} \in \mathcal{A}/W$ and $0 \leq j < e_\mathcal{C}$, we have

$$\mathbf{a}_1(\lambda t_{\mathcal{C},j}\mathbf{h}(b)) = \lambda^* t_{\mathcal{C},j}^{-1}\mathbf{h}(b^{-1}) .$$

• For any function $f \colon \mathcal{H}(W, \mathbf{u}) \to \mathbb{Z}_K[\mathbf{t}, \mathbf{t}^{-1}]$ we set (for $h \in \mathcal{H}$)

$$f^\vee(h) := f(\mathbf{a}_1(h))^\vee . \tag{3.27}$$

Thus, in particular, for $b \in B$ we have

$$f^\vee(\mathbf{h}(b)) := f(\mathbf{h}(b^{-1}))^\vee .$$

3.28. LEMMA. *For all* $\chi \in \operatorname{Irr}(W)$, *we have* $\chi_\mathbf{t}^\vee = \chi_\mathbf{t}$.

[It suffices to check that $\chi_\mathbf{t}^\vee$ specializes to χ for the specialization $t_{\mathcal{C},j} \mapsto 1$, which is clear.]

The opposite algebra and the Lusztig involution.

The opposite algebra $\mathcal{H}(W,\mathbf{u})^{\mathrm{op}}$ is defined as usual:

- as a $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$–module, we have $\mathcal{H}(W,\mathbf{u})^{\mathrm{op}} = \mathcal{H}(W,\mathbf{u})$;
- the multiplication in $\mathcal{H}(W,\mathbf{u})^{\mathrm{op}}$ is defined by $(h,h') \mapsto h'h$.

It is easy to check the following two properties.

3.29. *The opposite algebra $\mathcal{H}(W,\mathbf{u})^{\mathrm{op}}$ is*

(a) *isomorphic to $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]B^{\mathrm{op}}/\mathfrak{I}^{\mathrm{op}}$, where $\mathfrak{I}^{\mathrm{op}}$ denotes the ideal of the algebra $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]B^{\mathrm{op}}$ generated by all the elements*

$$(\mathbf{s}_H - u_{\mathcal{C},0})(\mathbf{s}_H - u_{\mathcal{C},1})\cdots(\mathbf{s}_H - u_{\mathcal{C},e_{\mathcal{C}}-1})$$

where $\mathcal{C} \in \mathcal{A}/W$, $H \in \mathcal{C}$, $\mathbf{s}_H$ is a distinguished braid reflection around H in B,

(b) *semi–linearly isomorphic to the algebra $\mathcal{H}(W,\mathbf{u})$ through the map $h \mapsto h^\vee$.*

We denote by

$$h \mapsto {}^r h\ ,\ \mathcal{H}(W,\mathbf{u}) \longrightarrow \mathcal{H}(W,\mathbf{u})^{\mathrm{op}}$$

the (linear) isomorphism such that ${}^r\mathbf{s} = \mathbf{s}$ whenever $\mathbf{s}$ is the image in $\mathcal{H}(W,\mathbf{u})$ of a distinguished braid reflection around an hyperplane.

DEFINITION. We call *Lusztig involution* of $\mathcal{H}(W,\mathbf{u})$ the involutive semi-linear automorphism ι of $\mathcal{H}(W,\mathbf{u})$ defined by $\iota(h) := {}^r h^\vee$.

Let $\mathbf{s}_1^{n_1}\mathbf{s}_2^{n_2}\cdots\mathbf{s}_l^{n_l} \in B$, where (for $1 \le j \le l$) $\mathbf{s}_j$ is a distinguished braid reflection and $n_j \in \mathbb{Z}$. Then for all $\lambda \in \mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$, we have

$$\iota(\lambda\mathbf{s}_1^{n_1}\mathbf{s}_2^{n_2}\cdots\mathbf{s}_l^{n_l}) = \lambda^\vee\mathbf{s}_1^{-n_1}\mathbf{s}_2^{-n_2}\cdots\mathbf{s}_l^{-n_l}\,.$$

Generic Hecke algebras as symmetric algebras

3A. The canonical trace form.

An element $P(\mathbf{u}) \in \mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ is called "multi–homogeneous" if, for each $\mathcal{C} \in \mathcal{A}/W$, it is homogeneous as a Laurent polynomial in the indeterminates $u_{\mathcal{C},j}$ for $j = 0,\ldots,e_{\mathcal{C}}-1$.

The following assertion is conjectured to be true for all reflection groups.

- It is proved in general ([**BMM2**], §2A) that if there exists a form satisfying all conditions (1),(a), (b), (c) of 3.30, then it is unique.
- It is proved in general for W a Coxeter group (*loc.cit.*, §2A).
- It is proved for all infinite families of non Coxeter complex reflection groups

  [Indeed, it is proved in *loc.cit.*, §4, under Malle's conjecture [**Mal**] about the Schur elements, which is now proved in [**GIM**]].

- Finally, Theorem–Assumption 2 is only partly checked for the non Coxeter exceptional groups. A "good" candidate for $t_{\mathbf{u}}$ is known for all groups: it satisfies all properties below but property (1)(a). Property (1)(a) is probably easily checkable on computer for some of the small groups, but it is still open for large exceptional non Coxeter reflection groups.

3.30. THEOREM–ASSUMPTION.

(0) *W satisfies Theorem-Assumption 3.16.*

(1) *There exists a unique linear form*

$$t_{\mathbf{u}} \colon \mathcal{H}(W, \mathbf{u}) \to \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$$

with the following properties.

(a) *$t_{\mathbf{u}}$ is a central form on the algebra $\mathcal{H}(W, \mathbf{u})$, i.e., for all h, $h' \in \mathcal{H}(W, \mathbf{u})$, we have $t_{\mathbf{u}}(hh') = t_{\mathbf{u}}(h'h)$, and it endows $\mathcal{H}(W, \mathbf{u})$ with a structure of symmetric algebra over the ring $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$, i.e., the map*

$$\widehat{t}_{\mathbf{u}} \colon \mathcal{H}(W, \mathbf{u}) \to \mathrm{Hom}(\mathcal{H}(W, \mathbf{u}), \mathbb{Z}[\mathbf{t}, \mathbf{t}^{-1}]) \ , \ h \mapsto (h' \mapsto t_{\mathbf{u}}(hh'))$$

is an isomorphism of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-modules between $\mathcal{H}(W, \mathbf{u})$ and its dual as a $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-module.

(b) *Through the specialization $u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j$, the form $t_{\mathbf{u}}$ becomes the canonical linear form on the group algebra.*

(c) *For all $b \in B$, we have*

$$t_{\mathbf{u}}(b^{-1})^{\vee} = \frac{t_{\mathbf{u}}(b\boldsymbol{\pi})}{t_{\mathbf{u}}(\boldsymbol{\pi})} \, .$$

(2) *The form $t_{\mathbf{u}}$ satisfies the following conditions.*

(a) *For $b \in B$, $t_{\mathbf{u}}(b)$ is invariant under the action of $\mathfrak{S}_W$.*

(b) *As an element of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$, $t_{\mathbf{u}}(b)$ is multi-homogeneous with degree $\ell_{\mathcal{C}}(b)$ in the indeterminates $u_{\mathcal{C},j}$ for $j = 0, 1, \ldots, e_{\mathcal{C}} - 1$. In particular, we have*

$$t_{\mathbf{u}}(1) = 1 \quad \textit{and} \quad t_{\mathbf{u}}(\boldsymbol{\pi}) = (-1)^{N^{\vee}} \prod_{\substack{H \in \mathcal{A} \\ 0 \le j \le e_H - 1}} u_{H,j} \, .$$

(c) *If W' is a parabolic subgroup of W, the restriction of $t_{\mathbf{u}}$ to a parabolic sub-algebra $\mathcal{H}(W', W, \mathbf{u})$ is the corresponding specialization of $t_{\mathbf{u}'}(W')$ (we recall that $\mathcal{H}(W', W, \mathbf{u})$ is a specialization of $\mathcal{H}(W', \mathbf{u}')$).*

EXAMPLE. Let us examine the value of $t_{\mathbf{u}}$ in the case where W is cyclic (see [**BrMa2**]). We denote by e the order of W, we set $\mathbf{u} = \{u_0, u_1, \ldots, u_{e-1}\}$, and we denote by $\mathbf{s}$ the "positive" generator of B.

- If $j > 0$, $t_{\mathbf{u}}(\mathbf{s}^j)$ is $(-1)^{e-1}$ times the sum of all monomials in $\mathbf{u}$ of degree j, where each indeterminate occurs with a strictly positive exponent.
- If $j \le 0$, $t_{\mathbf{u}}(\mathbf{s}^j)$ is the sum of all monomials in $\mathbf{u}$ of degree j (where each indeterminate occurs with a non-positive exponent).

Thus we have in particular

$$\begin{aligned} t_{\mathbf{u}}(\mathbf{s}^j) &= 0 \quad \text{for } 1 \le j < e \, , \\ t_{\mathbf{u}}(\mathbf{s}^e) &= (-1)^{e-1} u_0 u_1 \cdots u_{e-1} \, , \\ t_{\mathbf{u}}(\mathbf{s}^{-1}) &= \sum_{j=0}^{j=e-1} u_j^{-1} \, . \end{aligned}$$

Notice that, by (2)(c) in Theorem–Assumption above, it results from what precedes that, for W any complex reflection group, we have

$$t_{\mathbf{u}}(\mathbf{s}) = 0 \quad \text{whenever } \mathbf{s} \text{ is a distinguished braid reflection in } B \, .$$

Schur elements.

Since the ring $\mathbb{Z}_K[\mathbf{t},\mathbf{t}^{-1}]$ is integrally closed (see [**Bou2**], §3, Corollaire 3), the first assertion of the following proposition follows from Appendix 2, 8.14 below. Assertion (2) follows from the fact that the functions $\chi_{\mathbf{t}}$ (for $\chi \in \mathrm{Irr}(W)$) are linearly independent.

3.31. PROPOSITION. *Assume that $t_{\mathbf{u}}$ is a linear form on $\mathcal{H}(W,\mathbf{u})$ such that assertion (1)(a) of 3.30 holds, i.e., the map*

$$\widehat{t}_{\mathbf{u}}\colon \mathcal{H}(W,\mathbf{u}) \to \mathrm{Hom}(\mathcal{H}(W,\mathbf{u}),\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}])$$

is an isomorphism.

(1) *There exist elements $S_\chi(\mathbf{t}) \in \mathbb{Z}_K[\mathbf{t},\mathbf{t}^{-1}]$ for $\chi \in \mathrm{Irr}(W)$, such that*

$$t_{\mathbf{u}} = \sum_{\chi\in\mathrm{Irr}(W)} \frac{1}{S_\chi(\mathbf{t})}\,\chi_{\mathbf{t}}\,.$$

(2) *For all $g \in \mathrm{Gal}(K(\mathbf{t})/K(\mathbf{u}))$ we have $S_{g(\chi)}(\mathbf{t}) = g(S_\chi(\mathbf{t}))$.*

The element $S_\chi(\mathbf{t})$ is called *the Schur element* associated with χ.

It results from (3.20) that

$$S_{\det_{C,j}}(\mathbf{t}) \in \mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]\,. \tag{3.32}$$

The following palindromicity property of the Schur elements is the translation of the invariance property of $t_{\mathbf{u}}$ under the involution $x \mapsto x^\vee$.

3.33. PROPOSITION. *Assume W satisfies Theorem-Assumption 3.30. Whenever $\chi \in \mathrm{Irr}(W)$, its Schur element $S_\chi(\mathbf{t})$ satisfies the following property:*

$$S_\chi(\mathbf{t}^{-1})^* = \frac{t_{\mathbf{u}}(\boldsymbol{\pi})}{\omega_{\chi_{\mathbf{t}}}(\boldsymbol{\pi})} S_\chi(\mathbf{t})\,.$$

The following property of Schur elements, which in a sense "explains" their palindromicity, results from a case by case study of the groups W satisfying 3.16 (cf. [**BrMa**], [**GIM**], [**Ma2**], [**Ma3**], [**Ma5**]).

3.34. PROPOSITION. *Let $\chi \in \mathrm{Irr}(W)$. The Schur element $s_\chi(\mathbf{v})$ associated with the characters $\chi_{\mathbf{v}}$ of $K(\mathbf{v})\mathcal{H}$ is such that*

$$s_\chi(\mathbf{v}) = \xi_\chi \prod_{\Phi\in C_\chi} \Phi(\mathbf{v})^{n_\chi}\,,$$

where

- $\xi_\chi \in \mathbb{Z}_K$,
- *C_χ is a set of degree zero homogeneous elements of $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$, dividing (in $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$) a Laurent polynomial of the shape $M_1(\mathbf{v}) - M_2(\mathbf{v})$, where $M_1(\mathbf{v})$ and $M_2(\mathbf{v})$ are degree zero unitairy (Laurent) monomials,*
- $n_\chi \in \mathbb{N}$.

EXAMPLE. Consider the case where W is the Weyl group of type G_2. The generic Hecke algebra (an algebra over the ring $\mathbb{Z}[u_0, u_1, v_0, v_1, u_0^{-1}, u_1^{-1}, v_0^{-1}, v_1^{-1}]$) is generated by two elements $\mathbf{s}$ et $\mathbf{t}$ satisfying the relations

$$\mathbf{ststst} = \mathbf{tststs} \quad \text{and} \quad (\mathbf{s} - u_0)(\mathbf{s} - u_1) = (\mathbf{t} - v_0)(\mathbf{t} - v_1) = 0\,.$$

Let x_0, x_1, y_0, y_1 be indeterminates such that

$$x_0^2 = u_0\,,\ x_1^2 = -u_1\,,\ y_0^2 = v_0\,,\ y_1^2 = -v_1\,.$$

We set $\mathbf{x} := (x_0, x_1)\,, \mathbf{y} := (y_0, y_1)$, and

$$\begin{cases} S(\mathbf{x},\mathbf{y}) := (x_1y_1)^{-6}(x_0^2 + x_1^2)(y_0^2 + y_1^2)(x_0^2y_0^2 + x_0x_1y_0y_1 + x_1^2y_1^2)(x_0^2y_0^2 - x_0x_1y_0y_1 + x_1^2y_1^2) \\ T(\mathbf{x},\mathbf{y}) := 2(x_0x_1y_0y_1)^{-2}(x_0^2y_0^2 - x_0x_1y_0y_1 + x_1^2y_1^2)(x_0^2y_1^2 + x_0x_1y_0y_1 + x_1^2y_0^2) \end{cases}$$

Then the Schur elements of the algebra $\mathcal{H}(W)$ are

$$S(x_0, x_1, y_0, y_1)\,, S(x_0, x_1, y_1, y_0)\,,\ S(x_1, x_0, y_0, y_1)\,,\ S(x_1, x_0, y_1, y_0)\,,$$
$$T(x_0, x_1, y_0, y_1)\,,\ T(x_1, x_0, y_0, y_1)\,.$$

Spetsial and cyclotomic specializations of Hecke algebras

Cyclotomic Hecke algebras.

Let us start with some definitions and notation which will be justified later on.

Let W be a complex reflection group, with field of definition K. Let d be an integer, a divisor of the order $|ZW|$ of the center ZW of W. Let ζ be a root of the unity in K, of order d. Let y be an indeterminate. We set

$$x := \zeta y^{|\boldsymbol{\mu}(K)|}\,.$$

3.35. DEFINITION. A *ζ–cyclotomic specialization* of the set of generic indeterminates $\mathbf{t}$ is a morphism of $\mathbb{Z}_K$–algebras

$$\phi\colon \mathbb{Z}_K[\mathbf{t}, \mathbf{t}^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$$

with the following properties:

(a) $\phi\colon t_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$, with $n_{\mathcal{C},j} \in \mathbb{Z}$ for all $\mathcal{C}$ and all j.

(b) For all $\mathcal{C} \in \mathcal{A}/W$, the polynomial in t

$$\prod_{j=0}^{j=e_{\mathcal{C}}-1} \left(t - \zeta_{e_{\mathcal{C}}}^j y^{n_{\mathcal{C},j}}\right)$$

is invariant under the action of the Galois group $\mathrm{Gal}(K(y)/K(x))$, *i.e.*, belongs to $\mathbb{Z}_K[x, x^{-1}][t]$.

We shall write $m_{\mathcal{C},j} := n_{\mathcal{C},j}/|\boldsymbol{\mu}(K)|$ and then set $\phi\colon u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j(\zeta^{-1}x)^{m_{\mathcal{C},j}}$.

From now on, we assume that Theorem-Assumption 3.30 holds for W.

The corresponding *cyclotomic Hecke algebra* is the $\mathbb{Z}_K[x, x^{-1}]$–algebra $\mathcal{H}_\phi(W)$ obtained as the specialization of the algebra $\mathcal{H}(W, \mathbf{u})$ through the specialization ϕ, endowed with the symmetrizing form t_ϕ specialized from the form $t_{\mathbf{u}}$.

The algebra $\mathcal{H}_\phi(W)$ is an image of the group algebra $\mathbb{Z}_K[x, x^{-1}]B$ under a morphism which we still denote by $\phi\colon \mathbb{Z}_K[x, x^{-1}]B \to \mathcal{H}_\phi(W)$, and for $x = \zeta$ it specializes onto the group algebra $\mathbb{Z}_K W^{\mathrm{op}}$ (while its chosen symmetrizing form specializes onto the canonical symmetric form on $\mathbb{Z}_K W^{\mathrm{op}}$).

Schur elements.

Let y be an indeterminate. We call *K–cyclotomic polynomials* the monic minimal polynomials (over $K[y]$) of roots of unity. The K–cyclotomic polynomials are irreducible elements of $\mathbb{Z}_K[y]$. We denote by $\mathrm{Cycl}(K)$ the set of all K–cyclotomic polynomials.

The following property is an immediate consequence of 3.34 above.

3.36. Proposition. *Let $\mathcal{H}_\phi(W)$ be a ζ–cyclotomic Hecke algebra, defined by $\phi\colon t_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$. The Schur element of any irreducible character $\chi_\phi \in K(y)\mathcal{H}_\phi(W)$ has the following shape :*

$$s_{\chi_\phi}(y) = \psi_\chi\, y^{a_\chi} \prod_{\Phi(y)\in\mathrm{Cycl}(K)_\chi} \Phi(y)^{m_\chi}$$

where

- $\psi_\chi \in \mathbb{Z}_K$,
- $a_\chi \in \mathbb{Z}$,
- $\mathrm{Cycl}(K)_\chi$ *is a set of K–cyclotomic polynomials, and* $m_\chi \in \mathbb{N}$.

Linear characters of the cyclotomic Hecke algebra.

Let $\mathcal{H}_\phi(W)$ be a ζ–cyclotomic Hecke algebra. Let

$$\theta\colon \mathcal{H}_\phi(W) \longrightarrow \mathbb{Z}_K[y,y^{-1}]$$

be a linear character of $\mathcal{H}_\phi(W)$ (recall that $y^{|\boldsymbol{\mu}(K)|} = \zeta^{-1}x$). We denote by $\mathbf{j}_\theta := (j_\mathcal{C})_{\mathcal{C}\in\mathcal{A}/W}$ the family of integers ($0 \le j_\mathcal{C} < e_\mathcal{C}$) such that $\theta(\mathbf{s}_\mathcal{C}) = \zeta_{e_\mathcal{C}}^{j_\mathcal{C}}(\zeta^{-1}x)^{m_{\mathcal{C},j_\mathcal{C}}}$.

We set

$$D_\theta := \sum_{\mathcal{C}\in\mathcal{A}/W} m_{\mathcal{C},j_\mathcal{C}} N_\mathcal{C} e_\mathcal{C}\,.$$

Let us denote by $\overline{\theta}$ the linear character of W obtained through the specialization $x \mapsto \zeta$. Thus we have

$$\overline{\theta}(s_\mathcal{C}) = \zeta_{e_\mathcal{C}}^{j_\mathcal{C}}\,.$$

3.37. Lemma. *Let $\mathbf{w}$ be an element of B such that $\mathbf{w}^d = \boldsymbol{\pi}$, and let w be its image in W. We have*

$$\theta(\mathbf{w}) = \overline{\theta}(w)(\zeta^{-1}x)^{D_\theta/d}.$$

[This is a particular case of 3.22.] ☐

Principal cyclotomic Hecke algebras.

3.38. Definition. A ζ–cyclotomic algebra $\mathcal{H}_\phi(W)$ is said to be *principal* if there exists a linear character

$$\theta_0\colon \mathcal{H}_\phi(W) \longrightarrow \mathbb{Z}_K[x,x^{-1}] \quad , \quad \mathbf{s}_\mathcal{C} \mapsto \zeta_{e_\mathcal{C}}^{j_\mathcal{C}}(\zeta^{-1}x)^{m_\mathcal{C}}$$

such that

(p1) we have $m_\mathcal{C} \ge m_{\mathcal{C},j}$ for all j ($0 \le j < e_\mathcal{C}$),

(p2) whenever $\mathbf{w}$ is an element of B such that $\mathbf{w}^d = \boldsymbol{\pi}$, we have $\theta_0(\mathbf{w}) = x^{D_{\theta_0}/d}$.

The character θ_0 is then called a *principal character of $\mathcal{H}_\phi(W)$*.

EXAMPLES.

(1) The *spetsial Hecke algebra* $\mathcal{H}(W)$ is the principal cyclotomic Hecke algebra associated with the 1–cyclotomic specialization

$$\begin{cases} u_{\mathcal{C},0} \mapsto x\,, \\ u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^{j} \quad \text{for } j > 0\,. \end{cases} \tag{3.39}$$

In other words, the images of the elements $\mathbf{s}_H$ in the algebra $\mathcal{H}_x(W)$ satisfy the equations

$$(\mathbf{s}_H - x)(1 + \mathbf{s}_H + \mathbf{s}_H^2 + \cdots + \mathbf{s}_H^{e_H - 1}) = 0\,.$$

In particular, one sees that if W is a Coxeter group, the spetsial Hecke algebra $\mathcal{H}(W)$ is the ordinary generic Hecke algebra over $\mathbb{Z}_K[x, x^{-1}]$.

(2) Let $W = G(4,2,2)$ and let $\zeta = i$. Recall (see for example [**BMR**]) that the corresponding braid group B is generated by three distinguished braid reflections $\mathbf{s}$, $\mathbf{s}'$, $\mathbf{s}''$, such that $\mathbf{s}\mathbf{s}'\mathbf{s}'' = \mathbf{s}'\mathbf{s}''\mathbf{s} = \mathbf{s}''\mathbf{s}\mathbf{s}'$, corresponding to reflections s, s', s'' of order 2 in W which satisfy the relations $s''s's = s'ss'' = ss''s'$.

The relations

$$\begin{cases} \mathbf{s}\mathbf{s}'\mathbf{s}'' = \mathbf{s}'\mathbf{s}''\mathbf{s} = \mathbf{s}''\mathbf{s}\mathbf{s}' \\ (\mathbf{s}-1)(\mathbf{s}-x^2) = (\mathbf{s}'-1)(\mathbf{s}'-x^2) = (\mathbf{s}''-1)(\mathbf{s}''-x^2) = 0 \end{cases}$$

define a principal cyclotomic Hecke algebra associated with W.

Some definitions and notation about palindromicity.

Let $P(x) \in \mathbb{C}(x)$ (or more generally suppose that P is a meromorphic function on $\mathbb{C}$). We say that $P(x)$ is *semi-palindromic* if there exist an integer $m \in \mathbb{Z}$ and a norm one complex number ξ such that

$$P(1/x)^* = \xi x^{-m} P(x)\,.$$

The following statements are obvious to check:

1. A constant is semi-palindromic.
2. If $P(x)$ and $Q(x)$ are semi-palindromic, so is $P(x)Q(x)$.
3. If ζ is a norm one complex number and if $P(x)$ is semi-palindromic, then the element $P^{\zeta}(x) := P(\zeta^{-1}x)$ is semi-palindromic.

For $P(x) \neq 0$, define the *valuation* $\mathrm{val}_x(P)$ (also denoted by a) as the order of 0 as a zero of $P(x)$ ($a < 0$ if 0 is a pole of $P(x)$). Let c_a be the nonzero complex number defined by

$$c_a := \left(x^{-a}P(x)\right)_{|x=0}\,.$$

Define the *degree* $\deg_x(P)$ (also denoted by A) as the valuation of $P(1/x)$. Let c_A be the nonzero complex number defined by

$$c_A := \left(x^{A}P(1/x)\right)_{|x=0}\,.$$

Thus if $P(x) \in \mathbb{C}[x, x^{-1}]$ is a Laurent polynomial, we have

$$P(x) = c_a x^a + c_{a+1}x^{a+1} + \cdots + c_{A-1}c^{A-1} + c_A x^A$$

for some coefficients c_j. The following lemma is immediate.

3.40. LEMMA. *Let $P(x)$ such that $P(1/x)^* = \xi x^{-m} P(x)$.*

(1) *We have $m = a + A$, and $c_A^* = \xi c_a$. Moreover,*

$$mP(x) = xP'(x) + \xi^* x^{m-1} P'(1/x)^* .$$

(2) *Let ζ be a norm one complex number such that $P(\zeta) \neq 0$.*

(a) *We have*

$$\xi = \zeta^m \frac{P(\zeta)^*}{P(\zeta)} \quad \text{and} \quad m = \zeta \frac{P'(\zeta)}{P(\zeta)} + \zeta^* \frac{P'(\zeta)^*}{P(\zeta)^*} = 2\Re\left(\zeta \frac{P'(\zeta)}{P(\zeta)}\right) .$$

(b) *If moreover $P(\zeta) \in \mathbb{R}$, we have $\xi = \zeta^m$. In particular if $\zeta = 1$, we have then*

$$\xi = 1 \quad \text{and} \quad m = \frac{P'(1) + P'(1)^*}{P(1)} .$$

More generally, assume that $P(y) \in \mathbb{C}(y)$, and set $x := y^n$ for some natural integer n. Then by abuse of notation, we define

$$\deg_x(P) := \frac{\deg_y(P)}{n} \quad \text{and} \quad \operatorname{val}_x(P) := \frac{\operatorname{val}_y(P)}{n} .$$

On spetsial cyclotomic Hecke algebras.

The results in this paragraph are taken from [**BMM2**], §6C. They are designed to be applied to characters of finite reductive groups (see below chap. 5 and 6).

We recall that we are now assuming Theorem-Assumption 3.30 holds.

3.41. PROPOSITION. *Let us denote by $\mathcal{H}_x(W)$ the specialization of the generic Hecke algebra under (3.39) above. Then,*

(1) *the polynomial $P_W(\mathbf{u})$ specializes to a polynomial $P_W(x) \in \mathbb{Z}_K[x, x^{-1}]$ which is palindromic and satisfies the relation*

$$P_W(x) = x^{N^\vee} P_W(x^{-1}) ,$$

(2) *the absolute generic degree $\mathrm{Deg}_\chi(\mathbf{t})$ specializes to an element $\mathrm{Deg}_\chi(x) \in K(x^{1/|\mu(K)|})$ which satisfies the relation*

$$\mathrm{Deg}_\chi(x) = x^{\frac{N(\chi)+N(\chi^*)}{\chi(1)}} \mathrm{Deg}_\chi(x^{-1})^* ,$$

(3) *the character $\chi_\mathbf{t}$ specializes to a character χ_x which satisfies the relation*

$$\omega_{\chi_x}(\boldsymbol{\pi}) = x^{N + N^\vee - \frac{N(\chi)+N(\chi^*)}{\chi(1)}} .$$

3.42. COROLLARY. *The sum of the valuation a_χ and of the degree A_χ of the generic degree $\mathrm{Deg}_\chi(x)$ of χ is computable from its fake degree $\mathrm{Feg}_\chi(x)$: we have*

$$a_\chi + A_\chi = \frac{N(\chi) + N(\chi^*)}{\chi(1)} .$$

Indeed, this results from 3.41, (2) and 3.40, (1).

Palindromicity, fake degrees, generic degrees.

By [**Ma3**], 4.8, we know that the field $K(x^{1/|\mu(K)|})$ contains the values of the absolutely irreducible characters of the spetsial cyclotomic Hecke algebra $\mathcal{H}_x(W)$. Following *loc.cit.*, we denote by δ the generator of $\mathrm{Gal}(K(x^{1/|\mu(K)|})/K(x))$ defined by

$$\delta(x^{1/|\mu(K)|}) = \exp\left(2\pi i/|\mu(K)|\right) x^{1/|\mu(K)|} .$$

We denote by ι the automorphism of order 2 of $K(x^{1/|\mu(K)|})$ defined as the composition of the complex conjugation by the automorphism δ, *i.e.*,

$$\iota(\lambda) := \delta(\lambda^*) \quad \text{for } \lambda \in K(x^{1/|\mu(K)|}) .$$

We have an operation of $\mathrm{Gal}(K(x^{1/|\mu(K)|})/K(x))$ on the set $\mathrm{Irr}(W)$ of irreducible characters of W (see also above §1) defined as follows:

$$g(\chi)_x = g(\chi_x)$$

(for $g \in \mathrm{Gal}(K(x^{1/|\mu(K)|})/K(x))$ and $\chi \in \mathrm{Irr}(W)$).

The following property is proved in *loc.cit.*, (6.5), through a case by case analysis. Note that Opdam [**Op1**] has given a general proof for the case where W is a Coxeter group, and that his proof can be generalized to any complex reflection group W (see [**Op2**]) provided one assumes that presentations like in Appendix 1 hold for B (see above, comments before 3.18).

3.43. PROPOSITION. *Let W satisfy Theorem-Assumption 3.16. Then, for any $\chi \in \mathrm{Irr}(W)$, there exists an integer m such that*

$$\mathrm{Feg}_\chi(x) = x^m \mathrm{Feg}_{\iota(\chi)}(1/x) .$$

As in [**BMM2**], §6.D, one deduces easily from the preceding proposition a formula which was only known case by case for W a Weyl group.

Let us introduce the valuation b_χ and the degree B_χ of $\mathrm{Feg}_\chi(x)$.

3.44. COROLLARY. *We have*

$$\frac{(b_\chi + B_\chi) + (b_{\iota(\chi)} + B_{\iota(\chi)})}{2} = a_\chi + A_\chi .$$

Indeed, proposition 3.43 implies that the polynomial $P_\chi(x) := \mathrm{Feg}_\chi(x)\mathrm{Feg}_{\iota(\chi)}(x)$ satisfies the relation

$$P_\chi(x) = x^{2m} P_\chi(1/x) ,$$

hence is palindromic. Then lemma 3.40, (2)(b) above shows that

$$2m = 2\frac{P'_\chi(1)}{P_\chi(1)} ,$$

hence

$$m = \frac{N(\chi) + N(\iota(\chi))}{\chi(1)} .$$

By 3.24 above, we know that $N(\iota(\chi)) = N(\chi^*)$, which implies

$$m = \frac{N(\chi) + N(\chi^*)}{\chi(1)} .$$

Since the valuation and the degree of the polynomial $\mathrm{Feg}_\chi(x)\mathrm{Feg}_{\iota(\chi)}(x)$ are respectively $b_\chi + b_{\iota(\chi)}$ and $B_\chi + B_{\iota(\chi)}$, we have

$$\frac{(b_\chi + B_\chi) + (b_{\iota(\chi)} + B_{\iota(\chi)})}{2} = N^\vee - \sum_{\rho \in \mathrm{Ref}(W)} \frac{\chi(\rho)}{\chi(1)} = \frac{N(\chi) + N(\chi^*)}{\chi(1)} .$$

It results from 3.41 and 3.42 above that $\mathrm{Deg}_\chi(x)$ is semi-palindromic, and that, if a_χ and A_χ are respectively its valuation and degree in x, we have

$$a_\chi + A_\chi = \frac{N(\chi) + N(\chi^*)}{\chi(1)} .$$

Corollary 3.44 is now immediate.

On generic degrees of cyclotomic Hecke algebras.

Let ζ be a root of unity of order d, and let as above $\mathcal{H}_\phi(W)$ be a ζ-cyclotomic Hecke algebra defined by a cyclotomic specialization $\phi\colon t_{C,j} \mapsto y^{n_{C,j}}$ where $y^{|\mu(K)|} = \zeta^{-1}x$.

Let $\mathrm{Irr}(\mathcal{H}_\phi(W))$ denote the set of (absolutely) irreducible characters of the split semi-simple algebra $K(y)\mathcal{H}_\phi(W)$.

Let $P(y) \in K(y)$ be semi-palindromic.

Let $\psi \in \mathrm{Irr}(\mathcal{H}_\phi(W))$.

- We denote by $\overline{\psi}$ the character of W defined by ψ through the specialization $y \mapsto 1$. Thus in particular we have $\overline{\psi}(s_C) = (\psi(\mathbf{s}_C))_{y=1}$.
- The *P-generic degree* of ψ is the element $\mathrm{Deg}_\psi^{(P)}(y) \in K(y)$ defined as follows:

$$\mathrm{Deg}_\psi^{(P)}(y) := \frac{P(y)}{S_\psi(y)} ,$$

where we denote by $S_\psi(y)$ the Schur element of ψ relative to the form t_ϕ.

COMMENT. In the applications to finite reductive groups, we take $P(y) = \mathrm{Deg}(R_{\mathbf{L}}^{\mathbf{G}}(1))(x)\mathrm{Deg}(\lambda)(x)$.

- We denote by a_ψ and A_ψ respectively the (generalized) valuation and degree of $\mathrm{Deg}_\psi^{(P)}(y)$ in x.

Note that $S_\psi(y)$ is semi-palindromic, since by 3.33 it satisfies the equality

$$S_\chi(y^{-1})^* = \frac{t_\phi(\boldsymbol{\pi})}{\omega_\psi(\boldsymbol{\pi})} S_\chi(y) . \tag{3.45}$$

It follows that $\mathrm{Deg}_\psi^{(P)}(y)$ is semi-palindromic, so satisfies the identity

$$\mathrm{Deg}_\psi^{(P)}(y^{-1})^* = \xi_\psi x^{-(a_\psi + A_\psi)} \mathrm{Deg}_\psi^{(P)}(y)$$

for some root of the unity ξ_ψ.

In what follows, we choose a linear character θ_0 of $\mathcal{H}_\phi(W)$, such that $\theta_0(\mathbf{s}_C) = \zeta_{e_C}^{j_C}(\zeta^{-1}x)^{m_C}$. We set

$$\begin{cases} D_0 := D_{\theta_0} = \displaystyle\sum_{C \in \mathcal{A}/W} m_C N_C e_C , \\ a_0 := a_{\theta_0} \quad \text{and} \quad A_0 := A_{\theta_0} . \end{cases}$$

The following results are Propositions 6.15 and 6.16 in [**BMM2**].

3.46. PROPOSITION. *Let* $\mathbf{w}$ *be an element of* B *such that* $\mathbf{w}^d = \boldsymbol{\pi}$.

(1) *We have*

$$\omega_\psi(\mathbf{w}) = \omega_{\overline{\psi}}(w)(\zeta^{-1}x)^{\frac{D_0-((a_\psi+A_\psi)-(a_0+A_0))}{d}}\,.$$

(2) *Assume that* $\mathcal{H}_\phi(W)$ *is principal, and that* θ_0 *is its principal character. Assume moreover that* $\mathrm{Deg}^{(P)}_{\theta_0}(y) = 1$. *We have*

$$\omega_\psi(\mathbf{w}) = \omega_{\overline{\psi}}(w)(\zeta^{-1}x)^{\frac{D_0-(a_\psi+A_\psi)}{d}}\,.$$

For $\mathcal{C} \in \mathcal{A}/W$ and $0 \leq j < e_\mathcal{C}$, we denote by $\theta_{\mathcal{C},j}$ the linear character of $\mathcal{H}_\phi(W)$ defined by

$$\theta_{\mathcal{C},j}(\mathbf{s}_{\mathcal{C}'}) = \begin{cases} \zeta_{e_\mathcal{C}}^j(\zeta^{-1}x)^{m_{\mathcal{C},j}} & \text{if } \mathcal{C}' = \mathcal{C}\,, \\ \theta_0(\mathbf{s}_{\mathcal{C}'}) & \text{if } \mathcal{C}' \neq \mathcal{C}\,. \end{cases}$$

We set

$$\begin{cases} \mathrm{Deg}^{(P)}_{\mathcal{C},j}(y) := \mathrm{Deg}^{(P)}_{\theta_{\mathcal{C},j}}(y)\,, \\ \omega_{\mathcal{C},j}(y) := \omega_{\theta_{\mathcal{C},j}}(y)\,, \\ a_{\mathcal{C},j} := a_{\theta_{\mathcal{C},j}} \quad \text{and} \quad A_{\mathcal{C},j} := A_{\theta_{\mathcal{C},j}}\,. \end{cases}$$

3.47. PROPOSITION.

(1)

$$(m_\mathcal{C} - m_{\mathcal{C},j})N_\mathcal{C} e_\mathcal{C} = (a_{\mathcal{C},j} + A_{\mathcal{C},j}) - (a_0 + A_0)\,.$$

(2) *In particular, if* $\mathcal{H}_\phi(W)$ *is principal, if* θ_0 *is the principal character and if* $\mathrm{Deg}^{(P)}_{\theta_0} = 1$, *we have*

$$(m_\mathcal{C} - m_{\mathcal{C},j})N_\mathcal{C} e_\mathcal{C} = (a_{\mathcal{C},j} + A_{\mathcal{C},j})\,.$$

CHAPTER IV

REFLECTION DATA

Definitions

From now on we assume K *is a subfield of the field* $\mathbb{Q}(\mu_\infty)$ generated by all roots of unity. The complex conjugation $z \mapsto z^*$ induces an automorphism of K. We denote by $\mathbb{Z}_K$ the ring of integers of K.

4.1. DEFINITIONS.

1. A *reflection datum* $\mathbb{G}$ on K is a pair (V, Wf), where
- V is a finite dimensional vector space over K,
- W is a finite reflection group on V,
- f is an element of finite order of $\mathrm{GL}(V)$ which normalizes W.

Note that only the coset Wf (and not the automorphism f) is defined by $\mathbb{G}$.

The vector space V is called the space of the reflection datum and its dimension r is called *the rank of the reflection datum*. The group W is called the reflection group of the reflection datum. The image of Wf modulo W is denoted by $\overline{f}$ and called *the twist of the reflection datum*. Its order is denoted by $\delta(\mathbb{G})$: we have $f^m \in W$ if and only if $\delta(\mathbb{G})$ divides m.

A reflection datum is called *split* if $Wf = W$, *i.e.*, if $f \in W$.

2. A *sub–reflection datum* of $\mathbb{G} = (V, Wf)$ is a reflection datum of the shape $\mathbb{G}' = (V', W'(wf)_{|_{V'}})$, where V' is a subspace of V, W' is a reflection subgroup of $N_W(V')|_{V'}$ (the restriction to V' of the stabilizer $N_W(V')$ of V', isomorphic to $N_W(V')/C_W(V')$), and wf is an element of Wf which stabilizes V' and normalizes W'.

3. A *toric reflection datum (or torus)* is a reflection datum whose reflection group is trivial. A torus of $\mathbb{G} = (V, Wf)$ is a sub–reflection datum of the form $(V', (wf)_{|_{V'}})$, where V' is a subspace of V, and wf is an element of Wf which stabilizes V'.

4. If $\mathbb{T} = (V', (wf)_{|_{V'}})$ is a torus of $\mathbb{G}$, the centralizer of $\mathbb{T}$ in $\mathbb{G}$ is the reflection datum defined by $C_{\mathbb{G}}(\mathbb{T}) := (V, C_W(V')wf)$ (notice that $C_W(V')wf$ is the set of all elements in Wf which act like wf on V'). Such a reflection datum is also called a *Levi sub-reflection datum* of $\mathbb{G}$.

5. The *center of a reflection datum* $\mathbb{G} = (V, Wf)$ is the torus $Z(\mathbb{G}) := (V^W, f_{|_{(V^W)}})$ (notice that it does not depend on the choice of f in Wf).

6. We say that an extension K' of K *splits* $\mathbb{G}$ if it contains the m-th roots of unity, where m is the l.c.m. of the orders of the elements of Wf.

REMARKS.

1. The reflection group of a reflection datum over $\mathbb{Q}$ (resp. over a subfield of $\mathbb{R}$) is a Weyl group (resp. a Coxeter group).

2. Any torus of $\mathbb{G}$ is contained in a *maximal* torus of $\mathbb{G}$, a reflection datum of the form (V, wf) where $w \in W$.

4.2. LEMMA.

(1) *The Levi sub-reflection data of* $\mathbb{G} = (V, Wf)$ *are the reflection data of the form* $\mathbb{L} = (V, W'wf)$, *where* W' *is a parabolic subgroup of* W *and* wf *is an element of* Wf *which normalizes* W'.

(2) *If* $\mathbb{L}$ *is a Levi sub-reflection datum of* $\mathbb{G}$, *then* $C_{\mathbb{G}}(Z(\mathbb{L})) = \mathbb{L}$.

Indeed, by the above definition of Levi sub-reflection datum (4.1, 4) it is enough to prove that if $\mathbb{L}$ is defined as in (1), then $C_{\mathbb{G}}(Z(\mathbb{L})) = \mathbb{L}$. By 4.1, 5, we have $Z(\mathbb{L}) = (V^{W'}, (wf)_{|(V^{W'})})$, hence by 4.1, 3, we have $C_{\mathbb{G}}(Z(\mathbb{L})) = (V, C_W(V^{W'})wf)$, and the result follows from Steinberg's theorem 1.2. $\square$

Reflection data, generic groups and finite reductive groups

We start with a connected reductive algebraic group $\mathbf{G}$ over the algebraic closure of a finite field of characteristic $p > 0$. We assume that $\mathbf{G}$ is already defined over a finite field and let $F : \mathbf{G} \to \mathbf{G}$ be the corresponding Frobenius morphism. The group of fixed points $\mathbf{G}^F$ is then a finite reductive group. The choice of an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ and a Borel subgroup containing $\mathbf{T}$ gives rise to a root datum $(X, R, Y, R^\vee)$, consisting of the character and cocharacter groups X, Y of $\mathbf{T}$, the set of roots $R \subset X$ and the set of coroots $R^\vee \subset Y$. The Frobenius map F acts on $Y_{\mathbb{R}} := Y \otimes_{\mathbb{Z}} \mathbb{R}$ as qf where q is a power of p and f is an automorphism of finite order. Replacing the Borel subgroup by another one containing $\mathbf{T}$ changes f by an element of the Weyl group W of $\mathbf{G}$ with respect to $\mathbf{T}$. Hence f is uniquely determined as automorphism of $Y_{\mathbb{R}}$ up to elements of W.

We can thus naturally associate to $(\mathbf{G}, \mathbf{T}, F)$ the data $(X, R, Y, R^\vee, Wf)$. Here,

(i) X, Y are free $\mathbb{Z}$-modules of equal finite rank, endowed with a duality $X \times Y \to \mathbb{Z}$, $(x, y) \mapsto \langle x, y \rangle$,

(ii) $R \subset X$ and $R^\vee \subset Y$ are root systems with a bijection $R \to R^\vee$, $\alpha \mapsto \alpha^\vee$, such that $\langle \alpha, \alpha^\vee \rangle = 2$

(iii) W is the Weyl group of the root system $R^\vee$ in Y and f is an automorphism of Y of finite order stabilizing $R^\vee$.

A quintuple $(X, R, Y, R^\vee, Wf)$ satisfying these properties is called a *generic finite reductive group.*

Conversely, let's start from a generic group $\mathbb{G} = (X, R, Y, R^\vee, Wf)$. Then for any choice of a prime number p, $\mathbb{G}$ determines a pair $(\mathbf{G}, \mathbf{T})$ as above, up to inner automorphisms of $\mathbf{G}$ induced by $\mathbf{T}$. Moreover, the additional choice of a power q of p determines a triple $(\mathbf{G}, \mathbf{T}, F)$ as above. In this way the generic finite reductive group $\mathbb{G}$ gives rise to a whole series $\{\mathbb{G}(q) := \mathbf{G}^F \mid q \text{ a prime power}\}$ of finite reductive groups.

EXAMPLE. Let $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}_q})$ be the group of invertible $n \times n$-matrices over the algebraic closure of the finite field $\mathbb{F}_q$ with the maximal torus $\mathbf{T}$ consisting of the diagonal matrices in $\mathbf{G}$. Then $\mathbf{T}$ is F-stable for the Frobenius map $F : \mathbf{G} \to \mathbf{G}$ which raises every matrix entry to its qth power, as well as for the product F^- of F with the transpose-inverse map on $\mathbf{G}$. In the first case, the group of F-fixed points is the general linear group over $\mathbb{F}_q$, while for the second Frobenius map F^- we obtain the general unitary group $\mathrm{U}_n(q)$. One easily checks that $(\mathbf{G}, \mathbf{T}, F)$ gives rise to a generic finite reductive group of the form

$$\mathrm{GL}_n = (\mathbb{Z}^n, R, \mathbb{Z}^n, R^\vee, \mathfrak{S}_n \cdot \mathrm{Id})$$

with $R = R^\vee = \{e_i - e_j \mid i \neq j\}$, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{Z}^n$, while $(\mathbf{G}, \mathbf{T}, F^-)$ gives rise to $\mathrm{U}_n = (\mathbb{Z}^n, R, \mathbb{Z}^n, R^\vee, \mathfrak{S}_n(-\mathrm{Id}))$. In this sense we may think of $\mathrm{U}_n(q)$ as being $\mathrm{GL}_n(-q)$.

Now we relate the generic groups to reflection data.

For the simplicity of the exposition, we restrict ourselves to the case of generic groups over $\mathbb{Q}$ (for the "very twisted" cases ${}^2\mathrm{B}_2$, ${}^2\mathrm{F}_4$, ${}^2\mathrm{G}_2$, see for example [**BMM2**]*).*

Let $\mathbb{G} = (V, Wf)$ be a reflection datum on $\mathbb{Q}$, so that W is a Weyl group. Let $(R, R^\vee)$ be a root system for W (so in particular R is a finite subset of V, while $R^\vee$ is a finite subset of $V^\vee$). Let us denote by $Q(R)$ and $Q(R^\vee)$ the $\mathbb{Z}$-submodules of respectively $V^\vee$ and V generated respectively by R and $R^\vee$.

We denote by $V^\vee := \mathrm{Hom}(V, \mathbb{Q})$ the dual space of V, we still denote by W the image of W acting through the contragredient operation on the dual vector space $V^\vee$, and we set $\phi^\vee := {}^t\phi^{-1}$.

- Any choice of a pair of dual lattices X and Y in respectively $V^\vee$ and V such that

(a) X is $W.\phi^\vee$-stable and Y is Wf-stable,
(b) $Q(R) \subseteq X$ and $Q(R^\vee) \subseteq Y$

provides a generic group:

$$\mathbb{G}_{(X,Y)} := ((X, R, Y, R^\vee), Wf).$$

- Reciprocally, any generic group $((X, R, Y, R^\vee), Wf)$ defines a reflection datum

$$\mathbb{G} := (\mathbb{Q} \otimes_\mathbb{Z} Y, Wf),$$

and we have $\mathbb{G}_{(X,Y)} = ((X, R, Y, R^\vee), Wf)$.

Two generic groups $((X, R, Y, R^\vee), Wf)$, $((X_1, R_1, Y_1, R_1^\vee), W_1\phi_1)$ such that $\mathbb{Q}\otimes Y = \mathbb{Q}\otimes Y_1$ and $Wf = W_1\phi_1$ define the same reflection datum. Thus reflection data classify generic groups up to isogeny.

With the previous notation, the map $\mathbb{L}_{(X,Y)} \mapsto \mathbb{L}$ is then a bijection between Levi sub-reflection data of $\mathbb{G}$ and generic Levi subgroups of $\mathbb{G}_{(X,Y)}$ which respects all the invariants which will be introduced for reflection data and which were previously introduced (see [**BMM1**]) for generic groups (such as polynomial orders, signs, graded representation, class functions) or will be introduced later on (such as unipotent degrees). Moreover, it "commutes" with the usual constructions (adjoint, dual), and $\mathbb{L}$ is Φ_d-split if and only if $\mathbb{L}_{(X,Y)}$ is.

Comment. The motivation for the definition of generic groups was to formalize the properties of reductive algebraic groups over a finite field $\mathbb{F}_q$ which are independent of q.

The properties of reflection data that we will discuss in the remainder of this paper are motivated by the properties of unipotent class functions on reductive groups (which have been observed to depend only on the associated reflection datum). We will reflect on that by comments of the form "for $\mathbf{G}$ this means" and "for $\mathbf{G}^F$ this means" where by $\mathbf{G}$ we will mean *any* of the algebraic groups over $\overline{\mathbb{F}}_q$ with reflection datum $\mathbb{G}$ (in general, we will use the same letter as for the reflection datum, but in bold instead of blackboard bold font).

The order of a reflection datum

Let $\mathbb{G} = (V, Wf)$ be a reflection datum. We set $N^\vee(\mathbb{G}) := N^\vee(W)$ (or simply $N^\vee$), the number of pseudo–reflections in W, and we set $N(\mathbb{G}) := N(W)$ (or simply N), the number of reflecting hyperplanes of W.

Generalized degrees of a reflection datum.

Let K' be an extension of K which splits $\mathbb{G}$. We set $V' := K' \otimes_K V$.

Let S be the symmetric algebra of V. Similarly, let S' be the symmetric algebra of V'. Since f acts completely reducibly on the subspace of elements of $(S')^W$ with given degree, we can always choose a family $\{f_1, \ldots, f_r\}$ of basic homogeneous invariants which are all eigenvectors for the action of f (see [**St2**], 2.1), with eigenvalues respectively $\zeta_1, \ldots, \zeta_r$. Then the family of pairs $\{(d_1, \zeta_1), \ldots, (d_r, \zeta_r)\}$ depends only on $\mathbb{G}$ (cf. for example [**Sp**], 6.1), and we call them *the generalized degrees* of the reflection datum $\mathbb{G}$.

Generalized sign.

We recall that R_+ denotes the ideal of R consisting of elements without degree zero terms. The vector space $R_+/(R_+)^2$ has dimension r, and is endowed with an action of the image $\overline{f}$ of f modulo W. Any family of basic homogeneous invariants $\{f_1, \ldots, f_r\}$ as above provides a basis of $(S')^W_+/((S')^W_+)^2$ on which $\overline{f}$ is diagonal. The *generalized sign* (note that in general it is not a sign !) of the reflection datum $\mathbb{G}$ is by definition:

$$\varepsilon_{\mathbb{G}} := (-1)^r \zeta_1 \cdots \zeta_r \,.$$

One can prove (cf. [**Sp**], 6.5) that if f admits a fixed point in $V - \bigcup_{H \in \mathcal{A}} H$, then $\{\zeta_1, \ldots, \zeta_r\}$ is the spectrum of f (in its action on V'). In particular, we have then $\varepsilon_{\mathbb{G}} := (-1)^r \mathrm{det}_V(f)$.

It follows that

4.3. If $\mathbb{T} = (V, wf)$ is a maximal torus of $\mathbb{G}$, we have $\varepsilon_{\mathbb{T}} = (-1)^r \mathrm{det}_V(wf)$. Moreover, if f admits a fixed point in $V - \bigcup_{H \in \mathcal{A}} H$, then $\varepsilon_{\mathbb{T}} = \varepsilon_{\mathbb{G}} \mathrm{det}_V(w)$.

REMARK. Contrary to the case of real reflection data, the group $\langle Wf \rangle$ (subgroup of $\mathrm{GL}(V)$ generated by Wf) is not always a semi-direct product. For example, for $W = G(4,2,4)$, there exists an element f of order 4 in $N_{\mathrm{GL}(V)}(W)$ such that $\langle Wf \rangle = G(4,1,4)$, which is a non split extension of W (see proposition 4.12 below).

The graded regular representation.

We set $R\mathbb{G} := S_W = S/(S.R_+)$ the coinvariant algebra viewed as endowed with the natural action of the group $W\langle f \rangle$). Recall that this finite dimensional graded algebra is isomorphic, as a KW–module, to the regular representation of W. We call $R\mathbb{G}$ the *graded regular representation* of the reflection datum $\mathbb{G}$. We denote by $R^n\mathbb{G}$ the subspace of elements of degree n of $R\mathbb{G}$. Then we have (see 1.19):

$$R\mathbb{G} = \sum_{n=0}^{N^\vee(\mathbb{G})} R^n\mathbb{G} \,.$$

The polynomial order.

The *polynomial order of* $\mathbb{G}$ is the polynomial (in $K[x]$) denoted by $|\mathbb{G}|$ and defined by the formula

$$|\mathbb{G}| := \frac{\varepsilon_{\mathbb{G}} x^{N(\mathbb{G})}}{\frac{1}{|W|} \sum_{w\in W} \frac{1}{\det_V(1 - xwf)^*}} . \tag{4.4}$$

where $\det^*$ denotes the complex conjugate of the determinant (Note that in this case, we have $\det_V^*(1 - xwf) = \det_V(1 - x(wf)^{-1})$).

COMMENT. In the case of reductive groups, $|\mathbb{G}|(q)$ is the order of $\mathbf{G}^F$. See e.g. [**BrMa1**], th. 2.2.

4.5. PROPOSITION. *We have*

$$|\mathbb{G}| = \varepsilon_{\mathbb{G}} x^{N(\mathbb{G})} \prod_{j=1}^{j=r} (1 - \zeta_j^* x^{d_j}) = x^{N(\mathbb{G})} \prod_{j=1}^{j=r} (x^{d_j} - \zeta_j) \quad ,$$

and in particular $|\mathbb{G}| \in \mathbb{Z}_K[x]$.

Indeed, by Molien's formula (making free use of the notion of "graded character" of a graded module as in [**BrMa1**]) we have

$$\frac{1}{\det_V(1 - xwf)} = \operatorname{tr}(wf\,; S) \,,$$

Thus

$$\begin{aligned} \frac{1}{|W|} \sum_{w\in W} \frac{1}{\det_V(1 - xwf)} &= \frac{1}{|W|} \sum_{w\in W} \operatorname{tr}(wf; S) \\ &= \frac{1}{|W|} \sum_{w\in W} \operatorname{tr}(fw; S) = \operatorname{tr}(f; R) \,. \end{aligned}$$

It results from the definition of generalized degrees that, as an $\langle f\rangle$-module, R is isomorphic to $K[f_1] \otimes \ldots \otimes K[f_r]$ where the action of f is given by $f \cdot f_j = \zeta_j f_j$. We see that the first equality of the proposition results from this, and the second one results from the first one and from 1.10. $\square$

Recall that the *cyclotomic polynomials over* K are the minimal polynomials over K of roots of unity. We see that an irreducible divisor of $|\mathbb{G}|$ in $K[x]$ is either x or a cyclotomic polynomial over K. More precisely,

4.6. COROLLARY.

(1) *We have* $|\mathbb{G}| = x^{N(\mathbb{G})} \prod_\Phi \Phi(x)^{a(\Phi)}$, *where* Φ *runs over the set of cyclotomic polynomials over* K, *and where* $a(\Phi)$ *is the number of indices* j *such that* $\zeta^{d_j} = \zeta_j$ *(where* ζ *is a chosen root of* Φ*).*

(2) *The degree of the polynomial* $|\mathbb{G}|$ *is* $(N(\mathbb{G}) + N^\vee(\mathbb{G}) + r)$.

(3) *We have* $|\mathbb{G}|(1/x) = \varepsilon_{\mathbb{G}} x^{-(2N(\mathbb{G})+N^\vee(\mathbb{G})+r)} |\mathbb{G}|(x)^*$.

Various constructions with reflection data

Product of reflection data.

Let $\mathbb{G} = (V, Wf)$ and $\mathbb{G}' = (V', W'f')$ be two reflection data. Their product is the reflection datum

$$\mathbb{G} \times \mathbb{G}' := (V \times V', (W \times W')(f \times f')) .$$

COMMENT. This corresponds to the product of reductive groups.

Extending scalars.

Let $\mathbb{G} = (V, Wf)$ be a reflection datum on the subfield K of $\mathbb{Q}(\boldsymbol{\mu}_\infty)$, and let K' be an extension of K contained in $\mathbb{Q}(\boldsymbol{\mu}_\infty)$. The reflection datum $K'\mathbb{G}$ is then defined as the image of $\mathbb{G}$ under the functor $K' \otimes \cdot$:

$$K'\mathbb{G} := (K' \otimes V, 1 \otimes (Wf)) .$$

COMMENT. This construction has no analogue for reductive groups.

Lifting scalars.

Let $\mathbb{G} = (V, Wf)$ be a reflection datum and let $a \in \mathbb{N}$.

(4.7) We define the reflection datum $\mathbb{G}^{(a)} = (V^{(a)}, W^{(a)} f^{(a)})$ by the following rules:

(ls.1) *$V^{(a)} := V \times \cdots \times V$ (a times), and $W^{(a)} := W \times \cdots \times W$ (a times),*

(ls.2) *$f^{(a)}$ is the product of f (acting diagonally on $V \times \cdots \times V$) by the a–cycle which permutes cyclically the factors V of $V^{(a)}$.*

We have

$$|\mathbb{G}^{(a)}|(x) = |\mathbb{G}|(x^a) .$$

COMMENT. This corresponds to an extension of scalars from $\mathbb{F}_q$ to $\mathbb{F}_{q^a}$ for reductive groups.

Changing x into ζx and generalized Ennola duality.

Let $\mathbb{G} = (V, Wf)$ be a reflection datum. Let $\zeta \in K$ be a root of unity. We define the reflection datum $\mathbb{G}^\zeta$ by

$$\mathbb{G}^\zeta := (V, W\zeta f) . \tag{4.8}$$

From the definition of the polynomial order, we get

$$|\mathbb{G}^\zeta|(x) = \varepsilon_{\mathbb{G}^\zeta} \varepsilon_{\mathbb{G}}^{-1} \zeta^{N(\mathbb{G})} |\mathbb{G}|(\zeta^{-1} x)$$

and, since

$$\varepsilon_{\mathbb{G}^\zeta} = \varepsilon_{\mathbb{G}} \zeta^{d_1 + d_2 + \cdots + d_r} ,$$

we get (by 1.10)

$$|\mathbb{G}^\zeta|(x) = \zeta^{(N + N^\vee + r)} |\mathbb{G}|(\zeta^{-1} x) .$$

If $\zeta \mathrm{Id} \in W$, then $\mathbb{G}^\zeta = \mathbb{G}$, and all the invariants of $\mathbb{G}$ and of $\mathbb{G}^\zeta$ coincide.

COMMENT. A prototype for this construction is the *Ennola* duality between the linear and unitary groups, which corresponds to changing x into $-x$ in the reflection datum associated to linear groups.

The dual, the "semi–simple quotient" and adjoint reflection data.

• We recall that we denote by $V^\vee$ the dual space of V. We still denote by W the image in $\mathrm{GL}(V^\vee)$ of the group W acting through the contragredient representation $w \mapsto {}^t w^{-1}$ and we set $f^\vee := {}^t f^{-1}$. Then the *dual reflection datum* of $\mathbb{G}$ is

$$\mathbb{G}^\vee := (V^\vee, Wf^\vee)\,.$$

COMMENT. This corresponds to the Langlands dual for algebraic groups. However, since Weyl groups are real reflection groups, the reflection data associated to an algebraic group and its Langlands dual are isomorphic.

• The "semi–simple quotient" of $\mathbb{G}$ is "the quotient of $\mathbb{G}$ by its center $Z(\mathbb{G})$", namely

$$\mathbb{G}_{\mathrm{ss}} := (V/V^W, Wf)\,,$$

where we identify W and f with their images in the linear group of V/V^W.

REMARK. One may also define the *adjoint reflection datum* of $\mathbb{G}$ as

$$\mathbb{G}_{\mathrm{ad}} := (((V^\vee)^W)^\perp, Wf)\,,$$

where we identify W, f with their images in the linear group of $((V^\vee)^W)^\perp$. Denoting by pr_W the projector of V defined by $\mathrm{pr}_W := \dfrac{1}{|W|}\sum_{w\in W} w$, it is readily checked that the map $1-\mathrm{pr}_W$ induces a W–isomorphism from V/V^W onto $((V^\vee)^W)^\perp$, which shows that $\mathbb{G}_{\mathrm{ss}}$ and $\mathbb{G}_{\mathrm{ad}}$ are isomorphic.

COMMENT. As the chosen names reflect, this corresponds to the quotient by the radical (resp. to the adjoint group) for reductive groups.

Intersection of a sub-reflection datum of maximal rank with a Levi sub-reflection datum.

Let $\mathbb{L} = (V, W_{\mathbb{L}}vf)$ be a Levi sub-reflection datum of $\mathbb{G}$ and let $\mathbb{M} = (V, W_{\mathbb{M}}wf)$ be a sub-reflection datum of maximal rank of $\mathbb{G}$.

(4.9) We say that $\mathbb{L}\cap\mathbb{M}$ is defined if

$$W_{\mathbb{L}}vf \cap W_{\mathbb{M}}wf \neq \emptyset\,.$$

In that case, choosing an element $u \in W_{\mathbb{L}}v \cap W_{\mathbb{M}}w$, we define

$$\mathbb{L}\cap\mathbb{M} := (V, (W_{\mathbb{L}} \cap W_{\mathbb{M}})uf)\,.$$

Note that $W_{\mathbb{L}}vf = W_{\mathbb{L}}uf$ and $W_{\mathbb{M}}wf = W_{\mathbb{M}}uf$, hence $W_{\mathbb{L}}vf \cap W_{\mathbb{M}}wf = (W_{\mathbb{L}} \cap W_{\mathbb{M}})uf$. Notice also that $W_{\mathbb{L}} \cap W_{\mathbb{M}}$ is a parabolic subgroup of $W_{\mathbb{M}}$. It is easy to see that $\mathbb{L}\cap\mathbb{M}$ is a well–defined (*i.e.*, independent of the choice of u) Levi sub-reflection datum of $\mathbb{M}$, and a sub-reflection datum of maximal rank of $\mathbb{L}$.

By the preceding definition, it is clear that

(4.10) $\mathbb{L} \cap \mathbb{M}$ *is defined if and only if there exists a maximal sub-torus* $\mathbb{T}$ *of* $\mathbb{G}$ *contained in both* $\mathbb{L}$ *and* $\mathbb{M}$*. In this case,* $\mathbb{T}$ *is contained in* $\mathbb{L} \cap \mathbb{M}$*.*

For $w \in W$ and $\mathbb{L}$ a sub–reflection datum of maximal rank of $\mathbb{G}$, we denote by ${}^w\mathbb{L} := w\mathbb{L}w^{-1}$ the conjugate of $\mathbb{L}$ under w.

4.11. DEFINITION. *For* $\mathbb{L}$ *a Levi sub-reflection datum and* $\mathbb{M}$ *a sub-reflection datum of maximal rank of* $\mathbb{G}$*, we denote by* $\mathcal{T}_{W_\mathbb{G}}(\mathbb{L}, \mathbb{M})$ *the set of all* $w \in W_\mathbb{G}$ *such that* ${}^w\mathbb{L} \cap \mathbb{M}$ *is defined.*

Classification of reflection data

Using the classification of finite irreducible complex reflection groups by Shephard and Todd (cf. above 1.5) it is possible to give a complete description of all reflection data as follows.

Let $\mathbb{G} = (V, Wf)$ be a reflection datum. Let $V_1 \leq V$ be a $W\langle f\rangle$–invariant subspace of V and V_2 a $W\langle f\rangle$-invariant complement. Any reflection in W either acts trivially on V_1 or on V_2. Thus, as W is generated by reflections, the decomposition $V = V_1 \times V_2$ induces a f-invariant direct product decomposition $W = W_1 \times W_2$, such that W_i acts trivially on V_{3-i} $(i = 1, 2)$. This shows that $\mathbb{G}$ is a direct product

$$\mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_2, \qquad \text{with } \mathbb{G}_i = (V_i, W_i f|_{V_i}) \text{ for } i = 1, 2\,.$$

Now assume that $W\langle f\rangle$ acts irreducibly on V. Let $V = V_1 \times \ldots \times V_a$ be the decomposition of V into W-irreducible subspaces. As above this induces a direct product decomposition $W = W_1 \times \ldots \times W_a$ of W into reflection subgroups W_i such that W_i acts trivially on all V_j for $j \neq i$. Moreover, the (V_i, W_i) are permuted transitively by f. Thus $\mathbb{G}$ is the lifting of scalars

$$\mathbb{G} = (V_1, W_1\psi)^{(a)}, \qquad \text{with } \psi = f^a|_{V_1}$$

of the reflection datum $(V_1, W_1\psi)$ where W_1 acts irreducibly (hence absolutely irreducibly) on V_1.

It hence remains to classify those reflection data $\mathbb{G}$ where V is an (absolutely) irreducible W-module. For this, it is useful to first determine normal embeddings of irreducible reflection groups in the same dimension.

4.12 PROPOSITION. *Let* W *be an irreducible finite complex reflection group on the complex vector space* V *and* W' *a proper irreducible (normal) subgroup of* W *generated by a union of reflection classes of* W*, maximal in* W *with respect to these properties. Then* (W, W') *are in the following list (where we adopt the notation of*

[**ShTo**] *for irreducible reflection groups):*

W	W'
$G(dep, e, r)$	$G(dep, ep, r)$, p *prime*
$G(2de, 2e, 2)$	$G(de, e, 2)$
G_5	G_4
G_6	$G(4,2,2)$, G_4
G_7	G_6, G_5
G_8	$G(4,2,2)$
G_9	G_{13}, G_8
G_{10}	G_7, G_8
G_{11}	G_{15}, G_9, G_{10}

W	W'
G_{13}	$G(4,2,2)$, G_{12}
G_{14}	G_{12}, G_5
G_{15}	G_{13}, G_{14}, G_7
G_{17}	G_{22}, G_{16}
G_{18}	G_{20}, G_{16}
G_{19}	G_{21}, G_{17}, G_{18}
G_{21}	G_{22}, G_{20}
G_{26}	$G(3,3,3)$, G_{25}
G_{28}	$G(2,2,4)$

The proof is a routine case-by-case check, using for example the tables on p. 395 and p. 412 of [**Co**].

The following proposition is 3.13 in [**BMM2**].

4.13 PROPOSITION. *Let W be an irreducible complex reflection group on the n-dimensional complex vector space V and f an automorphism of finite order of V normalizing W. Then up to multiples of the identity, either $\overline{f} = 1$ or we are in one of the following cases :*

- *$W = G(de, e, r)$, with $e > 1$, and $\overline{f}$ of order dividing e comes from the embedding $G(de, e, r) < G(de, 1, r)$,*
- *$W = G(4,2,2)$ and f of order 3 comes from the embedding $G(4,2,2) < G_6$,*
- *$W = G(3,3,3)$ and f of order 4 comes from the embedding $G(3,3,3) < G_{26}$,*
- *$W = G(2,2,4)$ and f of order 3 comes from the embedding $W(D_4) < W(F_4)$,*
- *$W = G_5$ and f of order 2 comes from the embedding $G_5 < G_{14}$,*
- *$W = G_7$ and f of order 2 comes from the embedding $G_7 < G_{10}$,*
- *$W = G_{28}$ and f realizes the graph-automorphism of $W(F_4)$,*

Here $W(F_4)$, $W(D_4)$, denote the Weyl groups of type F_4, D_4 respectively.

Note that most of these can be visualized by graph–automorphisms of the corresponding diagram (see appendix 1).

The polynomial orders of the twisted reflection data corresponding to the above automorphisms f can easily be calculated from Proposition 4.5 in terms of the generalized degrees. For the infinite families ${}^tG(de, e, r)$, $t|e$, we have

$$|{}^tG(de, e, r)| = x^{N(G(de,e,r))}(x^{de} - 1)(x^{2de} - 1)\dots(x^{(r-1)de} - 1)(x^{rd} - \zeta)\,,$$

where ζ is a primitive t–th root of unity, while for the exceptional cases we obtain :

$$|{}^3G(4,2,2)| = x^6(x^4 - 1)(x^4 - \zeta_3^2)$$
$$|{}^4G(3,3,3)| = x^9(x^6 - 1)(x^6 + 1)$$
$$|{}^3G(2,2,4)| = x^{12}(x^2 - 1)(x^6 - 1)(x^8 + x^4 + 1)$$
$$|{}^2G_5| = x^8(x^6 - 1)(x^{12} + 1)$$
$$|{}^2G_7| = x^{14}(x^{12} - 1)(x^{12} + 1)$$
$$|{}^2G_{28}| = x^{24}(x^2 - 1)(x^6 + 1)(x^8 - 1)(x^{12} + 1)$$

(notice that ${}^3G(2,2,4) = {}^3D_4$ and ${}^2G_{28} = {}^2F_4$).

Uniform class functions on a reflection datum

Generalities, induction and restriction.

In this section, we assume that K *splits the reflection datum* $\mathbb{G} = (V, Wf)$.

Let $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$ be the $\mathbb{Z}_K$–module of all W–invariant functions on the coset Wf (for the natural action of W on Wf by conjugation) with values in $\mathbb{Z}_K$, called *uniform class functions on* $\mathbb{G}$. For $\alpha \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$, we denote by α^* its complex conjugate.

If α and $\alpha' \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$, we set $\langle \alpha, \alpha' \rangle_{\mathbb{G}} := \dfrac{1}{|W|} \sum_{w \in W} \alpha(wf)\alpha'(wf)^*$.

Notation.

- If $\mathbb{Z}_K \to \mathcal{O}$ is a ring morphism, we denote by $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G}, \mathcal{O})$ the $\mathcal{O}$–module of W–invariant functions on Wf with values in $\mathcal{O}$, which we call the module of *uniform class functions on* $\mathbb{G}$ with values in $\mathcal{O}$. We have $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G}, \mathcal{O}) = \mathcal{O} \otimes_{\mathbb{Z}_K} \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$.
- For $wf \in Wf$, we denote by $\mathrm{ch}^{\mathbb{G}}_{wf}$ (or simply ch_{wf}) the characteristic function of the orbit of wf under W. The family $\left(\mathrm{ch}^{\mathbb{G}}_{wf}\right)$ (where wf runs over a complete set of representatives of the orbits of W on Wf) is a basis of $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$.
- For $wf \in Wf$, we set

$$R^{\mathbb{G}}_{wf} := |C_W(wf)| \mathrm{ch}^{\mathbb{G}}_{wf}$$

(or simply R_{wf}). The $\mathbb{Z}_K$–module generated by the functions $R^{\mathbb{G}}_{wf}$ is denoted by $\mathrm{CF}_{\mathrm{uf}}{}^{\mathrm{pr}}(\mathbb{G})$. For $\alpha \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$ and $wf \in Wf$ we have $\langle \alpha, R_{wf} \rangle_{\mathbb{G}} = \alpha(wf)$, so $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$ is the $\mathbb{Z}_K$-dual of $\mathrm{CF}_{\mathrm{uf}}{}^{\mathrm{pr}}(\mathbb{G})$.

[The exponent "pr" stands for "projective", by analogy with the vocabulary of modular representation theory of finite groups]

COMMENT. In the case of reductive groups, we have $K = \mathbb{Q}$. Let $\mathrm{Uch}(\mathbf{G}^F)$ be the set of unipotent characters of $\mathbf{G}^F$: then the map which associates to $R^{\mathbb{G}}_{wf}$ the Deligne–Lusztig character $R^{\mathbf{G}}_{\mathbf{T}_{wF}}(\mathrm{Id})$ defines an isometric embedding (for the scalar products $\langle \alpha, \alpha' \rangle_{\mathbb{G}}$ and $\langle \alpha, \alpha' \rangle_{\mathbf{G}^F}$) from $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$ onto the sub-$\mathbb{Z}$-module of $\mathbb{Q}\mathrm{Uch}(\mathbf{G}^F)$ of the $\mathbb{Q}$-linear combinations of Deligne-Lusztig characters (i.e. "unipotent uniform functions") which have an integral scalar product with the Deligne-Lusztig characters.

- Let $\langle Wf \rangle$ be the subgroup of $\mathrm{GL}(V)$ generated by Wf. We recall that we denote by $\overline{f}$ the image of f in $\langle Wf \rangle / W$ — thus $\langle Wf \rangle / W$ is cyclic and generated by $\overline{f}$.

For $\psi \in \mathrm{Irr}(\langle Wf \rangle)$, we denote by $R^{\mathbb{G}}_{\psi}$ (or simply R_ψ) the restriction of ψ to the coset Wf. We have

$$R^{\mathbb{G}}_{\psi} = \frac{1}{|W|} \sum_{w \in W} \psi(wf) R^{\mathbb{G}}_{wf},$$

and we call such a function a *uniform almost character* of $\mathbb{G}$.

Let $\mathrm{Irr}(W)^{\overline{f}}$ denote the set of $\overline{f}$–stable irreducible characters of W. For $\chi \in \mathrm{Irr}(W)^{\overline{f}}$, we denote by $E_{\mathbb{G}}(\chi)$ (or simply $E(\chi)$) the set of restrictions to Wf of the extensions of χ to characters of $\langle Wf \rangle$. Since K contains the $\delta(\mathbb{G})$–th roots of unity (where $\delta(\mathbb{G})$ denotes the order of the twist $\overline{f}$), the group $\langle\overline{f}\rangle$ acts regularly on $E(\chi)$.

Each element of $E_{\mathbb{G}}(\chi)$ has norm 1, the sets $E_{\mathbb{G}}(\chi)$ for $\chi \in \operatorname{Irr}(W)^{\overline{f}}$ are mutually orthogonal, and we have

$$\mathrm{CF}_{\mathrm{uf}}(\mathbb{G}, K) = \overset{\perp}{\bigoplus_{\chi\in\operatorname{Irr}(W)^{\overline{f}}}} KE_{\mathbb{G}}(\chi)\,,$$

where we set $KE_{\mathbb{G}}(\chi) := KR_{\psi}^{\mathbb{G}}$ for some (any) $\psi \in E_{\mathbb{G}}(\chi)$.

Induction and restriction.

Let $\mathbb{L} = (V, W_{\mathbb{L}}wf)$ be a sub-reflection datum of maximal rank of $\mathbb{G}$, and let $\alpha \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$ and $\beta \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{L})$. We denote

- by $\operatorname{Res}_{\mathbb{L}}^{\mathbb{G}}\alpha$ the restriction of α to the coset $W_{\mathbb{L}}wf$,
- by $\operatorname{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta$ the uniform class function on $\mathbb{G}$ defined by

(4.14) $$\operatorname{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta(uf) := \frac{1}{|W_{\mathbb{L}}|}\sum_{v\in W}\tilde{\beta}(vufv^{-1}) \quad \text{for } uf \in W_{\mathbb{G}}f,$$

where $\tilde{\beta}(xf) = \beta(xf)$ if $x \in W_{\mathbb{L}}w$, and $\tilde{\beta}(xf) = 0$ if $x \notin W_{\mathbb{L}}w$. In other words, we have

(4.15) $$\operatorname{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta(uf) = \sum_{v\in W_{\mathbb{G}}/W_{\mathbb{L}},\, {}^{v}(uf)\in W_{\mathbb{L}}wf}\beta({}^{v}(uf))\,.$$

For any reflection datum $\mathbb{G}$ we denote by $1^{\mathbb{G}}$ the constant function on Wf with value 1. For $w \in W$, let us denote by $\mathbb{T}_{wf}$ the maximal torus of $\mathbb{G}$ defined by $\mathbb{T}_{wf} := (V, wf)$. It follows from the definitions that

(4.16) $$R_{wf}^{\mathbb{G}} = \operatorname{Ind}_{\mathbb{T}_{wf}}^{\mathbb{G}} 1^{\mathbb{T}_{wf}}\,.$$

For $\alpha \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$, $\beta \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{L})$ we have the *Frobenius reciprocity*:

(4.17) $$\langle \alpha, \operatorname{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta\rangle_{\mathbb{G}} = \langle \operatorname{Res}_{\mathbb{L}}^{\mathbb{G}}\alpha, \beta\rangle_{\mathbb{L}}\,.$$

COMMENT. In the case of reductive groups, assume that $\mathbb{L}$ is a Levi sub-reflection datum. Then $\operatorname{Ind}_{\mathbb{L}}^{\mathbb{G}}$ corresponds to Lusztig induction from $\mathbf{L}$ to $\mathbf{G}$ (this results from definition 4.14 applied to a Deligne–Lusztig character which, using the transitivity of Lusztig induction, agrees with Lusztig induction). Similarly, the Lusztig restriction of a uniform function is uniform by [**DeLu**], theorem 7, so by (4.17) $\operatorname{Res}_{\mathbb{L}}^{\mathbb{G}}$ corresponds to Lusztig restriction.

When $\mathbb{L}$ corresponds to a reductive subgroup of maximal rank which is *not* a Levi subgroup, then $\operatorname{Ind}_{\mathbb{L}}^{\mathbb{G}}$ corresponds to a generalization of Lusztig induction to that setting. However, this generalization does not necessarily map $\mathbb{Z}\mathrm{Uch}(\mathbf{L}^F)$ to $\mathbb{Z}\mathrm{Uch}(\mathbf{G}^F)$. For instance, when $\mathbf{L}$ is of type A_2 corresponding to the long roots of $\mathbf{G}$ of type G_2, then the image of $\mathbb{Z}\mathrm{Uch}(\mathbf{L}^F)$ by this generalized induction is only in $\mathbb{Z}[1/3]\mathrm{Uch}(\mathbf{G}^F)$.

The Mackey formula.

Let now $\mathbb{L}$ be a Levi sub-reflection datum and let $\mathbb{M}$ be a sub-reflection datum of maximal rank of $\mathbb{G}$.

It is clear that $W_{\mathbb{L}}$ acts on $\mathcal{T}_{W_{\mathbb{G}}}(\mathbb{L},\mathbb{M})$ (see definition 4.11 above) from the right, while $W_{\mathbb{M}}$ acts on $\mathcal{T}_{W_{\mathbb{G}}}(\mathbb{L},\mathbb{M})$ from the left. If we let w run over a chosen double coset $W_{\mathbb{M}}vW_{\mathbb{L}}$ for some $v \in \mathcal{T}_{W_{\mathbb{G}}}(\mathbb{L},\mathbb{M})$, we see that ${}^w\mathbb{L}\cap\mathbb{M}$ is defined up to $W_{{}^w\mathbb{L}}$–conjugation as a sub-reflection datum of ${}^w\mathbb{L}$ and up to $W_{\mathbb{M}}$–conjugation as a sub-reflection datum of $\mathbb{M}$, which proves that the operations $\mathrm{Ind}_{\mathbb{M}\cap{}^w\mathbb{L}}^{\mathbb{M}}$ and $\mathrm{Res}_{\mathbb{M}\cap{}^w\mathbb{L}}^{{}^w\mathbb{L}}$ depend only on the double coset of w. This gives sense to the following formula where $\mathrm{ad}(w)$ denotes the operator of conjugation by w

$$\mathrm{Res}_{\mathbb{M}}^{\mathbb{G}}\cdot\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} = \sum_{w\in W_{\mathbb{M}}\backslash\mathcal{T}_{W_{\mathbb{G}}}(\mathbb{L},\mathbb{M})/W_{\mathbb{L}}} \mathrm{Ind}_{\mathbb{M}\cap{}^w\mathbb{L}}^{\mathbb{M}}\cdot\mathrm{Res}_{\mathbb{M}\cap{}^w\mathbb{L}}^{{}^w\mathbb{L}}\cdot\mathrm{ad}(w)\,, \tag{4.18}$$

whose proof is a straightforward calculation as in the case of ordinary induction and restriction (note that $\mathcal{T}_{W_{\mathbb{G}}}(\mathbb{L},\mathbb{M})$ may be empty).

COMMENT. In the case of reductive groups, assuming that both $\mathbb{L}$ and $\mathbb{M}$ are Levi sub-reflection data, the Mackey formula corresponds to the Mackey formula for Lusztig induction and restriction (projected on uniform function).

Uniform class functions on $\mathbb{G}^\zeta$.

The map $\sigma_{\mathbb{G}}^{\zeta}\colon \mathrm{CF}_{\mathrm{uf}}(\mathbb{G}) \to \mathrm{CF}_{\mathrm{uf}}(\mathbb{G}^\zeta)$, given by $\sigma_{\mathbb{G}}^{\zeta}\alpha(w\zeta f) = \alpha(wf)$, is an isometry. The map $\mathbb{L}\mapsto\mathbb{L}^\zeta$ is a $W_{\mathbb{G}}$–equivariant bijection from the set of all sub-reflection data of maximal rank of $\mathbb{G}$ (resp. of all Levi sub-reflection data of $\mathbb{G}$) onto the set of all sub-reflection data of maximal rank of $\mathbb{G}^\zeta$ (resp. of all Levi sub-reflection data of $\mathbb{G}^\zeta$). It is clear that

$$\sigma_{\mathbb{G}}^{\zeta}\cdot\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} = \mathrm{Ind}_{\mathbb{L}^\zeta}^{\mathbb{G}^\zeta}\cdot\sigma_{\mathbb{L}}^{\zeta}\quad,\quad \sigma_{\mathbb{L}}^{\zeta}\cdot\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}} = \mathrm{Res}_{\mathbb{L}^\zeta}^{\mathbb{G}^\zeta}\cdot\sigma_{\mathbb{G}}^{\zeta}\,. \tag{4.19}$$

Uniform class functions on $\mathbb{G}^{(a)}$.

Let $a\in\mathbb{N}$. We have (see 4.7) $W_{\mathbb{G}^{(a)}} = (W_{\mathbb{G}})^a$, and the map

$$(w_1,w_2,\dots,w_a)f^{(a)} \mapsto w_1w_2\cdots w_af$$

defines a bijection between the set of classes of $W_{\mathbb{G}^{(a)}}f^{(a)}$ under $W_{\mathbb{G}^{(a)}}$–conjugacy and the set of classes of $W_{\mathbb{G}}f$ under $W_{\mathbb{G}}$–conjugacy. Thus it induces an isometry

$$\sigma_{\mathbb{G}}^{(a)}\colon \mathrm{CF}_{\mathrm{uf}}(\mathbb{G}^{(a)}) \xrightarrow{\sim} \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})\,.$$

4.20. PROPOSITION. *We have*

$$\begin{aligned}\sigma_{\mathbb{G}}^{(a)}\cdot\mathrm{Ind}_{\mathbb{L}^{(a)}}^{\mathbb{G}^{(a)}} &= \mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}\cdot\sigma_{\mathbb{L}}^{(a)}\\ \mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}\cdot\sigma_{\mathbb{G}}^{(a)} &= \sigma_{\mathbb{L}}^{(a)}\cdot\mathrm{Res}_{\mathbb{L}^{(a)}}^{\mathbb{G}^{(a)}}\,.\end{aligned}$$

Degrees.

• We denote by $\mathrm{tr}_{R\mathbb{G}}$ the uniform class function on $\mathbb{G}$ (with values in the polynomial ring $\mathbb{Z}_K[x]$) defined by the character of the "graded regular representation" $R\mathbb{G}$ (see above). Thus the value of the function $\mathrm{tr}_{R\mathbb{G}}$ on wf is

$$\mathrm{tr}_{R\mathbb{G}}(wf) := \sum_{n=0}^{N^\vee(\mathbb{G})} \mathrm{tr}(wf;R^n\mathbb{G})x^n\,.$$

We call $\mathrm{tr}_{R\mathbb{G}}$ *the regular character* of $\mathbb{G}$.

• We define the *degree*, a linear function

$$\mathrm{Deg}_{\mathbb{G}}\colon \mathrm{CF}_{\mathrm{uf}}(\mathbb{G}) \to K[x]\,,$$

as follows: for $\alpha \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$, we set

$$\mathrm{Deg}_{\mathbb{G}}(\alpha) := \langle \alpha, \mathrm{tr}_{R\mathbb{G}} \rangle_{\mathbb{G}} = \sum_{n=0}^{N^\vee(\mathbb{G})} \left(\frac{1}{|W|} \sum_{w\in W} \alpha(wf)\mathrm{tr}(wf; R^n\mathbb{G})^* \right) x^n\,. \tag{4.21}$$

We shall often omit the subscript $\mathbb{G}$ (writing then $\mathrm{Deg}(\alpha)$) when the context allows it.

Notice that

$$\mathrm{Deg}(R^{\mathbb{G}}_{wf}) = \mathrm{tr}_{R\mathbb{G}}(wf)\,, \tag{4.22}$$

and so in particular that

$$\mathrm{Deg}(R^{\mathbb{G}}_{wf}) \in \mathbb{Z}_K[x]\,. \tag{4.23}$$

Degrees of almost characters.

Let E be a $K\langle Wf\rangle$–module. Let χ_E be the restriction of the character of E (a uniform class function on $\langle Wf\rangle$) to Wf. Then the degree of R_{χ_E} is the "graded multiplicity" of E in the graded regular representation $R\mathbb{G}$:

$$\mathrm{Deg}_{\mathbb{G}}(R_{\chi_E}) = \mathrm{tr}(f\,;\,\mathrm{Hom}_{KW}(R\mathbb{G}, E))\,. \tag{4.24}$$

Notice that

$$\mathrm{Deg}_{\mathbb{G}}(R_{\chi_E}) \in \mathbb{Z}[\zeta_{\delta(\mathbb{G})}][x]\,, \tag{4.25}$$

(we recall that $\delta(\mathbb{G})$ is the order of the twist $\overline{f}$ of $\mathbb{G}$).

REMARK. Let $\chi \in \mathrm{Irr}(W)$. Then χ is a uniform function on the split reflection datum $\mathbb{G}_0 := (V, W)$, and we have (see above §1, 1.20)

$$\mathrm{Deg}_{\mathbb{G}_0}(R_\chi) = \mathrm{Feg}_\chi(x)\,.$$

Let $\chi \in \mathrm{Irr}(W)^{\overline{f}}$. If $\psi \in \mathrm{Irr}(\langle Wf\rangle)$ runs over the set of extensions of χ to $\langle Wf\rangle$, and since K contains the $\delta(\mathbb{G})$–th roots of unity, the set of degrees $\mathrm{Deg}(R_\psi)$ is an orbit of $\boldsymbol{\mu}_{\delta(\mathbb{G})}(K)$ on $\mathbb{Z}_K[x]$. The element $\mathrm{Deg}_{\mathbb{G}}(R_\psi)^* \cdot R_\psi$ depends only on χ and is the orthogonal projection of $\mathrm{tr}_{R\mathbb{G}}$ onto $K[x]E_{\mathbb{G}}(\chi)$. We set

$$\mathrm{Reg}^{\mathbb{G}}_\chi := \mathrm{Deg}_{\mathbb{G}}(R_\psi)^* \cdot R_\psi\,, \tag{4.26}$$

and in other words, we have

$$\mathrm{tr}_{R\mathbb{G}} = \sum_{\chi\in\mathrm{Irr}(W)^{\overline{f}}} \mathrm{Reg}^{\mathbb{G}}_\chi\,. \tag{4.27}$$

The proof of the following lemma is immediate.

4.28. LEMMA. *We have*

$$\mathrm{tr}_{R\mathbb{G}} = \frac{1}{|W|}\sum_{w\in W} \mathrm{Deg}_{\mathbb{G}}(R_{wf}^{\mathbb{G}})^* R_{wf}^{\mathbb{G}}\,,$$

and in particular

$$\mathrm{Deg}_{\mathbb{G}}(\mathrm{tr}_{R\mathbb{G}}) = \frac{|\mathbb{G}||\mathbb{G}|^*}{x^{2N(\mathbb{G})}}\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-xwf)\det_V(1-xwf)^*}\,.$$

Polynomial order and degrees.

From the isomorphism of $\langle Wf\rangle$–modules :

$$S \simeq R\mathbb{G}\otimes_K R\,,$$

we deduce

$$\frac{1}{|W|}\sum_{w\in W}\frac{\alpha(wf)}{\det_V(1-xwf)^*} = \mathrm{Deg}(\alpha)\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-xwf)^*}\,, \tag{4.29}$$

or, in other words

$$\langle\alpha,\mathrm{tr}_S\rangle_{\mathbb{G}} = \langle\alpha,\mathrm{tr}_{R\mathbb{G}}\rangle_{\mathbb{G}}\langle 1^{\mathbb{G}},\mathrm{tr}_S\rangle_{\mathbb{G}}\,. \tag{4.30}$$

Let us set $S_{\mathbb{G}}(\alpha) := \langle\alpha,\mathrm{tr}_S\rangle_{\mathbb{G}}$. Then (4.30) becomes:

$$S_{\mathbb{G}}(\alpha) = \mathrm{Deg}_{\mathbb{G}}(\alpha)S_{\mathbb{G}}(1^{\mathbb{G}})\,. \tag{4.31}$$

By (4.4) we get

$$|\mathbb{G}| = \varepsilon_{\mathbb{G}}\frac{x^{N(\mathbb{G})}}{S_{\mathbb{G}}(1^{\mathbb{G}})}\,,\quad \text{hence}$$

$$S_{\mathbb{G}}(\alpha)|\mathbb{G}| = \varepsilon_{\mathbb{G}}x^{N(\mathbb{G})}\mathrm{Deg}_{\mathbb{G}}(\alpha)\,. \tag{4.32}$$

• For a sub-reflection datum $\mathbb{L}$ of maximal rank of $\mathbb{G}$, by the Frobenius reciprocity (4.17), we have

$$\mathrm{Deg}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}1^{\mathbb{L}}) = \langle 1^{\mathbb{L}},\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}\mathrm{tr}_{R\mathbb{G}}\rangle_{\mathbb{L}} = \sum_{n=0}^{N^\vee(\mathbb{G})}\mathrm{tr}(wf;(R^n\mathbb{G})^{W_{\mathbb{L}}})^*x^n \tag{4.33}$$

where $W_{\mathbb{L}}wf$ is the coset associated to $\mathbb{L}$ and $(R^n\mathbb{G})^{W_{\mathbb{L}}}$ are the $W_{\mathbb{L}}$–invariants in $R^n\mathbb{G}$.

4.34. PROPOSITION. *We have*

$$|\mathbb{G}|/|\mathbb{L}| = \varepsilon_{\mathbb{G}}\varepsilon_{\mathbb{L}}^{-1}x^{N(\mathbb{G})-N(\mathbb{L})}\mathrm{Deg}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}1^{\mathbb{L}})\,,$$

and in particular $|\mathbb{L}|$ *divides* $|\mathbb{G}|$ *(in* $\mathbb{Z}_K[x]$*).*

• Let us recall that every element $wf\in Wf$ defines a maximal torus (or, equivalently, a minimal Levi sub-reflection datum) $\mathbb{T}_{wf} := (V,wf)$. By (4.16) and (4.22) we have

$$\mathrm{tr}_{R\mathbb{G}}(wf) = \mathrm{Deg}(\mathrm{Ind}_{\mathbb{T}_{wf}}^{\mathbb{G}}1^{\mathbb{T}_{wf}})^*\,.$$

So

$$|\mathbb{G}|/|\mathbb{T}_{wf}| = \varepsilon_{\mathbb{G}}\varepsilon_{\mathbb{T}_{wf}}^{-1}x^{N(\mathbb{G})}\mathrm{tr}_{R\mathbb{G}}(wf)^*\,. \tag{4.35}$$

It follows from 4.34 and (4.35) that

$$\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}\mathrm{tr}_{R\mathbb{G}}^{*} = \mathrm{Deg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}1^{\mathbb{L}})\mathrm{tr}_{R\mathbb{L}}^{*}\,. \tag{4.36}$$

(4.37) For $\beta \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{L})$, *we have* $\mathrm{Deg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta) = \mathrm{Deg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}1^{\mathbb{L}})\mathrm{Deg}_{\mathbb{L}}(\beta)$.

Indeed, $\mathrm{Deg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta) = \langle \beta, \mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}\mathrm{tr}_{R\mathbb{G}}\rangle_{\mathbb{L}} = \mathrm{Deg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}1^{\mathbb{L}})\langle \beta, \mathrm{tr}_{R\mathbb{L}}\rangle_{\mathbb{L}}$.

COMMENT. In the case of reductive groups, it follows from (4.22) and (4.35) that $\mathrm{Deg}(R_{wf})(q)$ is the degree of the Deligne–Lusztig character $R_{\mathbf{T}_{wf}}^{\mathbf{G}}$. Since the regular representation of $\mathbf{G}(q)$ is uniform, it follows that $\mathrm{tr}_{R\mathbf{G}}$ corresponds to a (graded by x) version of the unipotent part of the regular representation of $\mathbf{G}(q)$, and that Deg corresponds indeed to the (generic) degree for unipotent uniform functions on $\mathbf{G}(q)$.

Changing x *to* $1/x$.

As a particular uniform class function on $\mathbb{G}$, we can consider the function $\det_V$ restricted to $W_{\mathbb{G}}f$, which we still denote by $\det_V$. Notice that this restriction might also be denoted by $R_{\det_V}^{\mathbb{G}}$, since it is the almost character associated to the character of $\langle Wf\rangle$ defined by $\det_V$.

The following two results are 4.25 and 4.26 in [**BMM2**].

4.38. PROPOSITION. *Let* α *be a uniform class function on* $\mathbb{G}$. *We have*

$$S_{\mathbb{G}}(\alpha\det_V^*)(x) = (-1)^r x^{-r} S_{\mathbb{G}}(\alpha^*)(1/x)^*\,,$$
$$\mathrm{Deg}_{\mathbb{G}}(\alpha\det_V^*) = (-1)^r \varepsilon_{\mathbb{G}}^* x^{N^\vee(\mathbb{G})}\mathrm{Deg}_{\mathbb{G}}(\alpha^*)(1/x)^*\,.$$

4.39. COROLLARY. *We have*

$$\mathrm{Deg}_{\mathbb{G}}(\det_V^*) = (-1)^r \varepsilon_{\mathbb{G}}^* x^{N^\vee(\mathbb{G})}\,,$$
$$\mathrm{Deg}_{\mathbb{G}}(\det_V) = (-1)^r \varepsilon_{\mathbb{G}} x^{N(\mathbb{G})}\,.$$

Changing x *to* ξx.

Let E be a $K\langle Wf\rangle$-module, and let χ be the restriction of the character of E to Wf.

Let $\xi \in \boldsymbol{\mu}(K)$ such that $\xi\mathrm{Id} \in Z(\langle Wf\rangle)$.

Then we have

$$\mathrm{Deg}(\chi)(x) = \omega_\chi(\xi)\mathrm{Deg}(\chi)(\xi x)\,. \tag{4.40}$$

Indeed, by formula 4.29, we have

$$\frac{1}{|W|}\sum_{w\in W}\frac{\chi(wf)}{\det_V(1-xwf)^*} = \mathrm{Deg}(\alpha)(x)\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-xwf)^*}\,.$$

The preceding formula may be rewritten

$$\frac{1}{|W|}\sum_{w\in W}\frac{\chi(w\xi f)}{\det_V(1-xw\xi f)^*} = \mathrm{Deg}(\alpha)(x)\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-xw\xi f)^*}\,,$$

hence

$$\frac{\omega_\chi(\xi)}{|W|}\sum_{w\in W}\frac{\chi(wf)}{\det_V(1-xw\xi f)^*} = \mathrm{Deg}(\alpha)(x)\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-xw\xi f)^*}\,,$$

from which one deduces formula 4.40.

Φ–Sylow theory and Φ–split Levi sub-reflection data

The Sylow theorems.

Let $\mathbb{T} = (V, wf)$ be a maximal torus. It is easily checked that in this case the polynomial order $|\mathbb{T}|$ is $\det_V(x - wf)$.

Let $\Phi(x) \in K[x]$ be a cyclotomic polynomial. *A Φ–reflection datum* is a torus whose polynomial order is a power of Φ.

The following theorem is 5.1 in [**BMM2**].

4.41. THEOREM. *Let $\mathbb{G}$ be a reflection datum over K and let Φ be a cyclotomic polynomial on K.*

(1) *If Φ divides $|\mathbb{G}|$, there exist non trivial Φ-sub-reflection data of $\mathbb{G}$.*
(2) *Let $\mathbb{S}$ be a maximal Φ-sub-reflection datum of $\mathbb{G}$. Then $|\mathbb{S}| = \Phi^{a(\Phi)}$, contribution of Φ to $|\mathbb{G}|$.*
(3) *Two maximal Φ-sub-reflection data of $\mathbb{G}$ are conjugate under $W_\mathbb{G}$.*
(4) *Let $\mathbb{S}$ be a maximal Φ-sub-reflection datum of $\mathbb{G}$. We set $\mathbb{L} := C_\mathbb{G}(\mathbb{S})$ and $W_\mathbb{G}(\mathbb{L}) := N_{W_\mathbb{G}}(\mathbb{L})/W_\mathbb{L}$. Then*
$$|\mathbb{G}|/(|W_\mathbb{G}(\mathbb{L})||\mathbb{L}|) \equiv 1 \mod \Phi\,.$$

The maximal Φ–sub-reflection data of $\mathbb{G}$ are called the Sylow Φ–sub-reflection data.

EXAMPLE. Let $\mathbb{G} = (V, Wf)$ be the reflection datum attached to the series of Steinberg triality groups ${}^3D_4(q)$. Thus W is the Weyl group of type D_4 and f is a non-trivial automorphism of W of order 3 induced by the inclusion of W into the Weyl group of type F_4. The polynomial order of $\mathbb{G}$ is
$$|\mathbb{G}| = x^{12}(x^2-1)(x^8+x^4+1)(x^6-1) = x^{12}\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_{12}\,.$$
According to the Sylow theorems there exist (maximal) tori of $\mathbb{G}$ of orders respectively $\Phi_3^2, \Phi_6^2, \Phi_{12}$, namely the Sylow d-tori for $d = 3, 6, 12$. Furthermore, there exist Sylow tori of orders Φ_1^2, Φ_2^2, but these tori are not maximal.

Let $\mathbb{G}$ be a generic group and q a prime power. The Sylow Theorem 4.41 also translates to an assertion about Sylow ℓ-subgroups of $\mathbb{G}(q)$ for large primes ℓ not dividing q (see [**BMM2**], Cor. 3.13) :

4.42. PROPOSITION. *Let ℓ be a prime dividing $|\mathbb{G}(q)|$ but not dividing $q|W\langle f\rangle|$.*

(1) *There exists a unique d such that ℓ divides $\Phi_d(q)$ and Φ_d divides $|\mathbb{G}|$.*
(2) *Any Sylow ℓ-subgroup of $\mathbb{G}(q)$ is contained in the group $\mathbb{S}(q)$ for some Sylow d-torus $\mathbb{S}$ of $\mathbb{G}$.*
(3) *The Sylow ℓ-subgroups are isomorphic to a direct product of $a(d)$ cyclic groups of order ℓ^a, where $\Phi_d^{a(d)}$ is the precise power of Φ_d dividing $|\mathbb{G}|$ and where ℓ^a is the precise power of ℓ dividing $\Phi_d(q)$.*

Let us state and comment on the following fundamental supplementary property.

4.43. THEOREM. *Let $\mathbb{S}$ be a maximal Φ-sub-reflection datum of $\mathbb{G}$ and let $\mathbb{L} := C_\mathbb{G}(\mathbb{S})$. If $\mathbb{L} = (V, W_\mathbb{L} wf)$, we set $V(\mathbb{L}, \Phi) := \ker \Phi(wf) \cap V^{W_\mathbb{L}}$, viewed as a vector space over the field $K[x]/\Phi(x)$ through its natural structure of $K[wf]$-module. Then the pair $(V(\mathbb{L}, \Phi), W_\mathbb{G}(\mathbb{L}))$ is a reflection group.*

REMARK. This is a slight reformulation of the main result of [**LeSp**]. It had first been proved by Springer [**Sp**] in the particular case where $\mathbb{L}$ is a torus, *i.e.*, when wf is a regular element (see next paragraph below) and it had been checked case by case for many groups (see [**BMM1**]).

It should be noticed that, at least for W *spetsial* (see [**BMM2**]), 4.43 is a particular case of a more general result involving normalizers of "cuspidal Φ–pairs" (see below 5.7).

4.44. PROPOSITION. *Let $\mathbb{S}$ be a maximal Φ-sub-reflection datum of $\mathbb{G}$, and set $\mathbb{L} = C_{\mathbb{G}}(\mathbb{S})$. Let α be a class function on $\mathbb{G}$. We have:*

$$\mathrm{Deg}_{\mathbb{G}}(\alpha) \equiv \mathrm{Deg}_{\mathbb{L}}(\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}(\alpha)) \mod \Phi\,.$$

In particular, we have

$$\begin{aligned} \varepsilon_{\mathbb{G}} x^{N(\mathbb{G})} &\equiv \varepsilon_{\mathbb{L}} x^{N(\mathbb{L})} \mod \Phi\,, \\ \varepsilon_{\mathbb{G}}^{*} x^{N^{\vee}(\mathbb{G})} &\equiv \varepsilon_{\mathbb{L}}^{*} x^{N^{\vee}(\mathbb{L})} \mod \Phi\,, \\ x^{N(\mathbb{G})+N^{\vee}(\mathbb{G})} &\equiv x^{N(\mathbb{L})+N^{\vee}(\mathbb{L})} \mod \Phi\,. \end{aligned}$$

Φ-split Levi sub-reflection data.

We call *Φ–split Levi sub-reflection data* the centralizers in $\mathbb{G}$ of the Φ–sub-reflection data of $\mathbb{G}$. In particular, the centralizers of the Sylow Φ–sub-reflection data are the minimal Φ–split Levi sub-reflection data. It then follows that the preceding congruence holds for every Φ–split Levi sub-reflection datum of $\mathbb{G}$, from which we deduce by 4.34:

(4.45) For any Φ-split Levi sub-reflection datum $\mathbb{L}$ of $\mathbb{G}$, we have

$$|\mathbb{G}|/|\mathbb{L}| \equiv \mathrm{Deg}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) \mod \Phi\,.$$

Given a Levi sub-reflection datum $(V, W_{\mathbb{L}} wf)$, we define its image in $\mathbb{G}_{\mathrm{ss}}$ (resp. its image in $\mathbb{G}_{\mathrm{ad}}$) to be $(V/V^{W}, W_{\mathbb{L}} wf)$ (resp. $(((V^{\vee})^{W})^{\perp}, W_{\mathbb{L}} wf)$).

4.46. LEMMA. *A Levi sub-reflection datum is Φ-split if and only if its image in $\mathbb{G}_{\mathrm{ss}}$ (resp. in $\mathbb{G}_{\mathrm{ad}}$) is Φ-split.*

Mackey formula for Φ-split sub-reflection data.

The Mackey formula becomes particularly simple when we restrict ourselves to pairs of Φ–split Levi sub-reflection data (see [**BMM2**], 5.7).

4.47. PROPOSITION. *Let $\mathbb{M}_1$, $\mathbb{M}_2$ be Φ-split Levi sub-reflection data of $\mathbb{G}$.*

(1) *$\mathbb{M}_1 \cap \mathbb{M}_2$ is defined if and only if $\mathbb{M}_1$ and $\mathbb{M}_2$ contain a common Sylow Φ-sub-reflection datum of $\mathbb{G}$. In particular $\mathcal{T}_{W_{\mathbb{G}}}(\mathbb{M}_1, \mathbb{M}_2) \neq \emptyset$.*
(2) *If $\mathbb{M}_1 \cap \mathbb{M}_2$ is defined, then it is a Φ-split Levi sub-reflection datum of $\mathbb{G}$.*
(3) *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two Φ-split Levi sub-reflection data containing the minimal Φ-split Levi sub-reflection datum $\mathbb{L}$. Then we have*

$$\mathcal{T}_{W_{\mathbb{G}}}(\mathbb{M}_1, \mathbb{M}_2) = W_{\mathbb{M}_2} N_{W_{\mathbb{G}}}(\mathbb{L}) W_{\mathbb{M}_1}\,.$$

In particular, we have

$$\mathrm{Res}_{\mathbb{M}_2}^{\mathbb{G}} \cdot \mathrm{Ind}_{\mathbb{M}_1}^{\mathbb{G}} = \sum_{w \in W_{\mathbb{M}_2}(\mathbb{L}) \backslash W_{\mathbb{G}}(\mathbb{L}) / W_{\mathbb{M}_1}(\mathbb{L})} \mathrm{Ind}_{\mathbb{M}_2 \cap {}^{w}\mathbb{M}_1}^{\mathbb{M}_2} \cdot \mathrm{Res}_{\mathbb{M}_2 \cap {}^{w}\mathbb{M}_1}^{{}^{w}\mathbb{M}_1} \cdot \mathrm{ad}(w)\,,$$

where, for a Levi sub-reflection datum $\mathbb{L}$ of $\mathbb{G}$, we put $W_{\mathbb{G}}(\mathbb{L}) = N_{W_{\mathbb{G}}}(\mathbb{L})/W_{\mathbb{L}}$.

Regular elements and Sylow theorems.

This section is taken from [**BMM2**], §5B.

Let $\mathbb{G} = (V, Wf)$ be a reflection datum on K. Let K' be an extension of K which splits $\mathbb{G}$ and $V' := K' \otimes V$.

The following definition extend (cf. [**Sp**]) the definition of regular elements given in §1 above.

DEFINITION.

- We say that an element wf in the coset Wf is regular if it has a regular eigenvector.
- Moreover, if wf has a regular eigenvector for the eigenvalue ζ, we say that wf is ζ–regular.
- Note that if wf is ζ–regular, it is also ζ'–regular for any root ζ' of the minimal polynomial Φ of ζ over K. We then say that wf is Φ–regular.

For $wf \in Wf$ and ζ a root of unity in K', we denote by $V'(wf, \zeta)$ the eigenspace of wf corresponding to the eigenvalue ζ.

4.48. PROPOSITION. *Let ζ be a root of unity in K' and let Φ be its minimal polynomial over K. Let $\mathbb{G}$ be a reflection datum on K, and let $\Phi^{a(\Phi)}$ be the largest power of Φ which divides $|\mathbb{G}|$. For $wf \in Wf$, we set $\mathbb{S}(wf, \Phi) := (\ker(\Phi(wf)), (wf)_{|\ker(\Phi(wf))})$ (a Φ–sub-reflection datum of $\mathbb{G}$). Let $\mathbb{L}(wf, \Phi)$ denote its centralizer. The following assertions are equivalent.*

(i) *wf is ζ–regular,*
(ii) *$C_W(V'(wf, \zeta)) = \{1\}$,*
(iii) *$\mathbb{L}(wf, \Phi)$ is a torus,*
(iv) *for $w'f \in Wf$, $\Phi^{a(\Phi)}$ divides the characteristic polynomial of $w'f$ if and only if $w'f$ is W–conjugate to wf.*

4.49. LEMMA. *If a reflection datum $\mathbb{G} = (V, Wf)$ is not a torus (i.e., if $W \neq 1$), it has two maximal tori with different order (in other words, there are two elements wf and $w'f$ in Wf with different characteristic polynomials).*

Indeed, let $|\mathbb{G}| = x^{N(\mathbb{G})} \prod_\zeta (x - \zeta)^{a(\zeta)}$ be the factorization of the polynomial order $|\mathbb{G}|$ over $K'[x]$. By Sylow theorems (4.41) applied to the reflection datum defined over K', for each root of unity ζ there is $wf \in Wf$ whose characteristic polynomial is divisible by $(x - \zeta)^{a(\zeta)}$. Hence it suffices to check that, if W is not trivial, we have $\sum_\zeta a(\zeta) > \dim(V)$. Indeed, we have $\sum_\zeta a(\zeta) = \deg |\mathbb{G}| - N(\mathbb{G})$, *i.e.*, by 4.6, (2), $\sum_\zeta a(\zeta) = N^\vee(\mathbb{G}) + r$. □

We say that a cyclotomic polynomial Φ on K is *regular* for $\mathbb{G}$ if there is a Φ–regular element in Wf.

4.50. COROLLARY. *Let Φ be a cyclotomic polynomial on K. Let $\mathbb{G}$ be a reflection datum on K, and let $\Phi^{a(\Phi)}$ be the largest power of Φ which divides $|\mathbb{G}|$. The following assertions are equivalent:*

(i) *Φ is regular for $\mathbb{G}$,*
(ii) *the centralizer of a Sylow Φ–sub-reflection datum of $\mathbb{G}$ is a torus,*
(iii) *there is only one conjugacy class under W of maximal tori of $\mathbb{G}$ whose order is divisible by $\Phi^{a(\Phi)}$.*

It is an easy consequence of 4.48 and the fact (see the proof of 4.41 above) that any Sylow Φ–sub-reflection datum is of the form $\mathbb{S}(wf,\Phi)$.

Degrees and regular elements.

Let α be a class function on $\mathbb{G}$. Let wf be Φ–regular, so that the associated maximal toric sub-reflection datum $\mathbb{T}_{wf}$ of $\mathbb{G}$ is a Sylow Φ–sub-reflection datum by 4.48. Then the congruence given in 4.44 becomes

$$\mathrm{Deg}(R_\alpha) \equiv \alpha(wf) \quad \mathrm{mod}\ \Phi\,,$$

which can be reformulated into the following proposition.

4.51. PROPOSITION. *If wf is ζ–regular for some root of unity ζ, for any $\alpha \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$ we have*

$$\mathrm{Deg}(R_\alpha)(\zeta) = \alpha(wf)\,.$$

CHAPTER V

GENRALIZED HARISH–CHANDRA THEORIES

Generic unipotent characters

In the previous chapters we have introduced the concept of reflection datum to capture much of the generic nature of the subgroup structure and of uniform functions of finite reductive groups. We now turn to the description of the generic behaviour of irreducible characters.

From now on, we consider a reflection datum $\mathbb{G} = (V, Wf)$ which is associated with a finite reductive group $(\mathbf{G}, F)$.

An irreducible character $\gamma \in \mathrm{Irr}(\mathbf{G}^F)$ is called *unipotent* if there exists an F-stable maximal torus $\mathbf{S} \leq \mathbf{G}$ such that γ is a constituent of the Deligne–Lusztig induced character $R_{\mathbf{S}}^{\mathbf{G}}(1)$. Let $\mathrm{Uch}(\mathbf{G}^F)$ denote the set of unipotent characters of $\mathbf{G}^F$.

For any F-stable Levi subgroup $\mathbf{L} \leq \mathbf{G}$ Deligne and Lusztig (cf. for example [**Lu6**]) defined functors

$$R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\mathrm{Irr}(\mathbf{L}^F) \to \mathbb{Z}\mathrm{Irr}(\mathbf{G}^F)\,, \quad {}^*R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\mathrm{Irr}(\mathbf{G}^F) \to \mathbb{Z}\mathrm{Irr}(\mathbf{L}^F)\,,$$

between the character groups of $\mathbf{L}^F$ and $\mathbf{G}^F$, adjoint to each other with respect to the usual scalar product of characters.

More precisely, the definition of these functors also involves the choice of a parabolic subgroup $\mathbf{P}$ containing $\mathbf{L}$, but in our situation it turns out that they are in fact independent of this choice (see below §).

The results of Lusztig on unipotent characters (cf. for example [**Lu3**]), completed by some results of Shoji (see [**BMM1**], Ths. 1.26 and 1.33) may be rephrased as follows.

5.1. THEOREM. *There exists a set* $\mathrm{Uch}(\mathbb{G})$ *and a map*

$$\mathrm{Deg} : \mathrm{Uch}(\mathbb{G}) \to \mathbb{Q}[x], \quad \gamma \mapsto \mathrm{Deg}(\gamma)\,,$$

such that for any choice of p and q (and hence of $\mathbf{G}$ and F) there is a bijection

$$\psi_q^{\mathbb{G}} : \mathrm{Uch}(\mathbb{G}) \xrightarrow{\sim} \mathrm{Uch}(\mathbf{G}^F)$$

such that

(1) *The degree of $\psi_q^{\mathbb{G}}(\gamma)$ is $\psi_q^{\mathbb{G}}(\gamma)(1) = \mathrm{Deg}(\gamma)(q)$.*

(2) *For any generic Levi subgroup $\mathbb{L}$ of $\mathbb{G}$ there exist linear maps*

$$R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathrm{Uch}(\mathbb{L}) \to \mathbb{Z}\mathrm{Uch}(\mathbb{G}), \quad {}^*R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathrm{Uch}(\mathbb{G}) \to \mathbb{Z}\mathrm{Uch}(\mathbb{L})\,,$$

satisfying $\psi_q^{\mathbb{G}} R_{\mathbb{L}}^{\mathbb{G}} = R_{\mathbf{L}}^{\mathbf{G}} \psi_q^{\mathbb{L}}$ for all q (we extend $\psi_q^{\mathbb{G}}$ linearly to $\mathbb{Z}\mathrm{Uch}(\mathbb{G})$).

The set $\mathrm{Uch}(\mathbb{G})$ *is called the set of (generic) unipotent characters of* $\mathbb{G}$.

In this sense, the sets of unipotent characters together with the collection of functors $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ are generic for a series of finite reductive groups. of Lie type.

EXAMPLE. In the case of the finite reductive group $\mathbf{G}^F = \mathrm{GL}_n(q)$ the unipotent characters are just the constituents of the permutation character $\mathrm{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}$ on the F-fixed points of an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. The endomorphism algebra of this permutation module is the Hecke algebra of type A_{n-1} with parameter q. Its irreducible characters are in bijection with the irreducible characters of the symmetric group $\mathfrak{S}_n$, the Weyl group of $\mathbf{G}^F$. Since the latter are naturally indexed by partitions $\alpha \vdash n$, the same is true for the unipotent characters of $\mathbf{G}^F$. Hence in this case we have $\mathrm{Uch}(\mathrm{GL}_n) = \{\gamma_\alpha \mid \alpha \vdash n\}$. The function $\mathrm{Deg} : \mathrm{Uch}(\mathrm{GL}_n) \to \mathbb{Q}[x]$ can be described as follows: for a partition $\alpha = (\alpha_1 \leq \ldots \leq \alpha_m)$ of n let $\beta_i := \alpha_i + i - 1$, $1 \leq i \leq m$. Then

$$\mathrm{Deg}(\gamma_\alpha) = \frac{(x-1)\ldots(x^n-1)\prod_{j>i}(x^{\beta_j} - x^{\beta_i})}{x^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots}\prod_i \prod_{j=1}^{\beta_i}(x^j-1)}\,.$$

(This always is a polynomial.) Furthermore, there exists a bijection

$$\mathrm{Uch}(\mathrm{GL}_n) \to \mathrm{Uch}(\mathrm{U}_n), \quad \gamma \mapsto \gamma^-\,,$$

such that $\mathrm{Deg}(\gamma^-)(x) = \pm\mathrm{Deg}(\gamma)(-x)$. This is another consequence of the Ennola duality between GL_n and U_n.

Φ-Harish-Chandra theories

The idea that a generalized Harish-Chandra theory should exist for the unipotent characters of a finite reductive group was implicit in many instances of Lusztig's papers, and occurred also in the papers by Fong–Srinivasan [**FoSr**j] ($j = 1, 2, 3$) and Schewe [**Sch**] in particular cases. The general case was settled in [**BMM1**].

Φ-cuspidal characters.

Let $\mathbb{G}$ be a reflection datum and Φ a K–cyclotomic polynomial such that Φ divides $|\mathbb{G}|$. We first have to introduce a generalization of cuspidal characters.

DEFINITION. A generic unipotent character $\gamma \in \mathrm{Uch}(\mathbb{G})$ is called *Φ-cuspidal* if ${}^*R_{\mathbb{L}}^{\mathbb{G}}(\gamma) = 0$ for all Φ-split Levi subgroups $\mathbb{L}$ properly contained in $\mathbb{G}$.

Thus in the case $\Phi = x - 1$ the Φ-cuspidal characters of $\mathbb{G}$ are those whose image under $\psi_q^{\mathbb{G}}$ is a cuspidal unipotent character as defined in the ordinary Harish–Chandra theory. We have the following alternative characterization of Φ-cuspidality (see [**BMM1**], Prop. 2.9):

5.2. PROPOSITION. *A unipotent character* $\gamma \in \mathrm{Uch}(\mathbb{G})$ *is Φ-cuspidal if and only if*

$$\mathrm{Deg}(\gamma)_\Phi = |\mathbb{G}_{\mathrm{ss}}|_\Phi\,.$$

Here, we write f_Φ for the Φ-part of $f \in \mathbb{Q}[x]$. The semisimple quotient $\mathbb{G}_{\mathrm{ss}}$ of $\mathbb{G}$ was defined above (chap. iv).

EXAMPLE. We continue the example of GL_n (see above). We assume here that $\Phi = \Phi_d$, the d-th cyclotomic polynomial. Let $\alpha = (\alpha_1 \leq \ldots \leq \alpha_m)$ be a partition of n. A *d-hook of α* is a pair $h = (\nu, \beta)$ of integers $0 \leq \nu < \beta$ such that β occurs among the $\{\alpha_i + i - 1 \mid 1 \leq i \leq m\}$ but ν does not. The *length* of the hook h is $l(h) = \beta - \nu$ and h is also called an $l(h)$-hook. A moments thought shows that up to a power of x the degree given above of the unipotent character γ_α of the generic group of type GL_n is of the form

$$\frac{(x-1)\ldots(x^n-1)}{\prod_{h \text{ hook of } \alpha}(x^{l(h)}-1)}.$$

Note that the existence of an ad-hook of α for some $a \geq 1$ implies the existence of a d-hook. Thus, by Proposition 5.2 the unipotent character γ_α is Φ_d-cuspidal if and only if α has no d-hook. Such partitions α are also called *d-cores.* In particular, GL_n has a 1-cuspidal unipotent character if and only if $n = 0$, since the empty partition is the only 1-core. This shows that indeed all unipotent characters of GL_n occur in $R_{\mathbb{S}}^{\mathbb{G}}(1)$ for the maximally split torus $\mathbb{S}$.

Φ-Harish-Chandra series.

A pair $(\mathbb{L}, \lambda)$ consisting of a Φ-split Levi subgroup $\mathbb{L}$ of $\mathbb{G}$ and a unipotent character $\lambda \in \mathrm{Uch}(\mathbb{L})$ is called *Φ-split.* It is called *Φ-cuspidal* if moreover λ is Φ-cuspidal. We introduce the following relation on the set of Φ-split pairs:

DEFINITION. Let $(\mathbb{M}_1, \mu_1)$ and $(\mathbb{M}_2, \mu_2)$ be Φ-split in $\mathbb{G}$. Then we say that $(\mathbb{M}_1, \mu_1) \leq_\Phi (\mathbb{M}_2, \mu_2)$ if $\mathbb{M}_1$ is a Φ-split Levi subgroup of $\mathbb{M}_2$ and μ_2 occurs in $R_{\mathbb{M}_1}^{\mathbb{M}_2}(\mu_1)$.

For a Φ-cuspidal pair $(\mathbb{L}, \lambda)$ of $\mathbb{G}$ we write

$$\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \lambda)) := \{\gamma \in \mathrm{Uch}(\mathbb{G}) \mid (\mathbb{L}, \lambda) \leq_\Phi (\mathbb{G}, \gamma)\}$$

for the set of unipotent characters of $\mathbb{G}$ lying above $(\mathbb{L}, \lambda)$. We call $\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \lambda))$ the *Φ-Harish-Chandra series above* $(\mathbb{L}, \lambda)$ because of the following fundamental result, which shows that for any Φ we obtain a generalized Harish-Chandra theory.

5.3. THEOREM. *Let $\mathbb{G}$ be a reflection datum and $d \geq 1$ such that Φ divides* $|\mathbb{G}|$.

(1) (Disjointness) *The sets* $\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \lambda))$ *(where $(\mathbb{L}, \lambda)$ runs over a system of representatives of the $W_{\mathbb{G}}$-conjugacy classes of Φ-cuspidal pairs) form a partition of* $\mathrm{Uch}(\mathbb{G})$.

(2) (Transitivity) *Let $(\mathbb{L}, \lambda)$ be Φ-cuspidal and $(\mathbb{M}, \mu)$ be Φ-split. We assume that $(\mathbb{L}, \lambda) \leq_\Phi (\mathbb{M}, \mu)$ and $(\mathbb{M}, \mu) \leq_\Phi (\mathbb{G}, \gamma)$. Then $(\mathbb{L}, \lambda) \leq_\Phi (\mathbb{G}, \gamma)$.*

The statement of Theorem 5.3 has been checked case by case (except for the case $d = 1$ where a conceptual proof is known). It is a consequence of the more precise Theorem 5.6 about the decomposition of twisted induction of unipotent characters from Φ-split Levi subgroups.

EXAMPLE. We continue the example of GL_n, where we saw that the Φ_d-cuspidal characters of GL_n are indexed by partitions of n which are d-cores. Let $h = (\nu, \beta)$ be a d-hook of the partition $\alpha = (\alpha_1 \leq \ldots \leq \alpha_m)$, and let j be the index with $\alpha_j + j - 1 = \beta$. Then $\alpha' = (\alpha'_1 \leq \ldots \leq \alpha'_m)$ is called the partition obtained from α by *removing the d-hook h* if the set $\{\alpha'_i + i - 1\}$ coincides with the set

$$\{\alpha_i + i - 1 \mid i \neq j\} \cup \{\alpha_j + j - 1 - (\beta - \nu)\}.$$

The d-core obtained from α by successively removing all possible d-hooks is called the *d-core of α*. It can be shown that the Φ_d-Harish-Chandra series of GL_n above the Φ_d-cuspidal character indexed by the d-core α consists of the unipotent characters indexed by partitions of n whose d-core is α.

Generic blocks

One importance of Φ-Harish-Chandra series lies in their connection with ℓ-blocks of finite groups of Lie type for primes ℓ not dividing q. Before continuing the exposition of the generic theory we therefore briefly explain this application.

We fix $\mathbb{G}$ and a choice of a prime power q, hence a pair $(\mathbf{G}, F)$. A prime ℓ not dividing q is called *large for* $\mathbb{G}$ if ℓ does not divide the order of $W\langle f\rangle$. If ℓ is large then by Proposition4.42(a) there exists a unique Φ such that ℓ divides $\Phi(q)$ and Φ divides $|\mathbb{G}|$.

The following result is proved in [**BMM1**], Th. 5.24.

5.4. THEOREM. *Let ℓ be large for $\mathbb{G}$, and assume that ℓ divides $\Phi(q)$ and Φ divides $|\mathbb{G}|$. Then the partition of* $\mathrm{Uch}(\mathbf{G}^F)$ *into ℓ-blocks coincides with the image under $\psi_q^{\mathbb{G}}$ of the partition of* $\mathrm{Uch}(\mathbb{G})$ *into Φ-Harish-Chandra series.*

Thus the preceding theorem shows that distribution of unipotent characters of $\mathbf{G}^F$ into ℓ-blocks is generic.

Let us write $b_\ell(\mathbf{L}, \lambda)$ for the unipotent ℓ-block of $\mathbf{G}^F$ indexed by $(\mathbb{L}, \lambda)$. The unipotent blocks and their defect groups can be described more precisely.

DEFINITION. For $\gamma \in \mathrm{Uch}(\mathbb{G})$ let $S_\Phi(\gamma)$ denote the set of Φ-tori of $\mathbb{G}$ contained in a maximal torus $\mathbb{S}$ of $\mathbb{G}$ such that ${}^*R_{\mathbb{S}}^{\mathbb{G}}(\gamma) \neq 0$. The maximal elements of $S_\Phi(\gamma)$ are called the *Φ-defect tori of γ*.

The Φ-defect tori of a unipotent character can be characterized as follows (see [**BMM1**], Th. 4.8).

PROPOSITION. *Let $(\mathbb{L}, \lambda)$ be a Φ-cuspidal pair and $\gamma \in \mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \lambda))$. Then the Φ-defect tori of γ are conjugate to* $\mathrm{Rad}(\mathbb{L})_\Phi$.

Here $\mathrm{Rad}(\mathbb{L})_\Phi$ denotes the Sylow Φ-torus of the torus $\mathrm{Rad}(\mathbb{L})$. For an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ denote by $\mathrm{Ab}_\ell\mathrm{Irr}(\mathbf{L}^F)$ the group of characters (over a splitting field of characteristic 0) of ℓ-power order of the abelian group $\mathbf{L}^F/[\mathbf{L}^F, \mathbf{L}^F]$. Then [**BMM1**], Th. 5.24, gives the following sharpening of Theorem 5.4:

5.5. THEOREM. *Let ℓ be large for $\mathbb{G}$, and assume that ℓ divides $\Phi(q)$ and Φ divides $|\mathbb{G}|$. Let $(\mathbb{L}, \lambda)$ be a Φ-cuspidal pair of $\mathbb{G}$.*

(1) *The ℓ-block $b_\ell(\mathbf{L}, \lambda)$ of $\mathbf{G}^F$ consists of the irreducible constituents of the virtual characters $R_{\mathbf{L}}^{\mathbf{G}}(\theta\lambda)$, where $\theta \in \mathrm{Ab}_\ell\mathrm{Irr}(\mathbf{L}^F)$.*
(2) *The defect groups of $b_\ell(\mathbf{L}, \lambda)$ are the Sylow ℓ-subgroups of the groups of F-fixed points of the Φ-defect tori of unipotent characters in $b_\ell(\mathbf{L}, \lambda)$.*

The structure of the Sylow ℓ-subgroup of the group of F-fixed points of a Φ-torus was described in Proposition 4.42(c).

EXAMPLE. Theorem 5.5 and the description of Φ-Harish-Chandra series for GL_n in examples above provide the following description (first proved by Fong and Srinivasan) of unipotent ℓ-blocks of $\mathrm{GL}_n(q)$ for large primes ℓ dividing $\Phi_d(q)$. Two unipotent characters γ_α, $\gamma_{\alpha'}$ lie in the same ℓ-block if and only if the d-cores of α and α' coincide.

Relative Weyl groups

The Φ-Harish-Chandra theories presented above seem to be just the shadow of a much deeper theory describing the decomposition of the functor of twisted induction. Much of this is still conjectural (see below chap. VI).

The decomposition of $R_{\mathbb{L}}^{\mathbb{G}}$.

The only known proof of Theorem 5.3 in the case $\Phi \neq x-1$ consists in determining the decomposition of the Deligne-Lusztig induced of Φ-cuspidal unipotent characters. To state this result we need to introduce an important invariant of a Φ-Harish-Chandra series.

Let $(\mathbb{L}, \lambda)$ be a Φ-cuspidal pair in $\mathbb{G}$. Let $(\mathbf{G}, F)$ be a finite reductive group associated to $\mathbb{G}$, and let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ corresponding to $\mathbb{L}$. Then the relative Weyl group $W_{\mathbf{G}^F}(\mathbf{L}) = N_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F$ acts on $\mathrm{Irr}(\mathbf{L}^F)$. By results of Lusztig, this leaves the subset $\mathrm{Uch}(\mathbf{L}^F)$ of unipotent characters invariant. Moreover, the action on $\mathrm{Uch}(\mathbf{L}^F)$ is generic in the sense that it is possible to define an action of $W_{\mathbb{G}}(\mathbb{L}) = N_{W_{\mathbb{G}}}(W_{\mathbb{L}})/W_{\mathbb{L}}$ on $\mathrm{Uch}(\mathbb{L})$ which, under all $\psi_q^{\mathbb{G}}$, specializes to the action of $W_{\mathbf{G}^F}(\mathbf{L})$ on $\mathrm{Uch}(\mathbf{L}^F)$ (see [**BMM1**]). This gives sense to the definition

$$W_{\mathbb{G}}(\mathbb{L}, \lambda) := N_{W_{\mathbb{G}}}(W_{\mathbb{L}}, \lambda)/W_{\mathbb{L}}$$

of the *relative Weyl group of* $(\mathbb{L}, \lambda)$ *in* $\mathbb{G}$.

The following result was proved in [**BMM1**] (Th. 3.2) by using results of Asai on the decomposition of $R_{\mathbf{L}}^{\mathbf{G}}$ in the case of classical groups, and by explicit determination of these decompositions in the case of exceptional groups.

5.6. THEOREM. *For each Φ there exists a collection of isometries*

$$I_{(\mathbb{L},\lambda)}^{\mathbb{M}} : \mathbb{Z}\mathrm{Irr}(W_{\mathbb{M}}(\mathbb{L}, \lambda)) \to \mathbb{Z}\mathrm{Uch}(\mathbb{M}, (\mathbb{L}, \lambda)),$$

(where $\mathbb{M}$ runs over the Φ-split Levi subgroups of $\mathbb{G}$, and $(\mathbb{L}, \lambda)$ over the set of Φ-cuspidal pairs of $\mathbb{M}$) such that for all $\mathbb{M}$ and all $(\mathbb{L}, \lambda)$ we have

$$R_{\mathbb{M}}^{\mathbb{G}} I_{(\mathbb{L},\lambda)}^{\mathbb{M}} = I_{(\mathbb{L},\lambda)}^{\mathbb{G}} \mathrm{Ind}_{W_{\mathbb{M}}(\mathbb{L},\lambda)}^{W_{\mathbb{G}}(\mathbb{L},\lambda)}.$$

An isometry I from $\mathbb{Z}\mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L}, \lambda))$ to $\mathbb{Z}\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \lambda))$ is nothing else but a bijection

$$\mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L}, \lambda)) \xrightarrow{\sim} \mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \lambda)), \quad \chi \mapsto \gamma_{\mathrm{ch}},$$

together with a collection of signs

$$\{\epsilon(\gamma) \mid \gamma \in \mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \lambda))\},$$

such that

$$I(\chi) = \epsilon(\gamma_\chi)\, \gamma_\chi .$$

Thus, Theorem 5.6 states that up to an adjustment by suitable signs, twisted induction from Φ-split Levi subgroups is nothing but ordinary induction in relative

Weyl groups. As a consequence of [**BMM1**], Th. 5.24, we moreover have the following congruence of character degrees

$$\epsilon(\gamma_\chi)\operatorname{Deg}(\gamma_\chi) \equiv \chi(1) \pmod{\Phi} \quad \text{in } \mathbb{Q}[x]$$

in the situation of Theorem 5.6.

EXAMPLE. We continue the example of reflection datum of type ${}^3D_4$. The relative Weyl groups $W_{\mathbb{G}}(\mathbb{S}_\Phi)$ for the Sylow Φ-tori $\mathbb{S}_\Phi$ with $d \in \{3,6\}$ turn out to be isomorphic to $\mathrm{SL}_2(3)$. Since $\mathrm{SL}_2(3)$ has 7 irreducible characters Theorem 5.6 implies that $\mathrm{Uch}(\mathbb{G}, (\mathbb{S}_\Phi, 1))$ has cardinality 7 for $d \in \{3,6\}$. Moreover, the vector of degrees of the irreducible characters of $\mathrm{SL}_2(3)$ is $(1,1,1,2,2,2,3)$, so $R_{\mathbb{S}_\Phi}^{\mathbb{G}}(1)$ contains three constituents with multiplicity ± 1, three with multiplicity ± 2 and one with multiplicity ± 3.

Relative Weyl groups are pseudo-reflection groups.

Theorem 5.6 turns attention towards the relative Weyl groups of Φ-cuspidal pairs $(\mathbb{L}, \lambda)$. Lusztig proved that in the case $d = 1$ the relative Weyl groups are Coxeter groups.

If $\mathbb{L} = (V, W_{\mathbb{L}} wf)$, we denote by $Z(\mathbb{L})_\Phi$ the Sylow Φ–sub-reflection datum of the center of $\mathbb{L}$, defined by

$$Z(\mathbb{L})_\Phi = (\ker \Phi(wf) \cap V^{W_{\mathbb{L}}}, wf)\,.$$

In particular, we denote by $V(\mathbb{L}, \Phi)$ its vector space $\ker \Phi(wf) \cap V^{W_{\mathbb{L}}}$, viewed as a vector space over the field $K[x]/\Phi(x)$ through its natural structure of $K[wf]$–module.

Let us set

$$K_{\mathbb{G}}(\mathbb{L}, \lambda) := K[x]/\Phi(x)\,.$$

In a case-by-case analysis the following surprising fact can be verified (see [**BrMa2**]). It is a generalization, in this context, of the main result of [**LeSp2**], which concerns the case where $\mathbb{L}$ is a minimal split Levi reflection datum and λ is the trivial character.

5.7. THEOREM.
(1) *The pair* $(V(\mathbb{L}, \Phi), W_{\mathbb{G}}(\mathbb{L}, \lambda))$ *is a* $K_{\mathbb{G}}(\mathbb{L}, \lambda)$*-complex reflection group.*
(2) *The* $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ *is irreducible on* $V(\mathbb{L}, \Phi)$ *if* W *is irreducible on* V.

EXAMPLE. Let $\mathbb{G}$ be a generic group of type E_7 and let $d = 4$. A Sylow 4-torus $\mathbb{S}$ of $\mathbb{G}$ has order Φ_4^2 and its centralizer is a 4-split Levi subgroup $\mathbb{L} = C_{\mathbb{G}}(\mathbb{S})$ with semisimple part $(\mathrm{PGL}_2)^3$. Since $\mathbb{L}$ is minimal 4-split all its unipotent characters are 4-cuspidal. The relative Weyl group $W_{\mathbb{G}}(\mathbb{L})$ is a two-dimensional complex reflection group of order 96, denoted G_8 by Shephard and Todd. It has two orbits of length 3 and two fixed points on the set $\mathrm{Uch}(\mathbb{L})$. In particular, the relative Weyl group $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ for a (4-cuspidal) character λ in one of the orbits of length 3 is strictly smaller than $W_{\mathbb{G}}(\mathbb{L})$. It turns out to be the imprimitive complex reflection group denoted $G(4,1,2)$ of order 32.

The relative Weyl groups occurring in reflection data of exceptional type are collected in [**BMM1**], Tables 1 and 3, while those for classical types are described in [**BMM1**], Sec. 3, see also [**BrMa2**], 3B.

CHAPTER VI

THE ABELIAN DEFECT GROUP CONJECTURE FOR FINITE REDUCTIVE GROUPS AND SOME CONSEQUENCES

The origin of the following conjectures is a general conjecture about "abstract" finite groups (see [**Bro1**]), which concerns the structure (up to equivalence) of the derived bounded category of the ℓ–adic algebra of a block of any finite group with an abelian defect group. This "abstract" conjecture inspired more precise guesses in the particular case of finite reductive groups, which first appeared in [**Bro1**] (last paragraph). These conjectures have also been partly stated at the conference held in honor of Charlie Curtis at the University of Oregon in September 1991.

Notation.

Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraic closure $\overline{\mathbb{F}}_p$ of the prime field with p elements. Let q be a power of p and let $\mathbb{F}_q$ be the subfield of cardinal q of $\overline{\mathbb{F}}_p$. We assume that $\mathbf{G}$ is endowed with a Frobenius endomorphism F which defines a rational structure on $\mathbb{F}_q$. We denote by $2N$ the number of roots of $\mathbf{G}$.

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, with unipotent radical $\mathbf{U}$, and with F–stable Levi complement $\mathbf{L}$.

We denote by $\mathrm{Y}(\mathbf{U})$ the associated Deligne–Lusztig variety defined (cf. for example [**Lu6**]) by

$$\mathrm{Y}(\mathbf{U}) := \{g(\mathbf{U}\cap F(\mathbf{U})) \in \mathbf{G}/\mathbf{U}\cap F(\mathbf{U})\,;\, g^{-1}F(g) \in F(\mathbf{U})\}\,.$$

Notice that $\mathbf{G}^F$ acts on $\mathrm{Y}(\mathbf{U})$ by left multiplication while $\mathbf{L}^F$ acts on $\mathrm{Y}(\mathbf{U})$ by right multiplication.

It is known (cf. for example [**Lu6**]) that $\mathrm{Y}(\mathbf{U})$ is an $\mathbf{L}^F$–torsor on a variety $\mathrm{X}(\mathbf{U})$, which is smooth of pure dimension equal to $\dim(\mathbf{U}/\mathbf{U}\cap F(\mathbf{U}))$, and which is affine (at least if q is large enough). In particular $\mathrm{X}(\mathbf{U})$ is endowed with a left action of $\mathbf{G}^F$. If $\mathcal{O}$ is a commutative ring, the image of the constant sheaf $\mathcal{O}$ on $\mathrm{Y}(\mathbf{U})$ through the finite morphism $\pi\colon \mathrm{Y}(\mathbf{U}) \to \mathrm{X}(\mathbf{U})$ is a locally constant constant sheaf $\pi_*(\mathcal{O})$ on $\mathrm{X}(\mathbf{U})$. We denote this sheaf by $\mathcal{F}_{\mathcal{O}\mathbf{L}^F}$.

In the particular case where $\mathbf{B} = \mathbf{TU}$ is a Borel subgroup such that $\mathbf{B}$ and $F(\mathbf{B})$ are in relative position w for some $w \in W$ (cf. [**DeLu**], 1.2), we set $\mathrm{X}_w := \mathrm{X}(\mathbf{U})$.

Let ℓ be a prime number which does not divide q and let $\mathcal{O}$ be the ring of integers of a finite extension of the field $\mathbb{Q}_\ell$ of ℓ–adic numbers. For any $\mathbf{G}^F$–equivariant torsion free $\mathcal{O}$–sheaf $\mathcal{F}$ on $\mathrm{X}(\mathbf{U})$, we denote by $\mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}),\mathcal{F})$ the algebra of endomorphisms of the "ℓ–adic cohomology" complex $\mathrm{R}\Gamma_c(\mathrm{X}(\mathbf{U}),\mathcal{F})$ viewed as an element of the derived bounded category $\mathcal{D}^b(\mathcal{O}\mathbf{G}^F)$ of the category of finitely generated $\mathcal{O}\mathbf{G}^F$–modules.

We set

$$\mathrm{R}\Gamma_c(\mathrm{Y}(\mathbf{U})) := \mathrm{R}\Gamma_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F}) \quad \text{and} \quad \mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U})) := \mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F}) \,.$$

Note that the algebra $\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))$ contains the group algebra $\mathcal{O}\mathbf{L}^F$ as a subalgebra.

For K an extension of $\mathcal{O}$, we set $\mathcal{H}_{\mathrm{K}}(\mathrm{X}(\mathbf{U}), \mathcal{F}) := \mathrm{K} \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}), \mathcal{F})$.

Finally, if $\mathbf{\Gamma} = (\cdots \to \mathbf{\Gamma}^{n-1} \to \mathbf{\Gamma}^n \to \mathbf{\Gamma}^{n+1} \to \cdots)$ is a complex of $\mathcal{O}$–modules, we denote by $\mathbf{\Gamma}^\vee$ the "ordinary" $\mathcal{O}$–dual of $\mathbf{\Gamma}$: we have $\mathbf{\Gamma}^{\vee m} := \mathrm{Hom}_{\mathcal{O}}(\mathbf{\Gamma}^{-m}, \mathcal{O})$, and the differentials are defined by $\mathcal{O}$–transposition.

Conjectures about ℓ–blocks

The data.

(H1) Let $\ell \neq p$ be a prime number which does not divide $|Z(\mathbf{G})/Z^o(\mathbf{G})|$ nor $|Z(\mathbf{G}^*)/Z^o(\mathbf{G}^*)|$, and which is good for $\mathbf{G}$.

(H2) Let $\mathcal{O}$ be the ring of integers of a finite unramified extension K of the field of ℓ–adic numbers $\mathbb{Q}_\ell$, with residue field k, such that the finite group algebra $\mathrm{k}\mathbf{G}^F$ is split.

(H3) Let e be a primitive central idempotent of $\mathcal{O}\mathbf{G}^F$ (an "ℓ–block" of $\mathbf{G}^F$) with abelian defect group D. Let $\mathbf{L} := C_{\mathbf{G}}(D)$. Let f be a block of $\mathbf{L}^F$ such that (D, f) is an e–subpair of $\mathbf{G}^F$.

It results from (H1) that the group $\mathbf{L}$ is a rational Levi subgroup of $\mathbf{G}$.

We have $N_{\mathbf{G}^F}(D, f) = N_{\mathbf{G}^F}(\mathbf{L}, f)$, and we set $W_{\mathbf{G}^F}(\mathbf{L}, f) := N_{\mathbf{G}^F}(\mathbf{L}, f)/\mathbf{L}^F$. Note that by known properties of maximal subpairs (see for example [**AlBr**]), D is a Sylow ℓ–subgroup of $Z(\mathbf{L})^F$, and ℓ does not divide $|W_{\mathbf{G}^F}(\mathbf{L}, f)|$.

ℓ–Conjectures.

There exist

• a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and Levi complement $\mathbf{L}$,

• a finite complex $\mathbf{\Upsilon} = (\cdots \to \mathbf{\Upsilon}^{n-1} \to \mathbf{\Upsilon}^n \to \mathbf{\Upsilon}^{n+1} \to \cdots)$ of $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$–bimodules, which are finitely generated projective as $\mathcal{O}\mathbf{G}^F$–modules as well as $\mathcal{O}\mathbf{L}^F$–modules,

with the following properties.

(ℓ-C1) Viewed as an object of the derived bounded category of the category of $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$–bimodules, $\mathbf{\Upsilon}$ is isomorphic to $\mathrm{R}\Gamma_c(\mathrm{Y}(\mathbf{U}))$. In particular, for each n, the n-th homology group of $\mathbf{\Upsilon}$ is isomorphic, as an $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$–bimodule, to $\mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathrm{H}_c^n(\mathrm{Y}(\mathbf{U}), \mathbb{Z}_\ell)$.

(ℓ-C2) The idempotent e acts as the identity on the complex $\mathbf{\Upsilon}.f$,

(ℓ-C3) Let $\Delta(D)$ denote the diagonal embedding of D in $\mathbf{G}^F \times \mathbf{L}^F$. For each n, the $\mathcal{O}[\mathbf{G}^F \times \mathbf{L}^F]$–module $\mathbf{\Upsilon}^n.f$ is relatively $\Delta(D)$–projective, and its restriction to $\mathcal{O}\Delta(D)$ is a permutation module for $\Delta(D)$.

(ℓ-C4) • the structure of complex of $(\mathcal{O}\mathbf{G}^F e, \mathcal{O}\mathbf{L}^F f)$–bimodules of $\mathbf{\Upsilon}.f$ extends to a structure of complex of $(\mathcal{O}\mathbf{G}^F e, f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f)$–bimodules, all of which are projective as right $f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f$–modules,

• the complexes $(\mathbf{\Upsilon}.f \otimes_{f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f} f.\mathbf{\Upsilon}^\vee)$ and $\mathcal{O}\mathbf{G}^F e$ are homotopy equivalent as complexes of $(\mathcal{O}\mathbf{G}^F e, \mathcal{O}\mathbf{G}^F e)$–bimodules,

• the complexes $(f.\mathbf{\Upsilon}^\vee \otimes_{\mathcal{O}\mathbf{G}^F e} \mathbf{\Upsilon}.f)$ and $f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f$ are homotopy equivalent as complexes of $(f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f, f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f)$–bimodules.

(ℓ-C5) The algebra $f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f$ is isomorphic to the block algebra $\mathcal{O}N_{\mathbf{G}^F}(D, f)f$.

Let us make several remarks and draw some consequences of the preceding conjectures.

1. It can be proved, using some results of Jeremy Rickard [**Ri**], that there exists a complex $\boldsymbol{\Upsilon}$ of $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$–bimodules such that ($\ell$-C1) is satisfied, and such that for every integer n, the $\mathcal{O}[\mathbf{G}^F \times \mathbf{L}^F]$–module $\boldsymbol{\Upsilon}^n$ is relatively $\Delta(S)$–projective, and its restriction to $\mathcal{O}\Delta(S)$ is a permutation module for $\Delta(S)$, where $\Delta(S)$ is the diagonal embedding of a Sylow ℓ–subgroup S of $\mathbf{L}^F$ in $\mathbf{G}^F \times \mathbf{L}^F$. Then ($\ell$-C3) follows from ($\ell$-C2).

2. It follows from (ℓ-C4) that the categories $\mathcal{D}^b(\mathcal{O}\mathbf{G}^F e)$ and $\mathcal{D}^b(f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f)$ are equivalent, whence by (ℓ-C5) that

(ℓ-C6) the derived categories $\mathcal{D}^b(\mathcal{O}\mathbf{G}^F e)$ and $\mathcal{D}^b(\mathcal{O}N_{\mathbf{G}^F}(D,f)f)$ are equivalent.

This last equivalence is a particular case of a general conjecture about "abstract" finite groups stated in [**Bro1**].

3. For any chain map endomorphism α of $\boldsymbol{\Upsilon}$, we set

$$\mathrm{tr}_{\boldsymbol{\Upsilon}}(\alpha) := \sum_n (-1)^n \mathrm{tr}_{\boldsymbol{\Upsilon}^n}(\alpha)\,,$$

where $\mathrm{tr}_{\boldsymbol{\Upsilon}^n}(\alpha)$ denotes the trace of α as an endomorphism of the free $\mathcal{O}$–module $\boldsymbol{\Upsilon}^n$. Then it is easy to see that, since $f\mathcal{H}_{\mathcal{O}}(\mathbf{U})f$ is equal to $\mathrm{End}_{\mathcal{O}\mathbf{G}^F e}(\boldsymbol{\Upsilon})$,

(ℓ-C7) the linear form $\mathrm{tr}_{\boldsymbol{\Upsilon}}$ gives $f\mathcal{H}(\mathrm{Y}_{\mathcal{O}}(\mathbf{U}))f$ a structure of symmetric $\mathcal{O}$–algebra.

4. Set $\mathrm{H}^n_c(\mathrm{Y}(\mathbf{U}),\mathrm{K}) := \mathrm{K}\otimes_{\mathbb{Z}_\ell}\mathrm{H}^n_c(\mathrm{Y}(\mathbf{U}),\mathbb{Z}_\ell)$. We have

$$\mathrm{R}\Gamma_c(\mathrm{Y}(\mathbf{U}),\mathrm{K}) = \bigoplus_n \mathrm{H}^n_c(\mathrm{Y}(\mathbf{U}),\mathrm{K})[-n]\ ,$$

and it follows from (ℓ-C4) that

(ℓ-C8)
- The algebra $f\mathcal{H}_{\mathrm{K}}(\mathrm{Y}(\mathbf{U}))f$ is semi–simple,
- the $\mathrm{K}\mathbf{G}^F e$–modules $\mathrm{H}^n_c(\mathrm{Y}(\mathbf{U}),\mathrm{K})$ are all disjoint,
- $f\mathcal{H}_{\mathrm{K}}(\mathrm{Y}(\mathbf{U}))f$ is the algebra of endomorphisms of the finite graded $\mathrm{K}\mathbf{G}^F e$–module $\mathrm{R}\Gamma_c(\mathrm{Y}(\mathbf{U}),\mathrm{K})$.

5. Let S be a subgroup of D. We set $\mathbf{G}_S := C_{\mathbf{G}}(S)$, and $\mathbf{U}_S := \mathbf{U}\cap\mathbf{G}_S$. Let e_S be the block of $\mathbf{G}^F(S) = C_{\mathbf{G}^F}(S)$ such that $(S,e_S)\subseteq(D,f)$. Then the datum $(\mathbf{G}_S, e_S, \mathbf{L}^F, f, \mathbf{U}_S)$ owns the same properties as the datum $(\mathbf{G}, e, \mathbf{L}^F, f, \mathbf{U})$ and there exists a complex $\boldsymbol{\Upsilon}_S$ such that properties (ℓ-C$_i$) ($i = 1,\dots,5$) hold (with appropriate substitutions).

It is likely that $\mathrm{k}\otimes_{\mathcal{O}}\boldsymbol{\Upsilon}_S$ is the image of $\boldsymbol{\Upsilon}$ (viewed as a complex of $\mathcal{O}[\mathbf{G}^F\times\mathbf{L}^F]$–modules) through the Brauer morphism Br_S (see [**Ri**]).

The preceding considerations will probably be the starting point to explain the correspondance between e and f known as "isotypie" (cf. [**Bro1**] and [**BMM1**]).

Conjectures about Deligne–Lusztig induction

Assume now that ℓ is "large", and more precisely that

ℓ does not divide the product of the order of the Weyl group W of $\mathbf{G}$ by the order of the outer automorphism of W induced by F.

We still denote by $\mathcal{O}$ the ring of integers of a finite unramified extension K of the field of ℓ–adic numbers $\mathbb{Q}_\ell$, with residue field k, such that the finite group algebra $\mathrm{k}\mathbf{G}^F$ is split.

We denote by $e_\ell^{\mathbf{G}^F}$ the central idempotent of $\mathcal{O}\mathbf{G}^F$ associated with the subset $\mathcal{E}_\ell(\mathbf{G}^F,1)$ of the set of irreducible characters of $\mathbf{G}^F$ (see [**BrMi1**]). We call "unipotent blocks of $\mathcal{O}\mathbf{G}^F$" the primitive central idempotents e such that $ee_\ell^{\mathbf{G}^F} \neq 0$.

Then the following hold.

◉ (see [**BrMa1**], 3.13) There exists a unique integer d such that the d–th cyclotomic polynomial divides the polynomial order of $\mathbf{G}^F$ and ℓ divides $\Phi_d(q)$. The ℓ–subgroups of $\mathbf{G}^F$ ar all contained, up to $\mathbf{G}^F$–conjugation, in a Sylow Φ_d–subgroup of $\mathbf{G}$.

◉ (see [**BMM1**]) If e is a unipotent block with maximal subpair (D,f) (where D is a defect group of e, and f is a block of $\mathcal{O}C_{\mathbf{G}^F}(D)$), then $\mathbf{L} := C_{\mathbf{G}}(D)$ is a d–split Levi subgroup of $\mathbf{G}$ (*i.e.*, $\mathbf{L} = C_{\mathbf{G}}(Z^o(\mathbf{L})_d)$), D is a Sylow ℓ–subgroup of $Z(\mathbf{L}^F)$, and the canonical character of f is a d–cuspidal unipotent character λ of $\mathbf{L}^F$. Moreover, the set of irreducible characters of $\mathcal{O}\mathbf{G}^F e$ coincides with the set irreducible constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\tau\lambda)$, where τ runs over the set of characters of $\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F$ whose order is a power of ℓ.

Restricting ourselves to unipotent characters, we see that the "ℓ–conjectures" have the following particular consequences.

More notation and data.

Let d be an integer such that Φ_d divides the polynomial order of $\mathbf{G}^F$. From now on, we assume that ℓ divides $\Phi_d(q)$.

Let $(\mathbf{L},\lambda)$ be a d–cuspidal pair of $\mathbf{G}^F$. The character λ is a character with ℓ–defect zero of $\mathbf{L}^F/Z(\mathbf{L}^F)$. It follows that there exists a unique $\mathcal{O}$–free $\mathcal{O}\mathbf{L}^F$–module M_λ with character λ (M_λ is a projective $\mathcal{O}[\mathbf{L}^F/Z(\mathbf{L}^F)]$–module). The natural morphism $\mathcal{O}\mathbf{L}^F \to \mathrm{End}_{\mathcal{O}}(M_\lambda)$ is onto, and defines a torsion–free $\mathcal{O}$–sheaf on $\mathrm{X}(\mathbf{U})$ which we denote by $\mathcal{F}_\lambda$.

Let $\mathrm{Uch}(\mathbf{G}^F,(\mathbf{L},\lambda))$ be the set of all (unipotent) irreducible characters γ of $\mathbf{G}^F$ such that $\left(R_{\mathbf{L}}^{\mathbf{G}}(\lambda),\gamma\right)_{\mathbf{G}^F} \neq 0$. For each $\gamma \in \mathrm{Uch}(\mathbf{G}^F,(\mathbf{L},\lambda))$ we denote by $e_\gamma^{\mathbf{G}^F}$ the primitive central idempotent of $\mathrm{K}\mathbf{G}^F$ corresponding to γ.

We set

$$e_{(\mathbf{L},\lambda)}^{\mathbf{G}^F} := \sum_{\gamma \in \mathrm{Uch}(\mathbf{G}^F,(\mathbf{L},\lambda))} e_\gamma^{\mathbf{G}^F} .$$

d–Conjectures.

There exist

◉ a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and Levi complement $\mathbf{L}$,

◉ a finite complex $\Xi = (\cdots \to \Xi^{n-1} \to \Xi^n \to \Xi^{n+1} \to \cdots)$ of finitely generated projective $\mathcal{O}\mathbf{G}^F$–modules,

with the following properties.

(d-C1) Viewed as an object of the derived bounded category of the category of $\mathcal{O}\mathbf{G}^F$–modules, Ξ is isomorphic to $\mathrm{R}\Gamma_c(\mathrm{X}(\mathbf{U}),\mathcal{F}_\lambda)$. In particular, for each n, the n-th homology group of Ξ is isomorphic, as an $\mathcal{O}\mathbf{G}^F$–module, to $\mathrm{H}_c^n(\mathrm{X}(\mathbf{U}),\mathcal{F}_\lambda)$.

(d-C2) ◉ the structure of complex of $\mathcal{O}\mathbf{G}^F$–modules of $\Xi.f$ extends to a structure of complex of $(\mathcal{O}\mathbf{G}^F,\mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}),\mathcal{F}_\lambda))$–bimodules, all of which are projective as right $\mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}),\mathcal{F}_\lambda)$–modules,

◉ the complexes $(\Xi^\vee \otimes_{\mathcal{O}\mathbf{G}^F} \Xi)$ and $\mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}),\mathcal{F}_\lambda)$ are homotopy equivalent as complexes of $(\mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}),\mathcal{F}_\lambda),\mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}),\mathcal{F}_\lambda))$–bimodules.

(d-C3) The algebra $\mathcal{H}_{\mathrm{K}}(\mathrm{X}(\mathbf{U}), \mathcal{F}_\lambda)$ is isomorphic to the algebra $\mathrm{K}W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ (recall that we set $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) := N_{\mathbf{G}^F}(\mathbf{L}, \lambda)/\mathbf{L}^F$).

The following properties are consequences of the preceding conjectures.

(d-C4) We have

$$\mathrm{R}\Gamma_c(\mathrm{X}(\mathbf{U}, \mathcal{F}_\lambda; \mathrm{K}) = \bigoplus_n \mathrm{H}^n_c(\mathbf{U}, \mathcal{F}_\lambda; \mathrm{K})[-n] \,.$$

The cohomology groups $\mathrm{H}^n_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_\lambda; \mathrm{K})$ are all disjoint as $\mathrm{K}\mathbf{G}^F$–modules, and the algebra $\mathcal{H}_{\mathrm{K}}(\mathrm{X}(\mathbf{U}), \mathcal{F}_\lambda) := \mathrm{K}\otimes_{\mathcal{O}}\mathcal{H}(\mathrm{X}(\mathbf{U}), \mathcal{F}_\lambda)$ is equal to the algebra of $\mathbf{G}^F$–endomorphisms of the graded module $\mathrm{R}\Gamma_c(\mathrm{X}(\mathbf{U}, \mathcal{F}_\lambda; \mathrm{K})$.

(d-C5) The preceding graded module induces an equivalence between the categories of graded modules over the algebras $\mathrm{K}\mathbf{G}^F e^{\mathbf{G}^F}_{(\mathbf{L},\lambda)}$ and $\mathrm{K}W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ respectively. In particular (see [**BMM1**], fundamental theorem) there exists an isometry

$$(*) \qquad I^{\mathbf{G}^F}_{(\mathbf{L},\lambda)} : \mathbb{Z}\mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L}, \lambda)) \to \mathbb{Z}\mathrm{Uch}(\mathbf{G}^F, (\mathbf{L}, \lambda))$$

such that for all $\mathbf{G}$, we have

$$R^{\mathbf{G}}_{\mathbf{L}}(\lambda) = I^{\mathbf{G}^F}_{(\mathbf{L},\lambda)} \cdot \mathrm{Ind}^{W_{\mathbf{G}^F}(\mathbf{L},\lambda)}_{\{1\}}(1) \,.$$

Cyclotomic Hecke algebras conjectures

From now on, in order to settle things ready to be put in a more general setting, we denote by $\mathbb{G} = (V, Wf)$ a K–reflection datum. We assume that W acts irreducibly on V.

We assume that, for all choice of a prime number p, $\mathbb{G}$ defines to a reductive algebraic group $\mathbf{G}$ over an algebraic closure of a finite field together with a p–endomorphism F such that the fixed points group $\mathbf{G}^F$ is finite.

Let d be an integer, $d \geq 1$, and let Φ be an irreducible monic element of $\mathbb{Z}_K[x]$ which divides $x^d - 1$. Let ζ be a root of Φ in $\mathbb{C}$. In order to avoid trivialities, we assume that Φ divides the polynomial order $|\mathbb{G}|$ of $\mathbb{G}$.

Notation.

Here we use notation introduced in chap. 3 above in the section devoted to cyclotomic Hecke algebras.

We denote by $\mathrm{Uch}_\Phi(\mathbb{G})$ the set of all Φ–cuspidal unipotent characters of $\mathbb{G}$.

For a Φ–cuspidal pair $(\mathbb{L}, \lambda)$ of $\mathbb{G}$, where $\mathbb{L} = (V, W_{\mathbb{L}}wf)$ we recall that we set $W_{\mathbb{G}}(\mathbb{L}, \lambda) := N_W(\mathbb{L}, \lambda)/W_{\mathbb{L}}$ and $K_{\mathbb{G}}(\mathbb{L}, \lambda) := K[x]/\Phi(x)$, that we denote by $Z(\mathbb{L})_\Phi$ the Sylow Φ–sub-reflection datum of the center of $\mathbb{L}$, and by $V(\mathbb{L}, \Phi)$ its vector space $\ker\Phi(wf) \cap V^{W_{\mathbb{L}}}$, viewed as a vector space over the field $K[x]/\Phi(x)$ through its natural structure of $K[wf]$–module.

We recall that $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is a complex reflection group in its action on $V(\mathbb{L}, \Phi)$.

We denote by $\mathcal{A}_{\mathbb{G}}(\mathbb{L}, \lambda)$ the set of reflecting hyperplanes of $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ in its action on $V(\mathbb{L}, \Phi)$.

By the Φ–Harish-Chandra theories (see above), we have a partition

$$\mathrm{Uch}(\mathbb{G}) = \coprod_{[(\mathbb{L},\lambda)]} \mathrm{Uch}(\mathbb{G}; (\mathbb{L}, \lambda))$$

indexed by the orbits of W on the set of all Φ–cuspidal pairs of $\mathbb{G}$, and a pair of maps

$$\begin{cases} \mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda)) \longrightarrow \mathrm{Uch}(\mathbb{G};(\mathbb{L},\lambda)) \ , \ \chi \mapsto \gamma_\chi \\ \mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda)) \longrightarrow \{\pm 1\} \ , \ \chi \mapsto \varepsilon_\chi \end{cases}$$

satisfying the following properties :

For each $\gamma \in \mathrm{Uch}(\mathbb{G})$, let us denote by a_γ and A_γ respectively the valuation and the degree (in x) of $\mathrm{Deg}_\gamma(x)$.

For each orbit $\mathcal{C}$ of $W_{\mathbb{G}}(\mathbb{L},\lambda)$ on its set $\mathcal{A}_{\mathbb{G}}(\mathbb{L},\lambda)$ of reflecting hyperplanes, let $\det_{\mathcal{C}}^j$ be the j-th power of the character $\det_{\mathcal{C}}$ of $W_{\mathbb{G}}(\mathbb{L},\lambda)$ such that

$$\det{}_{\mathcal{C}}(s_H) = \begin{cases} \det_{V(\mathbb{L},\Phi)}(s_H) & \text{if } H \in \mathcal{C}\,, \\ 1 \quad \text{if } H \notin \mathcal{C}\,. \end{cases}$$

Let us denote by $\gamma_{\mathcal{C},j}$ the element of $\mathrm{Uch}(\mathbb{G})$ which corresponds to $\det_{\mathcal{C}}^j$, and by $a_{\mathcal{C},j}$ and $A_{\mathcal{C},j}$ respectively the valuation and the degree (in x) of $\mathrm{Deg}(\gamma_{\mathcal{C},j})$.

There exist

• an element $\gamma_0 \in \mathrm{Uch}(\mathbb{G};\ (\mathbb{L},\lambda)))$ such that, for $a_0 := a_{\gamma_0}$ and $A_0 := A_{\gamma_0}$, we have (for all $\gamma \in \mathrm{Uch}(\mathbb{G};\ (\mathbb{L},\lambda)))$:

$$(a_0 + A_0) \leq (a_\gamma + A_\gamma)\,,$$

• and (for each $\mathcal{C} \in \mathcal{A}_{\mathbb{G}}(\mathbb{L},\lambda)$) an integer $m_{\mathcal{C}} > 0$ allowing to define the rational numbers $m_{\mathcal{C},j}$ by the equalities

$$(m_{\mathcal{C}} - m_{\mathcal{C},j})N_{\mathcal{C}}e_{\mathcal{C}} = (a_{\mathcal{C},j} + A_{\mathcal{C},j}) - (a_0 + A_0)\,,$$

such that the cyclotomic Hecke algebra $\mathcal{H}(\mathbb{G};(\mathbb{L},\lambda))$ defined by the specialization

$$\mathcal{S}_{(\mathbb{G};(\mathbb{L},\lambda))} : u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j(\zeta^{-1}x)^{m_{\mathcal{C},j}}$$

satisfies the properties (U.5.3.j) below.

Let us first choose a suitable extension $K_{\mathbb{G}}(\mathbb{L},\lambda)(y)$ of the rational fraction field $K_{\mathbb{G}}(\mathbb{L},\lambda)(x)$ so that it induces a bijection

$$\mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda)) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}(\mathbb{G};(\mathbb{L},\lambda))) \quad , \quad \chi \mapsto \chi_x\,.$$

It follows that we have a bijection

$$\mathrm{Irr}(\mathcal{H}(\mathbb{G};(\mathbb{L},\lambda))) \xrightarrow{\sim} \mathrm{Uch}(\mathbb{G};\ (\mathbb{L},\lambda)) \quad , \quad \psi \mapsto \gamma_\psi\,.$$

Let us call θ_0 the element of $\mathrm{Irr}(\mathcal{H}(\mathbb{G};(\mathbb{L},\lambda)))$ which corresponds to γ_0.

(U5.3.1) The character θ_0 is a linear character, and for all $\mathcal{C} \in \mathcal{A}_{\mathbb{G}}(\mathbb{L},\lambda)$, there exists and integer $j_{\mathcal{C}}$ such that we have

$$\theta_0(\mathrm{s}_{\mathcal{C}}) = \zeta_{e_{\mathcal{C}}}^{j_{\mathcal{C}}}(\zeta^{-1}x)^{m_{\mathcal{C}}}\,.$$

Assume moreover that $\mathbb{L}$ is minimal (as a Φ–split Levi sub–reflection datum of $\mathbb{G}$) and that $\lambda = 1^{\mathbb{L}}$. Then

• the algebra $\mathcal{H}(\mathbb{G};(\mathbb{L},\lambda))$ is principal and θ_0 is a principal character,

• we have $\gamma_{\theta_0} = 1^{\mathbb{G}}$.

Assume moreover that the polynomial Φ is regular for $\mathbb{G}$. Then $\mathbb{L}$ is a (maximal) torus $\mathbb{T}$. We set

$$\mathcal{H}_{\mathbb{G}}(\mathbb{T}) := \mathcal{H}(\mathbb{G};(\mathbb{L},1))\,,\ W_{\mathbb{G}}(\mathbb{T}) := W_{\mathbb{G}}(\mathbb{L},1)\,,\ \mathcal{A}_{\mathbb{G}}(\mathbb{T}) := \mathcal{A}_{\mathbb{G}}(\mathbb{L},1)\,.$$

Then we have

$$N_{\mathbb{G}} + N_{\mathbb{G}}^{\vee} = \sum_{\mathcal{C}\in\mathcal{A}_{\mathbb{G}}(\mathbb{T})} N_{\mathcal{C}} e_{\mathcal{C}} m_{\mathcal{C}} .$$

(U5.3.2) For $\chi \in \operatorname{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda))$, let us denote by $S_\chi(x)$ the Schur element of χ_x, so that we have $t_\phi = \sum_{\chi\in\operatorname{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda))} \frac{1}{S_\chi(x)}\chi_x$, where t_ϕ is the specialisation of the canonical symmetrizing form $t_{\mathbf{u}}$ (see chap. III above). Then for all $\chi \in \operatorname{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda))$, we have

$$\varepsilon_\chi \operatorname{Deg}_{\mathbb{G}}(\gamma_\chi)(x) = \operatorname{Deg}_{\mathbb{L}}(\lambda)\operatorname{Deg}_{\mathbb{G}}(\operatorname{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}})\frac{1}{S_\chi(x)} .$$

Recall that $\operatorname{Deg}_{\mathbb{G}}(\operatorname{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) = \dfrac{|\mathbb{G}|/(\varepsilon_{\mathbb{G}} x^{N(\mathbb{G})})}{|\mathbb{L}|/(\varepsilon_{\mathbb{L}} x^{N(\mathbb{L})})}$.

(U5.3.3) Assume that ζ_d is a root of $\Phi(x)$.

The element $(wf)^{\delta_{\mathbb{G}}}$ belongs to the center of $W_{\mathbb{G}}(\mathbb{L},\lambda)$. Assume that it has order d, and that the center $Z(W_{\mathbb{G}}(\mathbb{L},\lambda))$ has order z.

Let $B_{\mathbb{G}}(\mathbb{L},\lambda)$ be a braid group associated with $W_{\mathbb{G}}(\mathbb{L},\lambda)$ (defined up to the choice of a base point), and let $P_{\mathbb{G}}(\mathbb{L},\lambda)$ be the corresponding pure braid group. We denote by $\boldsymbol{\pi}$ the positive generator of the center $Z(P_{\mathbb{G}}(\mathbb{L},\lambda))$ of the pure braid group, and by $\boldsymbol{\beta}$ the positive generator of the center $Z(B_{\mathbb{G}}(\mathbb{L},\lambda))$ of the braid group, so that we have $\boldsymbol{\beta}^z = \boldsymbol{\pi}$.

- There exists a choice of the base point such that $(wf)^{\delta_{\mathbb{G}}}$ is the image of the element $\boldsymbol{\beta}^{z/d}$.
- For all $\chi \in \operatorname{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda))$, we have

$$\frac{\varphi_{\mathbb{G}}(\gamma_\chi)}{\varphi_{\mathbb{L}}(\lambda)} x^{\frac{D_0 - ((a_{\gamma_\chi} + A_{\gamma_\chi}) - (a_0 + A_0))}{d}} = \omega_{\chi_x}(\boldsymbol{\beta}^{z/d}) ,$$

where we set (see above chap. III, notation before 3.46)

$$D_0 := \sum_{\mathcal{C}\in\mathcal{A}_{\mathbb{G}}(\mathbb{L},\lambda)} m_{\mathcal{C}} N_{\mathcal{C}} e_{\mathcal{C}} .$$

The following property is now a consequence of 3.46 above.

6.1. Proposition. *Assume as above that ζ_d is a root of $\Phi(x)$. For $\chi \in \operatorname{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda))$, we have*

$$\varphi_{\mathbb{G}}(\gamma_\chi) = \varphi_{\mathbb{L}}(\lambda) \exp\left(-2i\pi \frac{D_0 - ((a_{\gamma_\chi} + A_{\gamma_\chi}) - (a_0 + A_0))}{d^2}\right) \omega_\chi((wf)^\delta) .$$

(U5.3.4) Let $\mathbb{M}$ be a sub-reflection datum of $\mathbb{G}$ containing $\mathbb{L}$. Then the specialisation $\mathcal{S}_{(\mathbb{M};(\mathbb{L},\lambda))}$ is the restriction of the specialisation $\mathcal{S}_{(\mathbb{G};(\mathbb{L},\lambda))}$: whenever s is a distinguished pseudo–reflection in $W_{\mathbb{M}}(\mathbb{L},\lambda)$ and $0 \leq j \leq e_s - 1$, we have $\mathcal{S}_{(\mathbb{M};(\mathbb{L},\lambda))}(u_{s,j}) = \mathcal{S}_{(\mathbb{G};(\mathbb{L},\lambda))}(u_{s,j})$.

In particular, the Hecke algebra $\mathcal{H}(\mathbb{M};(\mathbb{L},\lambda))$ is naturally identified with a parabolic subalgebra of $\mathcal{H}(\mathbb{G};(\mathbb{L},\lambda))$.

Connection with the actual finite groups.

For any suitable element ξ, we denote by $\mathcal{H}(\mathbf{G}^F, (\mathbf{L}, \lambda); \xi)$ the $\mathbb{Z}_K[\xi, \xi^{-1}]$–algebra defined by the specialization $x \mapsto \xi$.

Let q be a power of a prime number p and let $(\mathbf{G}, F)$ ($\mathbf{G}$ a connected reductive algebraic group over $\overline{\mathbb{F}}_p$, F a p–endomorphism of $\mathbf{G}$ such that $\mathbf{G}^F$ is finite) be defined by $\mathbb{G}$ and q.

Let ℓ be a prime number which does not divide $|W\langle f\rangle|$ and which divides $\Phi(q)$.

Let $\mathcal{O}$ be a suitable extension of $\mathbb{Z}_\ell[\zeta, q, q^{-1}]$. We use notation introduced in d–Conjectures above — with suitable extensions of scalars.

(HC) the morphism $\mathbb{Z}_K[x, x^{-1}] \to \mathcal{O}$ defined by $x \mapsto q$ induces an $\mathcal{O}$–algebra isomorphism

$$\mathcal{O} \otimes \mathcal{H}(\mathbb{G}, (\mathbb{L}, \lambda); q) \xrightarrow{\sim} \mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}), \mathcal{F}_\lambda)$$

which send the canonical symmetrizing form t_x onto tr_Ξ.

More precise conjectures for roots of π

We shall present here more precise conjectures for the case where the cuspidal pair $(\mathbb{L}, \lambda)$ considered above is of the shape $(\mathbb{T}, 1)$, where $\mathbb{T}$ is a (maximal) torus.

For the simplicity of the exposition, we restrict ourselves to the split case (*i.e.*, where $\mathbb{G} = (V, W)$).

In this case, the variety $X(\mathbf{U})$ is isomorphic to the Deligne–Lusztig variety X_w for an element w in the conjugacy class of W defined by $\mathbb{T}$. The complex denoted above by $\mathrm{R}\Gamma_c(X(\mathbf{U}), \mathcal{F}_\lambda)$ is nothing but $\mathrm{R}\Gamma_c(X_w, \mathbb{Z}_\ell)$.

The "good choice" of $\mathbf{U}$ mentioned in the above conjectures amounts to a good choice of w in its conjugacy class. We shall see that this choice has to do with the roots of π in the corresponding braid monoid (see above 2.23 for the occurrence of d–th roots of π in B), and that we have indeed some actions of braids on the corresponding Deligne–Lusztig varieties.

The variety of Borel subgroups and the braid group.

Prerequisites.

Let $\mathbf{G}$ be a reductive connected algebraic group over an algebraically closed field. As in [**DeLu**], we consider *its* Weyl group W endowed with *its* fundamental chamber and hence with *its* set of distinguished generators S.

We denote by $\mathbf{B}$ the associated braid group, well defined by the choice of a base point in the fundamental chamber. We recall that there exists a set $\mathbf{S}$ and a bijection $S \xrightarrow{\sim} \mathbf{S}$ such that

- if $\mathbf{s} \in \mathbf{S}$ corresponds to $s \in S$, then $\mathbf{s}$ is an s–distinguished braid reflection (see 2.13 above),
- $\mathbf{S}$, together with the braid relations on S, defines a presentation of $\mathbf{B}$.

We denote by $\mathbf{B}^+$ the sub–monoid of $\mathbf{B}$ generated by $\mathbf{S}$.

We have an injective map $W \hookrightarrow \mathbf{B}^+$ defined as follows : for $s_1 s_2 \cdots s_n$ a reduced decomposition of $w \in W$, we set

$$\mathbf{w} := \mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_n \, .$$

We denote by $\mathbf{B}^+_{\mathrm{red}}$ the image of W under the preceding map.

Varieties of Borel subgroups and a theorem of Deligne.

Let $\mathcal{B}$ the variety of Borel subgroups of $\mathbf{G}$, a smooth projective homogeneous space for $\mathbf{G}$. Then (see for example [**DeLu**]), we have a natural bijection

$$(*) \qquad \mathcal{B} \times \mathcal{B}/\mathbf{G} \xrightarrow{\sim} W\,.$$

Moreover, for each choice of $B \in \mathcal{B}$, we have isomorphisms

$$\mathcal{B} \xrightarrow{\sim} \mathbf{G}/B \quad \text{and} \quad W \xrightarrow{\sim} B\backslash\mathbf{G}/B\,.$$

Let (B, B') be a pair of Borel subgroups. We say that B and B' are in relative position w (for $w \in W$) and we write $B \overset{w}{\to} B'$ if the orbit of (B, B') under $\mathbf{G}$ corresponds to the element w through the bijection (*) above.

We denote bu $\mathcal{O}(w)$ the variety of such pairs endowed with the left and right projections

$$\lambda\colon \mathcal{O}(w) \twoheadrightarrow \mathcal{B}\ ,\ (B, B') \mapsto B \quad \text{and} \quad \rho\colon \mathcal{O}(w) \twoheadrightarrow \mathcal{B}\ ,\ (B, B') \mapsto B'\,.$$

We define the composition

$$\mathcal{O}(w){\boldsymbol{\cdot}}\,\mathcal{O}(w') := \mathcal{O}(w) \underset{\mathcal{B}}{\times} \mathcal{O}(w')\,.$$

Whenever lengths of w and w' are additive, *i.e.*, $\ell(ww') = \ell(w) + \ell(w')$, Bruhat decomposition gives us an isomorphism

$$\mathcal{O}(ww') \xrightarrow{\sim} \mathcal{O}(w){\boldsymbol{\cdot}}\,\mathcal{O}(w')\,.$$

The main result of [**De3**] shows that one can extend the map $w \mapsto \mathcal{O}(w)$ (or, in other words, the map $\mathbf{w} \mapsto \mathcal{O}(\mathbf{w})$) to a construction which, to any $b \in \mathbf{B}^+$, associates a scheme $\mathcal{O}(b)$ on $\mathcal{B} \times \mathcal{B}$, defined up to a *unique* isomorphism, such that

$$\mathcal{O}(bb') = \mathcal{O}(b){\boldsymbol{\cdot}}\,\mathcal{O}(b') \quad \text{whenever } b, b' \in \mathbf{B}^+\,.$$

If $b = \mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_n$, where $\mathbf{w}_j \in \mathbf{B}^+_{\text{red}}$ for $1 \le j \le n$, then we may think of $\mathcal{O}(b)$ as defined by

$$\mathcal{O}(b) = \{(B_0, B_1, \ldots, B_n) \mid (B_j \in \mathcal{B} \text{ for } 0 \le j \le n)(B_{j-1} \overset{w_j}{\to} B_j \text{ for } 1 \le j \le n)\}\,.$$

Deligne–Lusztig generalized varieties and actions of braids.

From now on, we follow closely [**BrMi2**], §2.

We consider again the case where $\mathbf{G}$ is a connected reductive algebraic group defined over an algebraic closure of the prime field with p elements, endowed with a p–endomorphism $F\colon \mathbf{G} \longrightarrow \mathbf{G}$. In order to simplify the exposition, we assume that $(\mathbf{G}, F)$ is *split*, *i.e.*, that the automorphism of W induced by F is the identity.

We let F act on the right on $\mathcal{B}$, by setting

$$B \cdot F := F(B)\,.$$

The following definition has been inspired by the variety described by Lusztig in [**Lu0**], page 25.

DEFINITION. Let $b \in \mathbf{B}^+$. The Deligne–Lusztig variety associated with b is the part of $\mathcal{O}(b)$ lying over the graph of F :

$$X_b^{(F)} := \{x \in \mathcal{O}(b) \mid \lambda(x) = \rho(x) \cdot F\}.$$

When there is no ambiguity, we set $X_b = X_b^{(F)}$.

In other words, if $b = \mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_n$, we have

$$X_b^{(F)} = \{(B_0, B_1, \ldots, B_n) \mid (B_{j-1} \overset{w_j}{\to} B_j \text{ for } 1 \leq j \leq n) \text{ and } (B_n = B_0 \cdot F)\}.$$

DEFINITION. Let $b, b' \in \mathbf{B}^+$. We define an equivalence of étale sites

$$D_b^{(bb')} \colon X_{bb'} \longrightarrow X_{b'b}$$

as follows.

- We may think of an element of $X(bb')$ as of the shape (x, x') where $x \in \mathcal{O}(b)$, $x' \in \mathcal{O}(b')$, $\rho(x) = \lambda(x')$, $\rho(x') = \lambda(x) \cdot F$.
- We then set :

$$(x, x') \cdot D_b^{(bb')} := (x', x \cdot F)\,, \tag{6.2}$$

since we have $(x', x \cdot F) \in X_{b'b}$.

The following properties are immediate :

6.3. LEMMA.
(1) *For $b, b', c \in \mathbf{B}^+$, we have*

$$D_b^{(bb'c)} \cdot D_{b'}^{(b'cb)} = D_{bb'}^{(bb'c)} \colon X_{bb'c} \longrightarrow X_{cbb'}\,.$$

(2) *We have $D_b^{(b)} = F \colon X_b \longrightarrow X_b$.*

Varieties associated with roots of $\boldsymbol{\pi}$.

The variety $X_{\boldsymbol{\pi}}$.

Since $\boldsymbol{\pi} = \mathbf{w}_0^2$, any element $\mathbf{s}$ of $\mathbf{S}$ is a left divisor of $\boldsymbol{\pi}$ in $\mathbf{B}^+_{\mathrm{red}}$: there exists $\boldsymbol{\pi}_{\mathbf{s}} \in \mathbf{B}^+_{\mathrm{red}}$ such that $\boldsymbol{\pi} = \mathbf{s}\boldsymbol{\pi}_{\mathbf{s}}$.

Since $\boldsymbol{\pi}$ is central in the braid group, the definition 6.2 shows that $D_{\mathbf{s}}^{(\boldsymbol{\pi})}$ is an automorphism of étale site of $X_{\boldsymbol{\pi}}$. Then lemma 6.3 shows that the operators $(D_{\mathbf{s}}^{(\boldsymbol{\pi})})_{\mathbf{s} \in \mathbf{S}}$ satisfy the braid relations, hence we get

6.4. PROPOSITION. *The map $\mathbf{s} \mapsto D_{\mathbf{s}}^{(\boldsymbol{\pi})}$ extends to a group morphism*

$$B \mapsto \mathrm{Aut}(\mathrm{R}\Gamma_c(X_{\boldsymbol{\pi}}, \mathbb{Z}_\ell))\,.$$

Some arguments in favor of the following conjectures may be found in [**BrMi2**], §2.B, and in [**DMR**].

Let

$$\mathrm{R}\Gamma(X_{\boldsymbol{\pi}}, \overline{\mathbb{Q}}_\ell) := \bigoplus_{n=0}^{n=2N} \mathrm{H}^n(X_{\boldsymbol{\pi}}, \overline{\mathbb{Q}}_\ell)$$

be the graded module of the ℓ-adic cohomology of $X_{\boldsymbol{\pi}}$, seen as a graded $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$-module.

Let $\overline{\mathbb{Q}}_\ell\mathcal{H}_q(W)$ denote the ordinary Hecke algebra, *i.e.*, the algebra of $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$-endomorphisms of the module $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F/B_0^F]$, where B_0 denotes an F–stable Borel subgroup of $\mathbf{G}^F$.

6.5. CONJECTURES.

(1) *The operators D_b $(b \in B)$ generate the algebra of $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$-endomorphisms of* $\mathrm{R}\Gamma(X_{\boldsymbol{\pi}}, \overline{\mathbb{Q}}_\ell)$.

(2) *The morphism $b \mapsto D_b$ factorizes to an isomorphism*

$$\overline{\mathbb{Q}}_\ell\mathcal{H}_q(W)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{Q}}_\ell\mathbf{G}^F}(\mathrm{R}\Gamma_c(X_{\boldsymbol{\pi}}, \overline{\mathbb{Q}}_\ell))$$

which defines a graded version of the representation of $\overline{\mathbb{Q}}_\ell\mathcal{H}_q(W)$ on the module $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F/B_0^F]$.

(3) *The graded module $\mathrm{R}\Gamma(X_{\boldsymbol{\pi}}, \overline{\mathbb{Q}}_\ell)$ has only even degrees components. Two distinct (even degrees) components are disjoint as $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$–modules, i.e.,*

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell\mathbf{G}^F}(\mathrm{R}\Gamma(X_{\boldsymbol{\pi}}, \overline{\mathbb{Q}}_\ell), \mathrm{R}\Gamma_c(X_{\boldsymbol{\pi}}, \overline{\mathbb{Q}}_\ell)[n]) = 0 \quad \textit{if} \quad n \neq 0\,.$$

(4) *Given $\gamma \in \mathrm{Uch}(\mathbb{G})$, the degree n such that $\psi_q^{\mathbb{G}}(\gamma)$ is a constituent of the $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$–module $\mathrm{H}_c^n((X_{\boldsymbol{\pi}}, \overline{\mathbb{Q}}_\ell)$ does not depend on q (hence is "generic").*

The preceding construction and conjectures may be generalized to varieties associated to roots of $\boldsymbol{\pi}$.

The varieties $X_{\mathbf{w}}$ for $\mathbf{w}$ a root of $\boldsymbol{\pi}$.

Let $d \geq 1$ be an integer, and let $\mathbf{w} \in \mathbf{B}^+_{\mathrm{red}}$ such that $\mathbf{w}^d = \boldsymbol{\pi}$.

The following proposition is essentially due to [**BrMi2**], §5. Its proof is immediate.

6.6. PROPOSITION.

(1) *The map*

$$\begin{cases} X_{\mathbf{w}}^{(F)} \longrightarrow X_{\boldsymbol{\pi}}^{(F^d)} \\ x \mapsto (\lambda(x), \lambda(x).F, \lambda(x).F^2, \dots, \lambda(x).F^d) \end{cases}$$

is an embedding, which identifies $X_{\mathbf{w}}^{(F)}$ with the closed subvariety $X_{\mathbf{w}}^{(F,d)}$ of $X_{\boldsymbol{\pi}}^{(F^d)}$ defined by

$$X_{\mathbf{w}}^{(F,d)} := \{y \in X_{\boldsymbol{\pi}}^{(F^d)} \mid (y \cdot F = y \cdot D_{\mathbf{w}}^{(\boldsymbol{\pi})})\}\,.$$

(2) *The restriction $D_b^{(\boldsymbol{\pi})}) \mapsto D_b^{(\boldsymbol{\pi})})|_{X_{\mathbf{w}}^{(F,d)}}$ defines an operation of the centralizer $C_{\mathbf{B}^+}(\mathbf{w})$ of $\mathbf{w}$ in $\mathbf{B}^+$.*

Arguments in favor of the following conjectures may be found, for example, in [**Lu2**], [**BrMi2**] (§5), and [**DMR**].

For the first assertion below, see above 2.26.

6.7. CONJECTURES.

(0) *The monoid $C_{\mathbf{B}^+}(\mathbf{w})$ generates the centralizer $C_B(\mathbf{w})$ of $\mathbf{w}$ in the braid group, and the natural morphism $B(w) \mapsto C_B(\mathbf{w})$ is an isomorphism.*

(1) *The operators $D_b^{(\mathbf{w})}$ $(b \in B(\mathbf{w}))$ generate the algebra of $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$-endomorphisms of $\mathrm{R}\Gamma(X_{\boldsymbol{\pi}}, \overline{\mathbb{Q}}_\ell)$.*

(2) *There exists a ζ_d-cyclotomic specialization ϕ of the Hecke algebra of the group $W(w)$ such that the morphism $b \mapsto D_b^{(\mathbf{w})}$ factorizes to an isomorphism*

$$\overline{\mathbb{Q}}_\ell \mathcal{H}_\phi(W(w))^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}^F}(\mathrm{R}\Gamma_c(X_{\mathbf{w}}, \overline{\mathbb{Q}}_\ell)) \,.$$

(3) *Two distinct components are disjoint as $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$-modules, i.e.,*

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell \mathbf{G}^F}(\mathrm{R}\Gamma(X_{\mathbf{w}}, \overline{\mathbb{Q}}_\ell), \mathrm{R}\Gamma_c(X_{\mathbf{w}}, \overline{\mathbb{Q}}_\ell)[n]) = 0 \quad \textit{if} \quad n \neq 0 \,.$$

(4) *Given $\gamma \in \mathrm{Uch}(\mathbb{G}, \mathbb{T}_w)$, the degree n such that $\psi_q^{\mathbb{G}}(\gamma)$ is a constituent of $\mathrm{H}_c^n((X_{\mathbf{w}}, \overline{\mathbb{Q}}_\ell)$ does not depend on q (hence is "generic").*

APPENDIX 1

DIAGRAMS AND TABLES

Here are some definitions, notation, conventions, which will allow the reader to understand the diagrams.

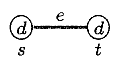
The groups have presentations given by diagrams $\mathcal{D}$ such that

- the nodes correspond to pseudo-reflections in W, the order of which is given inside the circle representing the node,
- two distinct nodes which do not commute are related by "homogeneous" relations with the same "support" (of cardinality 2 or 3), which are represented by links beween two or three nodes, or circles between three nodes, weighted with a number representing the degree of the relation (as in Coxeter diagrams, 3 is omitted, 4 is represented by a double line, 6 is represented by a triple line). These homogeneous relations are called the *braid relations* of $\mathcal{D}$.

More details are provided below.

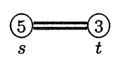
Meaning of the diagrams.

This paragraph provides a list of examples which illustrate the way in which diagrams provide presentations for the attached groups.

- The diagram 

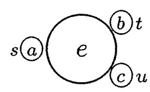
corresponds to the presentation

$$s^d = t^d = 1 \text{ and } \underbrace{ststs\cdots}_{e \text{ factors}} = \underbrace{tstst\cdots}_{e \text{ factors}}$$

- The diagram 

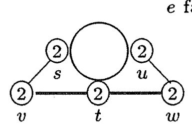
corresponds to the presentation

$$s^5 = t^3 = 1 \text{ and } stst = tsts\,.$$

- The diagram 

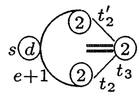
corresponds to the presentation

$$s^a = t^b = u^c = 1 \text{ and } \underbrace{stustu\cdots}_{e \text{ factors}} = \underbrace{tustus\cdots}_{e \text{ factors}} = \underbrace{ustust\cdots}_{e \text{ factors}}\,.$$

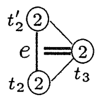
- The diagram 

corresponds to the presentation

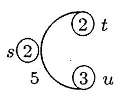
$$\begin{aligned}
&s^2 = t^2 = u^2 = v^2 = w^2 = 1\,,\\
&uv = vu\,,\ sw = ws\,,\ vw = wv\,,\\
&sut = uts = tsu\,,\\
&svs = vsv\,,\ tvt = vtv\,,\ twt = wtw\,, wuw = uwu\,.
\end{aligned}$$

- The diagram  corresponds to the presentation

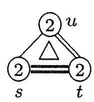
$$\begin{aligned}
&s^d = {t'_2}^2 = t_2^2 = t_3^2 = 1\,,\ st_3 = t_3 s\,,\\
&st'_2t_2 = t'_2t_2s\,,\\
&t'_2t_3t'_2 = t_3t'_2t_3\,,\ t_2t_3t_2 = t_3t_2t_3\,,\ t_3t'_2t_2t_3t'_2t_2 = t'_2t_2t_3t'_2t_2t_3\,,\\
&\underbrace{t_2st'_2t_2t'_2t_2t'_2\cdots}_{e+1 \text{ factors}} = \underbrace{st'_2t_2t'_2t_2t'_2t_2\cdots}_{e+1 \text{ factors}}\,.
\end{aligned}$$

- The diagram  corresponds to the presentation

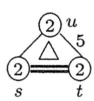
$$\begin{aligned}
&{t'_2}^2 = t_2^2 = t_3^2 = 1\,,\\
&t'_2t_3t'_2 = t_3t'_2t_3\,,\ t_2t_3t_2 = t_3t_2t_3\,,\ t_3t'_2t_2t_3t'_2t_2 = t'_2t_2t_3t'_2t_2t_3\,,\\
&\underbrace{t_2t'_2t_2t'_2t_2t'_2\cdots}_{e \text{ factors}} = \underbrace{t'_2t_2t'_2t_2t'_2t_2\cdots}_{e \text{ factors}}\,.
\end{aligned}$$

- The diagram  corresponds to the presentation

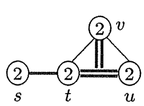
$$s^2 = t^2 = u^3 = 1\,,\ stu = tus\,,\ ustut = stutu\,.$$

- The diagram  corresponds to the presentation

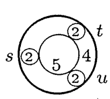
$$s^2 = t^2 = u^2 = 1\,,\ stst = tsts\,,\ tutu = utut\,,\ utusut = sutusu\,,\ sus = usu\,.$$

- The diagram  corresponds to the presentation

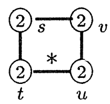
$$s^2 = t^2 = u^2 = 1\,,\ stst = tsts\,,\ tutut = ututu\,,\ utusut = sutusu\,,\ sus = usu\,.$$

- The diagram  corresponds to the presentation

$$\begin{aligned}
&s^2 = t^2 = u^2 = v^2 = 1\,,\ sv = vs\,,\ su = us\,,\\
&sts = tst\,,\ vtv = tvt\,,\ uvu = vuv\,,\ tutu = utut\,,\ vtuvtu = tuvtuv\,.
\end{aligned}$$

- The diagram  corresponds to the presentation

$$s^2 = t^2 = u^2 = 1\,,\ ustus = stust\,,\ tust = ustu\,.$$

- The diagram  corresponds to the presentation

$$s^2 = t^2 = u^2 = v^2 = 1\,,\ su = us\,,\ tv = vt\,,$$
$$sts = tst\,,\ tut = utu\,,\ uvu = vuv\,,\ vsv = svs\,,\ stuvstuvs = tuvstuvst\,.$$

In the following tables, we denote by $H \rtimes K$ a group which is a non-trivial split extension of K by H. We denote by $H \cdot K$ a group which is a non-split extension of K by H. We denote by p^n an elementary abelian group of order p^n.

A diagram where the orders of the nodes are "forgotten" and where only the braid relations are kept is called a *braid diagram* for the corresponding group.

The groups have been ordered by their diagrams, by collecting groups with the same braid diagram. Thus, for example,

- G_{15} has the same braid diagram as the groups $G(4d, 4, 2)$ for all $d \geq 2$,
- G_4, G_8, G_{16}, G_{25}, G_{32} all have the same braid diagrams as groups $\mathfrak{S}_3$, $\mathfrak{S}_4$ and $\mathfrak{S}_5$,
- G_5, G_{10}, G_{18} have the same braid diagram as the groups $G(d, 1, 2)$ for all $d \geq 2$,
- G_7, G_{11}, G_{19} have the same braid diagram as the groups $G(2d, 2, 2)$ for all $d \geq 2$,
- G_{26} has the same braid diagram as $G(d, 1, 3)$ for $d \geq 2$.

The element β (generator of $Z(W)$) is given in the last column of our tables. Notice that the knowledge of degrees and codegrees allows then to find the order of $Z(W)$, which is not explicitely provided in the tables.

The tables provide diagrams and data for all irreducible reflection groups.

- Tables 1 and 2 collect groups corresponding to infinite families of braid diagrams,
- Table 3 collects groups corresponding to exceptional braid diagrams (notice that the fact that the diagram for G_{31} provides a braid diagram is only conjectural), but G_{24}, G_{27}, G_{29}, G_{33}, G_{34},
- The last table (table 4) provides diagrams for the remaining cases (G_{24}, G_{27}, G_{29}, G_{33}, G_{34}). It is not known nor conjectural whether these diagrams provide braid diagrams for the corresponding braid groups.

Degrees and codegrees of a braid diagram.

The following property may be noticed on the tables. It generalizes a property already noticed by Orlik and Solomon for the case of Coxeter–Shephard groups (see [**OrSo3**], (3.7)).

7.1. THEOREM. *Let $\mathcal{D}$ be a braid diagram of rank r. There exist two families*

$$(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r) \quad \text{and} \quad (\mathbf{d}_1^\vee, \mathbf{d}_2^\vee, \ldots, \mathbf{d}_r^\vee)$$

of r integers, depending only on $\mathcal{D}$, and called respectively the degrees and the codegrees of $\mathcal{D}$, with the following property: whenever W is a complex reflection *group with $\mathcal{D}$ as a braid diagram, its degrees and codegrees are given by the formulae*

$$d_j = |Z(W)|\mathbf{d}_j \quad \textit{and} \quad d_j^\vee = |Z(W)|\mathbf{d}_j^\vee \quad (j = 1, 2, \ldots, r)\,.$$

The zeta function of a braid diagram.

In [**DeLo**], Denef and Loeser compute the *zeta function of local monodromy of the discriminant* of a complex reflection group W, which is the element of $\mathbb{Q}[q]$ defined by the formula

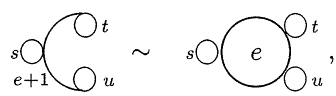
$$Z(q, W) := \prod_j \det(1 - q\mu, H^j(F_0, \mathbb{C}))^{(-1)^{j+1}}\,,$$

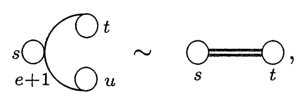
where F_0 denotes the Milnor fiber of the discriminant at 0 and μ denotes the monodromy automorphism (see [**DeLo**]).

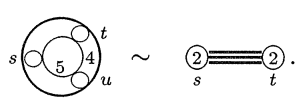
Putting together the tables of [**DeLo**] and our braid diagrams, one may notice the following fact.

7.2. THEOREM. *The zeta function of local monodromy of the discriminant of a complex reflection group W depends only on the braid diagram of W.*

REMARK. Two different braid diagrams may be associated to isomorphic braid groups. For example, this is the case for the following rank 2 diagrams (where the sign "$\sim$" means that the corresponding groups are isomorphic) :

For e even, 

for e odd, 

and 

It should be noticed, however, that the above pairs of diagrams do not have the same degrees and codegrees, nor do they have the same zeta function. Thus, degrees, codegrees and zeta functions are indeed attached to the braid diagrams, not to the braid groups.

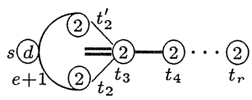
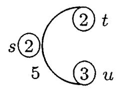
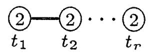
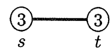
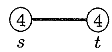
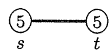
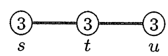
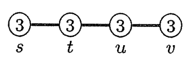
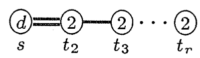
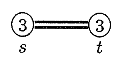
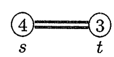
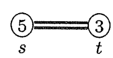
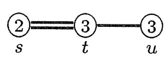
name	diagram	degrees	codegrees	β	field	$G/Z(G)$
$G(de,e,r)$ $e>2,d,r\geq 2$		$(ed,2ed,\ldots,(r-1)ed,rd)$	$(0,ed,\ldots,(r-1)ed)$	$s^{\frac{r}{(e\wedge r)}}(t_2't_2t_3\cdots t_r)^{\frac{e(r-1)}{(e\wedge r)}}$	$\mathbb{Q}(\zeta_{de})$	
G_{15}		$12,24$	$0,24$	$ustut=s(tu)^2$	$\mathbb{Q}(\zeta_{24})$	$\mathfrak{S}_4$
$\mathfrak{S}_{r+1}$		$(2,3,\ldots,\ldots,r+1)$	$(0,1,\ldots,\ldots,r-1)$	$(t_1\cdots t_r)^{r+1}$	$\mathbb{Q}$	
G_4		$4,6$	$0,2$	$(st)^3$	$\mathbb{Q}(\zeta_3)$	$\mathfrak{A}_4$
G_8		$8,12$	$0,4$	$(st)^3$	$\mathbb{Q}(i)$	$\mathfrak{S}_4$
G_{16}		$20,30$	$0,10$	$(st)^3$	$\mathbb{Q}(\zeta_5)$	$\mathfrak{A}_5$
G_{25}		$6,9,12$	$0,3,6$	$(stu)^4$	$\mathbb{Q}(\zeta_3)$	$3^2{\rtimes}SL_2(3)$
G_{32}		$12,18,24,30$	$0,6,12,18$	$(stuv)^5$	$\mathbb{Q}(\zeta_3)$	$PSp_4(3)$
$G(d,1,r)$ $d\geq 2$		$(d,2d,\ldots,\ldots,rd)$	$(0,d,\ldots,\ldots,(r-1)d)$	$(st_2t_3\cdots t_r)^r$	$\mathbb{Q}(\zeta_d)$	
G_5		$6,12$	$0,6$	$(st)^2$	$\mathbb{Q}(\zeta_3)$	$\mathfrak{A}_4$
G_{10}		$12,24$	$0,12$	$(st)^2$	$\mathbb{Q}(\zeta_{12})$	$\mathfrak{S}_4$
G_{18}		$30,60$	$0,30$	$(st)^2$	$\mathbb{Q}(\zeta_{15})$	$\mathfrak{A}_5$
G_{26}		$6,12,18$	$0,6,12$	$(stu)^3$	$\mathbb{Q}(\zeta_3)$	$3^2{\rtimes}SL_2(3)$

TABLE 1

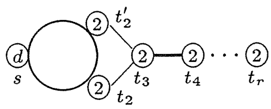
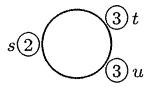
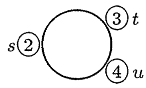
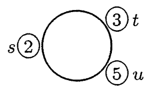
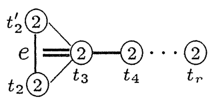
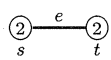
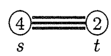
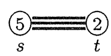
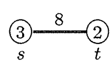
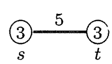
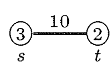
name	**diagram**	**degrees**	**codegrees**	β	**field**	$G/Z(G)$
$G(2d,2,r)$ $d,r\geq 2$		$(2d,4d,\ldots$ $2(r-1)d,rd)$	$(0,2d,\ldots$ $2(r-1)d)$	$s^{\frac{r}{(2\wedge r)}}(t_2't_2t_3\cdots t_r)^{\frac{2(r-1)}{(2\wedge r)}}$	$\mathbb{Q}(\zeta_{2d})$	
G_7		$12, 12$	$0, 12$	stu	$\mathbb{Q}(\zeta_{12})$	$\mathfrak{A}_4$
G_{11}		$24, 24$	$0, 24$	stu	$\mathbb{Q}(\zeta_{24})$	$\mathfrak{S}_4$
G_{19}		$60, 60$	$0, 60$	stu	$\mathbb{Q}(\zeta_{60})$	$\mathfrak{A}_5$
$G(e,e,r)$ $e\geq 2, r>2$		$(e,2e,\ldots,$ $(r-1)e,r)$	$(0,e,\ldots,(r-2)e,$ $(r-1)e-r)$	$(t_2't_2t_3\cdots t_r)^{\frac{e(r-1)}{(e\wedge r)}}$	$\mathbb{Q}(\zeta_e)$	
$G(e,e,2)$ $e\geq 3$		$2, e$	$0, e-2$	$(st)^{e/(e\wedge 2)}$	$\mathbb{Q}(\zeta_e+\zeta_e^{-1})$	
G_6		$4, 12$	$0, 8$	$(st)^3$	$\mathbb{Q}(\zeta_{12})$	$\mathfrak{A}_4$
G_9		$8, 24$	$0, 16$	$(st)^3$	$\mathbb{Q}(\zeta_8)$	$\mathfrak{S}_4$
G_{17}		$20, 60$	$0, 40$	$(st)^3$	$\mathbb{Q}(\zeta_{20})$	$\mathfrak{A}_5$
G_{14}		$6, 24$	$0, 18$	$(st)^4$	$\mathbb{Q}(\zeta_3,\sqrt{-2})$	$\mathfrak{S}_4$
G_{20}		$12, 30$	$0, 18$	$(st)^5$	$\mathbb{Q}(\zeta_3,\sqrt{5})$	$\mathfrak{A}_5$
G_{21}		$12, 60$	$0, 48$	$(st)^5$	$\mathbb{Q}(\zeta_{12},\sqrt{5})$	$\mathfrak{A}_5$

TABLE 2

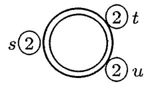
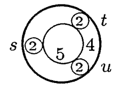
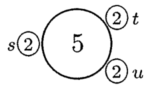
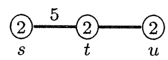
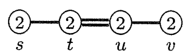
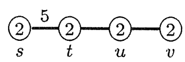
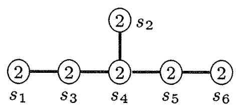
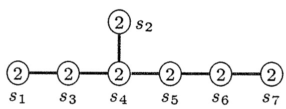
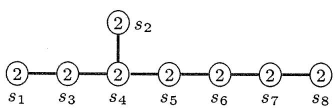
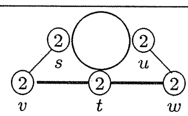
name	diagram	degrees	codegrees	β	field	$G/Z(G)$
G_{12}		6,8	0,10	$(stu)^4$	$\mathbb{Q}(\sqrt{-2})$	$\mathfrak{S}_4$
G_{13}		8,12	0,16	$(stu)^3$	$\mathbb{Q}(\zeta_8)$	$\mathfrak{S}_4$
G_{22}		12,20	0,28	$(stu)^5$	$\mathbb{Q}(i,\sqrt{5})$	$\mathfrak{A}_5$
G_{23}		2,6,10	0,4,8	$(stu)^5$	$\mathbb{Q}(\sqrt{5})$	$\mathfrak{A}_5$
G_{28}		2,6, 8,12	0,4, 6,10	$(stuv)^6$	$\mathbb{Q}$	$2^4\rtimes(\mathfrak{S}_3\times\mathfrak{S}_3)$ †
G_{30}		2,12, 20,30	0,10, 18,28	$(stuv)^{15}$	$\mathbb{Q}(\sqrt{5})$	$(\mathfrak{A}_5\times\mathfrak{A}_5)\rtimes 2$ ‡
G_{35}		2,5,6,8, 9,12	0,3,4,6, 7,10	$(s_1\cdots s_6)^{12}$	$\mathbb{Q}$	$SO_6^-(2)'$
G_{36}		2,6,8, 10,12, 14,18	0,4,6, 8,10, 12,16	$(s_1\cdots s_7)^9$	$\mathbb{Q}$	$SO_7(2)$
G_{37}		2,8,12, 14,18,20, 24,30	0,6,10, 12,16,18, 22,28	$(s_1\cdots s_8)^{15}$	$\mathbb{Q}$	$SO_8^+(2)$
G_{31}		8,12, 20,24	0,12, 16,28	$(stuvw)^6$	$\mathbb{Q}(i)$	$2^4\rtimes\mathfrak{S}_6$ ⋆

TABLE 3

It is still conjectural whether the corresponding braid diagram for G_{31} provides a presentation for the associated braid group.

† The action of $\mathfrak{S}_3\times\mathfrak{S}_3$ on 2^4 is irreducible.
‡ The automorphism of order 2 of $\mathfrak{A}_5\times\mathfrak{A}_5$ permutes the two factors.
⋆ The group $G_{31}/Z(G_{31})$ is not isomorphic to the quotient of the Weyl group D_6 by its center.

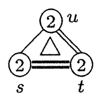
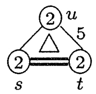
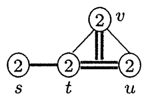
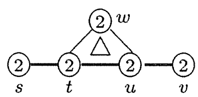
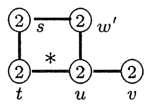
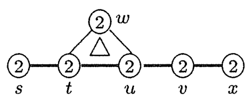
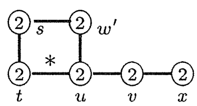
name	diagram	degrees	codegrees	β	field	$G/Z(G)$
G_{24}		4,6,14	0,8,10	$(stu)^7$	$\mathbb{Q}(\sqrt{-7})$	$GL_3(2)$
G_{27}		6,12,30	0,18,24	$(stu)^5$	$\mathbb{Q}(\zeta_3, \sqrt{5})$	$\mathfrak{A}_6$
G_{29}		4,8,12,20	0,8,12,16	$(stuv)^5$	$\mathbb{Q}(i)$	$2^4 \rtimes \mathfrak{S}_5$ †
G_{33}	 	4,6,10, 12,18	0,6,8, 12,14	$(ustvw)^9$	$\mathbb{Q}(\zeta_3)$	$SO_5(3)'$
G_{34}	 	6,12,18,24, 30,42	0,12,18,24, 30,36	$(stuvwx)^7$	$\mathbb{Q}(\zeta_3)$	$PSO_6^-(3)'{\cdot}2$

TABLE 4

These diagrams provide presentations for the corresponding finite groups. It is not known nor conjectural whether they provide presentations for the corresponding braid groups.

† The group $G_{29}/Z(G_{29})$ is not isomorphic to the Weyl group D_5.

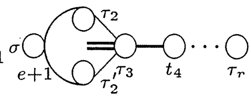
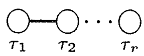
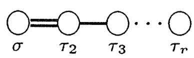
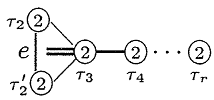
name	diagram	degrees	codegrees	β
$B(de,e,r)$ $e\geq 2, r\geq 2, d>1$		$e, 2e, \ldots,$ $(r-1)e, r$	$0, e, \ldots,$ $(r-1)e$	$\sigma^{\frac{r}{(e\wedge r)}}(\tau_2\tau_2'\tau_3\cdots\tau_r)^{\frac{e(r-1)}{(e\wedge r)}}$
$B(1,1,r)$		$2, 3, \ldots, r+1$	$0, 1, \ldots, r-1$	$(\tau_1\cdots\tau_r)^{r+1}$
$B(d,1,r)$ $d>1$		$1, 2, \ldots, r$	$0, , \ldots, (r-1)$	$(\sigma\tau_2\tau_3\cdots\tau_r)^r$
$B(e,e,r)$ $e\geq 2, r\geq 2$		$e, 2e, \ldots,$ $(r-1)e, r$	$0, e, \ldots, (r-2)e,$ $(r-1)e-r$	$(\tau_2\tau_2'\tau_3\cdots\tau_r)^{\frac{e(r-1)}{(e\wedge r)}}$

TABLE 5 : BRAID DIAGRAMS

This table provides a complete list of the infinite families of braid diagrams and corresponding data. Note that the braid diagram $B(de,e,r)$ for $e=2, d>1$ can also be described by a diagram as the one used for $G(2d,2,r)$ in Table 2. Similarly, the diagram for $B(e,e,r)$, $e=2$, can also be described by the Coxeter diagram of type D_r. The list of exceptional diagrams (but those associated with G_{24}, G_{27}, G_{29}, G_{33}, G_{34}) is identical with table 3.

APPENDIX 2

SYMMETRIC ALGEBRAS

For the convenience of the reader, we collect and prove here various results about specializations of algebras and about symmetric algebras. Most of these results are known. The presentation given here is inspired by various ideas in [**Bro1**], [**BrKi**], [**Ge**], [**GePf**], [**GeRo**] *and* [**Ro1**].

In what follows the following notation and assumptions will be in force:

- We denote by R an integrally closed commutative noetherian domain, with field of fractions denoted by F.
- We denote by $\mathcal{H}$ an R–algebra which is a free R–module of finite rank.
  [Actually, for most of the results proved below it would be enough to assume that $\mathcal{H}$ is a finitely generated projective R–module.]
- The R–dual of the module $\mathcal{H}$ is $\mathcal{H}^\vee := \mathrm{Hom}(\mathcal{H}, R)$. The map

$$\mathcal{H}^\vee \otimes \mathcal{H} \to \mathrm{End}_R(\mathcal{H}) \ , \ \phi \otimes h \mapsto (h' \mapsto \phi(h')h)$$

is an isomorphism. The trace $\mathrm{tr}_\mathcal{H} \colon \mathrm{End}_R(\mathcal{H}) \to R$ is defined, through the preceding isomorphism, by the natural pairing $\mathcal{H}^\vee \otimes \mathcal{H} \to R$. We still denote by

$$\mathrm{tr}_\mathcal{H} \colon \mathcal{H} \to R$$

the composition of the trace with the natural monomorphism $\mathcal{H} \hookrightarrow \mathrm{End}_R(\mathcal{H})$ defined by the left regular representation of $\mathcal{H}$.

- We set $F\mathcal{H} := F \otimes_R \mathcal{H}$. More generally, if $R \to R'$ is a ring morphism, we set $R'\mathcal{H} := R' \otimes_R \mathcal{H}$.

Values of characters.

8.1. PROPOSITION. *Let χ be the character of some finite dimensional $F\mathcal{H}$–module.*

(1) *For all $h \in \mathcal{H}$, we have $\chi(h) \in R$.*

(2) *Assume moreover that R is a $\mathbb{Z}$–graded ring, and that $\mathcal{H}$ is a graded R–module which has an R–basis consisting of homogeneous elements. Then, for all $h \in \mathcal{H}$ with degree n, $\chi(h)$ is an element with degree n of R.*

PROOF OF 8.1.

(1) For $h \in \mathcal{H}$, the characteristic polynomial $\mathrm{Cha}(h)(x)$ of the left multiplication by h in $\mathcal{H}$ is a unitary element of $R[x]$. Hence its roots are all integral over R.

In order to prove 8.1, we may assume that χ is the character of an irreducible $F\mathcal{H}$–module. In this case, the characteristic polynomial $\mathrm{Cha}_\chi(h)(x)$ of h acting on this module divides $\mathrm{Cha}(h)(x)$ and is unitary, from which it follows that its coefficients are integral over R. Since R is integrally closed, we see that $\mathrm{Cha}_\chi(h)(x) \in R[x]$. In particular, we have $\chi(h) \in R$.

(2) Keeping the same notation as above, we see now that, if

$$\mathrm{Cha}(h)(x) = x^d - a_{d-1}(h)x^{d-1} + \cdots + (-1)^d a_0(h) \, ,$$

then for each j the coefficient $a_j(h)$ is a homogeneous element of degree $n(d-j)$ in R (since the matrix of the left multiplication by h on a basis consisting of homogeneous elements must have homogeneous entries all of degree n).

Let us define a graduation on $R[x]$ by assigning the degree $\ell + nm$ to λx^m, for λ homogeneous of degree ℓ. Then we see that $\mathrm{Cha}(h)(x)$ is a homogeneous element of degree dn in $R[x]$.

Any divisor of a homogeneous element is again homogeneous. So $\mathrm{Cha}_\chi(h)(x)$ is homogeneous, and since its degree as a polynomial in x is $\chi(1)$, it is homogeneous of degree $n\chi(1)$. This proves in particular that the coefficient of $x^{\chi(1)-1}$, namely $\chi(h)$, is homogeneous of degree n. □

Let $p\colon R \twoheadrightarrow k$ be a morphism from R onto a field k. For $\varphi\colon \mathcal{H} \to R$ a linear form, we denote by $\varphi_k\colon k\mathcal{H} \to k$ the function such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\varphi"] \arrow[d, "p"'] & R \arrow[d, "p"'] \\ k\mathcal{H} \arrow[r, "\varphi_k"] & k\,. \end{tikzcd}$$

In particular, by 8.1 above, any character χ of a $F\mathcal{H}$–module defines a central function $\chi_k : k\mathcal{H} \to k$.

Under certain hypotheses, the function χ_k is again a character, as shown in particular by the following classical result, known as "Tits deformation theorem".

8.2. THEOREM. *Assume that $F\mathcal{H}$ and $k\mathcal{H}$ are split semi-simple. Then the map $\chi \mapsto \chi_k$ defines a bijection from the set $\mathrm{Irr}(F\mathcal{H})$ of irreducible characters of $F\mathcal{H}$ onto the set $\mathrm{Irr}(k\mathcal{H})$ of irreducible characters of $k\mathcal{H}$, which preserves the degrees of characters.*

PROOF OF 8.2. We only sketch a proof of this well known result.

Let n be the rank of $\mathcal{H}$ over R. We extend the scalars to the polynomial algebra $R[x_1, x_2, \dots, x_n]$ to get the algebra $R[x_1, x_2, \dots, x_n]\mathcal{H}$. Choose a basis $(h_1, h_2, \dots, h_n)$ of $\mathcal{H}$ over R and consider the generic element $g := \sum_{j=1}^{j=n} x_j h_j$. Let $\mathrm{Cha}(g)(x)$ be the characteristic polynomial of the left multiplication by g. Then we have the following factorization in $F(x_1, x_2, \dots, x_n)[x]$ (hence in $R[x_1, x_2, \dots, x_n][x]$ since $R[x_1, x_2, \dots, x_n][x]$ is integrally closed):

$$(*) \qquad \mathrm{Cha}(g)(x) = \prod_{\chi \in \mathrm{Irr}(F\mathcal{H})} \mathrm{Cha}_\chi(g)(x)^{d_\chi}\,,$$

where $\mathrm{Cha}_\chi(g)(x)$ is the characteristic polynomial of g in a representation with character χ and where d_χ is the degree of χ, *i.e.*, the degree (in x) of $\mathrm{Cha}_\chi(g)(x)$. It is easy to see that each $\mathrm{Cha}_\chi(g)(x)$ is irreducible.

Applying the morphism p, we get a decomposition of $p(\mathrm{Cha}(g))(x)$ in the algebra $k[x_1, x_2, \dots, x_n][x]$:

$$p(\mathrm{Cha}(g))(x) = \prod_{\chi \in \mathrm{Irr}(F\mathcal{H})} p(\mathrm{Cha}_\chi(g))(x)^{d_\chi}\,.$$

On the other hand, since $p(\mathrm{Cha}(g))(x)$ is the characteristic polynomial of the left multiplication by g in $k[x_1, x_2, \dots, x_n]\mathcal{H}$, this polynomial has a decomposition into irreducible factors in $k[x_1, x_2, \dots, x_n][x]$ which is analogous to (*) above:

$$p(\mathrm{Cha}(g))(x) = \prod_{\phi \in \mathrm{Irr}(k\mathcal{H})} \mathrm{Cha}_\phi(g)(x)^{d_\phi}\,,$$

where d_ϕ is the degree of $\mathrm{Cha}_\phi(g)(x)$. It is easy to see that the decompositions must coincide, whence the theorem 8.2 follows. □

Symmetric algebras.

8.3. DEFINITIONS.

- *A linear form* $t\colon \mathcal{H} \to R$ *is called central if* $t(hh') = t(h'h)$ *for all* $h, h' \in \mathcal{H}$.
- *A symmetrizing form on* $\mathcal{H}$ *is a central linear form* $t\colon \mathcal{H} \to R$ *such that the morphism*

$$\widehat{t}\colon \mathcal{H} \to \mathcal{H}^\vee \ , \ h \mapsto (h' \mapsto t(hh'))$$

is an isomorphism.

- *The algebra* $\mathcal{H}$ *is said to be symmetric if there exists a symmetrizing form.*
- *If the trace form* $\mathrm{tr}_\mathcal{H}\colon \mathcal{H} \to R$ *is symmetrizing, we say that* $\mathcal{H}$ *is trace symmetric.*

Let us note some elementary facts about specializations and symmetrizing forms.

8.4. PROPOSITION. *Let* $p\colon R \twoheadrightarrow k$ *be a morphism from* R *onto a field* k. *Assume that there is a central form* $t\colon \mathcal{H} \to R$ *which defines a symmetrizing form on* $k\mathcal{H}$. *Then* t *is a symmetrizing form for* $F\mathcal{H}$.

In particular, if $k\mathcal{H}$ *is trace symmetric, then* $F\mathcal{H}$ *is trace symmetric.*

PROOF OF 8.4. Let $\mathfrak{m}$ be the kernel of p. We set $\bar{t} := p \cdot t\colon \mathcal{H} \to k$. The discriminant of the form $\bar{t}$ on an R–basis of $\mathcal{H}$ does not belong to $\mathfrak{m}$, since its image through p has to be non zero. This proves that t is a symmetrizing form for the algebra $R_\mathfrak{m}\mathcal{H}$ (where $R_\mathfrak{m}$ is the localization of R at $\mathfrak{m}$), hence in particular is a symmetrizing form for $F\mathcal{H}$. ☐

Symmetrizing forms and separability.

If the algebra $F\mathcal{H}$ is trace symmetric, then it is separable, *i.e.*, whenever L is an extension of F, the algebra $L\mathcal{H}$ is semi–simple. This is a particular case of a more general result which we prove below (see 8.7).

Let us first introduce the notion of "pseudo–character" (see for example [**Ro1**]).

Whenever $\tau\colon \mathcal{H} \to R$ is a central function on $\mathcal{H}$, for all natural integers n we construct a symmetric function $S_n(\tau)\colon \mathcal{H}^n \to R$ by the formula

$$S_n(\tau) := \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)(\sigma \cdot \tau) \,,$$

where, for $\sigma = \sigma_1\sigma_2 \cdots \sigma_r$ the decomposition of σ into a product of disjoint cycles, and $\sigma_j = (j_1, \ldots, j_m)$, we set

$$(\sigma \cdot \tau)(x_1, x_2, \ldots, x_n) := \tau(\sigma_1(x))\tau(\sigma_2(x)) \cdots \tau(\sigma_r(x))$$

where

$$\sigma_j(x) := x_{j_1} \cdots x_{j_m} \,.$$

We say that the function τ is a *pseudo–character* of $\mathcal{H}$ if there exists an integer d such that $S_d(\tau) = 0$.

The following result (see [**Ro1**], 3.1) justifies the terminology.

8.5. LEMMA. *Any character of a* $F\mathcal{H}$*–module is a pseudo–character.*

Indeed, one reduces easily to the case where $\mathcal{H}$ is a matrix algebra over a characteristic zero field. Then the assertion becomes a formal property of traces of matrices.

Pseudo–characters inherit from characters the property of vanishing on nilpotent elements (see [**Ro1**], 2.6).

8.6. LEMMA. *Any pseudo–character vanishes on nilpotent elements of* $\mathcal{H}$.

8.7. PROPOSITION. *Assume that the algebra $F\mathcal{H}$ has a symmetrizing form which is a linear combination of pseudo–characters. Then $F\mathcal{H}$ is separable.*

PROOF OF 8.7. Let t be a symmetrizing form on $F\mathcal{H}$ which is a linear combination of pseudo–characters. Let L be an extension of F. Since the elements of the Jacobson radical of $L\mathcal{H}$ are nilpotent, they are contained in the kernel of all pseudo–characters, hence in the kernel of the map $\widehat{t}$, which proves that the Jacobson radical of $L\mathcal{H}$ equals $\{0\}$. □

The Casimir element.

From now on we assume that

$$\mathcal{H} \text{ is endowed with a symmetrizing form } t.$$

The isomorphisms $\mathcal{H}^\vee \otimes \mathcal{H} \xrightarrow{\sim} \mathrm{End}_R(\mathcal{H})$ and $\widehat{t}\colon \mathcal{H} \xrightarrow{\sim} \mathcal{H}^\vee$ define an isomorphism $\mathcal{H} \otimes \mathcal{H} \xrightarrow{\sim} \mathrm{End}_R(\mathcal{H})$.

We denote by $C_\mathcal{H}$ and call *the Casimir element* of $(\mathcal{H}, t)$ the element of $\mathcal{H}\otimes\mathcal{H}$ corresponding to the identity on $\mathcal{H}$ through the preceding isomorphism. If J is a finite set and if $(h_j)_{j\in J}$ and $(h'_j)_{j\in J}$ are two families of elements of $\mathcal{H}$ indexed by J such that $C_\mathcal{H} = \sum_j h'_j \otimes h_j$, we get the following characterization of $C_\mathcal{H}$:

$$h = \sum_{j\in J} t(hh'_j)h_j \quad \text{for all } h \in \mathcal{H}. \tag{8.8}$$

REMARK. If $(e_j)_j$ is an R–basis of $\mathcal{H}$, and if $(e'_j)_j$ is the dual basis (defined by the condition $t(e_j e'_{j'}) = \delta_{j,j'}$), then we have $C_\mathcal{H} = \sum_{j\in J} e'_j \otimes e_j$.

The following properties of the element $C_\mathcal{H}$ are straightforward.

8.9. LEMMA.
(1) *For all $h \in \mathcal{H}$, we have $\sum_{j\in J} hh'_j \otimes h_j = \sum_{j\in J} h_j \otimes h'_j h$.*
(2) *We have $\sum_{j\in J} h'_j \otimes h_j = \sum_{j\in J} h_j \otimes h'_j$, and $hC_\mathcal{H} = C_\mathcal{H} h$ for all $h \in \mathcal{H}$.*
(3) *For all $h \in \mathcal{H}$, we have*

$$h = \sum_{j\in J} t(hh'_j)h_j = \sum_{j\in J} t(hh_j)h'_j = \sum_{j\in J} t(h'_j)h_j h = \sum_{j\in J} t(h_j)h'_j h.$$

Let us prove (1). By 8.8, we have $hh'_j = \sum_{i\in J} t(hh'_j h'_i)h_i$, hence

$$\begin{aligned}
\sum_{j\in J} hh'_j \otimes h_j &= \sum_{i,j} t(hh'_j h'_i)h_i \otimes h_j \\
&= \sum_i h_i \otimes \sum_j t(h'_i hh'_j)h_j \\
&= \sum_i h_i \otimes h'_i h.
\end{aligned}$$

The assertions (2) and (3) follow immediately from (1).

For $\tau\colon \mathcal{H} \to R$ a linear form, we denote by $\tau^\vee$ the element of $\mathcal{H}$ defined by the condition

$$t(\tau^\vee h) = \tau(h) \quad \text{for all } h \in \mathcal{H}.$$

It is easy to check the following set of properties.

8.10. LEMMA.
(1) *τ is central if and only if $\tau^\vee$ is central in $\mathcal{H}$.*
(2) *We have $\tau^\vee = \sum_{j\in J} \tau(h'_j)h_j = \sum_{j\in J}\tau(h_j)h'_j$, and more generally, for all $h \in \mathcal{H}$, we have $\tau^\vee h = \sum_{j\in J}\tau(h'_j h)h_j = \sum_{j\in J}\tau(h_j h)h'_j$.*

Let χ_{reg} denote the character of the regular representation of $\mathcal{H}$, *i.e.*, the linear form on $\mathcal{H}$ defined by

$$\chi_{\mathrm{reg}}(h) := \mathrm{tr}_{\mathcal{H}/R}(\lambda_h)$$

where λ_h is the endormorphism of $\mathcal{H}$ defined by $\lambda_h \colon \mathcal{H} \longrightarrow \mathcal{H}\ ,\ x \mapsto hx$.

8.11. PROPOSITION. *For all $h \in \mathcal{H}$, we have*

$$\chi_{\mathrm{reg}}(h) = t(hc_{\mathcal{H}})\,,\quad \textit{or, in other words}\quad \chi_{\mathrm{reg}}^\vee = c_{\mathcal{H}}\,.$$

PROOF OF 8.11. The characterization of the Casimir element $C_{\mathcal{H}}$ shows that, through the isomorphism $\mathcal{H}^\vee \otimes \mathcal{H} \xrightarrow{\sim} \mathrm{Hom}_R(\mathcal{H},\mathcal{H})$, the endomorphism λ_h corresponds to the element $\sum_{j\in J}\widehat{t}(h'_j h)\otimes h_j$. Thus the trace of λ_h is

$$\mathrm{tr}(\lambda_h) = \sum_{j\in J}\widehat{t}(h'_j h)(h_j) = \sum_{j\in J} t(h'_j h h_j) = t(c_{\mathcal{H}}h)\,.$$

□

Casimir element and projectivity.

Let V be an $\mathcal{H}$–module, which is a projective R–module (hence, the natural morphism $\mathrm{Hom}_R(V,R)\otimes_R V \longrightarrow \mathrm{Hom}_R(V,V)$ is an isomorphism). The $\mathcal{H}$–morphism

$$t_V \colon \begin{cases} \mathrm{Hom}_{\mathcal{H}}(V,\mathcal{H}) \longrightarrow \mathrm{Hom}_R(V,R) \\ \phi \mapsto t\cdot\phi \end{cases}$$

is an isomorphism, whose inverse is the morphism $t_V^{-1} \colon \mathrm{Hom}_R(V,R) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(V,\mathcal{H})$ defined by

$$\forall h \in \mathcal{H}\,,\ v\in V\ ,\ \psi(hv) = t(ht_V^{-1}\psi)(v))\,.$$

As a consequence, the natural morphism $\mathrm{Hom}_{\mathcal{H}}(V,\mathcal{H})\otimes_R V \longrightarrow \mathrm{Hom}_{\mathcal{H}}(V,V)$ can be factorized as follows

$$\mathrm{Hom}_{\mathcal{H}}(V,\mathcal{H})\otimes_R V \xrightarrow{\sim} \mathrm{Hom}_R(V,R)\otimes_R V \xrightarrow{\sim} \mathrm{Hom}_R(V,V) \xrightarrow{H_V} \mathrm{Hom}_{\mathcal{H}}(V,V)$$

where the map H_V is defined by :

$$H_V(\alpha)(v) := \sum_{j\in J} h'_j\alpha(h_j v)\,.$$

8.12. PROPOSITION. *Let V be an $\mathcal{H}$–module which is a finitely generated projective R–module. The $\mathcal{H}$–module V is projective if and only if the image of the map*

$$H_V \colon \begin{cases} \mathrm{Hom}_R(V,V) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(V,V) \\ \alpha \mapsto \sum_{j\in J} h'_j\alpha(h_j v) \end{cases}$$

contains the identity endomorphism of V.

PROOF OF 8.12. Indeed we know that V is a projective $\mathcal{H}$–module if and only if the natural morphism $\mathrm{Hom}_{\mathcal{H}}(V,\mathcal{H})\otimes_R V \longrightarrow \mathrm{Hom}_{\mathcal{H}}(V,V)$ reaches the identity endomorphism of V. □

Schur elements.

Let χ be the character of an absolutely irreducible $F\mathcal{H}$–module V_χ. We denote by $\rho_\chi \colon F\mathcal{H} \twoheadrightarrow \mathrm{End}_F(V_\chi)$ the natural (epi)morphism. It restricts to a morphism $\omega_\chi \colon Z\mathcal{H} \to R$ (where we identify R with a subring of the center of $\mathrm{End}_F(V)$).

The *Schur element of* χ is by definition the element of R defined by

$$S_\chi := \omega_\chi(\chi^\vee)\,. \tag{8.13}$$

Notice that by 8.10, (2), we have

$$S_\chi \chi(1) = \sum_{j \in J} \chi(h'_j)\chi(h_j)\,.$$

8.14. PROPOSITION. *Assume that $F\mathcal{H}$ is split semi–simple.*

(1) *For each irreducible character χ of $F\mathcal{H}$, let e_χ be the primitive idempotent of the center $ZF\mathcal{H}$ of the algebra $F\mathcal{H}$ associated with χ. Then we have*

$$\chi^\vee = S_\chi e_\chi\,.$$

(2) *We have*

$$t = \sum_{\chi \in \mathrm{Irr}(F\mathcal{H})} \frac{1}{S_\chi}\chi\,.$$

PROOF OF 8.14.

(1) Since, for all $h \in \mathcal{H}$, we have $\chi(e_\chi h) = \chi(h)$, we see that $t(\chi^\vee e_\chi h) = t(\chi^\vee h)$, which proves that $\chi^\vee = \chi^\vee e_\chi$. The desired equality results from the fact that, for all $z \in ZF\mathcal{H}$, we have $z = \sum_{\chi \in \mathrm{Irr}(F\mathcal{H})} \omega_\chi(z) e_\chi$.

(2) Through the isomorphism between $\mathcal{H}$ and its dual, the equality

$$t = \sum_{\chi \in \mathrm{Irr}(F\mathcal{H})} \frac{1}{S_\chi}\chi$$

is equivalent to

$$1 = \sum_{\chi \in \mathrm{Irr}(F\mathcal{H})} \frac{1}{S_\chi}\chi^\vee\,,$$

which is obvious by (1) above. □

8.15. THEOREM. *Assume that $\mathcal{H}$ is endowed with a symmetrizing form t, and that $F\mathcal{H}$ is split semi-simple. For each $\chi \in \mathrm{Irr}(F\mathcal{H})$, let us denote by S_χ the corresponding Schur element.*

Let $p \colon R \twoheadrightarrow k$ be a morphism from R onto a field k. Assume that $k\mathcal{H}$ is split. Then the following assertions are equivalent:

(i) *For all $\chi \in \mathrm{Irr}(F\mathcal{H})$, we have $p(S_\chi) \neq 0$.*

(ii) *$k\mathcal{H}$ is (split) semi-simple.*

PROOF OF 8.15. Notice that the form t defines a symmetrizing form t_k on $k\mathcal{H}$.

(i) $\Rightarrow$ (ii): We have

$$t_k = \sum_{\chi \in \mathrm{Irr}(F\mathcal{H})} \frac{1}{p(S_\chi)}\chi_k\,.$$

In particular, we see that t_k is a linear combination of pseudo–characters of $k\mathcal{H}$, hence $k\mathcal{H}$ is separable by 8.7 above.

(ii) $\Rightarrow$ (i): By 8.2, we know that $\mathrm{Irr}(k\mathcal{H}) = \{\chi_k \mid \chi \in \mathrm{Irr}(F\mathcal{H})\}$. It is then clear that $S_{\chi_k} = p(S_\chi)$, which proves that $p(S_\chi) \neq 0$. □

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