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Open Problems in Geometry and Topology

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1. Thurston's Geometrization Conjecture: Let M be a closed, connected, oriented, irreducible 3-manifold. Then precisely one of the following holds:

   a) $M = S^3/\Gamma$, where $\Gamma \subset \mathrm{Isom}(S^3)$.
   b) $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$.
   c) $M = H^3/\Gamma$, where $\Gamma \subset \mathrm{Isom}(H^3)$.

   Here, S^3 is the 3-sphere with its round metric and H^3 is the open 3 -ball with its hyperbolic metric. Various portions of this conjecture are known to be true, see for example Gabai's article in *Surveys in Differential Geometry, 1996,* S. T. Yau ed., International Press, to appear.

   (This is a well known problem.)

2. Does there exist another knot in the 3-sphere with the same Jones polynomial as the unknot?

   (This is a well known problem.)

3. A generalized knot in S^3 is an oriented, immersed circle with only double point self intersections, and where the two tangent vectors at each double point are distinct. Now, let I denote and ambient, isotopy invariant of generalized knots in S^3 with at least n double points. Suppose that I integrates to an invariant of generalized knots with

$n-1$ double points. Does I then integrate to an invariant of honest knots?

Note that an invariant, I, immersed circles with n double points is said to integrate to an invariant of circles with $n-1$ double points when there exists and invariant of the latter, I', with the property that $I(K) = \sum_{p \in D}(I'(K_{p^+}) - I'(K_{p^-}))$. Here, the sum is over the set D of double points of K; and $K_{p^\pm}$ are the generalized knots with $n-1$ double points which are obtained from K as follows: First, view K so that a neighborhood of p in K looks like and $\times$ with both branches oriented to point towards the top of the page. Then, K_+,is obtained by resolving the cross with the branch/passing over the \ branch. Meanwhile, K is obtained by resolving the double point using the opposite crossing.

(Submitted by D. Bar-Natan.)

4. Is the smooth, 4-dimensional Poincare conjecture true? That is, if a compact, 4-dimensional manifold has vanishing π_1 and π_2, is it necessary diffeomorphic to S^4?

   Note that such a manifold is homeomorphic to S^4 by Freedman's theorem.

   (This is a well known problem.)

5. If $k > 2$, does every smooth, compact, $2k$-dimensional manifold which admits an almost complex structure have the structure of a complex manifold? (The almost complex structure as a complex manifold need not to be homotopic to the given one.)

   Note that when $k = 1$, the answer to the preceding is yes, and when $k = 2$, the answer is no.

   (This is one of S. T. Yau's problems from "Open Problems in Geometry", in *Differential Geometry: Partial Differential Equations on Manifolds*, S .T. Yau and R. Greene, ed., Symposia in Pure Math 154, Part 1, American Mathematical Society, Providence 1993.)

6. Let X be a smooth, projective variety and consider the Quillen-Segal group completion of the holomorphic mapping space, $\mathrm{Hol}(X, BU)^+$, where BU is viewed as a union of Grassmanninans. This space represents the "holomorphic K-theory" of X in that holomorphic maps from X to BU classify holomorphic vector bundles over X that are holomorphically embedded in trivial bundles. It is known that this

theory satisfies a kind of periodicity and sits between algebraic K-theory on the one hand and topological K-theory on the other. With the preceding understood, under what conditions (on X) is the inclusion $\mathrm{Hol}(X, BU)^+ \to \mathrm{Maps}(X, BU)$ a homotopy equivalence?

(Submitted by R. Cohen.)

7. The following is a question asked by John Moore about homotopy groups: Suppose that X is a simply connected, finite complex. If $\dim(\pi^*(X) \otimes \mathcal{Q})$ is finite, is there, for each prime p, and integer k for which p^k annihilates the p-torsion in $\pi^*(X)$?

   This conjecture and others of Moore are surveyed in the article "Moore conjectures" by Paul Sellick in *Algebraic Topology-Rational Homotopy Theory*, Springer Lecture Notes in Math 1318, Y. Felix ed. 1988, Springer-Verlag, Berlin.

   (Submitted by M. Hopkins.)

8. Does the mapping class group of a surface of genus $g \geq 2$ satisfy Kazhdan's Property T?

   What follows is the definition of Property T of a finitely generated group (as is the mapping class group). Fix a finite set of generators for the group. Then, there exists a positive number ε with the following significance: Suppose that $\mathcal{H}$ is a Hilbert space on which the group has a unitary representation. If there exists a vector $v \in \mathcal{H}$ with norm 1 such that $\|\gamma - v\| < \varepsilon$ for all elements γ from the generating set, then there exists a non-zero vector in $\mathcal{H}$ which is fixed by the group.

   For further reference, see "La propriete (T) de Kazhdan pour les groupes localement compacts" by P. de la Harpe and A. Valette in Asterisque **149**(1989).

   (Submitted by B. Farb.)

9. Let π be a Poincare duality group. That is, the Eilenberg-MacLane space $K(\pi, 1)$ satisfies Poincare duality with respect to a fundamental class in $H^m(K; \mathbb{Z})$ for some m. Prove or disprove the "Borel conjecture": $K(\pi, 1)$ is simple homotopy equivalent to a closed, topological m-manifold which is unique up to homeomorphism.

   (Unattributed.)

10. Settle the Euclidean no hair question: Does the four dimensional sphere have a unique Einstein metric (up to diffeomorphism)?

(Submitted by St. T. Yau.)