Quantum Chaos, Symmetry and Zeta Functions
Lecture II: Zeta Functions

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1 The Zeta Function

In the first lecture we discussed the role played by symmetry in local spacing statistics for quantizations of classical Hamiltonians. In this lecture we discuss the role of symmetry in the local spacing distribution between zeros of zeta functions.

We begin with Riemann Zeta Function $\zeta(s)$ and some phenomenology associated with it:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1} \tag{1.1}$$

which converges for $\mathrm{Re}(s) > 1$. $\zeta(s)$ has an analytic continuation and functional equation [1]:

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s) \tag{1.2}$$

$\xi(s)$ has simple poles at $s = 0$ and $s = 1$ and is otherwise analytic. Write the zeros ρ_j of $\xi(s)$ as:

$$\rho_j = \frac{1}{2} + i\gamma_j. \tag{1.3}$$

From (1.1) it is clear that $|\Im(\gamma_j)| \leq \frac{1}{2}$. The well known Riemann Hypothesis "R-H" asserts that $\gamma_j \in \mathbb{R}$. For the following questions of local spacings, let's assume RH (in numerical experiments that are quoted this has been verified for the zeros examined). Order the zeros

$$\cdots \leq \gamma_{-3} \leq \gamma_{-2} \leq \gamma_{-1} \leq \gamma_1 \leq \gamma_2 \cdots \tag{1.4}$$

here $\gamma_{-j} = -\gamma_j, j = 1, 2, \ldots$. It is well known [2] that:

$$|\{j \geq 1 | \gamma_j \leq T\}| \sim \frac{T \log T}{2\pi} \tag{1.5}$$

as $T \to \infty$. Hence we form the local spacings by unfolding:

$$\hat{\gamma}_j := \frac{\gamma_j \log \gamma_j}{2\pi} \tag{1.6}$$

The $\hat{\gamma}_j$'s have mean spacing one.

During the years 1980-1997 Odlyzko [3] has made an extensive and profound numerical study of these zeros and in particular of the local spacings of $\hat{\gamma}_j$. He found that these obey the GUE model perfectly. For example in Figure 1.14 of [4] of Lecture 1, the consecutive spacings for the 7x10^7 zeros beyond the 10^{20}-th zero are plotted against the density $\mu_1(\mathrm{GUE})$. At the phenomenological level this is perhaps the most striking discovery about the zeta function since Riemann. The big question is why is this so and what does it tell us about the nature (e.g. spectral) of the zeros. Also what is the symmetry which is responsible for this GUE or type II symmetric space statistics (cf. (2.4) of Lecture I).

Odlyzko's computations were inspired by the 1974 discovery of Montgomery [5] that the pair-correlation is, at least for a restricted class of test functions, equal to the GUE pair-correlation. Precisely he proves that as $N \to \infty$,

$$\frac{1}{N} \sum_{1 \leq j \neq k \leq N} \phi(\hat{\gamma}_j - \hat{\gamma}_k) \to \int_{-\infty}^{\infty} \phi(x) R_2(GUE)(x) dx \tag{1.7}$$

for any $\phi \in \mathcal{S}(\mathbb{R})$ for which the support of $\hat{\phi}$ is contained in $(-1,1)$ where $\hat{\phi}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi}\phi(x)dx$. Note that $\hat{R}_2(\xi)$ (see (2.8) of Lecture I) changes its analytic character at $\xi = \pm 1$ and indeed the terms contributing to (1.7) come from the "diagonal" [5, 6]. Extending (1.7) to $\hat{\phi}$'s whose support is not contained in [-1,1] involves new non-diagonal contributions and this has never been achieved (see Goldston Montgomery [7] for an equivalence). Note that this already (albeit assuming RH) goes far beyond anything that one can establish for the pair-correlation for the t_j's in Lecture I. The reason is that the unfolding of γ_j is $\gamma_j \log \gamma_j$ while for t_j it is t_j^2. This has the effect on the right hand side of the analogue of (5.1) of Lecture I for $\zeta(s)$ (known as the explicit formula see [6]) of facing $\log p \leq \log T$ terms when support $\hat{\phi} \subset (-1,1)$, while for the t_j case we always have e^T terms (this has been referred to as the exponential proliferation of periodic orbits in the latter case).

More recently Hejhal [8] used Montgomery's method to establish that the triple correlation is the GUE triple correlation as computed in Dyson [9]. Rudnick and Sarnak [6] by a somewhat different method (which in fact does not require RH) establish that all the $n \geq 2$ correlations are as predicted by GUE. All of these results are restricted as above, that is they are proven in the range of the Fourier transforms where only the "diagonal" contributions constitute the main term. An interesting heuristic derivation of the n-level correlations without any restrictions on the Fourier transforms has been given by Bogomolny and Keating [4].

The zeta function is but the first of the zoo of L-functions for which similar questions can be asked. There are the Dirichlet L-functions $L(s,\chi)$ defined as follows: $q \geq 1, \chi : (\mathbb{Z}/q\mathbb{Z})^* \to S^1$ is a character and extend χ to $\mathbb{Z}$ by setting χ to be periodic of period q and $\chi(m) = 0$ if $(m,q) \neq 1$. Then

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}. \tag{1.8}$$

The analogue of (1.2), that is the analytic continuation and functional equation are known for these. Even more generally we have for each automorphic cusp form f on $GL_m/\mathbb{Q}$ [10] an L-function $L(s,f)$, which satisfies similar properties [11]. A classical concrete form on $GL_2/\mathbb{Q}$ is the form $\Delta(q)$ [12],

$$\Delta(q) := q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n \tag{1.9}$$

$$L(s,\Delta) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{11/2}} n^{-s} = \prod_p (1 - \frac{\tau(p)}{p^{11/2}} p^{-s} + p^{-2s})^{-1}. \tag{1.10}$$

In general the L-function of an automorphic form on $GL_m/\mathbb{Q}$ is an Euler product of local factors of degree m in p^{-s}. In all these cases an R-H for $L(s,f)$ is expected to hold.

The results of Rudnick-Sarnak [6] were carried out in this context and they show that the $n \geq 2$ correlations (for ϕ's even more restricted as m grows) are universally the GUE ones! Moreover at the numerical level Rumely [13] has checked that the zeros of Dirichlet L-functions satisfy GUE statistics and Rubinstein [14] has checked various GL_2 L-functions and finds that they all satisfy GUE local spacing statistics. We call this phenomenon, that the high zeros of any L-function $L(s,f)$, f a cusp form on $GL_m/\mathbb{Q}$ obey GUE spacing laws, the "Montgomery - Odlyzko Law".

2 Function Fields

One can get much insight into the source of the Montgomery-Odlyzko Law by considering its function field analogue. The function field analogue of $\zeta(s)$ is due to Artin [15]. If k is a finite extension of the field $\mathbb{F}_q$ of rational functions with coefficients in the finite field $\mathbb{F}_q$, its zeta function $\zeta(k,T)$ is defined to be

$$\zeta(k,T) = \prod_v (1 - T^{\deg(v)})^{-1} \tag{2.1}$$

where the product is over all the places v of k [15]. One can also think of $\zeta(k,T)$ as the zeta function of a nonsingular curve $C/\mathbb{F}_q$ whose field of functions is k. This geometric point of view is very powerful. For example the Riemann-Roch Theorem on the curve plays the role of Poisson-Summation in (1.2) and it yields [16]

$$\zeta(k,T) = \frac{P(k,T)}{(1-T)(1-qT)} \tag{2.2}$$

where P is a polynomial of degree $2g, g$ being the genus of k, as well as a functional equation for $\zeta(k,T)$ when T is replaced by $1/qT$. The Riemann Hypothesis for $\zeta(k,T)$ asserts that all the zeros of P lie on the circle $|T| = q^{-1/2}$. This was established by Weil [17]. A key point in this proof is the interpretation of the zeros of $\zeta(k,T)$ as the reciprocals of the eigenvalues of Frobenius (which is the operation of raising the coordinates of points on the corresponding curve C to the power q) acting on the first cohomology groups of the curve C [17].

Turning to the distribution of the zeros of such a zeta function in (2.1), we write its zeros as:

$$\zeta_j = e^{i\theta_j} q^{-1/2}, j = 1, \ldots, 2g. \tag{2.3}$$

Form the local spacing measures as in (2.1) and (2.2) of Lecture I and denote them by $\mu_k(C/\mathbb{F}_q)$. For a fixed $\zeta(C/\mathbb{F}_q, T)$ there are only $2g(C)$ zeros and so we cannot have a spacing law. We therefore let the genus $g = g(C)$ go to infinity. However this alone will not allow one to deduce a unique limiting law since there are curves $C/\mathbb{F}_q$ of large genus which have a large number of symmetries and for which the local spacings are Poissonian, see [18]. In Katz-Sarnak [18] we therefore consider the typical curve of large genus g and over a large field $\mathbb{F}_q$. We show [18] that as q and g go to infinity the local spacings follow the GUE model, that is the Montgomery-Odlyzko Law is valid for these zetas. Precisely if $\mathcal{M}_g(\mathbb{F}_q)$ denotes the set of isomorphism classes of curves of genus g over $\mathbb{F}_q$, then $k \geq 1$:

$$\lim_{g\to\infty} \lim_{q\to\infty} \frac{1}{|\mathcal{M}_q(\mathbb{F}_q)|} \sum_{C \in M_q(\mathbb{F}_q)} D\left(\mu_k(C/\mathbb{F}_q), \mu_k(GUE)\right) = 0 \tag{2.4}$$

Note that the double limit must be carried out in the order indicated. The key ingredients in the proof of 2.4 are:

- The monodromy group of the family $\mathcal{M}_g$ (or more accurately a closely related family) [18] of curves of genus g, which arises through the representation of the fundamental group of the family on the first cohomology group at a base curve. The first homology group comes with an intersection pairing for cycles and the symplectic pairing is preserved by the monodromy. In fact the monodromy turns out to be the full symplectic group $Sp(2g)$ and this is the key point.

- The Equipartition Theorem of Deligne [19, 20] for the Frobenii for the family, in the monodromy, as $q \to \infty$.

- The Law of Large Numbers (2.5) of Lecture I for the scaling limits of $USp(2g)$ as $g \to \infty$.

Thus in the function field, the source of the GUE is clearly identified. In part it is due to the universality of the local statistics for type II symmetric spaces (2.4) of Lecture I. Also there is a symmetry behind the GUE law — it comes from the scaling limits of the monodromies of the family. We again see that it is more reasonable, at least to begin with, to examine these local spacing statistics for families. In this function field case, at least if the monodromies of the families and their scaling limits can be computed — then one has a complete understanding (at least on letting $q \to \infty$ as is done in (2.4)).

3 Families in the Global case

We return to global zeta or L-functions, that is $L(s,f)$ where f is an automorphic cusp form of $GL_m/\mathbb{Q}$, and consider families $\mathcal{F}$, of such. We do not offer a precise definition of what is meant by a family in this case, but rather (since this is all that we have at present) we give numerous examples of families. The set up is such that each $f \in \mathcal{F}$ has a "conductor" $c_f \in (0,\infty)$ (they are given explicitly in the examples below). For X a real parameter, we assume that the sets $\mathcal{F}_X = \{f \in \mathcal{F} | c_f \leq X\}$ are finite and that the asymptotics of $|\mathcal{F}_X|$ as $X \to \infty$, are known. The scaling statistics which we consider are the distributions of zeros of $L(s,f)$ near $s = 1/2$, as f varies over $\mathcal{F}$ ordered by conductor. That is we examine the numbers:

$$\frac{\gamma_f^{(j)} \log c_f}{2\pi}, 0 \leq \gamma_f^{(1)} \leq \gamma_f^{(2)} \ldots, \tag{3.1}$$

where $\frac{1}{2} + i\gamma_f^{(j)}$ are the nontrivial zeros of $L(s,f)$. That the scaling by $\frac{\log c_f}{2\pi}$ is appropriate will become clear from the results below. To measure these distributions we form for $\mathcal{F}$ the analogues of the measures ν_k (e.g. (2.11) of Lecture I) and the densities D_1 (e.g. (2.12) of Lecture I) as follows:

$$\nu_j(X,\mathcal{F})[a,b] = \frac{1}{|\mathcal{F}_X|} \# \left\{ f \in \mathcal{F} : c_f \leq X, \frac{\gamma_f^{(j)} \log c_f}{2\pi} \in [a,b] \right\} \tag{3.2}$$

and for $\phi \in \mathcal{S}(\mathcal{R})$ a test function set:

$$D(f,\phi) = \sum_{\gamma_f} \phi\left(\frac{\gamma_f \log c_f}{2\pi}\right) \tag{3.3}$$

and

$$W(X,\mathcal{F},\phi) = \frac{1}{|\mathcal{F}_X|} \sum_{c_f \leq X} D(f,\phi). \tag{3.4}$$

Thus $\nu_j(X,\mathcal{F})$ measures the distribution, as f varies over $\mathcal{F}$, of the j-th lowest zero of $L(s,f)$ normalized as in (3.1), while W measures the density the zeros which are within $O(1/\log c_f)$ of $s = 1/2$. One hopes that as $X \to \infty$ the measures $\nu_j(X,\mathcal{F})$ converge to measures $\nu_j(\mathcal{F})$ and the densities converge to $\int_{-\infty}^{\infty} \phi(x) W(\mathcal{F})(x)\, dx$, for a suitable density $W(\mathcal{F})(x)$. Indeed for the function field analogue of the above with various families $\mathcal{F}$, this is proven in [18] using the same methods mentioned in Section 2. For these cases the limiting measures $\nu_j(\mathcal{F})$ and the density $W(\mathcal{F})$ are determined by the "symmetry" $G(\mathcal{F})$ which is the scaling limit of the monodromy groups. They are determined (when G is one of the classical families) by (2.14) and

(2.15) of Lecture I. We now list some examples of families of such $\mathcal{F}$'s for which some results along these line have been established. In all cases we will assume RH for all L-functions (at the cost of restricting the test functions further one can remove this assumption).

I: The family $\mathcal{F}$ of Dirichlet L-functions $L(s,\chi)$ where χ is a primitive quadratic character (that is $\chi^2 = 1$), mod q. The conductor c_χ is equal to q.

- From the function field analogue we expect that $G(\mathcal{F}) = Sp(\infty)$, see [21].
- [21] (see also Ozluk-Snyder [22]) $W(X,\mathcal{F},\phi) \to \int_{-\infty}^{\infty} \phi(x)\omega_1(Sp,x)dx$ as $X \to \infty$ for $\phi \in \mathcal{S}(\mathcal{R})$ with support $\hat{\phi} \subset (-2,2)$. Here $\omega_1(Sp,x)$ is given in (2.16) of Lecture 1.
- Rubinstein [14] has investigated numerically the distributions of $\nu_j(\mathcal{F},X), j = 1,2$ and $W(X,\mathcal{F})$ for $X \approx 10^{12}$ and finds an excellent fit with the $Sp(\infty)$ predictions.
- The first to numerically compute zeros $L(s,\chi)$ in this family for moderate sized q appears to be Hazelgrove. He found that the zeros 'repel' the point $s = 1/2$ and this is sometimes called Hazelgrove's phenomenon. Now the density of $\nu_1(Sp)$ vanishes to second order at 0 (see [18]) and this is unique to the Sp symmetry! So this Hazelgrove phenomenon is a manifestation of the symplectic symmetry.

II: The family $\mathcal{F}$ of quadratic $L(s,\Delta \otimes \chi)$ of the GL_2 cusp form Δ of weight 12 for $\Gamma = SL_2(\mathbb{Z})$, see (1.9) above.

$$L(s,\Delta\otimes\chi) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{11/2}}\chi(n)n^{-s}. \tag{3.5}$$

The conductor $c_{\Delta\otimes\chi}$ is q^2 (where χ has conductor q). In this family half of the L-functions have even functional equations and half, odd functional equation, according to the sign of the "epsilon factor" $\epsilon_{\Delta\otimes\chi}$. We let $\mathcal{F}^{\pm}$ be the corresponding subfamilies.

- From the function field analogue we expect that $G(\mathcal{F}) = O(\infty)$. In particular $G(\mathcal{F})$ corresponds to the scaling limit through $O^+(even) = SO(even)$ or $O^-(odd)$ and $G(\mathcal{F}^-)$ to $O^+(odd) = SO(odd)$ or $O^-(even)$, see [21].
- ([21]) $W(X,\mathcal{F}^+,\phi) \to \int_{-\infty}^{\infty} \phi(x)\omega_1(SO(even),x)dx$,
  $W(X,\mathcal{F}^-,\phi) \to \int_{-\infty}^{\infty} \phi(x)\omega_1(SO(odd),x)dx$
  for ϕ with support $\hat{\phi} \subset (-1,1)$. The explicit densities $\omega_1(SO(even))$ of $\omega_1(SO(odd))$ are given in (2.16) of Lecture I.

- Numerical experimentations by Rubinstein [14] with $\nu_j(X, \mathcal{F}^{\pm}), j = 1,2$ and $W(X, \mathcal{F}^{\pm})$ with $X \approx 10^6$, agree well with the $O(\infty)$ predictions.

III: The family $\mathcal{F}$ of holomorphic (Hecke-eigen)-cusp forms of even integral weight k on $SL_2(\mathbb{Z}) \setminus \mathbb{H}^2$ (see [12]) as $k \to \infty$. For $f \in \mathcal{F}$, $L(s, f)$ is its L-function and its conductor is $C_f = k^2$. As in the last example half of these $L(s, f)$'s have even functional equations and half odd. In fact the sign ϵ_f is 1 if $k \equiv 0(4)$ and -1 if $k \equiv 2(4)$. Let $\mathcal{F}^{\pm}$ be the corresponding subfamilies.

- We expect, since the f's are generic GL_2 forms, that $G(\mathcal{F}) = O(\infty)$.
- Iwaniec-Luo-Sarnak [23] show that $W(X, \mathcal{F}^+, \phi) \to \int_{-\infty}^{\infty} \phi(x) \times$ $\times \omega_1(SO(even), x)dx$ and $W(X, \mathcal{F}^-, \phi) \to \int_{-\infty}^{\infty} \phi(x)\omega_1(SO(odd), x)dx$, for $\hat{\phi}$ supported in (-2,2).

IV: The family $\mathcal{F}$ of holomorphic new-forms of a fixed even integral weight $k \geq 2$ for $\Gamma_0(N) \setminus \mathbb{H}$ [24][1], with $N \to \infty$. We assume that the central character of f is trivial (i.e. trivial Nebentypus) and for simplicity we also assume that N is prime. This time we average over smaller families - that is over all f's above on $\Gamma_0(N)$, with $N \to \infty$. The conductor c_f is N and, as in the last two examples, approximately half of the signs ϵ_f are +1 and half -1. Let $H_k(N)$ denote the set of forms as above and $H_k^{\pm}(N)$ the subsets whose corresponding $\epsilon_f = \pm 1$.

- As in the last family we expect that $G(\mathcal{F}) = O(\infty)$.
- Iwaniec-Luo-Sarnak [23] prove that as $N \to \infty$

$$\frac{1}{|H_k^+(N)|} \sum_{f \in H_K^+(N)} D(\phi, f) \to \int_{-\infty}^{\infty} \phi(x)\omega_1(SO(even), x)dx.$$

$$\frac{1}{|H_k^-(N)|} \sum_{f \in H_K^-(N)} D(\phi, f) \to \int_{-\infty}^{\infty} \phi(x)\omega_1(SO(odd), x)dx.$$

for any $\phi \in \mathcal{S}(\mathbb{R})$ support $\hat{\phi} \subset (-2, 2)$.

V: The family of symmetric square L-functions, $L(S, \vee^2 f)$ (see [25]), where F is in family III. There are Euler products of degree three and, by a Theorem of Gelbart and Jacquet [10], they are L-functions of selfdual cusp froms on GL_3. The conductor $c_{\vee^2 f}$ is k^2. The sign of the functional equation $\epsilon_{\vee^2 f}$ is always equal to 1.

[1]Here

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : N|c \right\}$$

- Being generic selfdual forms on GL_3 we expect $G(\mathcal{F}) = Sp(\infty)$.
- In [23] it is proven that

$$W(X,\mathcal{F},\phi) \to \int_{-\infty}^{\infty} \phi(x)\omega_1(Sp,x)dx$$

as $X \to \infty$ for any $\phi \in \mathcal{S}(\mathbb{R})$ with support $\hat{\phi} \subset (-\frac{4}{3}, \frac{4}{3})$.

Remarks:

1. All of the above results confirm, to the extent that they apply, the predictions of the claimed symmetry $G(\mathcal{F})$. The Conjecture that the density sums $W(X,\mathcal{F},\phi)$ converge to the claimed density without any restrictions on $\hat{\phi}$, will be called the Density Conjecture for the family $\mathcal{F}$.

2. The proofs of the results about the densities all proceed by expressing $D(f,\phi)$ via the explicit formula in terms of sums involving the Hecke eigenvalues of f. What is then needed are techniques for averaging the latter over $f \in \mathcal{F}$. For the families III, IV and V we use heavily the tools developed in Iwaniec-Sarnak [26] (see below) for dealing with these averages.

3. With the exception of II, all the results allow for the support of $\hat{\phi}$ to be larger than [-1,1]. This is rather significant since $\hat{\omega}_1(Sp)(\xi)$, $\hat{\omega}_1(SO(even))(\xi)$ and $\hat{\omega}_1(SO(odd))(\xi)$ are all discontinious at $\xi = \pm 1$. This signals that new terms ("nondiagonal")enter into the main terms of the asymptotics as soon as support $\hat{\phi}(\xi)$ is larger than [-1,1]. Thus what is shown here goes beyond anything established for the correlations of high zeros (see the discussion following (1.7)), or for that matter the diagonal analysis of Berry [27] (see the discussion after (5.1) in Lecture I in the analogous analysis with the trace formula). For the families III, IV and V these new non-diagonal terms arise out of a far reaching analysis with Kloosterman sums (see [26] and [28] for related issues). That these fundamentally new nondiagonal contributions yield the conjectured $G(\mathcal{F})$ answers is very pleasing evidence for the conjectures.

4 Applications

The interest in the zeros of L-functions lies in their fundamental influence on arithmetical problems. For example the question of vanishing of an L-function at special points on the critical line arises in the Birch and

Swinnerton-Dyer Conjectures [29, 30], in the Shimura correspondence (see [31]) and in spectral deformation theory (Phillips-Sarnak [32]). The distribution of zeros near $s = 1/2$ (that is the central value) discussed in Section 2.3 has immediate application to nonvanishing at this point. By varying the test function ϕ in the Density Conjecture for any of the above families $\mathcal{F}$, together with the fact that $W(\mathcal{F})$ does not give positive mass to the point zero, implies (assuming the Density Conjecture) that as $X \to \infty$,

$$\frac{\#\{f \in \mathcal{F} | c_f \leq X, \epsilon_f = 1, L(\tfrac{1}{2}, f) \neq 0\}}{\#\{f \in \mathcal{F} | c_f \leq X, \epsilon_f = 1\}} \to 1, \tag{4.1}$$

and

$$\frac{\#\{f \in \mathcal{F} | c_f \leq X, \epsilon_f = -1, L'(\tfrac{1}{2}, f) \neq 0\}}{\#\{f \in \mathcal{F} | c_f \leq X, \epsilon_f = -1\}} \to 1. \tag{4.2}$$

The results of the last Section are approximations to the density Conjecture and give corresponding approximations to (4.1) and (4.2). We illustrate this with the family IV and with $k = 2$, this being perhaps the most interesting arithmetically. By choosing $\phi \in \mathcal{S}(\mathbb{R})$ so that $\phi(0) = 1$, $\phi(x) \geq 0$ and $\int_{-\infty}^{\infty} \phi(x) W(\mathcal{F}, x) dx$ is minimized (see [23]) we conclude from the density results in subsection 2.3 about family IV (which recall assume RH for automorphic L-functions) that for N, prime and large enough:

$$\frac{\#\{f \in H_2^+(N) | L(\tfrac{1}{2}, f) \neq 0\}}{\#\{f \in H_2^+(N)\}} > \frac{9}{16} \tag{4.3}$$

$$\frac{\#\{f \in H_2^-(N) | L'(\tfrac{1}{2}, f) \neq 0\}}{\#\{f \in H_2^-(N)\}} > \frac{15}{16} \tag{4.4}$$

and

$$\frac{|H_2(N)|}{2} + o(|H_2(N)|) \leq \sum_{f \in H_2(N)} ord(\frac{1}{2}, L(s, f)) < \frac{99}{100} |H_2(N)|, \tag{4.5}$$

where $ord(s_0, L(s, f))$ is the order of vanishing of $L(s, f)$ at $s = s_0$. Note that $|H_2(N)| \sim \frac{N}{12}$ and as Murty [33] shows (and this does not assume RH) that $|H_2^{\pm}(N)| \sim \frac{|H_2(N)|}{2}$, the lower bound in (4.5) is immediate. Concerning the upper bound in (4.5), Brumer [34] establishes such a result with 99/100 replaced by 3/2. One can reduce the 3/2 to 1 without appealing to the "off-diagonal" analysis of the last Section but to get anything below 1 already relies on extended ranges. A similar remark applies to (4.3), the off diagonal analysis allowing a lower bound bigger than 50%. This is significant as we will see below.

We can apply (4.3), (4.4), and (4.5) to the ranks of the Jacobians, $J_0(N)/\mathbb{Q}$, of the curves $X_0(N)$ (equal analytically to $\Gamma_0(N) \setminus \mathbb{H}$), by combining these results with known partial results to the Birch and Swinnerton-Dyer Conjecture (Kolyvagin [29] and Gross-Zagier [30]). Let $M_0(N)/\mathbb{Q}$ be the quotient of $J_0(N)$ considered by Merel [35]. It corresponds to the f's in $H_2^+(N)$ for which $L(\frac{1}{2}, f) \neq 0$ and is no doubt the largest quotient of $J_0(N)$ which is of rank zero. It is of great interest to know its size. Brumer [34] has computed these for $N \leq 10^4$ and based on his findings he conjectures that:

$$\lim_{N\to\infty} \frac{\dim M_0(N)}{|H_2^+(N)|} = 1 \tag{4.6}$$

$$\lim_{N\to\infty} \frac{\operatorname{rank} J_0(N)}{\dim J_0(N)} = \frac{1}{2}. \tag{4.7}$$

Note that the Density Conjectures for this family via (4.1) and (4.2), and [29] and [30] imply these Conjectures of Brumer. In the same way (4.3) and (4.4) imply (still under RH) that for N large:

$$\dim M_0(N) > \frac{9}{16}|H_2^+(N)| \tag{4.8}$$

and

$$\operatorname{rank} J_0(N) > \frac{15}{32}\dim(J_0(N)). \tag{4.9}$$

Moreover assuming the Birch and Swinnerton-Dyer Conjectures as well as (4.5) yields, that for N large

$$\frac{\dim J_0(N)}{2} + o(N) \leq \operatorname{rank} J_0(N) \leq \frac{99}{100}\dim J_0(N). \tag{4.10}$$

It is remarkable that the results (4.3), (4.4) and (4.5) can be established unconditionally with almost as good quality. The techniques to achieve this are quite different and more sophisticated than those used for the density results for the families III, IV and V, though they both make use of the methods for averaging developed in [26]. In [36], Duke examines the averages of $L(\frac{1}{2}, f)$ and $L^2(\frac{1}{2}, f)$ over the family $H_2(N)$ and this allows him to show that at least $N/(\log N)^2$ of the $L(\frac{1}{2}, f)$'s are not zero. Introducing "mollifiers" and other tools into the analysis of averages of $L(\frac{1}{2}, f)$ and $L^2(\frac{1}{2}, f)$, Iwaniec and Sarnak [26] show the following:

$$\lim_{N\to\infty} \frac{\#\{f \in H_2^+(N) | L(\frac{1}{2}, f) \geq (\log N)^{-2}\}}{\#\{f \in H_2^+(N)\}} \geq \frac{1}{2}. \tag{4.11}$$

This unconditional result is rather close to the conditional result (4.3) and moreover the 50% is of fundamental significance. In [26] it is shown that if (4.11) holds with any $C > 1/2$ in place of $1/2$ on the right hand side, then there are no Siegel zeros! Of course the conditional result (4.3) is of no relevance here since tautologically the RH's imply that there are no Siegel zeros. Using variations of the techniques above among many other ideas

Kowalski and Michel [37] and independently VanderKam [38] have shown that (4.3) and (4.4) hold unconditionally for some positive constants on the right hand sides. All of these unconditional results when combined with [29] and [30] lead to corresponding unconditional results towards Brumer's Conjectures. In another work, Kowalski and Michel [39] established that the upper bound in (4.5) holds unconditionally with 99/100 replaced by a large constant C.

5 Conclusion

P. Cohen once remarked to me that in a Colloquium talk, the first quarter should be understandable by everyone, the second by the experts, the third by the speaker and the end by no one. We now enter this final phase — at least as far as this speaker goes.

The results for function fields, the numerical experiments and the analytic results about densities all point convincingly to the fact that the distribution of zeros for families follow the $G(\mathcal{F})$ distributions. It is of course possible that $G(\mathcal{F})$ is simply an excellent model for predicting these densities. However based on what happens in the function field we believe that there is in fact a symmetry group in the global case which is the source of all of these phenomena. At a highly speculative level we expect (see [21]) that there is a natural spectral interpretation of the zeros of each $L(s, f)$ in terms of the eigenvalues of an operator $U(f)$ on a Hilbert space H (an interesting candidate for a spectral interpretation of the zeros of $L(s, \chi)$'s has been put forth by Connes [40]). Furthermore for one of our families $\mathcal{F}$ of such f's we expect that these $U(f)$'s can all be naturally defined on the same H. The symmetry $G(\mathcal{F})$ will then take the form that the corresponding operators $U(\mathcal{F})$ all preserve a corresponding structure on H (e.g. symplectic or orthogonal). The source of the distribution laws for families might then come from a grand "Chebotarev Theorem" asserting that as f varies over $\mathcal{F}$ with $c_f \leq X$, the $U(f)$'s become equidistributed in the corresponding space of operators. From this point of view it would follow from the Law of Large Numbers (2.5) of Lecture I and the universality of type II symmetric spaces, that for the typical member $f \in \mathcal{F}$, $L(s, f)$ satisfies the Montgomery-Odlyzko Law. That every $L(s, f)$ should satisfy this law, i.e. individually, is then special to the global L-functions (as mentioned before it does not apply in the function field or in the analogous Hamiltonian setting). In order to understand the symmetry of an individual $L(s, f)$ one should put the L-function in as small as possible family. For example, the Riemann Zeta function sits in the family I of Section 2.3 for which $G(\mathcal{F}) = Sp(\infty)$.We infer that in the proposed spectral interpretation of the zeros of the Riemann Zeta function the operator should preserve a natural symplectic structure!

The theme that there is a theory of families for global L-functions is a welcome one, since the proof by Deligne [19, 20] of the Weil Conjectures for zeta functions of varieties over finite fields (that is the generalization of the Riemann Hypothesis for function fields) uses the monodromy of families in a fundamental way.

To end we remark that one lesson that may be learned from this discussion on zeta functions that may apply to the case of Hamiltonians and in particular the Basic Conjectures is the following: In formulating the basic Conjecture for a family of Hamiltonians (i.e. that the measure theoretically typical member satisfies the Basic Conjecture) there should be a calculation which ensures that the family is large enough — just as the calculation of the monodromy being large, was crucial in the proof of (2.4).

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