Communications in
Analysis and Geometry
Volume 31, Number 9, 2255–2275, 2023

Kähler-Einstein metrics and eigenvalue gaps

Bin Guo, Duong H. Phong, and Jacob Sturm

The existence of Kähler-Einstein metrics on a Fano manifold is characterized in terms of a uniform gap between 0 and the first positive eigenvalue of the Cauchy-Riemann operator on smooth vector fields. It is also characterized by a similar gap between 0 and the first positive eigenvalue for Hamiltonian vector fields. The underlying tool is a compactness criteria for suitably bounded subsets of the space of Kähler potentials which implies a positive gap.

1. Introduction

Starting with the works of Calabi [4] and Yau [28], a central problem in Kähler geometry has been determining when a complex manifold admits a constant scalar curvature Kähler metric in a given Kähler class. One of the first obstructions to the existence of a cscK Kähler metric is the vanishing of the Futaki invariant, which is a character defined on the Lie algebra of holomorphic vector fields. The Yau-Tian-Donaldson conjecture [12, 26, 29] (see also [23] for a review) asserts that the existence of a Kähler metric with constant scalar curvature should be equivalent to the algebro-geometric notion of K-stability. Two recent major advances on this conjecture have been the solution of X.X. Chen, S. Donaldson, and S. Sun [7–9] of the case of Kähler-Einstein metrics on Fano manifolds, and the more recent works by X.X. Chen and J.R. Cheng [5, 6], which established the equivalence between the existence of a Kähler metric with constant scalar curvature and an analytic notion of K-stability.

The K-stability condition of a Kähler class is the requirement that the generalized Futaki invariant attached to a test configuration be non-negative, and vanish only if the test configuration is a product. It is only one possible characterization of the existence of a canonical metric, and for both

Work supported in part by the National Science Foundation under grants DMS-1855947 and DMS-1945869.

geometric and analytic reasons, it may be useful to have other characterizations as well. In the case of Kähler-Einstein metrics, which are the focus of the present paper, a notion of δ-invariant has been proposed by Fujita-Odaka [13] and Blum and Jonsson [3], and it has been shown by R. Berman, S. Boucksom, and M. Jonsson [1] that the existence of a Kähler-Einstein metric is equivalent to the δ-invariant being greater or equal to 1. In a different and even earlier direction, it had been shown in [21, 23, 30] that the Kähler-Ricci flow converges if the lowest strictly positive eigenvalue of the $\bar\partial$ operator on vector fields remains bounded uniformly away from 0 along the flow. It was suggested there [21–23] that it may be possible to characterize the existence of a Kähler-Einstein metric in terms of lower bounds for this eigenvalue, and this is the problem which we solve in the present paper.

More precisely, let X be a compact Kähler manifold with $c_1(X) > 0$. Fix a reference metric $\omega_0 \in c_1(X)$. For any $\omega \in c_1(X)$, let $K_{\omega_0}(\omega)$ be the K-energy[1] of ω with respect to the reference metric $\omega_0 \in c_1(X)$, and u_ω be the normalized Ricci potential of ω, as defined in (2.1) below. We define λ_ω to be the lowest strictly positive eigenvalue of the $\bar\partial$ operator on the space $T^{1,0}(X)$ of $(1,0)$-vector fields, i.e.,

$$\lambda_\omega = \inf_{V \in T^{1,0}(X), V \perp_\omega H^0(X, T^{1,0})} \frac{\|\bar\partial V\|_\omega^2}{\|V\|_\omega^2} \tag{1.1}$$

where the subindex denotes the L^2 norms taken with respect to the metric ω, and $\perp_\omega$ indicates the perpendicularity condition with respect to ω. Let R_ω be the scalar curvature of ω. For each $A > 0$, we introduce the following subset of the space of Kähler metrics in $c_1(X)$,

$$c_1(X; A) = \{\omega \in c_1(X); \|u_\omega\|_{C^0} + \|\nabla_\omega u_\omega\|_{C^0} + \|R_\omega\|_{C^0} \le A,\ K_{\omega_0}(\omega) \le A\}, \tag{1.2}$$

and the corresponding *eigenvalue gap* for the set $c_1(X; A)$ by

$$\lambda(X; A) = \inf_{\omega \in c_1(X; A)} \lambda_\omega. \tag{1.3}$$

If $c_1(X; A)$ is empty, we define $\lambda(X; A) = \infty$. Then we have the following characterizations of the existence of a Kähler-Einstein metric:

---

[1]We refer the readers to [24] for the definition of K-energy and other relevant functionals.

Theorem 1. *Let X be a compact Kähler manifold with $c_1(X) > 0$ and vanishing Futaki invariant. Then X admits a Kähler-Einstein metric if and only if $\lambda(X, A) > 0$ for any $A > 0$.*

Note that although the definition of the K-energy requires a choice of reference metric ω_0, under a change of reference metric, it just shifts by a constant. Thus the above condition is invariant under a change of reference metric, as it should be.

To explain the second characterization, we recall the following observations due by Futaki [14] (see also [21], Lemma 2). For any metric $\omega \in c_1(X)$, the differential operator operating on smooth functions $L_\omega f = -g^{i\bar{j}}\partial_i\partial_{\bar{j}} f + g^{i\bar{j}}\partial_i u_\omega\, \partial_{\bar{j}} f - f$ is non-negative, and its kernel is the space of functions f with ∇f a holomorphic vector field. Let μ_ω be the smallest positive eigenvalue of L_ω. Then the corresponding eigenfunctions f satisfy the identity

$$\int_X |\bar{\nabla}\bar{\nabla} f|^2 e^{-u_\omega}\omega^n = \mu_\omega \int_X |\bar{\nabla} f|^2 e^{-u_\omega}\omega^n. \tag{1.4}$$

Moreover

$$\mu_\omega = \inf_{f\in C^\infty(X),\, \int_X f e^{-u_\omega}\omega^n = 0} \frac{\int_X |\bar{\nabla}\bar{\nabla} f|^2 e^{-u_\omega}\omega^n}{\int_X |\bar{\nabla} f|^2 e^{-u_\omega}\omega^n} \tag{1.5}$$

We introduce, in analogy with (1.3), the eigenvalue gap for Hamiltonian vector fields by

$$\mu(X;A) = \inf_{\omega\in c_1(X;A)} \mu_\omega. \tag{1.6}$$

Theorem 2. *Let X be a compact Kähler manifold with $c_1(X) > 0$ and vanishing Futaki invariant. Then X admits a Kähler-Einstein metric if and only if $\mu(X, A) > 0$ for any $A > 0$.*

For each $A > 0$, we have the easy bound $\mu(X;A) \geq c_A\,\lambda(X;A)$ for some positive constant c_A. Thus the condition $\lambda(X;A) > 0$ in Theorem 1 implies the condition $\mu(X;A) > 0$ in Theorem 2. However, there does not appear to be a direct way to show that they are equivalent.

We now describe briefly our approach. One direction in Theorem 1 and Theorem 2 is known, by combining the work of Perelman on the Kähler-Ricci flow with the convergence results of [21] and [30]. The main problem is to establish the other direction, namely that the existence of a Kähler-Einstein metric on X implies that the gaps $\lambda(X;A)$ and $\mu(X;A)$ are strictly positive for any $A>0$. For each fixed ω, the eigenvalues λ_ω and μ_ω are positive by definition. So the desired statement can be interpreted as a compactness statement with respect to a suitable topology. Our strategy for such a statement is to view the Kähler potential φ of a metric $\omega \in c_1(X,A)$ as the solution of a Monge-Ampère equation with right hand side depending on the Ricci potential u_ω. The C^α estimates are derived by combining the theorem of Skoda-Zeriahi [31] with that of Kolodziej [16] following the idea of Guedj [2]. Then the $C^{3,\alpha}$ priori estimates can be obtained by combining methods for the Monge-Ampère equation together with the recent techniques introduced by Chen-Cheng [5] for the constant scalar curvature problem. Next, the $C^{2,\alpha}$ bounds imply the uniform equivalence of the metrics. This implies in turn uniform estimates of the corresponding eigenvalues on vector fields, using the arguments of [19] to handle the orthogonality condition with different metrics to holomorphic vector fields. The desired theorems follow.

2. $C^{1,\alpha}$ estimates for metrics in $c_1(X;A)$

First we set up the equation. Let n be the dimension of X. If ω is any metric in $c_1(X)$, we define its Ricci potential u_ω by

$$Ric(\omega) - \omega = -i\partial\bar\partial u_\omega, \qquad \int_X e^{-u_\omega}\omega^n = \int_X \omega^n \tag{2.1}$$

where $Ric(\omega) = -i\partial\bar\partial \log \omega^n$ is its Ricci curvature form. Fix now a reference metric $\omega_0 \in c_1(X)$, and let $Ric(\omega_0)$ and u_0 be its Ricci form and Ricci potential, respectively. We can then write $\omega = \omega_0 + i\partial\bar\partial\varphi$, where φ is normalized to satisfy $\sup_X \varphi = 0$. Since

$$-i\partial\bar\partial(u_\omega - u_0) + i\partial\bar\partial\varphi = Ric(\omega) - Ric(\omega_0) = -i\partial\bar\partial \log\frac{\omega^n}{\omega_0^n} \tag{2.2}$$

we find that φ satisfies the following complex Monge-Ampère equation

$$(\omega_0 + i\partial\bar\partial\varphi)^n = e^{u_\omega - u_0 - \varphi + c_\varphi}\omega_0^n \tag{2.3}$$

where c_φ is a specific constant, which is determined because φ, u_ω, and u_0 have all been normalized. It follows from the normalization of u_ω and φ that $c_\varphi \leq 0$.

2.1. The C^α estimates on potential

The first step is the following C^α estimate.

Lemma 1. *Assume that X admits a Kähler-Einstein metric $\omega_{\rm KE}$, which we take as reference metric $\omega_0 = \omega_{\rm KE}$. Then there exists $\alpha > 0$ with the following property. For any $\omega_\varphi \in c_1(X; A)$, there exists an automorphism g of X such that $\|\psi - \sup\psi\|_{C^\alpha(\omega_{\rm KE})} \leq C(A)$, where $g^*\omega_\varphi = \omega_{\rm KE} + i\partial\bar\partial\psi$.*

Proof. Since X admits a Kähler-Einstein metric, we can apply the Moser-Trudinger inequality. An early form of this inequality was first proved in [26], a sharp version subsequently in [20] in the case of manifolds without holomorphic vector fields, and the full sharp and general version in [11]. In this form, it asserts that there exists an $\epsilon > 0$ depending on X with the following property: for any $\omega_\varphi \in c_1(X; A)$, there exists a $g \in G$ (here G is the automorphism group of X) such that

$$A \;\geq\; K_{\omega_{\rm KE}}(\omega_\varphi) \;\geq\; \epsilon J_{\omega_{\rm KE}}(g^*\omega_\varphi) - \frac{1}{\epsilon} \tag{2.4}$$

where $J_{\omega_{\rm KE}}(\phi) = \int \varphi\omega^n_{\omega_{\rm KE}} - E_{\omega_{\rm KE}}(\phi)$ and $E_{\omega_{\rm KE}}$ is the Aubin-Yau functional ([24]) with reference metric $\omega_{\rm KE}$. Thus if we write $g^*\omega_\varphi = \omega_{\rm KE} + i\partial\bar\partial\psi$ we have

$$\begin{aligned}\psi - \sup\psi \;&\in\; S_{A_0}\\ &=\; \{\theta \in \mathcal{E}_1(X, \omega_{\rm KE}) \subseteq {\rm PSH}(X, \omega_{\rm KE}) \;:\; \sup_X \theta = 0 \;\text{ and }\; J_{\omega_{\rm KE}}(\omega_\theta) \leq A_0\}\end{aligned}$$

where $A_0 = (A + \epsilon^{-1})\epsilon^{-1}$, ${\rm PSH}(X, \omega_{\rm KE})$ is the space of plurisubharmonic functions and $\mathcal{E}_1(X, \omega_0) = \{\varphi \in {\rm PSH}(X, \omega_{\rm KE}) | \varphi \in L^1(X, \omega^n_\varphi)\}$ is the space of finite energy potentials. We now claim:

1) S_{A_0} is compact with respect to the weak $L^1(\omega_0^n)$ topology on ${\rm PSH}(X, \omega_0)$
2) Every element of S_{A_0} has zero Lelong number at z for all $z \in X$.
3) For every $p \geq 1$ there exists $C(p, \omega_0, A)$ such that

$$\int_X e^{-p\theta}\omega_0^n \;\leq\; C(p, \omega_{\rm KE}, A) \quad \text{for all } \theta \in S_{A_0}\ . \tag{2.5}$$

These follow as in [2] respectively from Lemma 4.13, Proposition 2.13 and Theorem 4.15 (due to Skoda and Zeriahi) of [10]. Next, applying (2.3) with $\varphi = \psi - \sup_X \psi$, we obtain for $\omega \in c_1(X; A)$,

(2.6)
$$(\omega_{\rm KE} + \sqrt{-1}\partial\bar\partial\psi)^n \;=\; e^{-(\psi - \sup\psi) + u_\psi + c_\psi}\omega_{\rm KE}^n \;\le\; C(A)e^{-(\psi-\sup\psi)}\omega_{\rm KE}^n$$

where we have used the fact that $|u_\psi| \le A$ and $c_\psi \le 0$. Now if we apply (2.5) to (2.6) we obtain that $\|\psi - \sup\psi\|_{C^\alpha(\omega_{\rm KE})} \le C(A)$ for some $\alpha = \alpha(n,p) \in (0,1)$ by the theorem of Kolodziej [16]. □

2.2. $C^{3,\alpha}$ estimates on potentials

We return to the study of the equation (2.3), for a general compact Kähler manifold X and reference metric ω_0, not necessarily Kähler-Einstein. The goal of the present subsection is to establish the following lemma:

Lemma 2. *Let φ be a smooth solution of the Monge-Ampère equation (2.3). Assume that $\|\varphi\|_{C^0} \le A$, $\|u_\omega\|_{C^0} + \|\nabla_\omega u_\omega\|_{C^0} + \|\Delta_\omega u_\omega\|_{C^0} \le A$. Then for any $\alpha \in (0,1)$, there exists a constant $C = C(n, A, \omega_0, \alpha) > 0$ so that*

$$\|\varphi\|_{C^{3,\alpha}(X,\omega_0)} \le C. \tag{2.7}$$

It is convenient to set $F = -\varphi + u_\omega - u_0 + c_\varphi$, so the equation can be written as

$$(\omega_0 + \partial\bar\partial\varphi)^n = e^F\omega_0^n, \quad \sup_X \varphi = 0, \tag{2.8}$$

and note that F depends on the Kähler potential φ. To simplify the notation, we shall denote u_ω by just u. Under the assumptions of the lemma, both φ and u are bounded, so it follows from the fact that ω and ω_0 have the same volume that $|c_\varphi|$ is bounded by a constant $C(A,\omega_0)$ as well. Thus we have

$$0 < \frac{1}{C(A,\omega_0)} \le e^F \le C(A,\omega_0). \tag{2.9}$$

We divide the proof of the lemma into the following steps:

1) Apply Chen-Cheng's argument [5] to show $\Delta_{\omega_0}\varphi$ is in $L^p(X,\omega_0)$ for any $p > 0$, hence $\varphi \in C^{1,\beta}(X,\omega_0)$ for any $\beta \in (0,1)$ by elliptic estimates. Here Δ_{ω_0} is the Laplacian with respect to the reference metric ω_0.
2) The Hölder continuity of φ and the assumption $\|\nabla u\|^2_{C^0(X,\omega)} \le C(A)$ implies that $u \in C^{0,\alpha'}(X,\omega_0)$ (see Lemma 6 below).

3) By a theorem of Li-Li-Zhang [17] (which is an improvement of a result of Yu Wang [27]), we get the $C^{2,\alpha'}(X,\omega_0)$ bound for φ.
4) After we show $u \in C^{1,\alpha''}(X,\omega_0)$ by elliptic estimates, we get the $C^{3,\alpha}$ estimate for φ by differentiating the Monge-Ampère equation (2.3).

We begin by modifying the arguments in Chen-Cheng [5] to derive the following estimates:

Lemma 3. *There exists a constant $C = C(A, n, \omega_0) > 0$ with*

$$\sup_X \|\nabla\varphi\|^2_{C^0(X,\omega_0)} \le C. \tag{2.10}$$

Proof. Denote $\Phi = -F - \lambda\varphi + \frac{1}{2}\varphi^2$ with a constant $\lambda > 0$ to be chosen later. We calculate

$$\begin{aligned}\Delta_\omega\big(e^\Phi(|\nabla\varphi|^2_{\omega_0} + 3)\big) = (|\nabla\varphi|^2_{\omega_0} + 3)\Delta_\omega e^\Phi + 2e^\Phi Re\langle\nabla\Phi, \bar\nabla|\nabla\varphi|^2_{\omega_0}\rangle_\omega \\ + e^\Phi\Delta_\omega|\nabla\varphi|^2_{\omega_0}.\end{aligned} \tag{2.11}$$

We consider the first term in (2.11).

$$\begin{aligned}\Delta_\omega e^\Phi &= e^\Phi(\Delta_\omega\Phi + |\nabla\Phi|^2_\omega) \\ &= e^\Phi\big((\lambda-1)\Delta_\omega(-\varphi) - \Delta_\omega u + \Delta_\omega u_0 + \varphi\Delta_\omega\varphi + |\nabla\varphi|^2_\omega + |\nabla\Phi|^2_\omega\big) \\ &\ge e^\Phi\big((\lambda - 1 - \varphi - C_0)\mathrm{tr}_\omega\omega_0 - C + |\nabla\varphi|^2_\omega + |\nabla\Phi|^2_\omega\big)\end{aligned}$$

where we used the assumption that $|\Delta_\omega u| \le C(A)$, and $C_0 = C_0(\omega_0) > 0$ is a constant satisfying $-C_0\omega_0 \le i\partial\bar\partial u_0 \le C_0\omega_0$.

To deal with the third term in (2.11), we introduce a normal coordinates system for ω_0 at the maximum point $x_0 \in X$ of $e^\Phi(3 + |\nabla\varphi|^2_{\omega_0})$ such that $g_0 = (\tilde g_{i\bar j}) = (\delta_{ij})$ and $dg_0 = 0$ at the point. Moreover, $\omega = (g_{i\bar i}\delta_{ij})$ is diagonal at x_0. We calculate at x_0,

$$\begin{aligned}
\Delta_\omega |\nabla\varphi|^2_{\omega_0} &= g^{p\bar p}\frac{\partial^2}{\partial z_p \partial \bar z_p}(\tilde g^{i\bar j}\varphi_{\bar j}\varphi_i)\\
&= g^{p\bar p}\frac{\partial^2 \tilde g^{i\bar j}}{\partial z_p\partial \bar z_p}\varphi_{\bar j}\varphi_i + g^{p\bar p}\varphi_{\bar i p}\varphi_{i\bar p} + g^{p\bar p}\varphi_{\bar i\bar p}\varphi_{ip}\\
&\quad + g^{p\bar p}\phi_i\frac{\partial^2\varphi_{\bar i}}{\partial z_p\partial\bar z_p} + g^{p\bar p}\varphi_{\bar i}\frac{\partial^2\varphi_i}{\partial z_p\partial \bar z_p}\\
&= \tilde R_{j\bar k p\bar p} g^{p\bar p}\varphi_k\varphi_{\bar j} + g^{p\bar p}\varphi_{\bar i p}\varphi_{i\bar p} + g^{p\bar p}\varphi_{\bar i\bar p}\varphi_{ip} + g^{p\bar p}\varphi_i\frac{\partial^2\varphi_{\bar i}}{\partial z_p\partial\bar z_p}\\
&\quad + g^{p\bar p}\varphi_{\bar i}\frac{\partial^2\varphi_i}{\partial z_p\partial\bar z_p}\\
&\geq -C_1\mathrm{tr}_\omega\omega_0|\nabla\varphi|^2_{\omega_0} + g^{p\bar p}\varphi_{\bar i p}\varphi_{i\bar p} + g^{p\bar p}\varphi_{\bar i\bar p}\varphi_{ip} + 2Re(\varphi_i F_{\bar i}),
\end{aligned}$$

where $\tilde R_{i\bar j k\bar l}$ is the bisectional curvature of the metric g_0, $-C_1$ is a lower bound of $\tilde R_{i\bar j k\bar l}$, and in the last inequality we have used the equation below by taking derivatives on both sides of (2.8)

$$g^{p\bar p}\frac{\partial^2\varphi_i}{\partial z_p\partial\bar z_p} = F_i, \text{ at } x_0.$$

Therefore, we get

$$\begin{aligned}
\Delta_\omega\big(e^\Phi(|\nabla\varphi|^2_{\omega_0}+3)\big) \geq e^\Phi\Big\{&(|\nabla\varphi|^2_{\omega_0}+3)\\
&\times\Big((\lambda-1-\varphi-C_0)\mathrm{tr}_\omega\omega_0 - C + |\nabla\varphi|^2_\omega + |\nabla\Phi|^2_\omega\Big)\\
&- C_1\mathrm{tr}_\omega\omega_0|\nabla\varphi|^2_{\omega_0} + g^{p\bar p}\varphi_{\bar i p}\varphi_{i\bar p} + g^{p\bar p}\varphi_{\bar i\bar p}\varphi_{ip}\\
&+ 2Re(\varphi_{\bar i}F_i) + 2Re\big(g^{i\bar i}\Phi_i(\varphi_j\varphi_{\bar j\bar i} + \varphi_{j\bar i}\varphi_{\bar j})\big)\Big\}. \qquad (2.12)
\end{aligned}$$

The last two terms are equal to (note that at x_0, $\varphi_{j\bar i} = \varphi_{i\bar i}\delta_{ij} = (g_{i\bar i}-1)\delta_{ij}$)

$$\begin{aligned}
&2Re\big(-\Phi_i\varphi_{\bar i} - (\lambda-\varphi)|\nabla\varphi|^2_{\omega_0} + g^{i\bar i}\Phi_i(g_{i\bar i}-1)\varphi_{\bar i}\big)\\
=& -2(\lambda-\varphi)|\nabla\varphi|^2_{\omega_0} - 2Re\big(\langle\nabla\Phi,\bar\nabla\varphi\rangle_\omega\big)\\
\geq& -2(\lambda-\varphi)|\nabla\varphi|^2_{\omega_0} - |\nabla\varphi|^2_\omega - |\nabla\Phi|^2_\omega,
\end{aligned}$$

the last two terms on the RHS can be absorbed by the corresponding terms in the first line on the right hand side in (2.12), while

$$2Re(g^{i\bar i}\Phi_i\varphi_j\varphi_{\bar i\bar j}) \geq -g^{i\bar i}\Phi_i\Phi_{\bar i}\varphi_j\varphi_{\bar j} - g^{i\bar i}\varphi_{ij}\varphi_{\bar i\bar j} = -|\nabla\varphi|^2_{\omega_0}|\nabla\Phi|^2_\omega - g^{i\bar i}\varphi_{ij}\varphi_{\bar i\bar j},$$

and the right hand side above can also be absorbed by terms in the first and second lines of the right hand side in (2.12). So we get by combining the above that at x_0

(2.13)
$$\begin{aligned} 0 \geq & \Delta_\omega\big(e^\Phi(|\nabla\varphi|^2_{\omega_0}+3)\big)\\ \geq & e^\Phi\Big\{(|\nabla\varphi|^2_{\omega_0}+3)\\ & \times\Big((\lambda-1-\varphi-C_0-C_1)\mathrm{tr}_\omega\omega_0 - C + \frac{1}{2}|\nabla\varphi|^2_\omega\Big) - 2(\lambda-\varphi)|\nabla\varphi|^2_{\omega_0}\Big\}\\ \geq & e^\Phi\Big(|\nabla\varphi|^2_{\omega_0}(\mathrm{tr}_\omega\omega_0+\frac{1}{2}|\nabla\varphi|^2_\omega) - C|\nabla\varphi|^2_{\omega_0} - C\Big)\\ \geq & e^\Phi\Big(c(n,A)|\nabla\varphi|^{2(1+\frac{1}{n})}_{\omega_0} - C|\nabla\varphi|^2_{\omega_0} - C\Big) \end{aligned}$$

where we choose $\lambda = 2 + \|\varphi\|_{L^\infty} + C_0 + C_1$. In the last step we apply the inequality below which follows from Young's inequality (i.e. for $a, b \geq 0$, $a^{\frac{1}{n}} b^{\frac{n-1}{n}} \leq c(n)(\frac{1}{2}a + b)$ for some $c(n) > 0$)

$$\begin{aligned} |\nabla\varphi|^2_{\omega_0} \leq |\nabla\varphi|^2_\omega \mathrm{tr}_{\omega_0}\omega \leq & |\nabla\varphi|^2_\omega(\mathrm{tr}_\omega\omega_0)^{n-1}\Big(\frac{\omega^n}{\omega_0^n}\Big)\\ = & e^F\Big(|\nabla\varphi|^{\frac{2}{n}}_\omega(\mathrm{tr}_\omega\omega_0)^{\frac{n-1}{n}}\Big)^n \leq c(n)e^F\Big(\frac{1}{2}|\nabla\varphi|^2_\omega + \mathrm{tr}_\omega\omega_0\Big)^n\\ \leq & C(n,A)\Big(\frac{1}{2}|\nabla\varphi|^2_\omega + \mathrm{tr}_\omega\omega_0\Big)^n. \end{aligned}$$

From (2.13) we conclude that at x_0, $|\nabla\varphi|^2_{\omega_0} \leq C(n,A)$. Since x_0 is a maximum point of $e^\Phi(|\nabla\varphi|^2_{\omega_0}+3)$, we see that $\sup_X |\nabla\varphi|^2_{\omega_0} \leq C(n,A)$. The lemma is proved. □

We next apply the argument in the proof of Theorem 3.1 of Chen-Cheng [5]. In our case the functions F and φ are bounded so we can simplify the proof a little bit.

Lemma 4. *For any $p > 0$, there exists a constant $C_p = C(n, A, \omega_0, p) > 0$ such that*
$$\int_X (\mathrm{tr}_{\omega_0}\omega)^p \omega_0^n \leq C_p.$$

Proof. We fix a constant $\alpha \geq 1$ which will be determined later. For notational simplicity, we write $\Psi = -\alpha F - \lambda\alpha\varphi$, and calculate $\Delta_\omega(e^\Psi \mathrm{tr}_{\omega_0}\omega)$,

$$(2.14)\quad \begin{aligned} \Delta_\omega(e^\Psi\mathrm{tr}_{\omega_0}\omega) = & e^\Psi\mathrm{tr}_{\omega_0}\omega(\Delta_\omega\Psi + |\nabla\Psi|^2_\omega) + 2e^\Psi Re\langle\nabla\Psi, \bar\nabla\mathrm{tr}_{\omega_0}\omega\rangle_\omega\\ & + e^\Psi\Delta_\omega\mathrm{tr}_{\omega_0}\omega. \end{aligned}$$

We use a normal coordinates system of ω_0, so that $\omega_0 = (\delta_{ij})$, $dg_0 = 0$ and g (i.e. ω) is diagonal at a given point. By the standard calculations as in Yau [28], the last term in (2.14) satisfies

$$e^{\Psi}\Delta_{\omega}\mathrm{tr}_{\omega_0}\omega \geq e^{\Psi}\big(- C_2\mathrm{tr}_{\omega}\omega_0\,\mathrm{tr}_{\omega_0}\omega + g^{i\bar i}g^{j\bar j}\varphi_{i\bar j k}\varphi_{j\bar i\bar k} + \Delta_{\omega_0}F - R_{\omega_0}\big)$$

where $-C_2$ is a lower bound of the bisectional curvature of ω_0, $\varphi_{i\bar j k}$ denotes the covariant derivative of φ under ∇_{ω_0} and R_{ω_0} is the scalar curvature of ω_0. We cannot apply the usual maximum principle here because apriori $\Delta_{\omega_0}F$ is not bounded.

The second term in (2.14) satisfies

$$\begin{aligned} 2e^{\Psi}Re\langle\nabla\Psi, \bar\nabla\mathrm{tr}_{\omega_0}\omega\rangle_{\omega} &\geq -2e^{\Psi}|\nabla\Psi|_{\omega}|\nabla\mathrm{tr}_{\omega_0}\omega|_{\omega} \\ &\geq -e^{\Psi}\mathrm{tr}_{\omega_0}\omega|\nabla\Psi|^2_{\omega} - e^{\Psi}\frac{|\nabla\mathrm{tr}_{\omega_0}\omega|^2_{\omega}}{\mathrm{tr}_{\omega_0}\omega} \\ &\geq -e^{\Psi}\mathrm{tr}_{\omega_0}\omega|\nabla\Psi|^2_{\omega} - e^{\Psi}g^{i\bar i}g^{j\bar j}\varphi_{i\bar j k}\varphi_{j\bar i\bar k}, \end{aligned}$$

where in the last step we use the inequality belowas in [28]

$$\begin{aligned} |\nabla\mathrm{tr}_{\omega_0}\omega|^2_{\omega} = \sum_i g^{i\bar i}\big|\sum_k \varphi_{k\bar k i}\big|^2 &\leq \mathrm{tr}_{\omega_0}\omega\sum_i g^{i\bar i}\sum_j g^{j\bar j}\varphi_{j\bar j i}\varphi_{\bar j j\bar i} \\ &\leq \mathrm{tr}_{\omega_0}\omega g^{i\bar i}g^{j\bar j}\,\varphi_{i\bar j k}\varphi_{j\bar i\bar k}. \end{aligned}$$

The first term in (2.14) is

$$\begin{aligned} e^{\Psi}\mathrm{tr}_{\omega_0}\omega\,\Delta_{\omega}\Psi &= e^{\Psi}\mathrm{tr}_{\omega_0}\omega\big(\alpha\Delta_{\omega}\varphi - \alpha\Delta_{\omega}u + \alpha\Delta_{\omega}u_0 - \lambda\alpha\Delta_{\omega}\varphi\big) \\ &\geq e^{\Psi}\mathrm{tr}_{\omega_0}\omega\big((\lambda\alpha - \alpha - C_0)\mathrm{tr}_{\omega}\omega_0 - C(n,A)\alpha\big), \end{aligned} \tag{2.15}$$

where as before $C_0 > 0$ satisfies $-C_0\omega_0 \leq i\partial\bar\partial u_0 \leq C_0\omega_0$. Combining the above inequalities we get

$$\begin{aligned} \Delta_{\omega}(e^{\Psi}\mathrm{tr}_{\omega_0}\omega) &\geq e^{\Psi}\Big((\lambda\alpha - \alpha - C_0 - C_2)\mathrm{tr}_{\omega_0}\omega\mathrm{tr}_{\omega}\omega_0 \\ &\qquad - C(n,A)\alpha\mathrm{tr}_{\omega_0}\omega + \Delta_{\omega_0}F - R_{\omega_0}\Big) \\ &\geq e^{\Psi}\Big(\alpha(\mathrm{tr}_{\omega_0}\omega)^{\frac{n}{n-1}}e^{-\frac{F}{n-1}} - C(n,A)\alpha\mathrm{tr}_{\omega_0}\omega + \Delta_{\omega_0}F - R_{\omega_0}\Big) \\ &\geq e^{\Psi}\Big(c_0\alpha(\mathrm{tr}_{\omega_0}\omega)^{\frac{n}{n-1}} + \Delta_{\omega_0}F - C(n,A)\alpha\Big) \end{aligned} \tag{2.16}$$

where we choose $\lambda = C_0 + C_2 + 2$, $c_0 = c_0(n, A, \omega_0) > 0$ depends on the lower bound of $e^{-\frac{F}{n-1}}$, and in the last step we apply Young's inequality $\mathrm{tr}_{\omega_0}\omega \leq \varepsilon(\mathrm{tr}_{\omega_0}\omega)^{\frac{n}{n-1}} + C(\varepsilon)$ for a suitable choice of small $\varepsilon > 0$.

We denote $v := e^{\Psi}\mathrm{tr}_{\omega_0}\omega > 0$ and for any $p \geq 1$ we have by (2.16)

$$\begin{aligned}\Delta_\omega v^p &= pv^{p-1}\Delta_\omega v + p(p-1)v^{p-2}|\nabla v|^2_\omega \\ &\geq pv^{p-1}e^{\Psi}\Big(c_0\alpha(\mathrm{tr}_{\omega_0}\omega)^{\frac{n}{n-1}} + \Delta_{\omega_0}F - C(n,A)\alpha\Big) \\ &\quad + p(p-1)v^{p-3}e^{\Psi}|\nabla v|^2_{\omega_0},\end{aligned} \tag{2.17}$$

where in the inequality we have applied the observation that $v|\nabla v|^2_\omega = e^{\Psi}\mathrm{tr}_{\omega_0}\omega|\nabla v|^2_\omega \geq e^{\Psi}|\nabla v|^2_{\omega_0}$. Integrating the inequality (2.17) over X against the volume form $\omega^n = e^F\omega_0^n$, we obtain

$$\begin{aligned}&\int_X \Big(v^{p-1}e^{\Psi+F}\big(c_0\alpha(\mathrm{tr}_{\omega_0}\omega)^{\frac{n}{n-1}} + \Delta_{\omega_0}F\big) + (p-1)v^{p-3}e^{\Psi+F}|\nabla v|^2_{\omega_0}\Big)\omega_0^n \\ &\leq C(n,A)\alpha\int_X v^{p-1}e^{\Psi+F}\omega_0^n.\end{aligned} \tag{2.18}$$

To deal with the term involving $\Delta_{\omega_0}F$, we will apply the integration by parts. We calculate

$$\begin{aligned}\int_X v^{p-1}e^{\Psi+F}\Delta_{\omega_0}F\omega_0^n &= \int_X v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}\Delta_{\omega_0}F\omega_0^n \\ &= \int_X \Big(-(p-1)v^{p-2}e^{-(\alpha-1)F-\lambda\alpha\varphi}\langle\nabla v, \bar\nabla F\rangle_{\omega_0} \\ &\quad + v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}(\alpha-1)|\nabla F|^2_{\omega_0} \\ &\quad + v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}\lambda\alpha\langle\nabla\varphi, \bar\nabla F\rangle_{\omega_0}\Big)\omega_0^n.\end{aligned} \tag{2.19}$$

The second term in the right hand side of (2.19) is good. The first term in (2.19) satisfies

$$\begin{aligned}&\int_X -(p-1)v^{p-2}e^{-(\alpha-1)F-\lambda\alpha\varphi}\langle\nabla v, \bar\nabla F\rangle_{\omega_0} \\ &\geq -\int_X (p-1)v^{p-2}e^{\Psi+F}|\nabla v|_{\omega_0}|\nabla F|_{\omega_0} \\ &\geq -\int_X \frac{\alpha-1}{4}v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}|\nabla F|^2_{\omega_0} - \int_X \frac{(p-1)^2}{\alpha-1}v^{p-3}e^{\Psi+F}|\nabla v|^2_{\omega_0} \\ &\geq -\int_X \frac{\alpha-1}{4}v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}|\nabla F|^2_{\omega_0} - \int_X v^{p-3}e^{\Psi+F}|\nabla v|^2_{\omega_0}\end{aligned}$$

if we take $\alpha = \alpha(p) \geq p+2$. These negative terms will be cancelled by the positive terms from (2.19) and (2.18). Next we look at the third term on the right hand side of (2.19). By Lemma 3 we have a bound on $\sup_X |\nabla\varphi|_{\omega_0}$, and thus

$$\begin{aligned}
&\int_X v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}\lambda\alpha\langle\nabla\varphi,\bar{\nabla}F\rangle_{\omega_0}\\
\geq& -C\lambda\alpha\int_X v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}|\nabla F|_{\omega_0}\\
\geq& -\frac{\alpha-1}{4}\int_X v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}|\nabla F|^2_{\omega_0}-\frac{C\alpha^2}{\alpha-1}\int_X v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}\\
\geq& -\frac{\alpha-1}{4}\int_X v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}|\nabla F|^2_{\omega_0}-C\alpha\int_X v^{p-1}e^{-(\alpha-1)F-\lambda\alpha\varphi}.
\end{aligned}$$

Plugging the above inequalities into (2.18) and re-organizing, it follows that

$$\int_X c_0\alpha v^{p-1}e^{\Psi+F}(\mathrm{tr}_{\omega_0}\omega)^{\frac{n}{n-1}}\omega_0^n\leq C(n,A)\alpha\int_X v^{p-1}e^{\Psi+F}\omega_0^n.$$

Note that Ψ and F are both bounded by $C(n,A)$, so we conclude that there exists a constant $C_p=C(n,A,\omega_0,p)>0$ such that

$$\int_X(\mathrm{tr}_{\omega_0}\omega)^{p-1+\frac{n}{n-1}}\omega_0^n\leq C_p\int_X(\mathrm{tr}_{\omega_0}\omega)^{p-1}\omega_0^n. \tag{2.20}$$

When $p=2$

$$\int_X \mathrm{tr}_{\omega_0}\omega\omega_0^n=\int_X(n+\Delta_{\omega_0}\varphi)\omega_0^n=n\int_X\omega_0^n$$

is clearly bounded. Now we define a sequence $\{p_k\}$ with $p_0=2$ and $p_k=2+\frac{n}{n-1}k$. Then (2.20) implies that

$$\int_X(\mathrm{tr}_{\omega_0}\omega)^{p_k}\omega_0^n\leq C_k\int_X(\mathrm{tr}_{\omega_0}\omega)^{p_{k-1}}\omega_0^n.$$

Since $p_k\to\infty$ as $k\to\infty$, iterating the inequality above gives that there exists a constant $C_k=C(n,A,\omega_0,k)>0$ such that

$$\int_X(\mathrm{tr}_{\omega_0}\omega)^{p_k}\omega_0^n\leq C_k.$$

Lemma 4 then follows from this inequality and the Hölder inequality. □

The next step is to establish $C^{3,\alpha}$ bounds on the potential φ. This can be deduced from the result of Theorem 1.6 in [6], but in our special setting we can provide a simpler and direct proof as follows.

Lemma 5. *For any $\beta \in (0,1)$, there exists a constant $C_\beta = C(n, A, \omega_0, \beta) > 0$ such that*

$$\|\varphi\|_{C^{1,\beta}(X,\omega_0)} \leq C_\beta.$$

Proof. By Lemma 4, $f := \Delta_{\omega_0}\varphi \in L^p(X, \omega_0^n)$ for any $p > 0$. By the $W^{2,p}$-estimates for linear elliptic equations (c.f. Theorem 9.11 in [15]), we have

$$\|\varphi\|_{W^{2,p}(X,\omega_0)} \leq C(\|\varphi\|_{L^p(X,\omega_0^n)} + \|f\|_{L^p(X,\omega_0^n)}) \leq C_p.$$

The $C^{1,\beta}(X, \omega_0)$ bound of φ then follows from the Sobolev embedding theorem (c.f. Corollary 7.11 in [15]) by taking $p > 1$ sufficiently large. ☐

Lemma 6. *The Ricci potential u of $\omega = \omega_0 + i\partial\bar{\partial}\varphi$ satisfies*

$$\|u\|_{C^\alpha(X,\omega_0)} \leq C_\alpha(n, A, \omega_0),$$

for any $\alpha \in (0,1)$.

Proof. Observe that

$$|\nabla u|^2_{\omega_0} \leq |\nabla u|^2_\omega \mathrm{tr}_{\omega_0}\omega \leq A\mathrm{tr}_{\omega_0}\omega.$$

By Lemma 4, it follows that $|\nabla u|_{\omega_0} \in W^{1,p}(X, \omega_0)$ for any $p > 1$. The lemma then follows from the Sobolev embedding theorem by taking $p > 1$ sufficiently large. ☐

To prove the $C^{2,\alpha}$-estimate of φ, we need the following recent result of Li-Li-Zhang, which weakens the condition of Y. Wang's result [27] on the regularity assumption of φ.

Lemma 7 ([17] Theorem 1.2). *Let $B_2 \subset \mathbb{C}^n$ be the Euclidean ball with radius 2 and center 0. Suppose $\varphi \in PSH(B_2) \cap C(B_2)$ solves the complex MA equation*

$$\det \varphi_{i\bar{j}} = f, \text{ in } B_2$$

with $f \geq \lambda > 0$ for some positive $\lambda \in \mathbb{R}$ and $f \in C^\alpha(B_2)$ for some $\alpha \in (0,1)$. If $\varphi \in C^{1,\beta}(B_2)$ for some $\beta > 1 - \frac{\alpha}{n(2+\alpha)-1}$, then $\varphi \in C^{2,\alpha}(B_1)$ and the $C^{2,\alpha}(B_1)$-norm of φ depends only on $n, \alpha, \beta, \lambda$, $\|\varphi\|_{C^{1,\beta}(B_2)}$ and $\|f\|_{C^\alpha(B_2)}$.

We arrive now at the $C^{2,\alpha}$ estimates for φ:

Lemma 8. *Under the conditions spelled out in the statement of Lemma 2, there exists $\alpha > 0$ with*

$$\|\varphi\|_{C^{2,\alpha}} \leq C(n, A, \alpha) \tag{2.21}$$

for some constant $C(n, A, \omega_0)$.

Proof. We note that by Lemma 1 and Lemma 6, the function on the right hand side of (2.3) has uniform $C^{0,\alpha'}(X, \omega_0)$ estimate. Lemma 5 provides the $C^{1,\beta}(X, \omega_0)$ estimates of the Kähler potential φ. Then Lemma 7 proves the $C^{2,\alpha}(X, \omega_0)$ estimates of φ. □

The following lemma is the key lemma that we shall need later for the proof of Theorem 1 and Theorem 2. It is an immediate consequence of the $C^{2,\alpha}(X, \omega_0)$-estimates of φ, and the fact that the right hand side e^F of the Monge-Ampère equation (2.3) is bounded above and below:

Lemma 9. *There exists a constant $C = C(n, A, \omega_0) \geq 1$ such that*

$$C^{-1}\omega_0 \leq \omega \leq C\omega_0,$$

and $u \in C^{1,\alpha}(X, \omega_0)$ for any $\alpha \in (0, 1)$.

Finally, we can complete the proof of Lemma 2. By Lemma 8, the metric $g_{\bar{j}i}$ has uniform $C^{2,\alpha}$ norm. By Lemma 6 and Cramer's rule, its inverse $g^{i\bar{j}}$ also has uniform $C^{0,\alpha}(X, \omega_0)$ norm. The equation that $R_\omega - n = \Delta_\omega u$ can be written locally in holomorphic coordinates as

$$g^{i\bar{j}} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = R_\omega - n \in L^\infty.$$

Then the $C^{1,\alpha}(X, \omega_0)$-norm of u follows from the $W^{2,p}$-estimates and Sobolev embedding theorem (c.f. [15]).

Finally, once we have the $C^{1,\alpha}(X, \omega_0)$-norm of u, we can take $\frac{\partial}{\partial z_i}$ on both sides of the equation (2.3) and apply local Schauder estimates to conclude that

$$\|\varphi\|_{C^{3,\alpha}(X,\omega_0)} \leq C(n, A, \omega_0, \alpha). \tag{2.22}$$

The proof of Lemma 2 is complete.

3. Proof of Theorem 1

One direction in Theorem 1 is a direct consequence of known results. Assume that $\lambda(X;A) > 0$ for any A. By the work of Perelman (see [25] for a detailed account), for any given initial data in $c_1(X)$, the orbit of the Kähler-Ricci flow lies in a set $c_1(X;A)$ for some $A > 0$. Thus a positive lower bound for $\lambda(X;A)$ implies a positive lower bound for the eigenvalue $\lambda(\omega)$ along the Kähler-Ricci flow. By the results of [21, 30], the flow converges then to a Kähler-Einstein metric.

The main issue in the present paper is to establish the other direction, namely that $\lambda(X;A) > 0$ for any $A > 0$ if a Kähler-Einstein metric is assumed to exist. But then for any fixed $A > 0$, and any $\omega \in c_1(X;A)$, after replacing ω by $g^*\omega$ for some $g \in G$,[2] we can apply Lemma 1 to conclude that these ω have potentials uniformly bounded in C^α-norm for some fixed $\alpha > 0$. By Lemma 9, they are all equivalent. The desired bound for $\lambda(X;A)$ is then a consequence of the following lemma, which was essentially proved in [19], Lemma 1:

Lemma 10. *Let ω, $\tilde\omega$ be two metrics in $c_1(X)$ which are equivalent, in the sense that*

$$\kappa^{-1}\omega \le \tilde\omega \le \kappa\omega \tag{3.1}$$

for some constant $\kappa > 0$. Let λ_ω and $\lambda_{\tilde\omega}$ be the corresponding eigenvalues, as defined in (1.1). Then

$$c(\kappa,n)^{-1}\lambda_\omega \le \lambda_{\tilde\omega} \le c(\kappa,n)\lambda_\omega \tag{3.2}$$

for some constant $c(\kappa,n) > 0$ depending only on κ and the dimension n.

Proof. Since this lemma is essential for our considerations and since its proof is short, we include the proof for the reader's convenience. In the definition (1.1) for λ_ω and $\lambda_{\tilde\omega}$, the norms $\|\bar\partial V\|_\omega$ and $\|\bar\partial V\|_{\tilde\omega}$ as well as the norms $\|V\|_\omega$ and $\|V\|_{\tilde\omega}$ are already equivalent, since the metrics ω and $\tilde\omega$ are equivalent, and so are their volume forms ω^n and $\tilde\omega^n$. The main issue is the difference in the orthogonality conditions $\perp_\omega$ and $\perp_{\tilde\omega}$. To address this issue, consider

[2] Note that $\lambda_\omega = \lambda_{g^*\omega}$ for $g \in G$.

any vector field V with $V \perp_\omega H^0(X, T^{1,0})$ and decompose it as

$$V = \tilde{V} + E \tag{3.3}$$

with $\tilde{V} \perp_{\tilde{\omega}} H^0(X, T^{1,0})$ and $E \in H^0(X, T^{1,0})$. Taking inner products with respect to the metric ω gives

$$0 = \langle \tilde{V}, E\rangle_\omega + \langle E, E\rangle_\omega \tag{3.4}$$

and hence by the Cauchy-Schwarz inequality,

$$\|E\|_\omega \leq \|\tilde{V}\|_\omega. \tag{3.5}$$

We can now write for some constant $c_1(\kappa, n)$

$$\|\bar{\partial} V\|_\omega^2 = \|\bar{\partial}\tilde{V}\|_\omega^2 \geq c_1(\kappa, n)\|\bar{\partial}\tilde{V}\|_{\tilde{\omega}}^2 \tag{3.6}$$

because ω and $\tilde{\omega}$ are equivalent, and at the same time, by the same equivalence and the triangle inequality,

$$\|V\|_\omega^2 \leq 2\|\tilde{V}\|_\omega^2 + 2\|E\|_\omega^2 \leq 4\|\tilde{V}\|_\omega \leq c_2(\kappa, n)\|\tilde{V}\|_{\tilde{\omega}}^2. \tag{3.7}$$

It follows that

$$\frac{\|\bar{\partial} V\|_\omega^2}{\|V\|_\omega^2} \geq \frac{c_1(\kappa, n)}{c_2(\kappa, n)} \frac{\|\bar{\partial}\tilde{V}\|_{\tilde{\omega}}^2}{\|\tilde{V}\|_{\tilde{\omega}}^2} \geq \frac{c_1(\kappa, n)}{c_2(\kappa, n)}\lambda_{\tilde{\omega}} \tag{3.8}$$

and hence $\lambda_\omega \geq \frac{c_1(\kappa,n)}{c_2(\kappa,n)}\lambda_{\tilde{\omega}}$. Reversing the roles of ω and $\tilde{\omega}$ gives the inequality in the opposite direction. The lemma is proved, completing the proof of Theorem 1. □

4. Proof of Theorem 2

Again, one direction of the theorem follows from the results of Perelman and [20, 30]. To prove the other direction, namely that the existence of a Kähler-Einstein metric implies a strictly positive gap $\mu(X; A)$ for any $A > 0$, we argue by contradiction. Recall the operator L_ω defined for a metric ω with Ricci potential u by $L_\omega f = -g^{j\bar{k}}\nabla_j\nabla_{\bar{k}} f + g^{j\bar{k}}\nabla_{\bar{k}} f \nabla_j u - f$ and whose eigenvalues and eigenfunctions satisfy the identity (1.4).

Assume then that X is Kähler-Einstein, and that there exists a sequence of metrics $\omega_j = \omega_0 + i\partial\bar{\partial}\varphi_j \in c_1(X; A)$ such that the eigenvalues μ_j of the operator L_{ω_j} goes to 0 as $j \to \infty$. For any fixed $A > 0$, and any $\omega \in c_1(X; A)$, after replacing ω by $g^*\omega$ for some $g \in G$, we can apply Lemma 1 to conclude that these ω have potentials uniformly bounded in C^α-norm for some fixed $\alpha > 0$. We take f_j to be eigenfunctions of L_{ω_j} with eigenvalues μ_j, normalized by $\|f_j\|_{L^2(X, e^{-u_j}\omega_j^n)} = 1$. It follows from straightforward calculation that for any holomorphic vector field $V \in H^0(X, T^{1,0}X)$

$$\int_X \langle \nabla_{\omega_j} f_j, V\rangle_{\omega_j} e^{-u_j}\omega_j^n = 0. \tag{4.1}$$

By Lemma 9 and Lemma 8, we can apply the elliptic estimates to f_j, which satisfies the linear equation

$$-g_j^{p\bar{q}}\nabla_p\nabla_{\bar{q}} f_j + g_j^{p\bar{q}}\nabla_{\bar{q}} f_j \nabla_p u_j - f_j = \mu_j f_j$$

to conclude that

$$\|f_j\|_{C^{2,\alpha}(X,\omega_0)} \le C(n, A), \quad \forall\, j.$$

Up to a subsequence, we may assume the Ricci potentials u_j converge in $C^{1,\alpha}$ to a function $u_\infty \in C^{1,\alpha}$, the metrics ω_j converge in $C^{1,\alpha}$ to a metric $\omega_\infty \in C^{1,\alpha}$, and the functions f_j converge in $C^{2,\alpha}$ to a function $f_\infty \in C^{2,\alpha}$. In particular, we have $\|f_\infty\|_{L^2(X, e^{-u_\infty}\omega_\infty^n)} = 1$. Passing to the limit, (4.1) gives that

$$\int_X \langle \nabla_{\omega_\infty} f_\infty, V\rangle_{\omega_\infty} e^{-u_\infty}\omega_\infty^n = 0, \quad \forall\, V \in H^0(X, T^{1,0}X). \tag{4.2}$$

Observe that the equations (i.e. (1.4))

$$\int_X |\bar{\nabla}\bar{\nabla} f_j|^2_{\omega_j} e^{-u_j}\omega_j^n + \int_X |\bar{\nabla} f_j|^2_{\omega_j} e^{-u_j}\omega_j^n = (1+\mu_j)\int_X |\bar{\nabla} f_j|^2_{\omega_j} e^{-u_j}\omega_j^n$$

hold for any j. Since $\mu_j \to 0$, passing to limit we get

$$\int_X |\bar{\nabla}\bar{\nabla} f_\infty|^2_{\omega_\infty} e^{-u_\infty}\omega_\infty^n = 0,$$

which implies $\nabla\nabla f_\infty = 0$, i.e. $\nabla_{\omega_\infty} f_\infty$ is a holomorphic vector field. From (4.2) we conclude that $\int_X |\nabla f_\infty|^2_{\omega_\infty} e^{-u_\infty}\omega_\infty^n = 0$. However, this contradicts the identity

$$1 = \int_X f_\infty^2 e^{-u_\infty}\omega_\infty^n = \int_X |\nabla f_\infty|^2_{\omega_\infty} e^{-u_\infty}\omega_\infty^n.$$

The proof of Theorem 2 is complete.

We observe that this argument could have been used also for the proof of Theorem 1. However, the argument there is more direct, and provides more precise information on the bounds for λ_ω.

5. Further remarks

We note that in Theorem 1, we cannot in general replace the gap $\lambda(X; A)$ for each $A > 0$ by the gap $\lambda(X) = \inf_{\omega \in c_1(X)} \lambda_\omega$ over all of $c_1(X)$. A simple counterexample is provided by the 2-dimensional sphere, which admits a Kähler-Einstein metric, but can be seen to have

$$\lambda(S^2) = 0 \tag{5.1}$$

as follows. Let $\eta : \mathbf{R} \to \mathbf{R}$ be a smooth increasing function such that $\eta = 0$ on $(-\infty, 1/3]$ and $\eta = 1$ on $[2/3, \infty)$. Let $a, N > 0$ and let $f : [0, 3N + 2] \to \mathbf{R}$ be a non-negative concave function, positive and smooth on $(0, 3N + 2)$ such that

1) $f(0) = f(3N + 2) = 0$
2) $f(x) = a$ for $x \in [1, 3N + 1]$

and let X be the surface obtained by revolving the graph of $y = f(x)$ around the x axis. Thus X is a smooth manifold (if we choose f so that its tangent line is vertical at 0 and $3N + 2$ and is tangent to the graph to infinite order), looks like a cigar, is flat between $x = 1$ and $x = 3N + 1$ and is diffeomorphic to S^2. Moreover, we can choose a so that the area of X is 1 (so a is roughly $\frac{1}{3N \cdot 2\pi}$). Let g_N be metric obtained by restricting the euclidean metric in $\mathbf{R}^3$. Let V_1 be a smooth vector field on X defined as follows.

$$V_1 = \eta(x - 1)\eta(N + 1 - x)\frac{\partial}{\partial x}$$

so V_1 is a smooth vector field on X compactly supported in $\{(x, y, z) \in M : x \in (1, N + 1)\}$. Similarly we define V_2 supported in $(N + 1, 2N + 1)$ and V_3 supported in $(2N + 1, 3N + 1)$.

Next we let

$$V = c_1V_1 + c_2V_2 + c_3V_3 \tag{5.2}$$

where the $c_i \in \mathbf{R}$ are chosen so that V is orthogonal to the 3-dimensional space of holomorphic vector fields and $c_1^2 + c_2^2 + c_3^2 = 1$. Now $|V_i|$ is roughly equal to 1 so $\|V_i\|_{L^2} \sim 1/3$ so $\|V\|_{L^2} \sim c_1^2\|V_1\|^2_{L^2} + c_2^2\|V_2\|^2_{L^2} + c_3^2\|V_3\|^2_{L^2} \sim$

$\frac{1}{9}$. On the other hand $\nabla V_1 = 0$ for $2 < x < N$ so $\|\nabla V_1\|^2_{L^2} = O(\frac{1}{N})$ which implies $\|\nabla V\|^2_{L^2} = O(\frac{1}{N})$. In particular, $\lambda_{\omega_N} \leq O(\frac{1}{N})$. This establishes our claim.

References

[1] Berman, R., S. Boucksom, and M. Jonsson, "A variational approach to the Yau-Tian-Donaldson conjecture". `arXiv:1509.04561`, to appear in J. Amer. Math. Soc.

[2] Boucksom, S., Eyssidieux, P., and V. Guedj, "Introduction to the Kähler-Ricci flow", Lecture Notes in Math., 2086, Springer, 2013.

[3] Blum, H. and M. Jonsson, "Thresholds, valuations, and K-stability". Adv. Math. 365 (2020), 107062, 57 pp.

[4] Calabi, E., "Extremal Kähler metrics", Seminar on Differential Geometry, Ann. Math. Studies 16 (1982) 259–290.

[5] Chen, X. and J. Cheng, "On the constant scalar curvature Kähler metrics (I), apriori estimates". J. Amer. Math. Soc. (2021) DOI: `https://doi.org/10.1090/jams/967`, `arXiv:1712.06697`.

[6] Chen, X. and J. Cheng, "On the constant scalar curvature Kähler metrics (II), existence results". J. Amer. Math. Soc. (2021) DOI: `https://doi.org/10.1090/jams/966`, `arXiv:1801.00656`.

[7] Chen, X., S. Donaldson, and S. Sun, "Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities". J. Amer. Math. Soc. 28 (2015), no. 1, 183–197.

[8] Chen, X., S. Donaldson, and S. Sun, "Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π". J. Amer. Math. Soc. 28 (2015), no. 1, 199–234.

[9] Chen, X., S. Donaldson, and S. Sun, "Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof". J. Amer. Math. Soc. 28 (2015), no. 1, 235–278.

[10] Darvas, T., "Geometric pluripotential theory on Kähler manifolds" Advances in complex geometry, 1–104, Contemp. Math., 735, Amer. Math. Soc., Providence, RI, 2019.

[11] Darvas, T. and Y. Rubinstein, "Tian's properness conjectures and Finsler geometry of the space of Kähler metrics", J. Amer. Math. Soc. 30 (2017), no. 2, 347–387.

[12] Donaldson, S.K., "Scalar curvature and stability of toric varieties", J. Differential Geom. 59 (2002) 289–349.

[13] Fujita, K., Odaka, Y., "On the K-stability of Fano varieties and anti-canonical divisors", Tohoku Math. J. (2) 70 (2018), no. 4, 511–521.

[14] Futaki, A. Kähler-Einstein metrics and integral invariants, Lecture Notes in Mathematics, 1314. Springer–Verlag, Berlin, 1988

[15] Gilbarg, D. and N. Trudinger, "Elliptic partial differential equations of second order". Second edition. Grundlehren der Mathematischen Wissenschaften, 224. Springer-Verlag, Berlin, 1983. xiii+513 pp.

[16] Kolodziej, S. "Hölder continuity of solutions to the complex Monge-Ampère equation with the right hand side in L^p. The case of compact Kähler manifolds". Math. Ann. 342, 379–386 (2008).

[17] Li C., J. Li, and X. Zhang, "A $C^{2,\alpha}$ estimate of the complex Monge-Ampère equation". `arXiv:1705.08634`.

[18] Perelman, G. "The entropy formula for the Ricci flow and its geometric applications." `arXiv:math/0211159`.

[19] Phong, D.H., J. Song, J. Sturm, and B. Weinkove, "The Moser-Trudinger inequality on Kähler-Einstein manifolds", Amer. J. Math. 130 no. 4 (2008) 1067–1085.

[20] Phong, D.H., J. Song, J. Sturm, and B. Weinkove, "The Kähler-Ricci flow with positive bisectional curvature". Invent. Math. 173 (2008), no. 3, 651–665.

[21] Phong, D. H., J. Song, J. Sturm, and B. Weinkove, "The Kähler-Ricci flow and the $\bar\partial$ operator on vector fields". J. Differential Geom. 81 (2009), no. 3, 631–647.

[22] Phong, D. H., J. Song, J. Sturm, and B. Weinkove, "On the convergence of the modified Kähler-Ricci flow and solitons", Comment. Math. Helv. 86 (2011), no. 1, 91–112.

[23] Phong, D.H. and J. Sturm, "On stability and the convergence of the Kähler-Ricci flow". J. Differential Geom. 72 (2006), no. 1, 149–168.

[24] Phong, D.H. and J. Sturm, "Lectures on stability and constant scalar curvature". Handbook of geometric analysis, No. 3, 357–436, Adv. Lect. Math. (ALM), 14, Int. Press, Somerville, MA, 2010.

[25] Sesum, N. and G. Tian, "Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman)". J. Inst. Math. Jussieu 7 (2008), no. 3, 575–587.

[26] Tian, G., "Kähler-Einstein metrics with positive scalar curvature", Invent. Math. 130 (1997) 1–37.

[27] Wang, Y., "On the $C^{2,\alpha}$-regularity of the complex Monge-Ampère equation", Math. Res. Lett. 19 (2012), no. 4, 939–946.

[28] Yau, S.-T., "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation", Comm. Pure Appl. Math. 31 (1978), 339–411.

[29] Yau, S.T., "Open problems in geometry", Proc. Symposia Pure Math., 54:1–28, 1993.

[30] Zhang, Z., "Kähler Ricci flow on Fano manifolds with vanished Futaki invariants", Math. Res. Lett. 18 (2011), no. 5, 969–982.

[31] Zeriahi, A., "Volume and capacity of sublevel sets of a Lelong class of plurisubharmonic functions", Indiana Univ. Math. J. 50 (2001), 671–703.

Department of Mathematics and Computer Sciences
Rutgers University, Newark, NJ 07102, USA
E-mail address: bguo@rutgers.edu

Department of Mathematics, Columbia University
New York, NY 10027, USA
E-mail address: phong@math.columbia.edu

Department of Mathematics and Computer Sciences
Rutgers University, Newark, NJ 07102, USA
E-mail address: sturm@andromeda.rutgers.edu

Received January 20, 2021
Accepted October 27, 2021