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On the existence of the conical Kähler-Einstein metrics on Fano manifolds

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In this paper, by using smooth approximation, we give a new proof of Donaldson's existence conjecture that there exist conical Kähler-Einstein metrics with positive Ricci curvatures on Fano manifolds.

1. Introduction

Since the conical Kähler-Einstein metrics play an important role in the solution of Yau-Tian-Donaldson's conjecture, the existence and geometry of these metrics have been widely concerned. The conical Kähler-Einstein metrics were studied on the Riemannian surfaces by McOwen [22] and Troyanov [31], and were first considered in higher dimensions by Tian [27]. The renewed interest has been sparked by Tian's series of work [5, 28, 30] etc. which aim to solve the smooth Kähler-Einstein problem on Fano manifolds and Donaldson's suggestions [11, 12] which introduce the continuity method by deforming the cone angles of the conical Kähler-Einstein metrics. Then by using this method, Chen-Donaldson-Sun [6–8] and Tian [29] proved the Yau-Tian-Donaldson's conjecture. There is by now a lot of results about these metrics, see the works of Berman [1], Brendle [3], Campana-Guenancia-Păun [4], Guenancia-Păun [15], Jeffres-Mazzeo-Rubinstein [16], Li-Sun [20], Mazzeo [23], Song-Wang [25] etc. For more details, readers can refer to Rubinstein's article [24].

When using the continuity method in the proof of Yau-Tian-Donaldson's conjecture, we should confirm that the set consisted of conical Kähler-Einstein metrics with positive Ricci curvatures is non-empty. The existence of conical Kähler-Einstein metrics with positive Ricci curvatureswas conjectured by Donaldson [10] and proved by Berman, Jeffres-Mazzeo-Rubinstein and Li-Sun (see Theorem 1.5 in [1], Corollary 1 in [16] and Theorem 1.1

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in [20]). They first deduced the properness of the log-Mabuchi (or Ding) energies by using the positivity of the log α-invariant. Then by using this properness and the continuity method, they proved this existence result. All of them considered the equations with conical singularities directly. In this paper, we prove this existence result by using smooth approximation.

Let M be a Fano manifold with complex dimension n, $\omega_0 \in c_1(M)$ be a smooth Kähler metric and $D \in |-\lambda K_M|$ ($0 < \lambda \in \mathbb{Q}$) be a smooth divisor. Denote $\gamma \in (0,1)$ and $\mu_\gamma = 1 - (1-\gamma)\lambda$. Let s be the definition section of D and h be a smooth Hermitian metric on L_D with curvature $\lambda\omega_0$, where L_D is the line bundle associated with divisor D. We assume that $|s|_h^2 < 1$ by rescaling h.

In this paper, we prove the following theorem which is the Li-Sun's theorem (Theorem 1.1 in [20]) in the case of $\lambda > 1$. In fact, it is meaningful to consider the case $\lambda > 1$ because Bertini's theorem implies that there always exists a smooth divisor $D \in |-mK_M|$ for $m \in \mathbb{N}^+$ sufficiently large.

Theorem 1.1. *Assume that $\lambda > 1$. There exists $\delta(\lambda) > 0$ such that there exists a unique conical Kähler-Einstein metric with Ricci curvature μ_γ and cone angle $2\pi\gamma$ along D for any $\gamma \in (0, 1 - \frac{1}{\lambda} + \delta(\lambda))$.*

In [20], Li-Sun proved this result by using the properness of log-Mabuchi energy, which follows from the positivity of the log α-invariant. The estimates they obtained for conical Kähler-Einstein metrics by using continuity method depend on μ_γ because there is no uniformly positive Ricci lower bound when μ_γ closing to 0. Here, by using the continuity of the smooth approximation sequence with respect to μ_γ, we prove uniform estimates for this sequence (Lemma 1.5), and then get the uniform estimates for all conical Kähler-Einstein metrics with $0 < \mu_\gamma \ll 1$.

Remark 1.2. When $\gamma \in (0, 1 - \frac{1}{\lambda}]$, that is, $\mu_\gamma \leqslant 0$, the existence follows from Eyssidieux-Guedj-Zeriahi's result (Theorem A and Theorem 4.1 in [14]) and Kołodziej's result [17], and the regularities follows from Guenancia-Păun's results (Theorem A in [15], see also Theorem 1.4 in [21]). In this paper, we pay attention to the case of $\mu_\gamma > 0$. The uniqueness follows form Berndtsson's uniqueness theorem [2] for the conical Kähler-Einstein metrics with bounded potentials. For the $\lambda = 1$ case, see Remark 2.4.

By saying that a closed positive $(1,1)$-current ω with locally bounded potentials is conical Kähler metric with cone angle $2\pi\beta$ ($0 < \beta < 1$) along D, we mean that ω is a smooth Kähler metric on $M \setminus D$. And near each point

$p \in D$, there exists a local holomorphic coordinate $(z_1, \cdots, z_n)$ in a neighborhood U of p such that $D = \{z_n = 0\}$ and ω is asymptotically equivalent to the model conical metric

$$\sqrt{-1}\sum_{j=1}^{n-1} dz_j \wedge d\overline{z}_j + \sqrt{-1}|z_n|^{2\beta-2} dz_n \wedge d\overline{z}_n \quad on \;\; U. \tag{1.1}$$

Definition 1.3. Let ω_0 be a smooth Kähler metric and $D \subset M$ be a smooth divisor which satisfies $c_1(M) = \mu[\omega_0] + (1-\beta)c_1(L_D)$ with $\mu \in \mathbb{R}$. We call ω a conical Kähler-Einstein metric with Ricci curvature μ and cone angle $2\pi\beta$ along D if it is a conical Kähler metric with cone angle $2\pi\beta$ along D and satisfies

$$Ric(\omega) = \mu\omega + (1-\beta)[D] \;\; on \;\; M. \tag{1.2}$$

Equation (1.2) is classical outside D and it holds in the sense of currents on M. In this paper, we approximate the conical Kähler-Einstein metrics by using the smooth twisted Kähler-Einstein metrics

$$Ric(\omega_{\gamma,\varepsilon}) = \mu_\gamma \omega_{\gamma,\varepsilon} + (1-\gamma)\theta_\varepsilon, \tag{1.3}$$

where $\theta_\varepsilon = \lambda\omega_0 + \sqrt{-1}\partial\overline{\partial}\log(\varepsilon^2 + |s|_h^2)$ with $\varepsilon > 0$ are smooth closed positive $(1,1)$-forms. Equation (1.3) can be written as the complex Monge-Ampère equation

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{\gamma,\varepsilon})^n = e^{-\mu_\gamma\varphi_{\gamma,\varepsilon} - F_0}\frac{\omega_0^n}{(\varepsilon^2 + |s|_h^2)^{1-\gamma}}, \tag{1.4}$$

where F_0 is the Ricci potential of ω_0 and satisfies $\frac{1}{V}\int_M e^{-F_0}\frac{dV_0}{|s|_h^{2(1-\beta)}} = 1$.

In the following arguments, we assume that $\mu_\beta = 0$, that is, $\beta = 1 - \frac{1}{\lambda}$. First, Kołodziej's results [18] imply that there exists a Hölder continuous solution φ_β to equation

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\beta)^n = e^{-F_0}\frac{\omega_0^n}{|s|_h^{2(1-\beta)}}. \tag{1.5}$$

It means that ω_{φ_β} is a Ricci flat conical Kähler-Einstein metric with cone angle $2\pi\beta$ along D. At the same time, by Yau's results [32], we know that there exist smooth solutions $\varphi_{\beta,\varepsilon}$ to equation

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{\beta,\varepsilon})^n = e^{-F_0+C_{\beta,\varepsilon}} \frac{\omega_0^n}{(\varepsilon^2 + |s|_h^2)^{1-\beta}}, \tag{1.6}$$

where $C_{\beta,\varepsilon}$ is the normalization constant and can be bounded uniformly. From Kołodziej's L^p-estimates [17], there exists uniform constant B_β such that

$$osc_M \varphi_{\beta,\varepsilon} \leqslant B_\beta \tag{1.7}$$

for any $\varepsilon \in (0,1)$. Equation (1.6) is equivalent to that there exists a smooth twisted Kähler-Einstein metric $\omega_{\varphi_{\beta,\varepsilon}} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{\beta,\varepsilon}$ satisfying

$$Ric(\omega_{\beta,\varepsilon}) = (1-\beta)\theta_\varepsilon. \tag{1.8}$$

When we prove the existence of the conical Kähler-Einstein metrics with Ricci curvatures μ_γ by using smooth approximation, we should confirm that there exist solutions to (1.3). Through using Székelyhidi's results (Theorem 1 in [26], see also Corollary 1.4 in [33]), we prove the following existence lemma for the twisted Kähler-Einstein metrics with Ricci curvatures μ_γ.

Lemma 1.4. *There exists δ_0 depending only on λ, n and ω_0, such that the equation* (1.3) *is solvable for any $\gamma \in [\beta, \beta+\delta_0)$ and $\varepsilon \in (0,1)$.*

Then by using the continuity of the solutions of equations (1.3) with respect to γ for fix $\varepsilon \in (0,1)$, we prove that

Lemma 1.5. *There exists $\delta(\lambda) > 0$ such that*

$$osc_M \varphi_{\gamma,\varepsilon} \leqslant \max\big(osc_M \varphi_\beta, B_\beta\big) + 1 \tag{1.9}$$

for any $\gamma \in [\beta, \beta+\delta(\lambda))$ and $\varepsilon \in (0, \delta(\lambda))$, where B_β is the constant in (1.7).

At last, by using these uniform estimates, for any $\gamma \in (\beta, \beta+\delta(\lambda))$, we deduce the existence of the conical Kähler-Einstein metric with Ricci curvature μ_γ and cone angle $2\pi\gamma$ along D.

2. Proof of Theorem 1.1

In this section, by using Székelyhidi's [26] existence theorem for the twisted Kähler-Einstein metrics (see also Zhang-Zhang's work [33]), we prove Lemma 1.4. Now we recall Aubin's functionals. Let ϕ_t be a path with $\phi_0 = c$ and

$\phi_1 = \phi$,

$$I_{\omega_0}(\phi) = \frac{1}{V}\int_M \phi(dV_0 - dV_\phi),$$
$$J_{\omega_0}(\phi) = \frac{1}{V}\int_0^1 \int_M \dot{\phi}_t(dV_0 - dV_{\phi_t})dt,$$

where $dV_0 = \frac{\omega_0^n}{n!}$ and $dV_\phi = \frac{\omega_\phi^n}{n!}$. They satisfy $0 \leqslant \frac{1}{n}J_{\omega_0} \leqslant \frac{1}{n+1}I_{\omega_0} \leqslant J_{\omega_0}$. The twisted Mabuchi energy is defined as

$$\begin{aligned}\mathcal{M}_{k,\theta}(\phi) = &-k(I_{\omega_0}(\phi) - J_{\omega_0}(\phi)) - \frac{1}{V}\int_M u_{\omega_0}(dV_0 - dV_\phi) \\ &+ \frac{1}{V}\int_M \log\frac{\omega_\phi^n}{\omega_0^n} dV_\phi,\end{aligned} \tag{2.1}$$

where u_{ω_0} satisfies $-Ric(\omega_0) + k\omega_0 + \theta = \sqrt{-1}\partial\bar{\partial}u_{\omega_0}$ and $\frac{1}{V}\int_M e^{-u_{\omega_0}}dV_0 = 1$.

Definition 2.1. By saying the twisted Mabuchi energy $\mathcal{M}_{k,\theta}$ is proper on $c_1(M)$, we mean that there exist constants A and B such that

$$\mathcal{M}_{k,\theta}(\phi) \geqslant AJ_{\omega_0}(\phi) - B$$

for any smooth strictly ω_0-plurisubharmonic function ϕ.

Now we recall Székelyhidi's [26] existence theorem.

Theorem 2.2 (Theorem 1 in [26] or Corollary 1.4 in [33]). *Given a Kähler metric $\tilde{\omega} \in c_1(M)$. The following are equivalent for $0 \leqslant k < 1$.*

- *There exists a unique solution to equation*

$$Ric(\omega) = k\omega + (1-k)\tilde{\omega};$$

- *There exists a Kähler metric $\hat{\omega} \in c_1(M)$ such that $Ric(\hat{\omega}) > k\hat{\omega}$;*
- *The twisted Mabuchi energy $\mathcal{M}_{k,(1-k)\tilde{\omega}}$ is proper on $c_1(M)$.*

Next, we prove Lemma 1.4.

Proof of Lemma 1.4. Fix $\varepsilon = 1$ in equation (1.8). Guenancia-Păun's arguments [15] imply that there exists a constant A depending only on β, n, ω_0

such that

$$\omega_{\beta,1} \leqslant A\frac{\omega_0}{(1+|s|_h^2)^{1-\beta}}. \tag{2.2}$$

When $0<\mu_\gamma<\frac{1}{2A}$. Calculations show that

$$\begin{aligned} Ric(\omega_{\beta,1}) &= (1-\beta)\big(\lambda\omega_0+\sqrt{-1}\partial\bar{\partial}\log(1+|s|_h^2)\big) \\ &= \omega_0+(1-\beta)\frac{\sqrt{-1}<D's,D's>_h}{(1+|s|_h^2)^2}-\frac{|s|_h^2}{1+|s|_h^2}\omega_0 \\ &\geqslant \mu_\gamma\omega_{\beta,1}-\mu_\gamma\omega_{\beta,1}+\omega_0-\frac{|s|_h^2}{1+|s|_h^2}\omega_0 \\ &\geqslant \mu_\gamma\omega_{\beta,1}-\frac{A\mu_\gamma\omega_0}{(1+|s|_h^2)^{1-\beta}}+\frac{1}{1+|s|_h^2}\omega_0 \\ &= \mu_\gamma\omega_{\beta,1}+\frac{1-A\mu_\gamma(1+|s|_h^2)^\beta}{1+|s|_h^2}\omega_0 \\ &\geqslant \mu_\gamma\omega_{\beta,1}+\frac{1-2A\mu_\gamma}{2}\omega_0>\mu_\gamma\omega_{\beta,1}, \end{aligned} \tag{2.3}$$

where D' is the $(1,0)$-part of the Chern connection associated to (L_D,h).

At the same time, we have

$$\begin{aligned} \frac{1}{\lambda}\theta_\varepsilon &= \omega_0+\frac{\varepsilon^2}{\lambda}\frac{\sqrt{-1}<D's,D's>_h}{(\varepsilon^2+|s|_h^2)^2}-\frac{|s|_h^2}{\varepsilon^2+|s|_h^2}\omega_0 \\ &\geqslant \frac{\varepsilon^2}{\varepsilon^2+|s|_h^2}\omega_0+\frac{\varepsilon^2}{\lambda}\frac{\sqrt{-1}<D's,D's>_h}{(\varepsilon^2+|s|_h^2)^2}>0. \end{aligned} \tag{2.4}$$

Let $\tilde{\omega}=\frac{1}{\lambda}\theta_\varepsilon$ and $k=\mu_\gamma$ in Theorem 2.2, for any $\mu_\gamma\in[0,\frac{1}{2A})$ and $\varepsilon\in(0,1)$, there exist solutions to equations (1.3). We complete the proof of Lemma 1.4. □

Remark 2.3. Fix $\varepsilon\in(0,1)$, the solution $\omega_{\gamma,\varepsilon}$ to equation (1.3) is a smooth path with respect to $\gamma\in[\beta,\beta+\delta_0)$ obtained in Lemma 1.4.

Proof of Lemma 1.5. Denote $L_\beta=\max\big(osc_M\varphi_\beta,B_\beta\big)$. If this lemma is not true. For $\delta_1<\delta_0$ with δ_0 obtained in Lemma 1.4, there exist $\varepsilon_1\in(0,\delta_1)$ and $\gamma_1'\in(\beta,\beta+\delta_1)$ such that

$$osc_M\varphi_{\gamma_1',\varepsilon_1}>L_\beta+1.$$

Remark 2.3 and (1.7) imply that there exists $\gamma_1 \in (\beta, \gamma_1')$ such that

$$osc_M \varphi_{\gamma_1,\varepsilon_1} = L_\beta + 1.$$

For $\delta_2 = \min(\frac{1}{2}, \varepsilon_1, \gamma_1 - \beta)$, there exist $\varepsilon_2 \in (0, \delta_2)$ and $\gamma_2' \in (\beta, \beta + \delta_2)$, such that

$$osc_M \varphi_{\gamma_2',\varepsilon_2} > L_\beta + 1.$$

Remark 2.3 and (1.7) imply that there exists $\gamma_2 \in (\beta, \gamma_2')$, such that

$$osc_M \varphi_{\gamma_2,\varepsilon_2} = L_\beta + 1.$$

After repeating the above process, we get a subsequence $\varphi_{\gamma_i,\varepsilon_i}$ with $\varepsilon_i \searrow 0$ and $\gamma_i \searrow \beta$ satisfying

$$osc_M \varphi_{\gamma_i,\varepsilon_i} = L_\beta + 1. \tag{2.5}$$

Let χ_γ with $\gamma \in (0,1)$ be the function $\chi_\gamma(\varepsilon^2 + |s|_h^2) = \frac{1}{\gamma}\int_0^{|s|_h^2} \frac{(\varepsilon^2+r)^\gamma - \varepsilon^{2\gamma}}{r} dr$ which is smooth on M and converge uniformly to $\frac{1}{\gamma^2}|s|_h^{2\gamma}$ as $\varepsilon \to 0$. Denote $F_{\gamma,\varepsilon} = F_0 + \log\left(\frac{(\omega_\varepsilon^\gamma)^n}{\omega_0^n} \cdot (\varepsilon^2 + |s|_h^2)^{1-\gamma}\right)$ and $\omega_\varepsilon^\gamma = \omega_0 + \sqrt{-1}k\partial\overline{\partial}\chi_\gamma$. From Campana-Guenancia-Păun's results (see (15) and (25) in [4]), $F_{\gamma,\varepsilon}$ and χ_γ are uniformly bounded (independent of γ and ε) for any $\varepsilon > 0$ and $\gamma \in [\beta, 1]$. Combining (2.5) with Guenancia-Păun's arguments (see section 2 and section 5 in [15]), we conclude that there exists uniform constant C such that

$$C^{-1}\omega_{\varepsilon_i}^{\gamma_i} \leqslant \omega_{\gamma_i,\varepsilon_i} \leqslant C\omega_{\varepsilon_i}^{\gamma_i} \tag{2.6}$$

for any γ_i and ε_i. For any $K \subset\subset M \setminus D$, there exists uniform constant C such that

$$C^{-1}\omega_0 \leqslant \omega_{\gamma_i,\varepsilon_i} \leqslant C\omega_0 \quad on \quad K. \tag{2.7}$$

Then for any $k \in \mathbb{N}^+$, Evans-Krylov's estimates (see [13] or [19]) imply that there exist uniform constants $C_{k,K}$ such that

$$\|\varphi_{\gamma_i,\varepsilon_i}\|_{C^k(K)} \leqslant C_{k,K} \tag{2.8}$$

for any γ_i and ε_i. Hence $\varphi_{\gamma_i,\varepsilon_i}$ (by taking a subsequence if necessary) converge to a function $\varphi_\infty \in C^\infty(M \setminus D)$ in C^∞_{loc}-topology in $M \setminus D$. By estimates (2.6), the Lebesgue Dominated Convergence theorem implies that

$\int_M(\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi_\infty)^n=\int_M\omega_0^n$. For any $K\subset\subset M\setminus D$, since $\omega_{\gamma_i,\varepsilon_i}$ are uniformly equivalent to ω_0 by (2.7) and μ_{γ_i} tend to 0, we conclude that

$$Ric(\omega_{\varphi_\infty})=(1-\beta)[D] \quad on \quad M. \tag{2.9}$$

This is equivalent to that φ_∞ satisfies equation (1.5). Then Dinew's uniqueness theorem (Theorem 1.2 in [9], see also Berndtsson's uniqueness theorem [2]) implies that $\varphi_\infty=\varphi_\beta+C$. Letting $i\to\infty$ in (2.5), we get

$$osc_M\varphi_\beta=osc_M\varphi_\infty=L_\beta+1>osc_M\varphi_\beta. \tag{2.10}$$

This leads to a contradiction. Thus Lemma 1.5 is proved. □

Proof of Theorem 1.1. For any $\gamma\in(\beta,\beta+\delta(\lambda))$ obtained in Lemma 1.5, there exists uniform constant C depending only on n, β, γ and ω_0 such that

$$\|\varphi_{\gamma,\varepsilon}\|_{C^0(M)}\leqslant C \tag{2.11}$$

for $\varepsilon\in(0,1)$. Then Guenancia-Păun's arguments (see Proposition 1 and section 5 in [15]) imply that there exists uniform constant C depending only on n, β, γ and ω_0 such that

$$C^{-1}\omega_\varepsilon^\gamma\leqslant\omega_{\gamma,\varepsilon}\leqslant C\omega_\varepsilon^\gamma \quad on \quad M. \tag{2.12}$$

Hence on any $K\subset\subset M\setminus D$, the metric $\omega_{\gamma,\varepsilon}$ are uniformly equivalent to ω_0. For any $k\in\mathbb{N}^+$, Evans-Krylov's estimates imply that there exist uniform constants $C_{k,K}$ depending only on n, β, γ, k, $dist_{\omega_0}(K,D)$ and ω_0 such that

$$\|\varphi_{\gamma,\varepsilon}\|_{C^k(K)}\leqslant C_{k,K} \tag{2.13}$$

for $\varepsilon\in(0,1)$. So we can choose a subsequence $\varphi_{\gamma,\varepsilon_i}$ which converge to a function $\varphi_\gamma\in C^\alpha(M)\bigcap C^\infty(M\setminus D)$ in C^α-sense globally and in C^∞_{loc}-topology in $M\setminus D$. Furthermore, ω_{φ_γ} is a conical Kähler-Einstein metric with Ricci curvature μ_γ and cone angle $2\pi\gamma$ along D. In fact, Berndtsson's uniqueness theorem implies that $\varphi_{\gamma,\varepsilon}$ converge to φ_γ in C^α-sense globally and in C^∞_{loc}-topology in $M\setminus D$. □

Remark 2.4. When $\lambda=1$, we can not get the uniform estimates as (1.7) when $\mu_\beta=0$ (that is, $\beta=0$). We denote $\kappa>0$ and $\nu_\gamma=1-(1-\gamma)(1+\kappa)$.

We consider the twisted conical Kähler-Einstein metrics

$$Ric(\omega) = \nu_\gamma \omega + (1-\gamma)\kappa\omega_0 + (1-\gamma)[D] \quad on\ M. \tag{2.14}$$

By similar arguments in this paper, for any $\gamma \in [1-\frac{1}{1+\kappa}, 1-\frac{1}{1+\kappa}+\delta(\kappa)]$, there exist twisted conical Kähler-Einstein metrics with positive Ricci curvatures ν_γ and cone angles $2\pi\gamma$ along D. If κ is sufficiently small, the cone angles are also small. We also remark that Donaldson's openness theorem (Theorem 2 in [12]), Chen-Donaldson-Sun [6–8] and Tian's [29] arguments still work for equation (2.14). When the cone angles evolve to 2π (that is, γ evolve to 1), this equation becomes the smooth Kähler-Einstein equation.

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