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Boundary unique continuation for the Laplace equation and the biharmonic operator

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We establish results on unique continuation at the boundary for the solutions of $\Delta u = f,\ f$ harmonic, and the biharmonic equation $\Delta^2 u = 0$. The work is motivated by analogous results proved for harmonic functions by X. Huang et al in [HK] and [HKMP] and by M. S. Baouendi and L. P. Rothschild in [BR1] and [BR2].

1. Introduction

The real part of $f(z) = e^{\frac{-1}{\sqrt{-iz}}}$ (use the main branch of the square root defined on $\mathbb{C} \setminus (-\infty, 0]$ is a harmonic function on the upper half plane $\mathbb{R}^2_+$ which is smooth up to the boundary and vanishes to infinite order at the origin. In the works [BR1] and [BR2] Baouendi and Rothschild proved results on unique continuation at the boundary and a local Hopf lemma for harmonic

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functions. Their papers extended earlier boundary unique continuation results for holomorphic functions obtained in the works [A2], [HKMP], and [L]. These latter results were applied to establish unique continuation for CR mappings between embedded CR submanifolds in the works [A1], [A2], [ABR], [BL], [BH1], [HK], and [HKMP].

This paper establishes analogues of the main results in [BR1] and [BR2] for solutions of the biharmonic equation $\Delta^2 u = 0$ and the Laplace equation $\Delta u = f$ where f is a harmonic function. We believe that the theorems for the biharmonic operator proved in this paper are the first boundary unique continuation results for operators of degree higher than two. The biharmonic equation arises in many areas of continuum mechanics. In solid mechanics it is used to model elasto-static deformation in the absence of body forces and the solution u may represent the Airy stress function for a two-dimensional, isotropic, linear elastic solid or the deflection of a clamped thin plate. In fluid mechanics, it can be used to describe the motion of an incompressible viscous fluid at low Reynolds number and its solution represents the stream function for Stokes flow.

Further extensions of the results of Baouendi and Rothschild in [BR1] were proved by V. Shklover ([Sh]) and H. S. Shapiro ([S]). See also [B] and [BH2]. In particular, Shklover showed that Theorem 3 in [BR1] (Theorem A in this paper) fails if the normal direction is replaced with a transverse direction. Shapiro used convolution transforms as discussed in [HW] to obtain new proofs and generalizations of the theorems in [BR1] and [BR2].

The Dirichlet problem for the biharmonic operator has been studied in numerous papers. In particular, in [DKV], the authors proved that if D is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $f \in L_1^2(\partial D)$, $g \in L^2(\partial D)$, there exists a unique biharmonic function u on D, which takes the boundary value f and whose normal derivative $\frac{\partial u}{\partial \nu}$ equals g on ∂D, both in the sense of non-tangential convergence, and such that the non-tangential maximal function of ∇u is in $L^2(\partial D)$.

In a forthcoming article, we will extend Theorems 2 and 3 in this work to polyharmonic functions of any finite order.

Section 2 contains the statements of our main results and examples. After some preliminary integral representations in Section 3, the proofs are presented in Sections 4, 5, and 6.

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2. Statements of the results and examples

We begin by recalling the results of [BR1] and [BR2]:
We will say that a continuous function u defined on a half ball

$$B_r^+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x| < r, x_n > 0\}$$

is flat at 0 if for every positive integer N, there is $c_N > 0$ such that

$$|u(x)| \leq c_N |x|^N.$$

Theorem A ([BR1]). Let u be harmonic on the half ball B_r^+, continuous on the closure. Suppose

(1) $u(s, 0) \geq 0$ for $|s| \leq r$, $s \in \mathbb{R}^{n-1}$;

(2) the function $x_n \longmapsto u(0', x_n)$ is flat at $x_n = 0$;

(3) for every positive integer N, the function $|s|^{-N} u(s, 0)$ is integrable on $\{s \in \mathbb{R}^{n-1} : |s| \leq r\}$. Then, $u(x', 0) \equiv 0$ on $|x'| \leq \epsilon$ for some $\epsilon > 0$.

The following corollary is an immediate consequence:

Corollary B. Let u be harmonic on the half ball B_r^+, continuous on the closure. Assume that

(1) $u(s, 0) \geq 0$ for $|s| \leq r$;

(2) $u(x)$ is flat at $x = 0$.

Then $u \equiv 0$.

When $n = 2$, Corollary B was proved in [HK]. Theorem A was generalized in [BR2] as follows:

Theorem C ([BR2]). Let u be harmonic on the half ball B_r^+, continuous on the closure. Assume that

(1) for some homogeneous polynomial $p(s)$ in $n - 1$ variables,

$$p(s)u(s, 0) \geq 0 \quad \text{for} \quad |s| \leq r;$$

(2) for every multi-index β in n variables, $|\beta| \leq d =$ degree of p,

$$x_n \longmapsto \partial_x^\beta u(0; x_n) \quad \text{is flat at} \quad x_n = 0;$$

(3) for every positive integer N, $|s|^{-N}p(s)u(s,0)$ is integrable on $\{s \in \mathbb{R}^{n-1} : |s| \leq r\}$. Then, $u(x',0) \equiv 0$ on $|x'| \leq \epsilon$ for some $\epsilon > 0$.

Corollary D. Let u be harmonic on the half ball B_r^+, continuous on the closure. Suppose

(1) for some homogeneous polynomial $p(s)$ in $n-1$ variables,

$$p(s)u(s,0) \geq 0 \quad for \quad |s| \leq r;$$

(2) $u(x)$ is flat at $x = 0$. Then $u \equiv 0$.

Inspired by the preceding results, in this paper we will prove:

Theorem 1. *Let u be a solution of the biharmonic equation $\Delta^2 u = 0$ in the half ball B_r^+, C^4 on the closure. Suppose that for some monomial $\alpha = (\alpha_1, ..., \alpha_{n-1})$,*

(1) *for every multi-index β in n variables with $|\beta| \leq d$, where $d = |\alpha|$ is the degree of s^α, the function $x_n \longmapsto (\partial_x^\beta)u(0, x_n)$ is flat at $x_n = 0$;*

(2) *for every positive integer N, the functions $|s|^{-N}s^\alpha u(s,0)$ and $|s|^{-N}s^\alpha u_{x_n}(s,0)$ are integrable on $\{s \in \mathbb{R}^{n-1} : |s| \leq r\}$. Then there exists $\epsilon > 0$ such that if $s^\alpha u(s,0) \geq 0$, then $u(x',0) \equiv 0$ for $|x'| \leq \epsilon$ and if $s^\alpha u_{x_n}(s,0) \geq 0$, $u_{x_n}(x',0) \equiv 0$ for $|x'| \leq \epsilon$.*

In particular, if $s^\alpha u(s,0) \geq 0$ and $s^\alpha u_{x_n}(s,0) \geq 0$, then u extends as a solution to a neighborhood of the origin.

Corollary 1. *Let u be a solution of $\Delta^2 u = 0$ in B_r^+, C^4 on the closure. Assume that for some monomial $\alpha = (\alpha_1, ..., \alpha_{n-1})$,*

(1) *$s^\alpha u(s,0) \geq 0$ and $s^\alpha u_{x_n}(s,0) \geq 0$ for $|s| \leq r$;*

(2) *$|s|^{-N}s^\alpha u_{x_n}(s,0)$ is integrable on $\{s \in \mathbb{R}^{n-1} : |s| \leq r\}$ for every N.*

(3) *$u(x)$ is flat at $x = 0$.*

Then, $u(x) \equiv 0$.

Observe that if in the preceding corollary, u is assumed to be smooth up to $x_n = 0$, then condition (2) is implied by (3).

Theorem 2. *Let u be a solution of $\Delta^2 u = 0$ in B_r^+, C^4 on the closure. Assume that*

(1) *for some homogeneous polynomial* $p(s)$ *in* $n-1$ *variables,* $p(s)u(s,0) \geq 0$ *for* $|s| \leq r$;

(2) *for every multi-index* β *in* n *variables with* $|\beta| \leq d$, *where* d *is the degree of* $p(s)$, *the function* $x_n \longmapsto (\partial_x^\beta)u(0,x_n)$ *is flat at* $x_n = 0$;

(3) *for every positive integer* N, $|s|^{-N}p(s)u(s,0)$ *is integrable on* $\{s \in \mathbb{R}^{n-1} : |s| \leq r\}$;

(4) $u_{x_n}(x',0) \equiv 0$.

Then, there exists $\epsilon > 0$ *such that* $u(x',0) \equiv 0$ *for* $|x'| \leq r$.

Corollary 2. *Let* u *be a solution of* $\Delta^2 u = 0$ *in* B_r^+, C^4 *on the closure. Assume that*

(1) *for some homogeneous polynomial* $p(s)$ *in* $n-1$ *variables,* $p(s)u(s,0) \geq 0$ *for* $|s| \leq r$;

(2) $u_{x_n}(x',0) \equiv 0$ *for* $|s| \leq r$;

(3) *for every multi-index* β *in* n *variables with* $|\beta| \leq d$, *where* d *is the degree of* $p(s)$, *the function* $x_n \longmapsto (\partial_x^\beta)u(0,x_n)$ *is flat at* $x_n = 0$;

(4) $u(x)$ *is flat at* $x = 0$.

Then, $u \equiv 0$.

We next state boundary uniqueness results for a different Dirichlet condition where we no longer assume integrability of $|s|^{-N}u(s,0)$ for any N.

Theorem 3. *Let* u *be a solution of the biharmonic equation* $\Delta^2 u = 0$ *in the half ball* B_r^+, C^4 *on the closure. Assume that*

(1) $u(s,0) \geq 0$ *and* $\Delta u(s,0) \leq 0$ *for* $|s| \leq r$ *where* Δ *is in* n *variables;*

(2) *the function* $x_n \longmapsto u(0,x_n)$ *is flat at* $x_n = 0$.

Then, $u(x',0) \equiv 0$ *and* $\Delta u(x',0) \equiv 0$ *on* $|x'| \leq \epsilon$ *for some* $\epsilon > 0$. *In particular,* u *extends as a solution to a neighborhood of origin.*

Corollary 3. *Let* u *be a solution of* $\Delta^2 u = 0$ *on* B_r^+, C^4 *on the closure. Suppose*

(1) $u(s,0) \geq 0$ *and* $\Delta u(s,0) \leq 0$ *for* $|s| \leq \epsilon$;

(2) $u(x)$ *is flat at* $x = 0$.

Then $u(x) \equiv 0$.

Theorem 3 leads to the following generalization of the main result in [BR1] (Theorem A in this paper) on boundary unique continuation for the Laplace operator:

Corollary 4. *Let* u *be* C^4 *on* $\overline{B_r^+}$ *and a solution of* $\Delta u = f$ *where* f *is harmonic on* B_r^+. *Suppose*

(1) $u(s, 0) \geq 0$ *and* $f(s, 0) \leq 0$ *for* $|s| \leq r$;

(2) *the function* $x_n \longrightarrow u(0, x_n)$ *is flat at* $x_n = 0$;

Then $u(x', 0) \equiv 0$ *and* $f(x', 0) \equiv 0$ *on* $|x'| \leq \epsilon$ *for some* $\epsilon > 0$. *In particular,* $u(x)$ *extends as a real analytic function past* $x_n = 0$.

Corollary 5. *Let* u *be* C^4 *on* $\overline{B_r^+}$ *and* $\Delta u = f$ *where* f *is harmonic on* B_r^+. *Suppose*

(1) $u(s, 0) \geq 0$ *and* $f(s, 0) \leq 0$ *for* $|s| \leq r$;

(2) $u(x)$ *is flat at* $x = 0$. *Then* $u \equiv 0$.

The following examples show that the theorems above may not hold if we drop some of the assumptions. In particular, Corollary 5 may not be valid if the harmonicity of f is replaced by real analyticity.

Example 1. In $\mathbb{R}^2$ let $u(x) = x_1^2 + x_1 x_2^2$. Then $\Delta^2 u = 0, u(x_1, 0) \geq 0$ and $u_{x_2}(x_1, 0) \equiv 0$. The function $x_2 \longmapsto u(0, x_2)$ is flat at $x_2 = 0$.

Example 2. Let $w(x_1) \in C^\infty(\mathbb{R})$ be flat at $x_1 = 0, w(x_1) > 0$ for $x_1 > 0$ and $w(x_1) < 0$ for $x_1 < 0$. Let v be a harmonic function in the half plane $x_2 > 0$ such that $v(x_1, 0) = w(x_1)$. Choose $M > 0$ large enough so that $M + v_{x_1 x_2}(0, 0) > 0$ and set $u(x) = x_1 v(x) - v_{x_2}(0, 0) x_1 x_2 + M x_1^2 x_2$. Then $\Delta^2 u(x) = 0$ for $x_2 > 0$, $u(x_1, 0)$ is flat at $x_1 = 0$ and so $|x_1|^{-N} u(x_1, 0)$ is locally integrable for all N. $u(x_1, 0) \geq 0$ and $x_2 \longmapsto u(0, x_2) \equiv 0$. Moreover, $u_{x_2}(x_1, 0) \geq 0$.

Example 3. Let

$$f(t) = \begin{cases} \exp\left(-\frac{1}{t^2}\right), & t > 0 \\ 0, & t = 0 \\ -\exp\left(-\frac{1}{t^2}\right), & t < 0. \end{cases}$$

The function f is C^∞ on $\mathbb{R}$, flat at $t = 0$ and $f'(t) \geq 0$. Let $v(x)$ be harmonic on $\mathbb{R}^2_+$ such that $v(x_1, 0) = f(x_1)$. Let $u(x_1, x_2) = x_1 v(x_1, x_2)$. Then,

$$\Delta^2 u(x) = 0 \quad \text{on} \quad \mathbb{R}^2_+, \quad u(x_1, 0) = x_1 f(x_1) \geq 0,$$

and

$$\Delta u(x_1, 0) = 2f'(x_1) \geq 0, \;\; u(0, x_2) \equiv 0.$$

The function $u(x_1, 0)$ is not identically zero in any neighborhood of the origin. This example shows that Theorem 3 may not hold if $u(x', 0)$ and $\Delta u(x', 0)$ have the same sign.

Example 4. In $\mathbb{R}^2_+$ let $u(x, y) = y^4 e^{\frac{-1}{x^2+y^2}}$. Then $\Delta u = f$ is real analytic in B_1^+, $u(x, 0) = \Delta u(x, 0) \equiv 0$, and $u(x, y)$ is flat at the origin. Thus Corolloary 5 may not be valid if f is merely real analytic in B_r^+.

3. Integral Representations And Preliminaries

In [Bo], Boggio showed that the Green's function G_n^B for Δ^2 with Dirichlet boundary conditions for the unit ball B in $\mathbb{R}^n$ is given by

$$G_n^B(x, y) = k_n |x - y|^{4-n} \int_1^{[XY]/[xy]} \frac{t^2 - 1}{t^{n-1}}\, dt$$

where k_n is a dimensional positive constant, $[XY] = \left| |x|y - \frac{x}{|x|} \right|$ and $[xy] = |x - y|$. In [BMZ], by replacing $[XY]$ in Boggio's formula by $|x - \bar{y}|$, the authors gave the following formula for the the Green's function

$$G_n(x, y) = c_n |x - y|^{4-n} \int_1^{\frac{|x-\bar{y}|}{|x-y|}} \frac{t^2 - 1}{t^{n-1}} dt$$

of Δ^2 on $\mathbb{R}^n_+$ with the Dirichlet boundary conditions:

$$G_n(x, y) = 0, \partial_{x_n} G_n(x, y) = 0 \;\; at \;\; x_n = 0.$$

Here, c_n is a constant and for $y = (y_1, ..., y_n), \bar{y} = (y_1, ..., -y_n)$. See also the paper [Be] for Green's functions for Δ^2 on the unit ball for other Dirichlet conditions.

We have:

$$G_n(x,y) = \begin{cases} c_n\left(\frac{1}{(4-n)|x-\bar{y}|^{n-4}} - \frac{|x-y|^2}{(2-n)|x-\bar{y}|^{n-2}} + \frac{2}{(n-4)(n-2)|x-y|^{n-4}}\right), \\ \qquad n \notin \{2,4\} \\ |x-y|^2 \log|\frac{x-y}{x-\bar{y}}|^2 + 4x_2y_2, \qquad n=2 \\ \frac{c_4}{2}\left(\log|\frac{x-\bar{y}}{x-y}|^2 + \frac{|x-y|^2}{|x-\bar{y}|^2} - 1\right), \qquad n=4. \end{cases}$$

In Green's identity

$$\int_{B_r^+} u(y)\Delta w(y)\,dy - \int_{B_r^+} \Delta u(y) w(y)\,dy = \int_{\partial B_r^+}\left(u(y)\frac{\partial w}{\partial \nu}(y) - \frac{\partial u}{\partial \nu}(y)w(y)\right)d\sigma,$$

we plug

$$w(y) = \Delta G_n(x,y) = \Delta_y G_n(x,y), \quad x \in B_r^+$$

to get:

$$u(x) - \int_{B_r^+} \Delta u(y)\Delta G_n(x,y)\,dy = \int_{\partial B_r^+}\left(u(y)\frac{\partial}{\partial \nu}\Delta G_n(x,y) - \Delta G_n(x,y)\frac{\partial u}{\partial \nu}(y)\right)d\sigma, \tag{1}$$

where $\frac{\partial}{\partial \nu}$ is the outer normal derivative and Δ acts in the y variable. Applying Green's identity again, we have:

$$\int_{B_r^+} \Delta G_n(x,y)\Delta u(y)\,dy = -\int_{\partial B_r^+}\left(G_n(x,y)\frac{\partial}{\partial \nu}\Delta u(y) - \Delta u(y)\frac{\partial G_n}{\partial \nu}(x,y)\right)d\sigma + \int_{B_r^+} G_n(x,y)\Delta^2 u(y)\,dy. \tag{2}$$

From (1) and (2), using $\Delta^2 u = 0$, and $G_n(x,y) = \frac{\partial G_n}{\partial \nu}(x,y) = 0$, when $y_n = 0$, we get the following representation formula for u in B_r^+ :

$$u(x) = -\int_{\partial B_r^+\setminus\Sigma}\left(G_n(x,y)\frac{\partial}{\partial \nu}\Delta u(y) - \Delta u(y)\frac{\partial G_n}{\partial \nu}(x,y)\right)d\sigma + \int_{\partial B_r^+}\left(u(y)\frac{\partial}{\partial \nu}\Delta G_n(x,y) - \Delta G_n(x,y)\frac{\partial u}{\partial \nu}(y)\right)d\sigma, \tag{3}$$

where $\Sigma = \{y \in \partial B_r^+ : \ y_n = 0\}$.
For $x \in B_r^+$, define

$$v(x) = \int_\Sigma \left(u(y)\frac{\partial}{\partial \nu}\Delta G_n(x,y) - \Delta G_n(x,y)\frac{\partial u}{\partial \nu}(y)\right) d\sigma.$$

For $x \in \Sigma, y \in \partial B_r^+ \setminus \Sigma$, since $G_n(x,y) = \partial_{x_n} G_n(x,y) = 0$ (because $G_n(x,y) = G_n(y,x)$) and $\frac{\partial}{\partial \nu}, \Delta$ in the integrals above act in the y variable, we have:

$$\frac{\partial}{\partial \nu}\Delta G_n(x,y) = \Delta G_n(x,y) = 0.$$

It follows that

$$v(x',0) = u(x',0) \ \text{ and } \ \frac{\partial v}{\partial x_n}(x',0) = \frac{\partial u}{\partial x_n}(x',0). \tag{4}$$

4. Proof of Theorem 1

We first assume that the multi-index $\alpha = 0$.

Case 1. Suppose $n \notin \{2,4\}$.
Since $\frac{\partial}{\partial \nu}$ is the outer normal derivative,

$$v(x) = \int_\Sigma \left(\Delta G_n(x,y')\frac{\partial u}{\partial y_n}(y',0) - u(y',0)\frac{\partial}{\partial y_n}\Delta G_n(x,y')\right) dy'$$

where $\Delta = \Delta_y$ and $\frac{\partial}{\partial y_n}\Delta G_n(x,y') = \frac{\partial}{\partial y_n}\Delta_y G_n(x,y)\Big|_{y_n=0}$.

We will need certain high order derivative of $v(x)$. We begin by computing $\Delta G_n(x,y)$ and $\frac{\partial}{\partial y_n}\Delta G_n(x,y)$ at $y_n = 0$.
Recall that when $n \notin \{2,4\}$,

$$G_n(x,y) = \\ c_n \left(\frac{1}{(4-n)|x-\bar{y}|^{n-4}} - \frac{|x-y|^2}{(n-2)|x-\bar{y}|^{n-2}} + \frac{2}{(n-4)(n-2)|x-y|^{n-4}} \right).$$

When $1 \le j \le n-1$,

$$\frac{\partial}{\partial y_j}|x-\bar{y}|^{4-n} = (n-4)|x-\bar{y}|^{2-n}(x_j - y_j), \text{and so}$$

$$\frac{\partial^2}{\partial y_j^2}|x-\bar{y}|^{4-n} = (4-n)\left((2-n)|x-\bar{y}|^{-n}(x_j-y_j)^2 + |x-\bar{y}|^{2-n}\right),$$

while

$$\frac{\partial}{\partial y_n}|x-\bar{y}|^{4-n} = (4-n)|x-\bar{y}|^{2-n}(x_n+y_n), \quad \text{and}$$
$$\frac{\partial^2}{\partial y_n^2}|x-\bar{y}|^{4-n} = (4-n)\left((2-n)|x-\bar{y}|^{-n}(x_n+y_n)^2 + |x-\bar{y}|^{2-n}\right).$$

It follows that

$$\begin{aligned} &\Delta_y|x-\bar{y}|^{4-n} = 2(4-n)|x-\bar{y}|^{2-n}, \\ &\text{and } \Delta_y|x-y|^{4-n} = 2(4-n)|x-y|^{2-n}. \end{aligned} \tag{5}$$

For $1 \le j \le n-1$,

$$\frac{\partial}{\partial y_j}|x-\bar{y}|^{2-n} = (n-2)|x-\bar{y}|^{-n}(x_j-y_j),$$
$$\frac{\partial^2}{\partial y_j^2}|x-\bar{y}|^{2-n} = (2-n)\left(|x-\bar{y}|^{-n} - n|x-\bar{y}|^{-n-2}(x_j-y_j)^2\right),$$
$$\frac{\partial}{\partial y_n}|x-\bar{y}|^{2-n} = (2-n)|x-\bar{y}|^{-n}(x_n+y_n), \quad \text{and}$$
$$\frac{\partial^2}{\partial y_n^2}|x-\bar{y}|^{2-n} = (n-2)\left(|x-\bar{y}|^{-n} - n|x-\bar{y}|^{-n-2}(x_n+y_n)^2\right).$$

Hence

$$\begin{aligned} \Delta_y|x-y|^2|x-\bar{y}|^{2-n} &= |x-y|^2\nabla_y|x-\bar{y}|^{2-n} + \Delta(|x-y|^2)|x-\bar{y}|^{2-n} \\ &\quad + 2\langle\nabla_y|x-y|^2, \nabla_y|x-\bar{y}|^{2-n}\rangle \\ &= \frac{2n}{|x-\bar{y}|^{n-2}} + 4(2-n)\frac{|x'-y'|^2}{|x-\bar{y}|^n} \\ &\quad + 4(n-2)\left(\frac{x_n^2-y_n^2}{|x-\bar{y}|^n}\right). \end{aligned} \tag{6}$$

From (5) and (6),

$$\begin{aligned} \Delta_y G_n(x,y) &= \frac{D_n}{|x-\bar{y}|^{n-2}} - 4c_n\left(\frac{|x'-y'|^2}{|x-\bar{y}|^n}\right) \\ &\quad + 4c_n\left(\frac{x_n^2-y_n^2}{|x-\bar{y}|^n}\right) - \frac{4c_n}{n-2}\left(\frac{1}{|x-y|^{n-2}}\right). \end{aligned} \tag{7}$$

where $D_n = \frac{2nc_n}{n-2} + 2c_n$.

It follows that

$$\Delta_y G_n(x,y') = \Delta_y G_n(x,y)|_{y_n=0} = \frac{8c_n x_n^2}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}}. \tag{8}$$

We next compute $\frac{\partial}{\partial y_n}\Delta_y G_n(x,y)$ at $y_n = 0$.

$$\frac{\partial}{\partial y_n}\left(\frac{1}{|x-\bar{y}|^{n-2}}\right)\bigg|_{y_n=0} = \frac{(2-n)x_n}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}},$$

$$\frac{\partial}{\partial y_n}\left(\frac{1}{|x-\bar{y}|^{n}}\right)\bigg|_{y_n=0} = \frac{-nx_n}{(|x'-y'|^2+x_n^2)^{\frac{n+2}{2}}},$$

$$\frac{\partial}{\partial y_n}\left(\frac{x_n^2-y_n^2}{|x-\bar{y}|^{n}}\right)\bigg|_{y_n=0} = \frac{-nx_n^3}{(|x'-y'|^2+x_n^2)^{\frac{n+2}{2}}}, \text{ and}$$

$$\frac{\partial}{\partial y_n}\left(\frac{1}{|x-y|^{n-2}}\right)\bigg|_{y_n=0} = \frac{(n-2)x_n}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}};$$

Therefore,

$$\begin{aligned} &\frac{\partial}{\partial y_n}\Delta_y G_n(x,y)|_{y_n=0} \\ &= \frac{(2-n)D_n x_n}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}} + \frac{4c_n n x_n |x'-y'|^2 - 4c_n n x_n^3}{(|x'-y'|^2+x_n^2)^{\frac{n+2}{2}}} \\ &\quad - \frac{4c_n x_n}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}} \\ &= \frac{(2-n)D_n x_n(|x'-y'|^2+x_n^2)+4c_n n x_n|x'-y'|^2-4c_n n x_n^3-4c_n x_n(|x'-y'|^2+x_n^2)}{(|x'-y'|^2+x_n^2)^{\frac{n+2}{2}}} \\ &= \frac{-8nc_n x_n^3}{(|x'-y'|^2+x_n^2)^{\frac{n+2}{2}}}. \end{aligned} \tag{9}$$

From (8) and (9), for $0 < x_n < r$,

$$v(0',x_n) = \int_\Sigma \left[\frac{8c_n x_n^2}{(|y'|^2+x_n^2)^{\frac{n}{2}}}\frac{\partial u}{\partial y_n}(y',0) + \frac{8nc_n x_n^3}{(|y'|^2+x_n^2)^{\frac{n+2}{2}}}u(y',0)\right]dy'. \tag{10}$$

Set $w(x) = u(x) - v(x),\ x \in B_r^+$. The function w is a solution of $\Delta^2 w = 0$ on B_r^+ , and on Σ, from (4),

$$w(x',0) = 0 \quad \text{and} \quad \frac{\partial w}{\partial x_n}(x',0) = 0.$$

By the reflection principle for Δ^2, it follows that $w(x)$ extends past $x_n = 0$ as a solution of Δ^2, and hence as a real analytic function. Indeed, for $x_n < 0$, the extension which we still denote by w is defined by (see [Hu])

$$w(x', -x_n) = -w(x) + 2x_n \frac{\partial w}{\partial x_n}(x) - x_n^2 \Delta w(x).$$

By the local integrability of $|y'|^{-N} u(y', 0)$ and $|y'|^{-N} \frac{\partial u}{\partial y_n}(y', 0)$ for all N, formula (10) implies that the function $x_n \longmapsto v(0^{'}, x_n)$ is smooth up to $x_n = 0$. Therefore,

$$x_n \longmapsto u(0^{'}, x_n) = v(0^{'}, x_n) + w(0^{'}, x_n)$$

is also a smooth on $[0, r)$. Because it is flat at $x_n = 0$,

$$\partial_{x_n}^k v(0) = -\partial_{x_n}^k w(0) \ \text{ for all } k.$$

Since w is analytic on a neighbourhood of the origin, there exists $c > 0$ such that

$$\left| \partial_{x_n}^k v(0) \right| \leq c^{k+1} k! \tag{11}$$

We next estimate the derivatives $\partial_{x_n}^k v(0)$ using formula (10). Let k be a positive integer, and consider

$$\begin{aligned} &\partial_{x_n}^k \left\{ \frac{8nc_n x_n^3}{(|y'|^2 + x_n^2)^{\frac{n+2}{2}}} \right\} \Bigg|_{x_n=0} \\ &\qquad = 8nc_n k(k-1)(k-2) \partial_{x_n}^{k-3} \left\{ \frac{1}{(|y'|^2 + x_n^2)^{\frac{n+2}{2}}} \right\} \Bigg|_{x_n=0}. \end{aligned}$$

To compute the later derivative, we use Faà di Bruno's formula

$$\frac{d^m}{dt^m} F(g(t)) = \sum \frac{m!}{m_1! \cdots m_m!} F^{(m_1+\cdots+m_m)}(g(t)) \prod_{j=1}^{m} \left(\frac{g^{(j)}(t)}{j!} \right)^{m_j}$$

where the sum is taken over all m-tuples of nonnegative integers $m_1, \cdots, m_m$ that satisfy the constraint

$$m_1 + 2m_2 + \cdots + m m_m = m.$$

Setting $g(t) = |y'|^2 + t^2$ and $F(s) = \frac{1}{s^{\frac{n+2}{2}}}$, the formula leads to

$$\left(\frac{d}{dt}\right)^m F(g(t))\Big|_{t=0} = 0, \quad \text{when } m \text{ is odd, and}$$

$$\left(\frac{d}{dt}\right)^{2N} F(g(t))\Big|_{t=0} = \frac{(2N)!}{N!} F^{(N)}(|y'|^2).$$

Since

$$\begin{aligned} F^{(N)}(s) &= \frac{(-1)^N(n+2)(n+4)\cdots(n+2N)}{2^N s^{\frac{n}{2}+N+1}}, \\ \partial_{x_n}^{2N+1} &\left\{ \frac{8nc_n x_n^3}{(|y'|^2+x_n^2)^{\frac{n+2}{2}}} \right\}\Big|_{x_n=0} \\ &= 8nc_n(2N+1)(2N)(2N-1)\partial_{x_n}^{2N-2} \left\{ \frac{1}{(|y'|^2+x_n^2)^{\frac{n+2}{2}}} \right\}\Big|_{x_n=0} \\ &= \frac{(-1)^{N-1}8nc_n(2N+1)!(n+2)(n+4)\cdots(n+2N-2)}{(N-1)!2^{N-1}|y'|^{n+2N}} \\ &= \frac{(-1)^{N-1}8nc_n(2N+1)!(\frac{n}{2}+1)(\frac{n}{2}+2)\cdots(\frac{n}{2}+N-1)}{(N-1)!|y'|^{n+2N}}. \end{aligned}$$

Thus,

$$\left| \partial_{x_n}^{2N+1} \left\{ \frac{8nc_n x_n^3}{(|y'|^2+x_n^2)^{\frac{n+2}{2}}} \right\}\Big|_{x_n=0} \right| \geq \frac{8nc_n(2N+1)!}{|y'|^{n+2N}}. \tag{12}$$

Consider next

$$\partial_{x_n}^k \left\{ \frac{8c_n x_n^2}{(|y'|^2+x_n^2)^{\frac{n}{2}}} \right\}\Big|_{x_n=0} = 8k(k-1)c_n \partial_{x_n}^{k-2} \left\{ \frac{1}{(|y'|^2+x_n^2)^{\frac{n}{2}}} \right\}\Big|_{x_n=0}.$$

Faà di Bruno's formula this time implies that

$$\partial_{x_n}^k \left\{ \frac{8c_n x_n^2}{(|y'|^2+x_n^2)^{\frac{n}{2}}} \right\}\Big|_{x_n=0} = 0 \quad \text{when } k \text{ is odd, and}$$

$$\begin{aligned} \partial_{x_n}^{2N} &\left\{ \frac{8c_n x_n^2}{(|y'|^2+x_n^2)^{\frac{n}{2}}} \right\}\Big|_{x_n=0} \\ &= \frac{(-1)^{N-1}\ 8c_n(2N)!\ n(n+2)\cdots(n+2N-4)}{(N-1)!\ \ 2^{N-1}\ \ |y'|^{n+2N-2}} \end{aligned}$$

and so

$$\left| \partial_{x_n}^{2N} \left\{ \frac{8c_n x_n^2}{(|y'|^2 + x_n^2)^{\frac{n}{2}}} \right\} \Big|_{x_n=0} \right| \geq \frac{8c_n (2N)!}{|y'|^{n+2N-2}} \tag{13}$$

If $u(x',0) \geq 0$, from (10), (12) and the fact that the odd order derivatives

$$\partial_{x_n}^k \left\{ \frac{x_n^2}{(|y'|^2 + x_n^2)^{\frac{n}{2}}} \right\} \Big|_{x_n=0} = 0,$$

we conclude that

$$\left|\partial_{x_n}^{2N+1} v(0)\right| \geq 8nc_n (2N+1)! \int_\Sigma \frac{u(y',0)}{|y'|^{n+2N}}\, dy'$$

and hence using (11), for any $0 < \epsilon < r$,

$$c^{2N+1} \geq 8nc_n \int_\Sigma \frac{u(y',0)}{|y'|^{n+2N}}\, dy' \geq \frac{8nc_n}{\epsilon^{n+2N}} \int_{|y'|<\epsilon} u(y',0)\, dy'.$$

Choosing ϵ so that $\epsilon < \frac{1}{c}$, taking the $(2N+1)^{th}$ root and letting $n \longrightarrow \infty$, we conclude that

$$u(x',0) \equiv 0, \quad \text{for} \quad |x'| \leq \epsilon.$$

Likewise, if $\frac{\partial u}{\partial x_n}(x',0) \geq 0$, since the even order derivatives

$$\partial_{x_n}^k \left\{ \frac{x_n^3}{(|y'|^2 + x_n^2)^{\frac{n+2}{2}}} \right\} \Big|_{x_n=0} = 0,$$

using (10), (11) and (13), we get

$$c^{2N} \geq 8c_n \int_\Sigma \frac{\frac{\partial u}{\partial y_n}(y',0)}{|y'|^{n+2N-2}}\, dy' \geq \frac{8nc_n}{\epsilon^{n+2N-2}} \int_{|y'|<\epsilon} \frac{\partial u}{\partial y_n}(y',0)\, dy',$$

and therefore,

$$\frac{\partial u}{\partial y_n}(x',0) \equiv 0 \quad \text{for} \quad |x'| \leq \epsilon.$$

Case 2. Assume $n = 4$. Recall that

$$G_4(x,y) = \frac{c_4}{2} \left(\log \left| \frac{x-\bar{y}}{x-y} \right|^2 + \frac{|x-y|^2}{|x-\bar{y}|^2} - 1 \right).$$

As before, we first compute $\Delta_y G_4(x,y)$ and

$$\frac{\partial}{\partial y_4}\Delta_y G_4(x,y) \quad \text{at} \quad y_4 = 0.$$

For $1 \le j \le 3$, we have:

$$\begin{aligned}
\frac{\partial}{\partial y_j}\log\left|\frac{x-\bar{y}}{x-y}\right|^2 &= \left|\frac{x-y}{x-\bar{y}}\right|^2 \frac{\partial}{\partial y_j}\left(\left|\frac{x-\bar{y}}{x-y}\right|^2\right)\\
&= \frac{2(x_j-y_j)(|x-\bar{y}|^2-|x-y|^2)}{|x-\bar{y}|^2|x-y|^2}, \quad \text{and}\\
\frac{\partial^2}{\partial y_j^2}\log\left|\frac{x-\bar{y}}{x-y}\right|^2 &= \frac{-2(|x-\bar{y}|^2-|x-y|^2)}{|x-\bar{y}|^2|x-y|^2}\\
&\quad + \frac{4(x_j-y_j)^2(|x-\bar{y}|^2-|x-y|^2)}{|x-\bar{y}|^4|x-y|^2}\\
&\quad + \frac{4(x_j-y_j)^2(|x-\bar{y}|^2-|x-y|^2)}{|x-\bar{y}|^2|x-y|^4}.\\
\frac{\partial}{\partial y_4}\log\left|\frac{x-\bar{y}}{x-y}\right|^2 &= \frac{|x-y|^2}{|x-\bar{y}|^2}\frac{\partial}{\partial y_4}\left(\left|\frac{x-\bar{y}}{x-y}\right|^2\right)\\
&= \frac{2(x_4+y_4)}{|x-\bar{y}|^2} + \frac{2(x_4-y_4)}{|x-y|^2}, \quad \text{and}\\
\frac{\partial^2}{\partial y_4^2}\log\left|\frac{x-\bar{y}}{x-y}\right|^2 &= \frac{2|x-\bar{y}|^2-4(x_4+y_4)^2}{|x-\bar{y}|^4}\\
&\quad + \frac{4(x_4-y_4)^2-2|x-y|^2}{|x-y|^4}.
\end{aligned}$$

For $1 \le j \le 3$,

$$\begin{aligned}
\frac{\partial}{\partial y_j}\frac{|x-y|^2}{|x-\bar{y}|^2} &= \frac{2(x_j-y_j)\left(|x-y|^2-|x-\bar{y}|^2\right)}{|x-\bar{y}|^4}, \quad \text{and}\\
\frac{\partial^2}{\partial y_j^2}\left(\frac{|x-y|^2}{|x-\bar{y}|^2}\right) &= \frac{2\left(|x-\bar{y}|^2-|x-y|^2\right)}{|x-\bar{y}|^4}\\
&\quad + \frac{8(x_j-y_j)^2\left(|x-y|^2-|x-\bar{y}|^2\right)}{|x-\bar{y}|^6}.\\
\frac{\partial}{\partial y_4}\left(\frac{|x-y|^2}{|x-\bar{y}|^2}\right) &= \frac{2(x_4-y_4)|x-\bar{y}|^2-2(x_4+y_4)|x-y|^2}{|x-\bar{y}|^4}, \quad \text{and}
\end{aligned}$$

$$\frac{\partial^2}{\partial y_4^2}\left(\frac{|x-y|^2}{|x-\bar{y}|^2}\right) = \frac{2}{|x-\bar{y}|^2} - \frac{2|x-y|^2}{|x-\bar{y}|^4} + \frac{8(x_4-y_4)(x_4+y_4)}{|x-\bar{y}|^4} + \frac{8(x_4+y_4)^2|x-y|^2}{|x-\bar{y}|^6}.$$

It follows that

$$\Delta_y G_4(x,y) = \frac{c_4}{2}\left\{\frac{12}{|x-\bar{y}|^2} - \frac{4}{|x-y|^2} - \frac{8|x'-y'|^2}{|x-\bar{y}|^4} + \frac{8(x_4^2-y_4^2)}{|x-\bar{y}|^4}\right\}, \quad \text{and}$$

$$\frac{\partial}{\partial y_4}\Delta_y G_4(x,y) = \frac{c_4}{2}\left\{\frac{-24(x_4+y_4)}{|x-\bar{y}|^4} - \frac{8(x_4-y_4)}{|x-y|^4} + \frac{32(x_4+y_4)|x'-y'|^2}{|x-\bar{y}|^6} - \frac{16y_4}{|x-\bar{y}|^4} - \frac{32(x_4-y_4)(x_4+y_4)^2)}{|x-\bar{y}|^6}\right\}.$$

Thus,

$$\Delta_y G_4(x,y)\Big|_{y_4=0} = \frac{c_4}{2}\left\{\frac{8}{|x-y'|^2} - \frac{8|x'-y'|^2}{|x-y'|^4} + \frac{8x_4^2}{|x-y'|^4}\right\}$$
$$= \frac{c_4}{2}\left\{\frac{16x_4^2}{(|x'-y'|^2+x_4^2)^2}\right\}, \quad \text{and}$$
$$\frac{\partial}{\partial y_n}\Delta_y G_4(x,y)\Big|_{y_4=0} = \frac{c_4}{2}\left\{\frac{-64x_4^3}{(|x'-y'|^2+x_4^2)^3}\right\}.$$

It follows that

$$v(0',x_4) = \int_\Sigma \left[\frac{8c_4x_4^2}{(|y'|^2+x_4^2)^2}\frac{\partial u}{\partial y_4}(y',0) + \frac{32c_4x_4^3}{(|y'|^2+x_4^2)^3}u(y',0)\right]dy'.$$

This formula is the same as (10) when $n=4$ and hence we reach the same conclusions, namely, for some $\epsilon > 0$: if $u(x',0)\geq 0$, then $u(x',0)\equiv 0$ for $|x'|\leq\epsilon$, and if $\frac{\partial u}{\partial x_n}(x',0)\geq 0$, then $\frac{\partial u}{\partial x_n}(x',0)\equiv 0$ for $|x'|\leq\epsilon$.

Case 3. Suppose $n=2$. In this case the Green's function

$$G_2(x,y) = \frac{c_2}{2}\left[|x-\bar{y}|^2 - |x-y|^2 - |x-y|^2\log\left|\frac{x-\bar{y}}{x-y}\right|^2\right].$$

We have

$$\Delta_y G_2(x,y)\Big|_{y_2=0} = \frac{Ax_2^2}{(x_1-y_1)^2+x_2^2}, \quad \text{and}$$
$$\frac{\partial}{\partial y_2}\Delta_y G_2(x,y)\Big|_{y_2=0} = \frac{Bx_2^3}{((x_1-y_1)^2+x_2^2)^2}$$

for some constants A and B. This means that $v(0^{'}, x_2)$ has the same form as in (10) which leads to the same conclusion for $u(x_1, 0)$ and $\frac{\partial u}{\partial y_2}(x_1, 0)$.

Suppose now $\alpha = (\alpha_1, ..., \alpha_{n-1}), \quad |\alpha| > 0$, and either

$$s^\alpha u(s,0) \geq 0 \quad \text{or} \quad s^\alpha \frac{\partial u}{\partial y_n}(s,0) \geq 0 \ \text{ for } \ |s| \leq r.$$

Then there is a multi-index which we will still denote by $\alpha = (\alpha_1, ..., \alpha_{n-1})$ such that

$$s^\alpha u(s,0) \geq 0 \quad \text{or} \quad s^\alpha \frac{\partial u}{\partial y_n}(s,0) \geq 0 \ \text{ for } \ |s| \leq r$$

with each $\alpha_i \in \{0, 1\}$. The proof will show that without loss of generality, we may assume that $\alpha = (1, ..., 1)$. Recall the formula $v(x)$ from (10) which we saw holds for all n :

$$v(x) = \int_\Sigma \left[\frac{8c_n x_n^2}{(|x' - y'|^2 + x_n^2)^{\frac{n}{2}}} \frac{\partial u}{\partial y_n}(y', 0) + \frac{8nc_n x_n^3}{(|x' - y'|^2 + x_n^2)^{\frac{n+2}{2}}} u(y', 0) \right] dy'$$

We have

$$\begin{aligned} (14) \quad \partial_{x_1} ... \partial_{x_{n-1}} v(x) \Big|_{x'=0} &= \int_\Sigma \Bigg[\frac{8c_n n(n+2)...(n+2(n-2))x_n^2}{\Big(|y'|^2 + x_n^2\Big)^{\frac{n}{2}+n-1}} y_1 ... y_{n-1} \frac{\partial u}{\partial y_n}(y', 0) \\ &+ \frac{8nc_n(n+2)(n+4)...(n+2(n+1))x_n^3}{\Big(|y'|^2 + x_n^2\Big)^{\frac{n}{2}+n}} y_1 ... y_{n-1} \, u(y', 0) \Bigg] dy'. \end{aligned}$$

We next estimate $\partial_{x_n}^k \left(\partial_{x_1} ... \partial_{x_{n-1}} v(x) \Big|_{x'=0} \right) \Big|_{x_n=0}$. From (13) with n replaced by $3n - 2$, we have:

$$(15) \qquad \partial_{x_n}^{2N} \left\{ \frac{x_n^2}{(|y'|^2 + x_n^2)^{\frac{n}{2}+n-1}} \right\} \Bigg|_{x_n=0} \geq \frac{(2N)!}{|y'|^{3n+2N-4}}$$

while $\partial_{x_n}^{2N+1} \left\{ \frac{x_n^2}{(|y'|^2 + x_n^2)^{\frac{n}{2}+n-1}} \right\} \Big|_{x_n=0} = 0.$

Likewise, (12) implies that

$$\partial_{x_n}^{2N+1}\left\{\frac{x_n^3}{(|y'|^2+x_n^2)^{\frac{n}{2}+n}}\right\}\Big|_{x_n=0} \geq \frac{(2N+1)!}{|y'|^{3n-2+2N}} \tag{16}$$

and $\partial_{x_n}^{2N}\left\{\frac{x_n^3}{(|y'|^2+x_n^2)^{\frac{n}{2}+n}}\right\}\Big|_{x_n=0} = 0$. It follows that if $y_1...y_{n-1}u(y',0) \geq 0$,

$$\begin{aligned}\frac{1}{(2N)!}\partial_{x_n}^{2N}\left(\partial_{x_1}...\partial_{x_{n-1}}v(x)\Big|_{x'=0}\right)\Big|_{x_n=0} \\ \geq 8c_n n(n+2)...(n+2(n-2))\int_\Sigma \frac{y_1...y_{n-1}u(y',0)}{|y'|^{3n+2N-4}}\,dy'\end{aligned} \tag{17}$$

and if $y_1...y_{n-1}\frac{\partial}{\partial y_n}u(y',0) \geq 0$,

$$\begin{aligned}\frac{1}{(2N+1)!}\partial_{x_n}^{2N+1}\left(\partial_{x_1}...\partial_{x_{n-1}}v(x)\Big|_{x'=0}\right)\Big|_{x_n=0} \\ \geq 8nc_n(n+2)(n+4)...(n+2(n+1))\int_\Sigma \frac{y_1...y_{n-1}\frac{\partial u}{\partial y_n}(y',0)}{|y'|^{3n+2N-2}}\,dy'.\end{aligned} \tag{18}$$

Estimates (17),(18), together with the analyticity of $u(x)-v(x)$ at 0, and the flatness of $x_n \longmapsto \partial_{x_1}\cdots\partial_{x_{n-1}}u(0',x_n)$ imply as in the case $\alpha=0$ the following conclusion:
$\exists\,\epsilon>0$ such that if $y_1...y_{n-1}u(y',0)\geq 0$ for $|y'|\leq r$, $u(y',0)\equiv 0$ for $|y'|\leq \epsilon$ and if $y_1...y_{n-1}\frac{\partial u}{\partial y_n}(y',0)\geq 0$ for $|y'|\leq r$, then $\frac{\partial u}{\partial y_n}(y',0)\equiv 0$.

5. Proof of Theorem 2

Let $P_d(\mathbb{R}^n)$ denote the space of homogeneous polynomials of degree d in $\mathbb{R}^n$. Consider the linear map

$$T:\quad P_d(\mathbb{R}^n)\longrightarrow P_d(\mathbb{R}^n)$$

defined by $T(q_1)=q_2$ where for $x\in\mathbb{R}^n$,

$$q_2(x)=|x|^{n+2+2d}q_1(\partial_x)\left(\frac{1}{|x|^{n+2}}\right).$$

and $\partial_x=(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n})$. We will show that T is a bijection. Suppose $T(q_1)=$

0. Since $\frac{1}{|x|^{n+2}}$ is homogeneous on $\mathbb{R}^n \setminus 0$, by Theorem 3.2.4 in [Ho], there is a distribution S on $\mathbb{R}^n$ such that

$$S = \frac{1}{|x|^{n+2}} \quad \text{on} \quad \mathbb{R}^n \setminus 0.$$

We then have $q_1(\partial_x)S$ is supported at the origin and hence has the form

$$q_1(\partial_x)S = \sum_{|\alpha|\le m} a_\alpha \delta_0^{(\alpha)}.$$

Taking the Fourier transform, we get

$$q_1(-i\xi)\hat{S}(\xi) = \sum_{|\alpha|\le m} a_\alpha (-i\xi)^\alpha.$$

By the arguments on page 169 in [Ho],

$$\hat{S}(\xi) = U_1(\xi) + U_2(\xi)\log|\xi|,$$

where $U_2(\xi)$ is a homogeneous polynomial of degree 2 and $U_1(\xi)$ is a homogeneous distribution of degree 2 that is C^∞ in $\mathbb{R}^n \setminus 0$. The function $U_2(\xi)$ is given by

$$U_2(\xi) = -\int_{|w|=1} \left(\sum_{j=1}^n w_j \xi_j \right)^2 dw$$

and hence is not identically zero. Therefore, the equation

$$q_1(-i\xi)(U_1(\xi) + U_2(\xi)\log|\xi|) = \sum_{|\alpha|\le m} a_\alpha (-i\xi)^\alpha$$

implies that $q_1 \equiv 0$ which shows that the map T is injective and hence a bijection. A hypothesis of Theorem 2 and (10) tell us that for $x \in B_r^+$,

$$v(x) = \int_\Sigma \frac{8nc_n x_n^3}{(|x'-y'|^2 + x_n^2)^{\frac{n+2}{2}}} u(y',0)\,dy'.$$

Given the homogeneous polynomial $p(x')$ of degree d as in the statement of Theorem 2, viewing it as an element of $P_d(\mathbb{R}^n)$, let $q(x) \in P_d(\mathbb{R}^n)$ such that

$T(q)(x) = p(x')$. We have:

$$q(\partial_x)v(x)\Big|_{x'=0} = \int_\Sigma \frac{8nc_n x_n^3}{(|y'|^2 + x_n^2)^{\frac{n+2+2d}{2}}} p(y')u(y',0)\,dy'.$$

Replacing n with $n+2d$ in (12), we have:

$$\left|\partial_{x_n}^{2N+1}\left\{\frac{8nc_n x_n^3}{(|y'|^2 + x_n^2)^{\frac{n+2+2d}{2}}}\right\}\Big|_{x_n=0}\right| \geq \frac{8(n+2d)c_n(2N+1)!}{|y'|^{n+2d+2N}}. \tag{19}$$

Since $|y'|^{-k}p(y')u(y',0)$ is locally integrable for all k, the function $q(\partial_x)v(0,x_n)$ is smooth on $[0,r)$. Therefore, by the analyticity of $v(x)-u(x)$ and the flatness of $u(0,x_n)$ at $x_n=0$, for some $C>0$,

$$\begin{aligned} C^{2N+d+2}(2N+d+1)! &\geq \left|\partial_{x_n}^{2N+1}\left(q(\partial_x)v(x)|_{x'=0}\right)\Big|_{x_n=0}\right| \\ &\geq 8(n+2d)c_n(2N+1)!\int_\Sigma \frac{p(y')u(y',0)}{|y'|^{n+2d+2N}}\,dy'. \end{aligned}$$

Hence, using the inequality $(2N+1+d)! \leq 2^{2N+1+d}(2N+1)!d!$, we get:

$$C(2C)^{2N+d+1}d! \geq 8(n+2d)c_n\int_\Sigma \frac{p(y')u(y',0)}{|y'|^{n+2d+2N}}\,dy'.$$

As before, since $p(y')u(y',0)\geq 0$, this leads to $u(y',0)\equiv 0$.

6. Proof of Theorem 3

Let $\eta \in C_0^\infty(\mathbb{R}^{n-1})$, $\eta(x)\equiv 1$ for $|x|\leq \frac{r}{2}$ and $\eta(x)\equiv 0$ when $|x|\geq r$. Define $v(x)$ on B_r^+ by

$$v(x) = \begin{cases} \frac{2}{n\omega_n}\int_\Sigma \frac{x_n}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}}\,\eta(y')\,u(y',0)\,dy' \\ \quad -\frac{1}{n(n-1)\omega_n}\int_\Sigma \frac{x_n}{(|x'-y'|^2+x_n^2)^{\frac{n-2}{2}}}\,\eta(y')\,\Delta u(y',0)\,dy',\ n\neq 2 \\ \frac{1}{\omega_2}\int \frac{x_2}{(x_1-y_1)^2+x_2^2}\,\eta(y_1)\,u(y_1,0)\,dy_1 \\ \quad +\frac{1}{4\omega_2}\int x_2\log[(x_1-y_1)^2+x_2^2]\,\eta(y_1)\,\Delta u(y_1,0)\,dy_1,\ n=2; \end{cases}$$

where $\omega_n =$ the volume of the unit ball in $\mathbb{R}^n$. It is clear that

$$v(x',0) = \lim_{x_n\longrightarrow 0^+} v(x) = u(x',0) \quad \text{for } |x'|\leq \frac{r}{2}.$$

We claim that

$$\Delta v(x',0) = \lim_{x_n \longrightarrow 0^+} \Delta v(x) = \Delta u(x',0) \quad \text{for } |x'| \le \frac{r}{2}.$$

To see this note first that $\frac{2}{n\omega_n}\frac{x_n}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}}$ is the Poisson kernel in $\mathbb{R}^n$ and so it is harmonic in x. Next, observe that for $n \neq 2$,

$$\begin{aligned}
\Delta_x \left\{ \frac{x_n}{(|x'-y'|^2+x_n^2)^{\frac{n-2}{2}}} \right\} &= \frac{1}{4-n}\Delta_x \partial_{x_n} \left\{ \frac{1}{(|x'-y'|^2+x_n^2)^{\frac{n-4}{2}}} \right\} \\
&= \frac{1}{4-n}\partial_{x_n} \Delta_x \left\{ \frac{1}{(|x'-y'|^2+x_n^2)^{\frac{n-4}{2}}} \right\} \\
&= 2\frac{\partial}{\partial x_n} \left\{ \frac{1}{(|x'-y'|^2+x_n^2)^{\frac{n-2}{2}}} \right\} \\
&= 2(2-n)\frac{x_n}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}}.
\end{aligned}$$

Thus,

$$\Delta_x \left\{ \frac{-1}{n(n-2)\omega_n} \left(\frac{x_n}{(|x'-y'|^2+x_n^2)^{\frac{n-2}{2}}} \right) \right\} = \frac{2}{n\omega_n} \left(\frac{x_n}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}} \right)$$

for $n \neq 2$, which is the Poisson kernel.
Likewise, when $n = 2$,

$$\Delta_x \left\{ \frac{1}{\omega_2} \left(x - 2\log\left[(x_1-y_1)^2 + x_2^2\right] \right) \right\} = \frac{1}{\omega_2} \left(\frac{x_2}{((x_1-y_1)^2 + x_2^2} \right)$$

which is also the Poisson kernel. It now follows that

$$\Delta v(x',0) = \lim_{x_n \longrightarrow 0^+} \Delta v(x) = \Delta u(x',0) \quad \text{for } |x'| \le \frac{r}{2}.$$

Define $w(x) = u(x) - v(x)$ for $x \in B_r^+$. From the preceding observations, $\Delta^2 w(x) = 0$ on B_r^+, $w(x',0) = 0$, and $\Delta w(x',0) = 0$. We will show that $w(x)$ extends past $x_n = 0$ as a solution of Δ^2. Let $h(x) = \Delta w(x)$, $x \in B_r^+$. Then $\Delta h = 0$ in B_r^+ and $h(x',0) = 0$ and so h extends to a real anatytic function $\tilde{h}$ on a ball B_r. Let $\tilde{w}$ be a solution of $\Delta \tilde{w} = \tilde{h}$ in B_r and $\tilde{w}(x',0) = 0$. $\tilde{w}$ is real analytic on B_r. Since $\Delta(\tilde{w} - w) = 0$ on B_r^+ and $\tilde{w}(x',0) - w(x',0) = 0$, $\tilde{w} - w$ extends as a real analytic function past $x_n = 0$ and hence $w(x)$

extends also past $x_n = 0$. Using also Lemma 1 below, as before,

$$\partial_{x_n}^k v(0) = -\partial_{x_n}^k w(0), \forall k$$

and so for some $c > 0$,

$$\left|\partial_{x_n}^k v(0)\right| \leq c^{k+1} k! \tag{20}$$

We next estimate these derivatives using the integral formula for $v(x)$. From the steps between (16) and (17), when $n \neq 2$,

$$\begin{aligned}\partial_{x_n}^k \left\{ \frac{x_n}{(|y'|^2 + x_n^2)^{\frac{n}{2}}} \right\}\Bigg|_{x_n=0} &= k\partial_{x_n}^{k-1} \left\{ \frac{1}{(|y'|^2 + x_n^2)^{\frac{n}{2}}} \right\}\Bigg|_{x_n=0} \\ &= 0, \quad \text{when } k \text{ is even,}\end{aligned}$$

and when $k = 2N + 1$,

$$\begin{aligned}&\partial_{x_n}^{2N+1} \left\{ \frac{x_n}{(|y'|^2 + x_n^2)^{\frac{n}{2}}} \right\}\Bigg|_{x_n=0} \\ &= (2N+1)\partial_{x_n}^{2N} \left\{ \frac{1}{(|y'|^2 + x_n^2)^{\frac{n}{2}}} \right\}\Bigg|_{x_n=0} \\ &= \frac{(-1)^N (2N+1)! n(n+1)\cdots(n+2N-2)}{N! 2^N |y'|^{n+2N}}.\end{aligned} \tag{21}$$

Likewise,

$$\partial_{x_n}^k \left\{ \frac{x_n}{(|y'|^2 + x_n^2)^{\frac{n-2}{2}}} \right\}\Bigg|_{x_n=0} = 0, \quad \text{when } k \text{ is even,}$$

and when $k = 2N + 1$,

$$\begin{aligned}&\partial_{x_n}^{2N+1} \left\{ \frac{x_n}{(|y'|^2 + x_n^2)^{\frac{n-2}{2}}} \right\}\Bigg|_{x_n=0} \\ &= \frac{(-1)^N (2N+1)!(n-2)(n-1)\cdots(n+2N-4)}{N! 2^N |y'|^{n-2+2N}}.\end{aligned} \tag{22}$$

We also have

$$\partial_{x_2}^k \left\{ x_2 \log(y_1^2 + x_2^2) \right\}\Big|_{x_2=0} = k\partial_{x_2}^{k-1} \log(y_1^2 + x_2^2)\Big|_{x_2=0}$$
$$= 2k\partial_{x_2}^{k-2} \left\{ \frac{x_2}{y_1^2 + x_2^2} \right\}\Big|_{x_2=0}$$
$$= 0 \quad \text{when } k \text{ is even,}$$

and if $k = 2N + 1$,

$$\partial_{x_2}^{2N+1} \left\{ x_2 \log(y_1^2 + x_2^2) \right\}\Big|_{x_2=0} = \frac{(-1)^{N-1}(2N-1)!2.3...(2N-2)}{(N-1)!2^{N-1}|y'|^{2N}}. \tag{23}$$

Inequality (20) and the estimates (21),(22),(23) imply as before that for some $\epsilon > 0$,

$$u(x', 0) \equiv \Delta u(x', 0) \equiv 0 \quad \text{for} \quad |x'| \leq \epsilon.$$

As shown already, this in turn implies that $u(x)$ extends as a real analytic function to $B_\epsilon(0)$.

In the preceding proof, we used the following lemma:

Lemma 1. *Let u be a solution of $\Delta^2 u = 0$ in the half ball B_r^+, C^4 on the closure. Assume that*

(1) *$u(s, 0) \geq 0$ and $\Delta u(s, 0) \leq 0$ for $|s| \leq r$ where Δ is in n variables;*

(2) *the function $x_n \longmapsto u(0, x_n)$ is flat at $x_n = 0$;*

Then for every positive integer N, $|s|^{-N} u(s, 0)$ and $|s|^{-N} \Delta u(s, 0)$ are integrable on $\{s \in \mathbb{R}^{n-1} : |s| \leq r\}$.

Proof. Suppose first $n \geq 2$. Recall that for $x_n > 0$,

$$\frac{v(0, x_n)}{x_n} = \frac{2}{n\omega_n} \int_\Sigma \frac{\eta(y')u(y', 0)}{(|y'|^2 + x_n^2)^{\frac{n}{2}}}\, dy' - \frac{1}{n(n-2)\omega_n} \int_\Sigma \frac{\eta(y')\Delta u(y', 0)}{(|y'|^2 + x_n^2)^{\frac{n-2}{2}}}\, dy'. \tag{24}$$

We will use Taylor's remainder formulas:

$$(25)\quad \frac{1}{(|y'|^2+x_n^2)^{\frac{n}{2}}} = \sum_{k=0}^{N} \frac{A_k}{|y'|^n}\left(\frac{x_n}{|y'|}\right)^{2k} + (N+1)A_{N+1}\left(\int_\Sigma\int_0^1 \frac{(1-t)^N}{(|y'|^2+tx_n^2)^{\frac{n+2N+2}{2}}}\,dt\,dy'\right)x_n^{2N+2}$$

and

$$(26)\quad \frac{1}{(|y'|^2+x_n^2)^{\frac{n-2}{2}}} = \sum_{k=0}^{N} \frac{B_k}{|y'|^{n-2}}\left(\frac{x_n}{|y'|}\right)^{2k} + (N+1)B_{N+1}\left(\int_\Sigma\int_0^1 \frac{(1-t)^N}{(|y'|^2+x_n^2)^{\frac{n+2N}{2}}}\,dt\,dy'\right)x_n^{2N+2}$$

where for each k,

$$A_k = \frac{(-1)^k n(n+2)\cdots(n+2k-2)}{2^k k!},$$

and

$$B_k = \frac{(-1)^k (n-2)n\cdots(n+2k-4)}{2^k k!}.$$

Let $w(x)=v(x)-u(x)$. Then $w(x',0)=0$, $\Delta w(x',0)=0$, and $\Delta^2 w(x)=0$ for $x\in B^+_{\frac{r}{2}}$, and so as we saw before, w has a real analytic extension past $x_n=0$ which we still denote by w. Observe that $\partial^{2j}_{x_n} w(x',0)=0$ for all j.

Since $w(0,0)=0$, the function $\frac{w(0,x_n)}{x_n}$ is real analytic at $x_n=0$. Write

$$\frac{w(0,x_n)}{x_n} = \sum_{k=0}^{\infty} c_k x_n^{2k}$$

Using the flatness of $x_n \longmapsto u(0,x_n)$, for each N, we get

$$\frac{v(0,x_n)}{x_n} = \sum_{k=0}^{N} c_k x_n^{2k} + O(x_n^{2N+2}).$$

We have

$$\lim_{x_n\longrightarrow 0^+} \frac{w(0,x_n)}{x_n} = c_0$$

and so since $u(y',0) \geq 0$, by the Monotone Convergence Theorem, (24) tells us that $|y'|^{-n}u(y',0) \in L^1_{\text{loc}}$, $|y'|^{2-n}\Delta u(y',0) \in L^1_{\text{loc}}$and

$$c_0 = \frac{2}{n\omega_n}\int_\Sigma \frac{\eta(y')u(y',0)}{|y'|^n}\,dy' - \frac{1}{n(n-2)\omega_n}\int_\Sigma \frac{\eta(y')\Delta u(y',0)}{|y'|^{n-2}}\,dy' \tag{27}$$

Observe next that

$$\lim_{x_n \longrightarrow 0^+} \frac{1}{x_n^2}\left(\frac{v(0,x_n)}{x_n} - c_0\right) = c_1 \tag{28}$$

Therefore, from (24), (25), and (26), for $x_n > 0$,

$$\begin{aligned} \frac{v(0,x_n)}{x_n} = {} & \frac{2}{n\omega_n}\int_\Sigma \frac{\eta(y')u(y',0)}{|y'|^n}\,dy' \\ & - \frac{1}{n(n-2)\omega_n}\int_\Sigma \frac{\eta(y')\Delta u(y',0)}{|y'|^{n-2}}\,dy' \\ & - \frac{1}{\omega_n}\left(\int_\Sigma\int_0^1 \frac{\eta(y')u(y',0)}{(|y'|^2+tx_n^2)^{\frac{n+2}{2}}}\,dt\,dy'\right)x_n^2 \\ & + \frac{1}{2n\omega_n}\left(\int_\Sigma\int_0^1 \frac{\eta(y')\Delta u(y',0)}{(|y'|^2+tx_n^2)^{\frac{n}{2}}}\,dt\,dy'\right)x_n^2. \end{aligned} \tag{29}$$

From (25), (26), and (27),

$$\begin{aligned} c_1 &= \lim_{x_n \longrightarrow 0^+} \frac{1}{x_n^2}\left(\frac{v(0,x_n)}{x_n} - \frac{2}{n\omega_n}\int_\Sigma \frac{\eta(y')u(y',0)}{|y'|^n}\,dy' \right. \\ &\qquad \left. + \frac{1}{n(n-2)\omega_n}\int_\Sigma \frac{\eta(y')\Delta u(y',0)}{|y'|^{n-2}}\,dy'\right) \\ &= \lim_{x_n \longrightarrow 0^+}\left(\frac{-1}{\omega_n}\int_\Sigma\int_0^1 \frac{\eta(y')u(y',0)}{(|y'|^2+tx_n^2)^{\frac{n+2}{2}}}\,dt\,dy' \right. \\ &\qquad \left. + \frac{1}{2n\omega_n}\int_\Sigma\int_0^1 \frac{\eta(y')\Delta u(y',0)}{(|y'|^2+tx_n^2)^{\frac{n}{2}}}\,dt\,dy'\right). \end{aligned}$$

Since $u(y',0) \geq 0$ and $\Delta u(y',0) \leq 0$, by the Monotone Convergence Theorem, we conclude that

$$c_1 = \frac{-1}{\omega_n}\int_\Sigma \frac{\eta(y')u(y',0)}{|y'|^{n+2}}\,dy' + \frac{1}{2n\omega_n}\int_\Sigma \frac{\eta(y')\Delta u(y',0)}{|y'|^n}\,dy',$$

and thus $|y'|^{-n-2}u(y',0) \in L^1_{\text{loc}}$ and $|y'|^{-n}\Delta u(y',0) \in L^1_{\text{loc}}$. Suppose now that for some $N \geq 2$, $|y'|^{-n-2N}u(y',0)$ and $|y'|^{-n-2N+2}\Delta u(y',0) \in L^1_{loc}$ and that for $j \geq N$,

$$c_j = \frac{2A_j}{n\omega_n}\int_\Sigma \frac{\eta(y')u(y',0)}{|y'|^{n+2j}}dy' - \frac{B_j}{n(n-2)\omega_n}\int_\Sigma \frac{\eta(y')\Delta u(y',0)}{|y'|^{n+2j-2}}dy'.$$

Then

$$\begin{aligned}
\frac{v(0,x_n)}{x_n} &= \sum_{k=0}^{N+1} c_k x_n^{2k} + O(x_n^{2k+4}) \\
&= \sum_{k=0}^{N}\left(\frac{2A_k}{n\omega_n}\int_\Sigma \frac{\eta(y')u(y',0)}{|y'|^{n+2k}}\,dy' - \frac{B_k}{n(n-2)\omega_n}\int_\Sigma \frac{\eta(y')\Delta u(y',0)}{|y'|^{n+2k-2}}\,dy'\right)x_n^{2k} \\
&\quad + \frac{2(N+1)A_{N+1}}{n\omega_n}\left(\int_\Sigma\int_0^1 \frac{(1-t)^N\eta(y')u(y',0)}{(|y'|^2+tx_n^2)^{\frac{n+2N+2}{2}}}\,dt\,dy'\right)x_n^{2N+2} \\
&\quad - \frac{(N+1)B_{N+1}}{n(n-2)\omega_n}\left(\int_\Sigma\int_0^1 \frac{(1-t)^N\eta(y')\Delta u(y',0)}{(|y'|^2+tx_n^2)^{\frac{n+2N}{2}}}\,dt\,dy'\right)x_n^{2N+2}.
\end{aligned}$$

Using the induction assumption, it follows that

$$\begin{aligned}
c_{N+1} &= \frac{2(N+1)A_{N+1}}{n\omega_n}\int_\Sigma\int_0^1 \frac{(1-t)^N\eta(y')u(y',0)}{(|y'|^2+tx_n^2)^{\frac{n+2N+2}{2}}}\,dt\,dy' \\
&\quad - \frac{(N+1)B_{N+1}}{n(n-2)\omega_n}\int_\Sigma\int_0^1 \frac{(1-t)^N\eta(y')\Delta u(y',0)}{(|y'|^2+tx_n^2)^{\frac{n+2N}{2}}}\,dt\,dy' + O(x_n^2).
\end{aligned}$$

Since $u(y',0) \geq 0$ and $\Delta u(y',0) \leq 0$, we can let $x_n \longrightarrow 0^+$ and arrive at

$$c_{N+1} = \frac{2A_{N+1}}{n\omega_n}\int_\Sigma \frac{u(y',0)}{|y'|^{n+2N+2}}\,dy' - \frac{B_{N+1}}{n(n-2)\omega_n}\int_\Sigma \frac{\Delta u(y',0)}{|y'|^{n+2N}}\,dy'.$$

Thus $|y'|^{-n-2N-2}u(y',0)$ and $|y'|^{-n-2N}\Delta u(y',0) \in L^1_{\mathrm{loc}}$ and the lemma is proved for $n \neq 2$.

The case $n = 2$ is proved the same way using this time the Taylor formula

$$\begin{aligned}
\log(y_1^2+x_2^2) &= \log y_1^2 + \sum_{j=0}^{N}\frac{(-1)^{j+1}}{j}\left(\frac{x_2}{y_1}\right)^{2j} \\
&\quad + (-1)^{N+2}\left(\int_0^1 \frac{(1-t)^N}{(y_1^2+tx_2^2)^{N+1}}\,dt\right)x_2^{2N+2}.
\end{aligned}$$

$\square$

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