Price inequalities and Betti number growth on manifolds without conjugate points

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We derive Price inequalities for harmonic forms on manifolds without conjugate points and with a negative Ricci upper bound. The techniques employed in the proof work particularly well for manifolds of non-positive sectional curvature, and in this case we prove a strengthened Price inequality. We employ these inequalities to study the asymptotic behavior of the Betti numbers of coverings of Riemannian manifolds without conjugate points. Finally, we give a vanishing result for L^2-Betti numbers of closed manifolds without conjugate points.

1. Introduction

In the early 1980's, Donnelly and Xavier introduced an integral inequality [DX84, Theorem 2.2] which became a standard tool in the proof of cohomology vanishing theorems and spectral bounds on simply connected negatively curved manifolds. Around the same time Price introduced an inequality [Pri83] which became ubiquitous in the study of singularities of harmonic maps and Yang-Mills fields. Price's inequality can be proved in a manner which generalizes the proof of Donnelly and Xavier's inequality. In this work, we further this circle of ideas by applying Price's generalization of the Donnelly-Xavier inequality to study cohomology, obtaining new bounds on the Betti numbers of Riemannian manifolds with negative Ricci curvature and no conjugate points. In particular, we obtain asymptotic bounds on the growth of Betti numbers in towers of regular coverings of such manifolds. The range of Betti numbers to which these results apply depends on how negative the Ricci curvature is assumed to be.

Our approach is quite robust and works without assuming pinched negative curvature. In particular, the hypotheses of the main technical estimate, Theorem 39, allow some positive sectional curvature. Of course, when we impose stronger pinched negative sectional curvature assumptions, we derive a strengthened Price inequality, Theorem 87.

Lohkamp ([Loh94, Theorem A]) proved that every manifold of dimension ≥ 3 admits a metric of negative Ricci curvature. Hence there are no topological consequences of the negative Ricci curvature assumption. Lohkamp's theorem does not, however, provide a mechanism for producing the large geodesic balls required for our estimates to be useful. With the no conjugate point hypothesis, we can find large geodesic balls by passing to finite covers with large injectivity radius.

The study of the asymptotic behavior of Betti numbers has attracted considerable interest in the last four decades, especially in the context of coverings of locally symmetric spaces of non-compact type. See, for example, [DW78], [DW79], [Xue91], [SX91], [ABBGNRS17], [Mar14]. Many of the algebraic techniques employed in these works are not obviously amenable to generalization outside the locally symmetric context.

Given a co-compact torsion free lattice Γ acting on a symmetric space G/K of non-compact type, we say a sequence of nested, normal, finite index subgroups $\{\Gamma_i\}$ of Γ is a *cofinal filtration* of Γ if $\bigcap_i \Gamma_i$ is the identity element. Any such Γ is known to be *residually finite* (see [Bor63, Proposition 2.3]), so that cofinal filtrations always exists. Denote by M_i the finite index regular cover of $\Gamma\backslash G/K$ associated to Γ_i. It follows from the results of [DW78],

[DW79], that for any cofinal filtration of Γ, for $k \neq \frac{1}{2}dim(G/K)$,

$$\text{(1)} \qquad \lim_{i \to \infty} \frac{b_k(M_i)}{Vol(M_i)} = 0,$$

where $b_k(M_i)$ denotes the k-th Betti number of M_i.

Our techniques provide new methods for proving such results and quantifying the sub volume order growth of certain Betti numbers. Because our techniques do not rely on representation theory or the trace formula, it is natural to consider these questions on manifolds which are *not* locally symmetric. The Price inequalities given in Theorem 39 and Proposition 66 are tailored to address such problems. For example, we have the following *non-locally symmetric* analog of the DeGeorge-Wallach result (1).

Theorem 2 (See Corollary 78). *Let (M^n, g) be a closed Riemannian manifold without conjugate points with $-1 \leq \sec_g \leq 1$. Assume there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g.$$

Let $\pi_i : M_i \to M$ be a sequence of Riemannian covers of M with injectivity radii, denoted $\gamma_{g_i}(M_i)$, satisfying $\gamma_{g_i}(M_i) \to \infty$. Then there exists $b(n, k, \delta) > 0$ so that for $\gamma_{g_i}(M_i)$ sufficiently large,

$$\text{(3)} \qquad \frac{b_k(M_i)}{Vol(M_i)} \leq b(n, k, \delta) e^{-(\sqrt{\delta} - 2k)\gamma_{g_i}(M_i)}.$$

In particular,

$$\text{(4)} \qquad \lim_{i \to \infty} \frac{b_k(M_i)}{Vol(M_i)} = 0.$$

Observe that the range of Betti numbers covered by Theorem 2 grows as the square root of the lower bound δ. Under our curvature normalization, δ can at most be $n - 1$ with equality if and only if the underlying Riemannian manifold is real hyperbolic with sectional curvature -1. Thus, Theorem 2 does not address the full range of Betti numbers achieved by DeGeorge and Wallach in the locally symmetric case. On the other hand, Theorem 2 is not only free of any homogeneity requirement on the metric, but also does not require any direct assumption on the sectional curvature.

Given a closed manifold (M, g) without conjugate points and infinite residually finite fundamental group, the injectivity radius goes to infinity in

any tower of regular Riemannian covers associated to a cofinal filtration of its fundamental group. (See [DW78, Theorem 2.1].) Hence Theorem 2 not only implies (1) for certain k-th Betti numbers, but also implies $\frac{b_k(M_i)}{Vol_g(M_i)}$ decays exponentially in the injectivity radii $\gamma_{g_i}(M_i)$. Such decay results have been obtained by numerous authors in the locally symmetric context. See for example [Xue91], [SX91], [Mar14]. Outside the locally symmetric space context, related results have also been obtained by [BLLS14] for $p-$adically defined towers, for which they show that if $\lim_{i\to\infty} \frac{b_k(M_i)}{Vol_g(M_i)} = 0$, then $\frac{b_k(M_i)}{Vol_g(M_i)}$ decays like $Vol(M_i)^{-d}$, for a specific dimensional constant d.

There is overlap between this paper and the results of [CW03] on Betti number bounds. The results in [CW03] are derived under some combination of a vanishing assumption on L^2-Betti numbers, a spectral gap assumption, and a positivity assumption for the Novikov-Shubin invariants. Our techniques not only do not require these latter assumptions but can be used both to verify the requisite L^2-Betti number vanishing or spectral gap hypotheses of [CW03], and to give a new analytical proof of the consequences of these assumptions. Price inequalities can also be applied to obtain information about Novikov-Shubin invariants. We hope to explore this direction in future work.

We can reexpress the decay results of Theorem 2 in terms of volumes, under suitable hypotheses on the injectivity radii of Riemannian coverings associated to a cofinal filtration. We introduce one such an assumption on the cofinal filtration which we call "congruence type", see Definition 81. Under this assumption, Theorem 2 immediately yields the following corollary.

Corollary 5. *Let (M^n, g) be a Riemannian manifold without conjugate points with $-1 \leq \sec_g \leq 1$ and residually finite fundamental group Γ. Assume*

$$-Ric \geq \delta g, \quad \delta > 4k^2.$$

For any congruence type cofinal filtration $\{\Gamma_i\}$ of Γ of exponent α, if we denote by $\pi_i : M_i \to M$ the regular Riemannian cover of M associated to Γ_i, we have for $\gamma_g(M)$ sufficiently large,

$$b_k(M_i) \leq d(n, k, \delta, \Gamma)Vol(M_i)^{1-2\alpha(\frac{\sqrt{\delta}}{2}-k)}.$$

where $d(n, k, \Gamma)$ is a positive constant.

If one assumes the sectional curvature to be negative and suitably pinched, Corollary 5 can be considerably strengthened. See, for example, Corollary 106.

We obtain our sharpest results for real hyperbolic space. See Theorem 116. The pinched cases, of course, have already been well studied. See for example [DX84] and [CW03]. We provide a self-contained treatment of this important class of examples. In a subsequent paper, we will show how to modify our techniques to obtain stronger estimates for Betti numbers of the complex and quaternionic hyperbolic spaces than follow from our results in this paper.

Finally, our techniques can also be applied to L^2-cohomology problems. For example, we have the following vanishing result for L^2-Betti numbers of closed manifolds without conjugate points.

Theorem 6 (See Theorem 122). *Let (M^n, g) be a closed Riemannian manifold without conjugate points and $-1 \leq \sec_g \leq 1$. If there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g,$$

then the k-th L^2-Betti number of M vanishes.

Theorem 6 provides further evidence towards the Singer Conjecture for aspherical manifolds. For the details of the proof and more on the Singer Conjecture we refer to Section 8.

We conclude this introduction by noting that Theorem 6 applies to large classes of manifolds which are not locally symmetric. These include, for example, the exotic locally symmetric spaces constructed by Farrell-Jones [FJ89], the Gromov-Thurston manifolds [GT87], the Mostow-Siu surfaces [MS80], and the manifolds obtained by Anderson's higher dimensional Dehn filling [And06].

2. Some integral equalities for harmonic forms

In this section, we fix notation and derive the Price/Donnelly-Xavier inequality for harmonic k-forms.

Let (M^n, g) be a complete n-dimensional Riemannian manifold, with injectivity radius $\gamma_g(M)$. Given a point $p \in M$, denote by $B_R(p)$ the geodesic ball of radius $0 < R \leq \gamma_g(M)$ centered at p. In such a ball, let $g = dr^2 + g_r$ be the expression of the metric g in geodesic polar coordinates, and let ∂_r denote the unit radial vector field.

Given a 1-form ϕ, let $e(\phi)$ denote exterior multiplication on the left by ϕ. Let $e^*(\phi)$ denote the adjoint operator. Fix a local orthonormal frame $\{e_j\}_j$ and coframe $\{\omega^j\}_j$. Acting on forms of arbitrary degree, the Lie derivative in the radial direction can be written as

$$(7) \qquad L_{\partial_r} = \{d, e^*(dr)\} = \nabla_{\partial_r} + e(\omega^j)e^*(\nabla_{e_j} dr).$$

Choosing the orthonormal frame so that $e_n = \partial_r$, we may express $\nabla_{e_j} dr$ in terms of the the second fundamental form h of $S_r(p)$. Recall for X and Y tangent to S_r,

$$h(X, Y) := g(\nabla_X \partial_r, Y).$$

Thus, we write

$$\nabla_{e_i} dr = h_{i1}\omega^1 + \ldots + h_{in-1}\omega^{n-1},$$

and

$$(8) \qquad e(\omega^j)e^*(\nabla_{e_j} dr) = \sum_{j,k<n} h_{jk} e(\omega^j) e^*(\omega^k) =: Q.$$

The operator Q defined by (8) is the natural extension of the second fundamental form to an endomorphism of forms of arbitrary degree. As usual, we set the mean curvature of the geodesic sphere S_r to be the trace $H = \sum_k h_{kk}$.

Let $\mathcal{H}_g^k(M)$ denote the strongly harmonic k-forms on M. Given $\alpha \in \mathcal{H}_g^k(M)$, we have

$$\int_{B_R(p)} \langle L_{\partial_r}\alpha, \alpha\rangle dv = \int_{B_R(p)} \langle d(e^*(dr)\alpha), \alpha\rangle dv$$

$$= \int_{B_R(p)} \langle e^*(dr)\alpha, d^*\alpha\rangle dv + \int_{S_R(p)} |e^*(dr)\alpha|^2 d\sigma$$

$$(9) \qquad = \int_{S_R(p)} |e^*(dr)\alpha|^2 d\sigma,$$

where $d\sigma$ is the volume element on the geodesic sphere $S_R(p)$. Using (7) for L_{∂_r} gives the alternate expression

$$\int_{B_R(p)} \langle L_{\partial_r}\alpha, \alpha\rangle dv = \int_{B_R(p)} \langle(\nabla_{\partial_r} + Q)\alpha, \alpha\rangle dv$$

$$= \int_{B_R(p)} \frac{1}{2}(L_{\partial_r} - Q)(|\alpha|^2 dv) + \int_{B_R} \langle Q\alpha, \alpha\rangle dv$$

$$(10) \qquad = \int_{B_R(p)} \langle(Q - \frac{H}{2})\alpha, \alpha\rangle dv + \int_{S_R(p)} \frac{1}{2}|\alpha|^2 d\sigma.$$

Next, let us define the functions

$$(11) \qquad \mu(r) := \frac{\int_{S_r(p)} |e^*(dr)\alpha|^2 d\sigma}{\int_{S_r(p)} |\alpha|^2 d\sigma},$$

and

$$(12) \qquad q(r) := \frac{\int_{S_r(p)} \langle(\frac{H}{2} - Q)\alpha, \alpha\rangle d\sigma}{\int_{S_r(p)} |\alpha|^2 d\sigma}.$$

Equating the expressions in (9) and (10) yields

$$(13) \qquad \int_{B_R(p)} q(r)|\alpha|^2 dv = \frac{1}{2} \int_{S_R(p)} (1 - 2\mu(R))|\alpha|^2 d\sigma.$$

Now if we multiply (13) by $\phi'(R)$, ϕ to be determined, and integrate from σ to $\tau \leq \gamma_g(M)$ we get

$$\phi(\tau) \int_{B_\tau(p)} q(r)|\alpha|^2 dv - \phi(\sigma) \int_{B_\sigma(p)} q(r)|\alpha|^2 dv$$

$$(14) \qquad = \int_{B_\tau(p)\backslash B_\sigma(p)} [\phi(r)q(r) + \frac{1}{2}\phi'(r)(1 - 2\mu(r))]|\alpha|^2 dv.$$

Next, we choose

$$(15) \qquad \phi(r) := e^{-\int_\sigma^r \frac{q(s)ds}{\frac{1}{2} - \mu(s)}}.$$

in order to eliminate the last line of (14). Let us summarize this discussion into a proposition.

Proposition 16. *Let (M, g) be a Riemannian manifold. For any strongly harmonic k-form $\alpha \in \mathcal{H}^k_g(M)$ and for any $\sigma < \tau \leq \gamma_g(M)$, we have the Price equality*

$$(17) \qquad \phi(\sigma) \int_{B_\sigma(p)} q(r)|\alpha|^2 dv = \phi(\tau) \int_{B_\tau(p)} q(r)|\alpha|^2 dv,$$

where $\mu(r)$, $q(r)$, and $\phi(r)$ are respectively defined as in (11), (12) and (15).

We conclude this section by studying the behavior of the function $\mu(r)$ for r close to zero.

Lemma 18. *Let (M, g) be a Riemannian manifold and $p \in M$. Given a k-form α such that $\alpha(p) \neq 0$, we have*

$$(19) \qquad \lim_{r \to 0} \mu(r) = \frac{k}{n}.$$

Proof. This lemma does not require α to be harmonic. Fix geodesic coordinates with p at the origin. Write

$$|e^*(dr)\alpha|^2(x) = \sum_{i,j} \frac{x^i x^j}{r^2} \langle e^*(dx^i)\alpha(x), e^*(dx^j)\alpha(x) \rangle$$

$$= \sum_{i,j} \frac{x^i x^j}{r^2} \langle e^*(dx^i)\alpha(0), e^*(dx^j)\alpha(0) \rangle + o(1).$$

Using

$$\int_{S_r} \frac{x^i x^j}{r^2} d\sigma = \frac{\delta_{ij}}{n} Vol(S_r),$$

and

$$\sum_j |e^*(dx^j)\alpha|^2(0) = k|\alpha|^2(0),$$

we see

$$\mu(r) = \frac{k}{n} + o(1).$$

$\square$

In order to extract geometric information out of Proposition 16, we need to understand the positivity properties of $q(r)$. This is a problem in comparison geometry which we address in the next section.

3. Controlling the second fundamental form

Let $\lambda_1 \geq \cdots \geq \lambda_{n-1}$ denote the eigenvalues of h_{ij}. Q commutes with the decomposition of a $k-$form α as $\alpha = e^*(dr)e(dr)\alpha + e(dr)e^*(dr)\alpha$. Hence

$$\langle(\frac{H}{2} - Q)\alpha, \alpha\rangle \geq \frac{1}{2}(-\lambda_1 - \cdots - \lambda_k + \lambda_{k+1} + \cdots + \lambda_{n-1})|e(dr)\alpha|^2$$

(20)
$$+ \frac{1}{2}(-\lambda_1 - \cdots - \lambda_{k-1} + \lambda_k + \cdots + \lambda_{n-1})|e^*(dr)\alpha|^2.$$

With this notation,

$$(21) \qquad \int_{S_R(p)} q(r)|\alpha|^2 d\sigma \geq \int_{S_R(p)} ((\frac{H}{2} - \sum_{i=1}^{k} \lambda_i)|\alpha|^2 + \mu\lambda_k|\alpha|^2)d\sigma.$$

In order to extract information from this inequality, we use the Rauch comparison theorem (see eg. [Pet15, p. 255]) and the Riccati Equation for the mean curvature of a geodesic sphere (see eg. [Pet15, Chapter 5]) to control the second fundamental form terms arising in the main inequality of Proposition 21. We first recall those results.

Let $\gamma(K)$ denote the injectivity radius of the space form of constant curvature K. Let

$$s_K(r) := \begin{cases} \frac{1}{\sqrt{K}}\sin(\sqrt{K}r) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}r) & \text{if } K < 0. \end{cases}$$

Theorem 22 (Rauch Comparison). *If (M^n, g) satisfies $K_1 \leq \sec_g \leq K_2$, $h(r)$ is the second fundamental form of the geodesic sphere of radius r, and $g = dr^2 + g_r$ denotes the metric in geodesic polar coordinates, then*

$$\frac{s'_{K_2}(r_1)}{s_{K_2}(r_1)}g_{r_1} \leq h(r_1), \quad and \quad h(r_2) \leq \frac{s'_{K_1}(r_2)}{s_{K_1}(r_2)}g_{r_2},$$

for $0 < r_1 \leq \min\{\gamma_g(M), \gamma(K_2)\}$, and $0 < r_2 \leq \min\{\gamma_g(M), \gamma(K_1)\}$.

If we assume

$$(23) \qquad\qquad\qquad -1 \leq \sec_g \leq \kappa,$$

Rauch's comparison becomes:

$$(24) \qquad \sqrt{\kappa}\cot(\sqrt{\kappa}r_1)g_{r_1} \leq h(r_1), \text{ and } h(r_2) \leq \coth(r_2)g_{r_2},$$

for $0 < r_1 \leq \min\{\gamma_g(M), \frac{\pi}{2\sqrt{\kappa}}\}$, and $0 < r_2 \leq \gamma_g(M)$, which then implies the following bounds on the mean curvature $H(r, \sigma)$ of any geodesic sphere S_r.

$$(25) \quad (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r_1) \leq H(r_1, \sigma), \text{ and } H(r_2, \sigma) \leq (n-1)\coth(r_2).$$

The Riccati equation for the mean curvature H is:

$$(26) \qquad\qquad \partial_r H + |h|^2 = -Ric(\partial_r, \partial_r).$$

Since $H^2 \geq |h|^2$, we have

$$(27) \qquad\qquad \partial_r H + H^2 \geq -Ric(\partial_r, \partial_r).$$

If we assume the Ricci curvature is negative and bounded away from zero, say

$$(28) \qquad\qquad -Ric \geq \delta g, \quad (n-1) \geq \delta > 0,$$

then (27) implies

$$(29) \qquad\qquad \partial_r H + H^2 \geq \delta.$$

(The upper bound $(n-1)$ on δ follows from the normalization (23) of the sectional curvature.) Next consider the ordinary differential equation saturating the inequality given in equation (29)

$$u' + u^2 = \delta.$$

One solution to this equation is given by

$$(30) \qquad\qquad u(r) := \sqrt{\delta}\coth(\sqrt{\delta}\,r).$$

We now use a Riccati type comparison argument in conjunction with Rauch's comparison.

Lemma 31. *Let (M^n, g), $n \geq 3$, be a closed Riemannian manifold with $-1 \leq \sec_g \leq \kappa$, and*

$$-Ric \geq \delta g, \quad (n-1) \geq \delta > 0.$$

For any $p \in M$ and geodesic sphere $S_r(p)$ with $r \leq \gamma_g(M)$, we have

$$H(r, \sigma) \geq \sqrt{\delta}\coth(\sqrt{\delta}\,r), \ \forall \sigma \in S_r(p).$$

Proof. On any common interval of definition for H and u, equations (29) and (30) imply the following first inequality:

$$\left((u - H)e^{\int (u+H)}\right)' \leq 0.$$

Thus, if we can find r_0 such that $H(r_0, \sigma) \geq u(r_0)$, we then have that $H(r, \sigma) \geq u(r)$ for any $r \geq r_0$. Now, recall from equation (25) that

(32) $$(n - 1)\sqrt{\kappa} \cot(\sqrt{\kappa}r) \leq H(r, \sigma) \leq (n - 1)\coth(r),$$

for $r \leq \max(\gamma_g(M), \frac{\pi}{2\sqrt{\kappa}})$. Observe $(n - 1)\sqrt{\kappa}\cot(\sqrt{\kappa}r) = \frac{n-1}{r} + O(r)$ and $\sqrt{\delta}\coth(\sqrt{\delta}r) = \frac{1}{r} + O(r)$. Thus, for r' sufficiently close to zero, we have $H(r', \sigma) \geq \sqrt{\delta}\coth(\sqrt{\delta}r')$, $\forall \sigma \in S_{r'}$, and the result follows. $\square$

4. An integral inequality for harmonic forms

The main result of this section is the inequality given in Theorem 39. This inequality controls the the L^2-norm of a harmonic k-form on a ball of fixed radius, in terms of the L^2-norm on the complement of the given ball. We begin with a lemma giving pointwise and integral bounds on the geometric quantity $q(r)$ appearing in the Price equality given in Proposition 16. Key to this lemma are the Riccati-Rauch arguments of Section 3.

Lemma 33. *Let (M^n, g) be a Riemannian manifold with $-1 \leq \sec_g \leq \kappa$ and $0 \leq \kappa \leq 1$. Let α be a harmonic $k-$form on M. Assume further that*

$$-Ric \geq \delta g, \quad (n - 1) \geq \delta > 4k^2.$$

Then for $0 \leq \sigma < r < \gamma_g(M)$,

(34) $$\int_\sigma^r q(s)ds \geq (r - \sigma)\left(\frac{\sqrt{\delta}}{2} - k\right) + k\ln(1 - e^{-2\sigma}).$$

Define $r_0 := \frac{1}{\sqrt{\kappa}}arccot(\frac{\sqrt{\delta}}{\sqrt{\kappa}(n-1)}) \in (0, \frac{\pi}{2\sqrt{\kappa}})$. If

(35) $$\frac{\sqrt{\delta}}{2k} \geq \coth(r_0) + \frac{\epsilon}{k},$$

then

(36) $$\epsilon \leq q(r) \leq (n - 1)\coth(r),$$

for any $p \in M$ and $r \leq \gamma_g(M)$.

Proof. Recall equation (24) gives the upper bound on the second fundamental form h of the geodesic sphere S_r:

$$h \leq \coth(r)g_r,$$

for all $r \leq \gamma_g(M)$. Then Proposition 21 yields

$$q(r) \geq \frac{H}{2} - k\coth(r) + \mu(r)\coth(r) \geq \frac{H}{2} - k\coth(r).$$

The lower bound on h in (24) and the monotonicity of $\frac{\cot(\sqrt{\kappa}\,r)}{\coth(r)}$ on $(0, \frac{\pi}{2\sqrt{\kappa}})$, implies that given $r_0 < \frac{\pi}{2\sqrt{\kappa}}$ for any $r \in (0, r_0]$ we have

$$\frac{H(r,\sigma)}{2} - k\coth(r) \geq \frac{(n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r)}{2} - k\coth(r)$$

(37)
$$\geq k\coth(r_0)\Big(\frac{(n-1)}{2k}\frac{\sqrt{\kappa}\cot(\sqrt{\kappa}r_0)}{\coth(r_0)} - 1\Big) \geq \epsilon.$$

On the other hand by Lemma 31 and the monotonicity of $\coth(r)$ on $(0, \infty)$, we have for all $r > r_0$,

$$\frac{H(r,\sigma)}{2} - k\coth(r) \geq \frac{\sqrt{\delta}}{2}\coth(\sqrt{\delta}\,r) - k\coth(r)$$

(38)
$$\geq k\coth(r_0)\Big(\frac{\sqrt{\delta}}{2k\coth(r_0)} - 1\Big) \geq \epsilon.$$

Expanding $\coth(t) = 1 + O(e^{-2t})$ as a power series in e^{-2t} and estimating error terms, the first line in (38) implies (34). The last line in (38) combined with (37) implies the lower bound in (36). The upper bound is a simple consequence of (32). Finally we remark that when $\kappa = 0$ (i.e., the sectional curvature is non-positive), one has the freedom of choosing any sufficiently large r_0 for which (35) is satisfied. □

We are now ready to prove the main technical result of this section.

Theorem 39. *Let (M^n, g) be a Riemannian manifold with $-1 \leq \sec_g \leq \kappa$, $0 \leq \kappa \leq 1$ and dimension $n \geq 6$. Let α be a harmonic k-form. Assume*

$$-Ric \geq \delta g, \quad \delta > 4k^2.$$

Let again $r_0 = \frac{1}{\sqrt{\kappa}} arccot\left(\frac{\sqrt{\delta}}{\sqrt{\kappa(n-1)}}\right)$. *Let* $C_{\sigma,k} = \coth(\sigma)^{2k}$. *If for some* $\epsilon > 0$,

(40)
$$\frac{\sqrt{\delta}}{2k} \geq \coth(r_0) + \frac{\epsilon}{k},$$

then for any $0 < \sigma < \tau \leq \gamma_g(M)$ *and* $p \in M$,

(41)
$$\int_{B_\sigma(p)} |\alpha|^2 dv \leq \frac{(n-1)C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)}}{\epsilon(1 - C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)})} \int_{B_\tau(p) \smallsetminus B_\sigma(p)} |\alpha|^2 dv.$$

Proof. By Lemma 33, $q(r) \geq \epsilon > 0$. Equation (13) then implies $1 - 2\mu(s) > 0$ for $s \in [0, \gamma_g(M)]$. Then the weight function ϕ defined in (15) satisfies

(42)
$$\phi(r) \leq e^{-2\int_\sigma^r q(s)ds} \leq C_{\sigma,k}e^{-2(r-\sigma)(\frac{\sqrt{\delta}}{2}-k)},$$

where we have used (34) for the last inequality, and we set $C_{\sigma,k} = \coth(\sigma)^{2k}$ (not sharp). Inserting this inequality into (17) yields

(43)
$$(1 - C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)})\int_{B_\sigma(p)} q(r)|\alpha|^2 dv$$
$$\leq C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)} \int_{B_\tau(p) \smallsetminus B_\sigma(p)} q(r)|\alpha|^2 dv.$$

The pointwise bounds on $q(r)$ given in Lemma 33 now yield

(44)
$$(1 - C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)})\epsilon \int_{B_\sigma(p)} |\alpha|^2 dv$$
$$\leq C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)}(n-1)\coth(\sigma) \int_{B_\tau(p) \smallsetminus B_\sigma(p)} |\alpha|^2 dv,$$

which then concludes the proof. Once again when $\kappa = 0$, one has the freedom of choosing any sufficiently large r_0 for which (35) is satisfied. $\square$

5. From integral inequalities to dimension estimates

In this section, we use the integral inequalities of the previous section to extract dimension estimates. In the following section, we will sharpen these estimates using cohomological techniques. We include the slightly weaker results here for the reader who may be interested in applying them to related problems - such as bounding the dimensions of eigenspaces with very

small eigenvalue - to which these techniques, but not their cohomological improvement, readily apply.

In order to extract dimension estimates from (41), we first need the following standard lemma.

Lemma 45. *Let (M^n, g) be a closed Riemannian manifold. There exists $\alpha \in \mathcal{H}_g^k(M)$, with $\|\alpha\|_{L^2} = 1$, such that*

$$\max_{p \in M} |\alpha|^2 \geq \frac{k!(n-k)!b_k(M)}{n!Vol(M)},$$

where $b_k(M) = \dim_{\mathbb{R}} \mathcal{H}_g^k(M)$ is the k-th Betti number of M.

Proof. Let $K(x, y)$ denote the Schwartz kernel of the L^2−orthogonal projection onto $\mathcal{H}_g^k(M)$. Then

$$(46) \qquad \int_M \operatorname{tr} K(x, x) dv = dim\mathcal{H}_g^k(M).$$

Hence, there exists $p \in M$ such that

$$(47) \qquad \operatorname{tr} K(p, p) \geq \frac{b_k(M)}{Vol(M)}.$$

Then there exists a unit eigenvector z of $K(p, p)$ with eigenvalue

$$\lambda \geq \frac{k!(n-k)!b_k(M)}{n!Vol(M)}.$$

Since K is the Schwartz kernel of an L^2−orthogonal projection,

$$(48) \qquad \|K(x, p)z\|_{L^2}^2 = \langle K(p, p)z, z \rangle = \lambda.$$

Set

$$(49) \qquad \alpha(x) := \frac{K(x, p)z}{\sqrt{\lambda}}.$$

Then $\|\alpha\|_{L^2} = 1$, and

$$(50) \qquad |\alpha(p)|^2 = \frac{|K(p, p)z|^2}{\lambda} = \lambda \geq \frac{k!(n-k)!b_k(M)}{n!Vol(M)}.$$

$\square$

The following lemma allows us to pass from integral inequalities to pointwise estimates.

Lemma 51. *Let (M^n, g) be a closed Riemannian manifold with*

$$-1 \leq \sec_g \leq 1.$$

Given a harmonic k-form $\alpha \in \mathcal{H}_g^k(M)$, for any

$$p \in M \quad and \quad R < \min(\gamma_g(M), 1)$$

there exists a strictly positive constant $d(n, R) := d(n)(1 + \frac{1}{R})^n$ such that

$$||\alpha||^2_{L^\infty(B_{\frac{R}{2}}(p))} \leq d(n, R)||\alpha||^2_{L^2(B_R(p))}.$$

Proof. This proof is a standard application of Moser iteration. See for example [LS18, Proposition 2.2], where the theorem is proved for hyperbolic manifolds. The extension to our context follows immediately substituting the bounded variable curvature for constant curvature and using Lemma 2.24 of [Aubin, Chapter 2] to replace [LS18, (2.1)]. More precisely, one replaces [LS18, (2.1)] by the statement that there exists $S_M > 0$, such that for all $p \in M$, $R < \max\{\frac{\delta(M)}{2}, 1\}$, and all compactly supported smooth functions on the ball $\xi \in C_c^\infty(B_R(p))$, one has

$$(52) \qquad S_M||d\xi||^2_{L^2(B_R(p))} \geq ||\xi||^2_{L^{\frac{2n}{n-2}}(B_R(p))}.$$

We refer to [Aubin, Chapter 2] for this well-known statement. $\square$

We close this section with our main application.

Theorem 53. *Let (M^n, g) be a Riemannian manifold with $-1 \leq \sec_g \leq \kappa$. Assume*

$$-Ric \geq \delta g, \quad \delta \geq 4k^2.$$

Let $r_0 = \frac{1}{\sqrt{\kappa}} arccot(\frac{\sqrt{\delta}}{\sqrt{\kappa}(n-1)})$. If

$$(54) \qquad \frac{\sqrt{\delta}}{2k} \geq \coth(r_0) + \frac{\epsilon}{k},$$

then there exists a positive constant $c(n)$ depending on the dimension only such that for $\gamma_g(M)$ large

$$\frac{b_k(M)}{Vol_g(M)} \leq c(n)\epsilon^{-1}e^{-2\gamma_g(M)(\frac{\sqrt{\delta}}{2}-k)}.$$

Proof. By Lemma 45, there exists $\alpha \in \mathcal{H}_g^k(M)$ and $p \in M$, with $||\alpha||_{L^2(M)} = 1$, and

$$|\alpha(p)|^2 \geq \frac{k!(n-k)!b_k(M)}{n!Vol(M)}. \tag{55}$$

On the other hand, Lemma 51 and equation (41) yield

$$
\begin{aligned}
|\alpha|^2(p) &\leq d(n, 1/2)||\alpha||^2_{L^2(B_p(1))} \\
&\leq \frac{\tilde{c}(n)}{\epsilon} e^{-2(\gamma_g(M)-1)(\frac{\sqrt{\delta}}{2}-k)}||\alpha||^2_{L^2(B_p(\tau))},
\end{aligned} \tag{56}
$$

where $d(n, 1/2) > 0$ is the constant of Lemma 51, and

$$\tilde{c}(n) := 2d(n, 1/2)(n-1)C_{1,k+\frac{1}{2}},$$

with $\gamma_g(M)$ large enough so that

$$1 - C_{1,k}e^{-2(\gamma_g(M)-1)(\frac{\sqrt{\delta}}{2}-k)} \geq \frac{1}{2}.$$

$$\square$$

6. Excising geodesic balls

In Section 3, we estimate $q(r)$ from below by combining Rauch's comparison with a Riccati type argument. Under the curvature constraint $\frac{\sqrt{\delta}}{2} > k \coth(r_0)$, with $r_0 := \frac{1}{\sqrt{\kappa}} arccot\left(\frac{\sqrt{\delta}}{\sqrt{\kappa}(n-1)}\right)$, these arguments suffice to provide a positive uniform lower bound on $q(r)$ for any r. The deleterious effects of positive curvature in the comparison arguments diminish at large radius. In fact, it only affects Rauch's comparison for small values of r. On the other hand, under our usual Ricci curvature assumption, the positive curvature does not affect the Riccati argument for large values of r. Thus it is natural to work in the complement of a large ball. In this section, we use cohomological arguments to remove the application of Rauch's comparison for small values of r. We begin with the following reformulation of Equation (13) with two boundary components.

Proposition 57. *Let (M^n, g) be complete and satisfy $-1 \leq \sec_g$. Given a point $p \in M$, $\alpha \in \mathcal{H}_g^k(M)$, and $\sigma \leq \tau \leq \gamma_g(M)$, we have*

$$\int_{S_\tau(p)} \left(\frac{1}{2} - \mu(\tau)\right) |\alpha|^2 d\sigma = \int_{S_\sigma(p)} \left(\frac{1}{2} - \mu(\sigma)\right) |\alpha|^2 d\sigma$$

(58)
$$+ \int_{B_\tau(p) \smallsetminus B_\sigma(p)} q(r) |\alpha|^2 dv.$$

Moreover, with ϕ as defined in (15),

$$(59) \quad \phi(\tau) \int_{B_\tau(p) \smallsetminus B_\sigma(p)} q(r) |\alpha|^2 dv = (1 - \phi(\tau)) \int_{S_\sigma(p)} \left(\frac{1}{2} - \mu(\sigma)\right) |\alpha|^2 d\sigma.$$

Proof. This follows from the same arguments as (13) and (17). $\qquad\square$

In Section 5, we showed that if ϕ decays to zero, it is possible to give effective estimates on the size of Betti numbers normalized by the Riemannian volume. In order to control the size of ϕ, it is necessary to understand the sign not only of $q(r)$ but also of $\frac{1}{2} - \mu(r)$, see Equation (15). In particular, we require $\mu(r) < \frac{1}{2}$ for all r sufficiently large. This inequality follows from (58), if $\mu(\sigma) < \frac{1}{2}$. Thus, it is natural to work with harmonic forms with Neumann boundary condition.

Given a complete Riemannian manifold with boundary, Ω, let $\mathcal{H}_{2,N}^k(\Omega)$ denote the L^2-harmonic k-forms on Ω satisfying Neumann boundary conditions on $\partial\Omega$. Let $H_{2,N}^k(\Omega)$ denote the (absolute) L^2- cohomology of Ω. This is the cohomology of the complex (C_2^*, d), where C_2^k denotes the closure in the L^2 graph norm $(\|\phi\|_{\text{graph}}^2 := \|\phi\|_{L^2}^2 + \|d\phi^2\|_{L^2})$ of the smooth L^2 $k-$ forms on Ω.

It is easy to see that, if 0 is not in the essential spectrum of Δ_k, then

$$(60) \qquad\qquad \mathcal{H}_{2,N}^k(\Omega) \simeq H_{2,N}^k(\Omega).$$

When Ω is compact (possibly with boundary), then elliptic regularity plus the de-Rham isomorphism implies $H_{2,N}^k(\Omega) \simeq H^k(\Omega)$. For more details on L^2-cohomology we refer to Section 8.

The next lemma is the natural analogue of Theorem 39 for Neumann harmonic forms.

Lemma 61. *Let (M^n, g) be a complete Riemannian manifold with $-1 \leq \sec_g \leq 1$. Let $p \in M$, $\rho < \gamma_g(M)$, and $\alpha \in \mathcal{H}_{2,N}^k(M \smallsetminus B_\rho(p))$. Assume there*

exists $\delta > 4k^2$ *and* $\epsilon > 0$ *such that*

$$-Ric \geq \delta g,$$

and

(62) $$\frac{\sqrt{\delta}}{2} \coth(\sqrt{\delta}\rho) - k \coth(\rho) \geq \epsilon.$$

Then for any $\sigma \in (\rho, \gamma_g(M))$, $\tau \in (\sigma, \gamma_g(M))$ *and* $\sigma < R < \tau$,

(63)
$$\int_{B_R(p) \smallsetminus B_\sigma(p)} |\alpha|^2 dv$$
$$\leq \frac{n-1}{\epsilon} \coth(\sigma)^{2k+1} e^{-(\sqrt{\delta}-2k)(\tau-R)} \int_{B_\tau(p) \smallsetminus B_\sigma(p)} |\alpha|^2 dv.$$

Proof. Recall that the Neumann boundary condition implies $i_{\partial_r}\alpha = 0$ on $S_\rho(p)$. Equivalently, $\mu(\rho) = 0$. Hence (58) with $\sigma = \rho$ implies $\mu(r) \leq \frac{1}{2}$, $\forall r \in [\rho, \gamma_g(M)]$, as long as $q(r) \geq 0$, for $\forall r \in [\rho, \gamma_g(M)]$. Taking (59) for two different values of τ and then taking the difference of the two equations gives

(64)
$$\phi(\tau) \int_{B_\tau(p) \smallsetminus B_\sigma(p)} q(r)|\alpha|^2 dv = \phi(R) \int_{B_R(p) \smallsetminus B_\sigma(p)} q(r)|\alpha|^2 dv$$
$$+ (\phi(R) - \phi(\tau)) \int_{S_\sigma(p)} \left(\frac{1}{2} - \mu(\sigma)\right)|\alpha|^2 d\sigma.$$

The hypotheses imply ϕ is monotonically decreasing. Our curvature estimates imply $q(s) > \frac{\sqrt{\delta}}{2} - k \coth(s)$, and the desired estimate follows from (15). $\qquad\square$

The Price inequality of Lemma 61 can now be applied to derive a vanishing for certain spaces of Neumann harmonic forms.

Corollary 65. *Let* (M^n, g) *be a simply connected non-compact complete Riemannian manifold without conjugate points and* $-1 \leq \sec_g \leq 1$. *Assume there exists* $\delta > 4k^2 > 0$ *such that*

$$-Ric \geq \delta g.$$

Let $p \in M$. *Then for* ρ *sufficiently large,* $\mathcal{H}^k_{2,N}(M \smallsetminus B_\rho(p)) = 0$.

Proof. Taking $\tau \to \infty$ in the preceding lemma, we see that α vanishes identically. $\qquad\square$

Next, we consider closed manifolds without conjugate points.

Proposition 66. *Let (M^n, g) be a closed manifold without conjugate points with $-1 \leq \sec_g \leq 1$. Assume there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g.$$

Let $\pi_i : M_i \to M$ be a sequence of Riemannian covers of M with $\gamma_g(M_i) \to \infty$. Let $h_i \in \mathcal{H}^k(M_i)$, and $p_i \in M_i$. Then there exists $c(n, k, \delta) > 0$ so that for $\gamma_{g_i}(M_i)$ sufficiently large,

$$(67) \qquad \int_{B_\rho(p_i)} |h_i|^2 dv \leq c(n, k, \delta) e^{-(\sqrt{\delta} - 2k)\gamma_g(M_i)} \|h_i\|^2.$$

Proof. Choose ρ sufficiently large so that

$$\frac{\sqrt{\delta}}{2} \coth(\sqrt{\delta}\rho) - k \coth(\rho) \geq \frac{\sqrt{\delta} - 2k}{4} =: \epsilon > 0.$$

Consider i sufficiently large so that $\rho < \gamma_g(M_i)$. Then for $k < n - 1$, we have

$$\mathcal{H}^k(M_i) \simeq H^k(M_i) \simeq H_N^k(M_i \setminus B_\rho(p_i)) \simeq \mathcal{H}_N^k(M_i \setminus B_\rho(p_i)).$$

Consider the map from $\mathcal{H}^k(M_i)$ obtained as follows. Given $h \in \mathcal{H}^k(M_i)$, let $J_1(h)$ denote the $H_N^k(M_i \setminus B_\rho(p_i))$ harmonic representative of the restriction of h to $M_i \setminus B_\rho(p_i)$. Let b denote a coexact primitive (with respect to the complex $(C_2^*(B_{\rho+2}(p_i) \setminus B_\rho(p_i)), d)$) for the restriction of $J_1(h)$ to $B_{\rho+2}(p_i) \setminus B_\rho(p_i)$. Let η be a smooth cutoff function with $|d\eta| < 2$, supported on $B_{\rho+2}(p_i) \setminus B_\rho(p_i)$, identically zero near $\partial B_{\rho+2}(p_i)$ and identically one in $B_{\rho+1}(p_i)$. Then $J_2(h) := J_1(h) - d(\eta b)$ defines an element in $H^k(M_i)$. Because every cycle in $H_k(M_i)$ has a representative disjoint from $B_\rho(p_i)$ and the integral of $J_2(h)$ over every such cycle is equal to the integral of h over the cycle, $J_2(h)$ is cohomologous to h. Let now $J_3(h)$ denote the M_i harmonic projection of $J_2(h)$. Then we have shown $J_3(h) = h$. Restriction and harmonic projection are norm nonincreasing. In particular, we have

$$(68) \qquad \|J_1(h)\|_{L^2(M_i \setminus B_\rho(p_i))}^2 \leq \|h\|_{L^2(M_i)}^2 - \int_{B_\rho} |h|^2 dv.$$

On the other hand, we have

$$\|h\|^2_{L^2(M_i)} = \|J_3(h)\|^2_{L^2(M_i)} \leq \|J_2(h)\|^2_{L^2(M_i)}$$

(69)
$$\leq \|h\|^2_{L^2(M_i)} - \int_{B_\rho} |h|^2 dv + \|d(\eta b)\|^2_{L^2(B_{\rho+2}(p_i) \smallsetminus B_\rho(p_i))}.$$

Hence

$$\int_{B_\rho} |h|^2 dv \leq \|d(\eta b)\|^2_{L^2(B_{\rho+2}(p_i) \smallsetminus B_\rho(p_i))}$$

(70)
$$\leq 2\|\eta J_1(h)\|^2_{L^2(M_i \smallsetminus B_\rho(p_i))} + 2\|d\eta \wedge b\|^2_{L^2(B_{\rho+2}(p_i) \smallsetminus B_\rho(p_i))}$$

(71)
$$\leq 2\|J_1(h)\|^2_{L^2(B_{\rho+2} \smallsetminus B_\rho)} + 8\|b\|^2_{L^2(B_{\rho+2} \smallsetminus B_\rho)}.$$

In order to bound $\|b\|^2_{L^2(B_{\rho+2} \smallsetminus B_\rho)}$, it suffices to estimate the L^2 norm of any primitive for the restriction of $J_1(h)$ to the annular region, since the coexact primitive has smallest L^2 norm among all primitives. We now construct one such primitive. Use normal coordinates to fix a diffeomorphism $\zeta : B_{\rho+2}(p_i) \setminus B_\rho(p_i) \to S^{n-1} \times [0,2]$. Write $H := \zeta^* J_1(h) = H_0 + dt \wedge H_1$, with $i_{\frac{\partial}{\partial t}} H_j = 0$, t the coordinate on $[0,2]$. Fix $L \in [0,2]$ minimizing $\int_{S_L} |H|^2 d\sigma$. Set

(72)
$$\beta_L(r) := \int_L^r H_1(s)ds.$$

Then since $d(\zeta^* J_1(h)) = 0$

$$d\beta_L = H - H_0(L).$$

In the product metric g_0 on $S^{n-1} \times [0,2]$, we have

$$\|\beta_L\|^2_{L^2(S^{n-1}\times[0,2],g_0)} = \int_0^2 \int_{S^{n-1}} \left(\int_L^r H_1(s)ds \right)^2 d\sigma dr$$

$$\leq 2 \int_{S^{n-1}\times[0,2]} |H_1|^2 dv$$

(73)
$$\leq 2\|H\|^2_{L^2(S^{n-1}\times[0,2],g_0)}.$$

(We now indicate the metric in our L^2 norms when confusion may occur.) Let ξ denote a coexact primitive for $H_0(L)$ viewed as an exact form on S^{n-1}.

Then in the product metric,

$$\|\xi\|_{L^2(S^{n-1}\times[0,2],g_0)}^2 \leq W_n\|H\|_{L^2(S^{n-1}\times[0,2],g_0)}^2,$$

with $W_n = O(\lambda_{1,k}^{-1})$, where $\lambda_{1,k}$ denotes the first eigenvalue of the Laplace Beltrami operator for k-forms on the sphere. (Our hypotheses imply $k \neq 0, n-1$.) By construction,

$$d(\beta_L + \xi) = H.$$

There exists $C_1(\rho) > 0$ so that

(74)
$$C_1^{-1}\|\cdot\|_{L^2(B_{\rho+2}(p)\smallsetminus B_\rho(p),g)}^2 \leq \|\cdot\|_{L^2(S^{n-1}\times[0,2],g_0)}^2$$
$$\leq C_1\|\cdot\|_{L^2(B_{\rho+2}(p)\smallsetminus B_\rho(p),g)}^2.$$

Then we have

(75) $\quad \|\beta_L + \xi\|_{L^2(S^{n-1}\times[0,2],g_0)}^2 \leq (W_n + 2\sqrt{2}\sqrt{W_n} + 2)\|H\|_{L^2(S^{n-1}\times[0,2],g_0)}^2.$

Equation (75) combined with (70) and (74) gives

(76)
$$\int_{B_\rho} |h|^2 dv \leq d(W_n, C_1)\|J_1(h)\|_{L^2(B_{\rho+2}(p)\smallsetminus B_\rho(p),g)}^2,$$

where $d(W_n, C_1)$ is a positive constant depending on W_n and C_1. Using (63), we have

(77)
$$\int_{B_\rho} |h|^2 dv \leq d(W_n, C_1)\frac{n-1}{\epsilon}\coth(\rho)^{2k+1}e^{-(\sqrt{\delta}-2k)(\tau-\rho-2)}$$
$$\times \int_{B_\tau\smallsetminus B_\rho} |J_1(h)|^2 dv,$$

and the result follows. $\square$

We can now prove the main result of this section.

Corollary 78. *Let (M^n, g) be a closed Riemannian manifold without conjugate points and with $-1 \leq \sec_g \leq 1$. Assume there exists $\delta > 4k^2 > 0$ such*

that

$$-Ric \geq \delta g.$$

Let $\pi_i : M_i \to M$ be a sequence of Riemannian covers of M with $\gamma_g(M_i) \to \infty$. Then there exists $b(n, k) > 0$ so that for $\gamma_g(M_i)$ sufficiently large,

$$(79) \qquad \frac{b_k(M_i)}{Vol(M_i)} \leq b(n, k, \delta)e^{-(\sqrt{\delta}-2k)\gamma_{g_i}(M_i)}.$$

In particular,

$$(80) \qquad \lim_{i \to \infty} \frac{b_k(M_i)}{Vol(M_i)} = 0.$$

Proof. Apply Lemmas 45 and 51 to Proposition 66. $\qquad\qquad\square$

Congruence subgroups of arithmetic groups of $\mathbb{Q}$ rank 0 algebraic groups provide an important and widely studied class of examples of towers of covers. These lattices have large injectivity radii relative to their covolumes with respect to the natural locally symmetric metric. For more details see [SX91], [Yeu94], and [Mar14]. The following definition abstracts the injectivity radius properties of such lattices.

Definition 81. Let (M^n, g) be a closed Riemannian manifold with infinite residually finite fundamental group $\Gamma := \pi_1(M^n)$. Given a cofinal filtration $\{\Gamma_i\}$ of Γ, for any index i let us set

$$r_i := \inf\{d_{\tilde{g}}(z, \gamma_i z)/2 \mid z \in \tilde{X}, \quad \gamma_i \in \Gamma_i, \quad \gamma_i \neq 1\}$$

where $(\tilde{X}, \tilde{g})$ is the Riemannian universal cover. We say that the cofinal filtration is of **congruence type** if there exist constants, $0 < \alpha(n, \{\Gamma_i\}) < 1$, $g(n, \{\Gamma_i\}) > 0$ such that

$$e^{r_i} \geq g(n, \{\Gamma_i\})[\Gamma : \Gamma_i]^{\alpha(n, \{\Gamma_i\})}$$

for any $i \geq 1$, where $[\Gamma : \Gamma_i]$ is the index of Γ_i in Γ. We call the constant α the **exponent** of the cofinal filtration.

We restate our Betti number asymptotics under the additional hypothesis of a congruence type cofinal filtration with exponent α.

Corollary 82. *Let (M^n, g) be a Riemannian manifold without conjugate points with $-1 \leq \sec_g \leq \kappa$ and residually finite fundamental group Γ. Assume*

$$-Ric \geq \delta g, \quad \delta > 4k^2.$$

Let $\{\Gamma_i\}_i$ be a congruence type cofinal filtration of Γ of exponent α. Denote by $\pi_i : M_i \to M$ the regular Riemannian cover of M associated to Γ_i. Then

$$b_k(M_i) \leq d(n, k, \delta, \Gamma)Vol(M_i)^{1-2\alpha(\frac{\sqrt{\delta}}{2}-k)}.$$

where $d(n, k, \Gamma)$ is a positive constant.

Proof. Since (M^n, g) has no conjugate points, for any $i \geq 1$ the numerical invariant $r_i > 0$ given in Definition 81 is simply the injectivity radius $\gamma_{g_i}(M_i)$. Since

$$Vol(M_i) = Vol(M)[\Gamma : \Gamma_i],$$

by Corollary 78 we have

$$b_k(M_i) \leq \frac{b(n, k)}{e^{2r_i(\frac{\sqrt{\delta}}{2}-k)}}Vol(M_i) \leq \frac{b(n, k)Vol(M)}{g(n, \Gamma)^{2(\frac{\sqrt{\delta}}{2}-k)}}[\Gamma : \Gamma_i]^{1-2(\frac{\sqrt{\delta}}{2}-k)\alpha},$$

and the proof is complete. $\square$

7. Inequalities for negatively pinched manifolds in dimensions $n \geq 4$

Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$ such that

$$(83) \qquad -b^2 \leq \sec_g \leq -a^2,$$

for some $a, b \in (0, \infty)$. Let $p \in M$, and denote by B_R the ball of radius $0 < R < \gamma_g(M)$ centered at p. In B_R we write the metric in geodesic spherical coordinates as $g = dr^2 + g_r$. With the curvature assumptions (83), Rauch's comparison (see Theorem 22) gives the following two sided bound on the second fundamental form of any geodesic sphere:

$$(84) \qquad a\coth(ar)g_r(u) \leq \mathrm{Hess}(r, u) \leq b\coth(br)g_r(u),$$

for any $r \leq \gamma_g(M)$ and for any point u on the geodesic sphere S_r. Taking the trace of (84) gives the corresponding mean curvature bound:

$$(85) \qquad (n-1)a\coth(ar) \leq H(r, u) \leq (n-1)b\coth(br),$$

for any $u \in S_r(p)$, given any $p \in M$. This implies that for any $r \leq \gamma_g(M)$, we have

$$\frac{(n-1)}{2} a \coth(ar) - kb \coth(br)$$

(86)
$$\leq q(r) \leq \frac{(n-1)}{2} b \coth(br) - ka \coth(ar).$$

We now specialize our Price inequalities to harmonic forms on closed pinched negatively curved manifolds of dimension $n \geq 4$. This is the main result of this section.

Theorem 87. *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 4$. Assume the sectional curvature is ϵ-pinched :*

$$-(1+\epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon \geq 0$. Let k be a non-negative integer such that

$$\epsilon_{n,k} := (n-1) - 2k(1+\epsilon) > 0.$$

For $\alpha \in \mathcal{H}_g^k(M)$, and $0 < \sigma < \tau \leq \gamma_g(M)$, we have

(88)
$$\int_{B_\sigma(p)} |\alpha|^2 dv \leq \frac{e^{-(\tau-\sigma)\epsilon_{n,k}}}{1 - e^{-(\tau-\sigma)\epsilon_{n,k}}} \frac{(n-1)}{\epsilon_{n,k}}$$
$$\times (1+\epsilon) \coth(\sigma) \int_{B_\tau(p) \smallsetminus B_\sigma(p)} |\alpha|^2 dv.$$

Proof. Recall the Price equality given in (17)

(89)
$$\int_{B_\sigma(p)} q(r)|\alpha|^2 dv = \phi(\tau) \int_{B_\tau(p)} q(r)|\alpha|^2 dv.$$

By (86) we have

$$q(r) \geq \frac{(n-1)}{2} \coth(r) - k(1+\epsilon) \coth((1+\epsilon)r)$$

(90)
$$> \frac{(n-1)}{2} - k(1+\epsilon).$$

Hence by (15), we have

(91)
$$\phi(r) \leq e^{-(r-\sigma)(n-1-2k(1+\epsilon))},$$

which combined with (17) gives

$$(92) \quad \int_{B_\sigma(p)} q(r)|\alpha|^2 dv \leq \frac{e^{-(r-\sigma)(n-1-2k(1+\epsilon))}}{1 - e^{-(r-\sigma)(n-1-2k(1+\epsilon))}} \int_{B_\tau(p) \smallsetminus B_\sigma(p)} q(r)|\alpha|^2 dv.$$

The upper bound of Equation (86) implies

$$q(r) \leq \frac{(n-1)}{2}(1+\epsilon)\coth((1+\epsilon)r) - k\coth(r),$$

so that for any $r \geq \sigma$, we have

$$(93) \quad q(r) \leq \frac{n-1}{2}(1+\epsilon)\coth(r).$$

The lower bound on q given in (86) combined with (93) then yields

$$(94) \quad \int_{B_\sigma(p)} |\alpha|^2 dv \leq \frac{e^{-(\tau-\sigma)\epsilon_{n,k}}}{1 - e^{-(\tau-\sigma)\epsilon_{n,k}}} \frac{(n-1)}{\epsilon_{n,k}}$$
$$\times (1+\epsilon)\coth(\sigma) \int_{B_\tau(p) \smallsetminus B_\sigma(p)} |\alpha|^2 dv,$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 95. *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 4$. Assume*

$$-(1+\epsilon)^2 \leq \sec_g \leq -1$$

with $\epsilon \geq 0$. There exists a constant $c(n,k) > 0$ so that for each non-negative integer k such that

$$\epsilon_{n,k} := (n-1) - 2k(1+\epsilon) > 0,$$

$$\frac{b_k(M)}{Vol_g(M)} \leq c(n,k)e^{-\epsilon_{n,k}\gamma_g(M)},$$

for $\gamma_g(M) > 1 + \frac{\ln(2)}{\epsilon_{n,k}}$.

Proof. Apply Lemmas 45 and 51 to Proposition 87. The (nonsharp) constraint on $\gamma_g(M)$ serves to control denominators. $\qquad\qquad\square$

Finally, we study the borderline case where $\epsilon_{n,k} = 0$ and $\epsilon > 0$.

Theorem 96. *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 4$. Assume the sectional curvature is ϵ-pinched :*

$$-(1 + \epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon > 0$. Let k be a positive integer such that

$$\epsilon_{n,k} := (n - 1) - 2k(1 + \epsilon) = 0.$$

For $\alpha \in \mathcal{H}_g^k(M)$ and $0 < 1 \leq \tau \leq \gamma_g(M)$, we have

$$(97) \qquad \int_{B_1(p)} |\alpha|^2 dv \leq \frac{\sinh^2(1 + \epsilon)}{2k(1 + \epsilon)\epsilon(\tau - 1)} \int_{B_\tau(p) \smallsetminus B_1(p)} |\alpha|^2 dv.$$

Proof. Given the assumption that $\epsilon_{n,k} = 0$, we write the first inequality of (90) as

$$q(r) \geq k(1 + \epsilon)(\coth(r) - \coth((1 + \epsilon)r)).$$

Using the mean value theorem, we may estimate the lower bound by

$$(98) \qquad q(r) \geq \frac{k(1 + \epsilon)r\epsilon}{\sinh^2(r + \epsilon r)}.$$

The righthand side of this inequality is monotonically decreasing in r. Hence for $r \in [0, 1]$ we have

$$(99) \qquad q(r) \geq \frac{k(1 + \epsilon)\epsilon}{\sinh^2(1 + \epsilon)}.$$

Recall (17) and (13): for $1 < \tau \leq \gamma_g(M)$,

$$\int_{B_\sigma(p)} q(r)|\alpha|^2 dv = \phi(\tau) \int_{B_\tau(p)} q(r)|\alpha|^2 dv$$
$$(100) \qquad\qquad = \phi(\tau) \int_{S_\tau(p)} \left(\frac{1}{2} - \mu(\tau)\right)|\alpha|^2 d\sigma.$$

When $\epsilon_{n,k} = 0$, ϕ need not decay exponentially, but by definition (15),

$$(101) \qquad\qquad\qquad 0 < \phi(r) \leq 1.$$

Thus, for any $s \geq 1$

$$\frac{k(1+\epsilon)\epsilon}{\sinh^2(1+\epsilon)} \int_{B_1(p)} |\alpha|^2 dv \leq \int_{B_1(p)} q(r)|\alpha|^2 dv$$

$$\leq 1 \cdot \int_{B_s(p)} q(r)|\alpha|^2 dv$$

(102)
$$= \int_{S_s(p)} \left(\frac{1}{2} - \mu(s)\right) |\alpha|^2 d\sigma.$$

Integrating Equation (102) from 1 to $1 < \tau \leq \gamma_g(M)$, yields

$$\frac{k(1+\epsilon)\epsilon}{\sinh^2(1+\epsilon)}(\tau - 1) \int_{B_1(p)} |\alpha|^2 dv \leq \frac{1}{2} \int_{B_\tau(p) \smallsetminus B_1(p)} |\alpha|^2 dv,$$

which concludes the proof. □

Corollary 103. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$. Assume*

$$-(1+\epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon > 0$ satisfying $k := \frac{n-1}{2(1+\epsilon)}$ is an integer. If $\gamma_g(M) > 1$, there exists a positive constant $c(n, k, \epsilon)$ such that

$$\frac{b_k(M)}{Vol_g(M)} \leq \frac{c(n, k, \epsilon)}{\gamma_g(M) - 1}.$$

Proof. Choose a normalized $\alpha \in \mathcal{H}_g^k(M)$ as in Lemma 45, and let $p \in M$ be a point where the norm of α achieves its maximum. Since $||\alpha||_{L^2} = 1$, for $1 < \tau \leq \gamma_g(M)$, Equation (97) combined with Lemma 45 and with Lemma 51 with $R = 1$ then gives

$$\frac{b_k(M)}{Vol(M)} \leq \binom{n}{k} |\alpha|_p^2 \leq \binom{n}{k} \frac{d(n,1)\sinh^2(1+\epsilon)}{2k(1+\epsilon)\epsilon(\tau - 1)} ||\alpha||^2_{L^2(B_1)}$$

$$\leq \binom{n}{k} \frac{d(n,1)\sinh^2(1+\epsilon)}{2k(1+\epsilon)\epsilon(\tau - 1)}.$$

Take $\tau = \gamma_g(M)$ to complete the proof. □

We can now study the asymptotic behavior of Betti numbers with respect to coverings of negatively pinched Riemannian manifolds.

Corollary 104. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$ with residually finite fundamental group $\Gamma := \pi_1(M)$. Assume*

$$-(1+\epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon \geq 0$. Given a cofinal filtration $\{\Gamma_i\}$ of Γ, denote by $\pi_i : M_i \to M$ the regular Riemannian cover of M associated to Γ_i. For each non-negative integer k such that $\epsilon_{n,k} = (n-1) - 2k(1+\epsilon) > 0$,

(105)
$$\lim_{i\to\infty} \frac{b_k(M_i)}{Vol(M_i)} = 0.$$

Moreover, if $\epsilon > 0$, the same holds true for k-forms of critical degree such that $\epsilon_{n,k} = 0$.

Proof. Given a cofinal filtration $\{\Gamma_i\}$ of Γ,

$$\lim_{i\to\infty} \gamma_{g_i}(M_i) = \infty$$

where M_i is equipped with the pull back metric $g_i := \pi_i^*(g)$. For a proof, see Theorem 2.1 in [DW78]. The corollary is now a consequence of Corollary 95 if $\epsilon_{n,k} > 0$. If $\epsilon_{n,k} = 0$, we instead appeal to Corollary 103. $\square$

We now consider cofinal filtrations of congruence type (see Definition 81).

Corollary 106. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$ with residually finite fundamental group $\Gamma := \pi_1(M)$. Assume*

$$-(1+\epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon \geq 0$. Given a cofinal filtration $\{\Gamma_i\}$ of Γ of congruence type with exponent $\alpha \in (0,1)$, denote by $\pi_i : M_i \to M$ the regular Riemannian cover of M associated to Γ_i. For any non-negative integer k such that

$$\epsilon_{n,k} := (n-1) - 2k(1+\epsilon) > 0,$$

and for all i such that

$$\gamma_{g_i}(M_i) > 1 + \frac{\ln(2)}{\epsilon_{n,k}},$$

we have

$$b_k(M_i) \leq k(n, k, \epsilon, \Gamma) Vol(M_i)^{1-\epsilon_{n,k}\alpha},$$

where $k(n, k, , \epsilon, \Gamma)$ is a positive constant. Moreover, if $\epsilon > 0$, then for k-forms of the critical degree such that $\epsilon_{n,k} = 0$, we have

$$b_k(M_i) \leq l(n, k, \epsilon, \Gamma) \frac{Vol_{g_i}(M_i)}{\alpha \ln(Vol_{g_i}(M_i))},$$

for some positive constant $l(n, k, \epsilon, \Gamma)$.

Proof. Re-express the functions of injectivity radius in Theorems 95 and 96 as functions of $Vol(M)$. $\square$

Remark 107. We note that Corollaries 104 and 106 can alternatively be derived by combining Theorem 3.2 in [DX84] with Theorem 0.1 in [CW03]. Finally, we point out that Corollary 104 can also be derived by combining Lück approximation result with the Donnelly-Xavier vanishing.

When $M = \mathbb{H}^n_\mathbb{R}/\Gamma$ is a compact hyperbolic manifold of dimension $n = 2k + 1$, we can also extend the results of Corollary 103 to estimate $b_k(M)$. In this case, in order to apply the usual Price inequality approach we need to understand the magnitude of μ.

Corollary 108. *Let M be a compact hyperbolic manifold of dimension $2k + 1$ with $\gamma_g(M) > 1$. There exists $C_k > 0$ such that*

(109) $$b_k(M) \leq C_k \frac{Vol(M)}{\gamma_g(M) - 1}.$$

Proof. The proof is the same as Corollary 103, except that we now have

$$q(r) = \mu(r) \coth(r),$$

so that we need to estimate the magnitude of μ from below. Let h be a harmonic k form, $k \neq 0, n$, not vanishing at $p \in M$. Recall that we have $\mu(0) = \frac{k}{2k+1}$, see Lemma 18. In hyperbolic geometry, it is possible to show there is a k dependent positive lower bound $\tilde{\mu}$ for μ on $[0, 1]$. This lower bound can be explicitly estimated using separation of variables. Relying less on the special geometry, we may also estimate μ as follows. Let $L \in (0, 1]$. Identify $B_R \setminus \{0\}$ with $(0, R) \times S^{n-1}$ in the usual manner, and write $h = h_0 + dr \wedge h_1$, with $i_{\partial_r} h_j = 0$. Use the product structure to identify the h_j

with a one parameter family of forms on S^{n-1}. Set

$$(110) \qquad b_L(r) = \int_L^r h_1(s)ds.$$

Then since $dh = 0$ we have

$$db_L(r) = h(r) - h_0(L).$$

The closed form $h_0(L)$ is exact on S^{n-1}. Hence there exists a $(k-1)$-form β_0 on S^{n-1} such that

$$d\beta_0 = h_0(L).$$

Moreover,

$$\int_{S_L} |\beta_0|^2 d\sigma \leq c_{n,k}^2 \sinh^2(L) \int_{S_L} |h_0|^2 d\sigma,$$

for some $c_{n,k} > 0$. On the other hand, we have

$$\int_{B_L(p)} |h|^2 dv = \int_{B_L(p)} \langle d(b_L + \beta_0), h \rangle dv$$

$$= \int_{S_L(p)} \langle \beta_0, e^*(dr)h \rangle d\sigma \leq \sqrt{\mu(L)} c_{n,k} \sinh(L) \|h\|_{S_L}^2$$

$$(111) \qquad = \frac{2\sqrt{\mu(L)}}{1 - 2\mu(L)} c_{n,k} \sinh(L) \int_{B_L(p)} q(r)|h|^2 dv,$$

where we have used (13) for the last equality. By Lemma 51

$$\sinh(L) \int_{B_{L/2}(p)} q(r)|h|^2 dv \leq d(n) \left(1 + \frac{2}{L}\right)^n$$

$$\times \sinh(L) \int_{B_L(p)} |h|^2 dv \int_{B_{L/2}(p)} q(r) dv$$

$$(112) \qquad \leq d(n, L) \int_{B_L(p)} |h|^2 dv.$$

where

$$d(n, L) := d(n) \left(1 + \frac{2}{L}\right)^n \sinh(L) Vol(S^{n-1}) \sinh(L/2)^{n-1}.$$

On the other hand

$$(113) \quad \sinh(L) \int_{B_L \setminus B_{L/2}} q(r)|h|^2 dv \leq (n-1)\sinh(L)\coth(L/2) \int_{B_L} |h|^2 dv.$$

By combining Equations (112) and (113)

$$\sinh(L) \int_{B_L(p)} q(r)|h|^2 dv \leq C_1(n,L) \int_{B_L(p)} |h|^2 dv,$$

where $C_1(n,L)$ is a constant depending only on n and L. Inserting this estimate back into (111) gives

$$(114) \qquad\qquad 1 \leq \frac{2\sqrt{\mu(L)}}{1 - 2\mu(L)} c_{n,k} C_1(n,L).$$

Hence $\mu(L)$ is bounded below for L bounded. Given this lower bound, the proof proceeds exactly as in Corollary 103 and Theorem 96. $\square$

Remark 115. One of the referees brought to our attention the fact that a slightly weaker version of Corollary 108 could be derived from the computation of the Novikov-Shubin invariant for real hyperbolic manifolds given in [Olb14].

We summarize our Betti number estimates for real hyperbolic manifolds with the following corollary.

Corollary 116. *Let $X^n = \mathbb{H}_{\mathbb{R}}^n/\Gamma$ be a closed real hyperbolic manifold with $sec_g = -1$ and injectivity radius $\gamma_g(X) > 1$. Given a cofinal filtration $\{\Gamma_i\}$ of Γ, let us denote by $\pi_i : X_i \to X$ the regular Riemannian cover of X associated to Γ_i. If $n = 2m$, for any integer $1 \leq k < m$, there exists a positive constant $c_1(n,k)$ such that*

$$\frac{b_k(X_i)}{Vol_{g_i}(X_i)} \leq c_1(n,k) e^{-(1+2(m-k))\gamma_{g_i}(X_i)}.$$

On the other hand, if $n = 2m+1$, for any integer $k < m$ there exists a positive constant $c_2(n,k)$ such that

$$\frac{b_k(X_i)}{Vol_{g_i}(X_i)} \leq c_2(n,k) e^{-(2(m-k))\gamma_{g_i}(X_i)}.$$

In both cases, the sub volume growth of the Betti numbers along the tower of coverings is exponential in the injectivity radius. Finally, for $n = 2k+1$ we have the existence of a positive constant $c_3(n,k)$ such that

$$\frac{b_k(M_i)}{Vol_{g_i}(M_i)} \leq \frac{c_3(n,k)}{\gamma_{g_i}(M_i) - 1}.$$

Proof. For $n = 2m$, notice that $\epsilon_{n,k} = \epsilon_{2m,m-1} = 1$. For $n = 2m + 1$, we have $\epsilon_{n,k} = \epsilon_{2m+1,m} = 0$. The statement in this case then follows from Corollary 95. For the critical case of k-forms in dimension $n = 2k + 1$, we use Corollary 108. $\qquad\square$

This result can also be obtained by trace formula techniques, but we consider our proof to be significantly simpler. For more details see again [Xue91], [SX91] and [Mar14].

8. L^2-cohomology and L^2-Betti numbers

Let $H_{2,N}^k(\Omega)$ denote the (absolute) L^2- cohomology of Ω. Let Δ_k denote the Laplace Beltrami operator on $k-$forms. If 0 is not in the essential spectrum of Δ_k, then

$$(117) \qquad \mathcal{H}_{2,N}^k(\Omega) \simeq H_2^k(\Omega).$$

Lemma 118. *Let (M^n, g) be a simply connected non-compact complete Riemannian manifold without conjugate points and $-1 \leq \sec_g \leq 1$. If there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g,$$

then zero is not in the essential spectrum of Δ_k.

Proof. It is well-known that zero is not in the essential spectrum of Δ_k if and only if there is a compact set $K \subset M$ and a constant $\gamma > 0$ such that

$$||d\alpha||_{L^2}^2 + ||d^*\alpha||_{L^2}^2 \geq \gamma ||\alpha||_{L^2}^2,$$

for any smooth k-form α compactly supported in $M \smallsetminus K$. (See for example [Ang93].) Fix $p \in M$. Choose ρ large enough so that $\left(\frac{\sqrt{\delta}}{2} - k \coth(\rho)\right) =: \epsilon$ is strictly positive. Given $\alpha \in C_c^\infty(\Lambda^k T^*(M \smallsetminus \overline{B_\rho(p)}))$, choose R sufficiently large so that the support of α is contained in $B_R(p)$. Now, in the absence of the harmonicity assumption and with the addition of the support assumption, (13) becomes

$$\int_{B_R} (i_{-\partial_r}\alpha, d^*\alpha)dv + \int_{B_R} (i_{-\partial_r}d\alpha, \alpha)dv \geq \int_{B_R} q(r)|\alpha|^2 dv.$$

Since $|\partial_r| = 1$,

$$\int_{B_R} (i_{-\partial_r}\alpha, d^*\alpha)dv + \int_{B_R} (i_{-\partial_r}d\alpha, \alpha)dv \leq \sqrt{2}\Big(||d\alpha||_{L^2}^2 + ||d^*\alpha||_{L^2}^2\Big)^{1/2}||\alpha||_{L^2}.$$

On the other hand, since the support of α does not intersect the closure of the ball $B_{r_\epsilon}(p)$ we have that

$$\int_{B_R(p)} q(r)|\alpha|^2 dv \geq \epsilon \int_{B_R(p)} |\alpha|^2 dv.$$

Setting $\gamma = \epsilon^2/2$ gives the desired lower bound on the spectrum. □

Corollary 119. *Let (M^n, g) be a simply connected non-compact complete Riemannian manifold without conjugate points and $-1 \leq \sec_g \leq 1$. If there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g,$$

then $\mathcal{H}_2^k(M) = H_2^k(M) = 0$.

Proof. By Lemma 118, zero is not in the essential spectrum of Δ_k. Thus, $\mathcal{H}_2^k(M) = H_2^k(M)$. The vanishing follows by applying Lemma 61 and standard long exact sequences. □

We now collect some consequences of Corollary 119 regarding the vanishing of L^2-Betti numbers of certain classes of manifolds without conjugate points and with negative Ricci curvature. The L^2-Betti numbers are non-negative *real* valued numerical invariants associated to a closed Riemannian manifolds. They were originally introduced by Atiyah in [Ati76] in connection with L^2-index theorems. Let us briefly recall their definition.

Definition 120. [Ati76, p. 44] Let (M^n, g) be a closed aspherical manifold. Let $\pi : (\tilde{M}, \pi^*(g)) \to (M, g)$ be the Riemannian universal cover. Thus, $M = \tilde{M}/\Gamma$ where Γ is a torsion free infinite group of isometries. The L^2-Betti numbers of M are the von Neumann dimension of the Γ-module $\mathcal{H}_2^k(\tilde{M})$:

$$b_k^{(2)}(M) := \dim_\Gamma(\mathcal{H}_2^k(\tilde{M})).$$

We do not enter here into a detailed discussion of the theory of von Neumann dimension of Hilbert spaces with group actions. For our purposes, it suffices to recall (see Atiyah [Ati76], see also [Lüc02, Chapter I]) that

(121) $$\dim_\Gamma \mathcal{H}_2^k(\tilde{M}) = 0 \quad \Leftrightarrow \quad \mathcal{H}_2^k(\tilde{M}) = 0.$$

We apply our Price inequality to prove a vanishing result for L^2-Betti numbers of manifolds without conjugate points and negative Ricci curvature.

Theorem 122. *Let (M^n, g) be a closed Riemannian manifold without conjugate points and $-1 \leq \sec_g \leq 1$. If there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g,$$

then $b_k^{(2)}(M) = 0$.

Proof. The claimed vanishing follows from Corollary 119 combined with (121). □

Theorem 122 provides new evidence for the Singer Conjecture. Let us recall its statement.

Conjecture 123 (Singer Conjecture). *If M^n is a closed aspherical manifold, then*

$$b_k^{(2)}(M) = 0, \quad if \quad 2k \neq n.$$

This conjecture is still open, even under the assumption that M^n admits a metric with *strictly negative* sectional curvature. While Theorem 122 does not assume the sectional curvature to be negative, it covers an insufficient range of Betti numbers to settle the conjecture. For more on the Singer Conjecture we refer to [Lüc02, Chapter XI].

Remark 124. It is interesting to observe that Theorem 122 when combined with the Lück approximation theorem [Lüc94] can be used to give an alternative proof of Equation (4) in Theorem 2. More precisely, we need to apply first Theorem 122 to closed manifolds with residually finite fundamental group, and then appeal to Lück's approximation theorem. Nevertheless, this L^2-cohomology approach to Theorem 2 has the disadvantage of not estimating how fast the ratio in (4) converges to zero. On the other hand, our original Price inequality approach to Theorem 2 provides directly an effective estimate for this convergence.

We conclude with an L^2-Betti number vanishing result for ϵ-pinched negatively curved closed manifolds. This result extends to the case $\epsilon_{n,k} = 0$ a vanishing theorem for ϵ-pinched negatively curved manifolds given by Donnelly-Xavier in [DX84, Proposition 4.1].

Corollary 125. *Let (M^n, g) be a complete simply connected Riemannian manifold of dimension $n_{\mathbb{R}} \geq 4$. Assume the sectional curvature is ϵ-pinched*

$$-(1 + \epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon > 0$. Let k be a positive integer such that

$$\epsilon_{n,k} = (n - 1) - 2k(1 + \epsilon) = 0.$$

Then, there are no L^2-harmonic k-forms $\mathcal{H}_2^k(M) = 0$.

Proof. This vanishing result is an immediate consequence of the Price inequality given in Theorem 96 combined with the fact that $\gamma_g(M) = \infty$. $\square$

As before, this result implies a vanishing result for L^2-Betti numbers of certain negatively curved manifolds.

Proposition 126. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$. Assume the sectional curvature is ϵ-pinched*

$$-(1 + \epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon > 0$. Let k be a positive integer such that

$$\epsilon_{n,k} = (n - 1) - 2k(1 + \epsilon) = 0,$$

we have $b_k^{(2)}(M) = 0$.

This vanishing result complements the one proved by Donnelly-Xavier in [DX84] by extending it to the critical equality case. More precisely, they prove a vanishing for L^2-Betti numbers of any degree k such that $\epsilon_{n,k} > 0$. The vanishing in the hyperbolic case ($\epsilon = 0$) for the critical degree $k = \frac{n-1}{2}$ was treated earlier by Dodziuk in [Dod79]. Alternatively, if one wishes, Corollary 108 can be used to give an alternative proof of Dodziuk's vanishing in the critical degree. Once again, the vanishing in Proposition 126 supports Conjecture 123.

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