Uniqueness theorems for non-compact mean curvature flow with possibly unbounded curvatures

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In this paper, we discuss uniqueness and backward uniqueness for mean curvature flow of non-compact manifolds. We use an energy argument to prove two uniqueness theorems for mean curvature flow with possibly unbounded curvatures. These generalize the results in [5]. Using similar method, we also obtain a uniqueness result on Ricci flows. A backward uniqueness theorem is also proved for mean curvature flow with bounded curvatures.

1. Introduction

Given an immersion $F_0 : \Sigma \to M$, the mean curvature flow (MCF) starting at F_0 is a family of immersions which moves along the mean curvature vector.

The MCF is the negative gradient flow of the area functional, and has been studied extensively for the past 40 years.

When Σ is compact, the MCF starting at an immersion F_0 always exists and is unique up to a maximal time interval $[0, T)$. On the other hand, if Σ is non-compact, the general existence and uniqueness problem is still not solved.

The first existence result in the non-compact setting is by Ecker and Huisken [8], where they prove the existence of MCF starting at a hypersurface M_0 in $\mathbb{R}^{n+1}$ with uniform Lipschitz bound. If M_0 is an entire graph, they also show the long time existence when M_0 is merely locally Lipschitz.

We remark that the minimal Lipschitz cone constructed by Lawson and Osserman [17] may serve as an obstruction to the apriori estimates in [8] in higher codimension. Existence of non-compact MCF has only been obtained for entire graph with assumptions on smallness of Lipschitz norms [3], [4], [11], [24], [20].

Next we discuss the uniqueness of MCF. Koch and Lamm show uniqueness of MCF [11] for entire graph with small Lipschitz bound in any codimension. Chen and Peng prove in [7] that any viscosity solution of the graphical Lagrangian MCF with a continuous initial data is unique. For general immersions, Chen and Yin show in [5] the uniqueness of MCF among flows with uniformly bounded second fundamental forms. Together with a pseudolocality theorem, they prove uniqueness of MCF starting from an proper embedding with bounded second fundamental form and satisfying an uniform graphic condition.

The first goal of this paper is to prove the following uniqueness theorem which generalizes Chen and Yin's uniqueness result to the case of possibly unbounded curvatures.

Theorem 1.1. *Let (M, h) be a non-compact complete Riemannian manifold with positive injectivity radius lower bound i_0 such that*

$$(1.1) \qquad |\bar{R}| \leq B_0, |\bar{\nabla}\bar{R}| \leq B_1 \quad and \quad |\bar{\nabla}^2\bar{R}|^2 \leq Lr^{2-\epsilon} \text{ for } r \gg 1,$$

where $\bar{\nabla}, \bar{R}$ are respectively the Levi-Civita connection and the Riemann curvature tensor of (M, h), $r(y) = d_M(y, y_0)$ for some $y_0 \in M$ and $B_0, B_1, L, \epsilon > 0$. Let $F_0 : \Sigma \to M$ be a smooth proper immersion so that

$$(1.2) \qquad\qquad Vol_\Sigma(F_0^{-1}B_M(y_0, \bar{r})) \leq De^{D\bar{r}^2}$$

for some constant $D > 0$ and for all $\bar{r} \gg 1$. Let $F, \widetilde{F}$ be smooth solutions to the MCF starting at F_0, which satisfy the following conditions:

1) $F, \widetilde{F}$ are uniformly continuous with respect to t,

2) The induced metric $g(t), \tilde{g}(t)$ on Σ are uniformly equivalent to g_0, and

3) the second fundamental forms $A, \widetilde{A}$ satisfy

$$(1.3) \qquad |A|^2(t, x) + |\widetilde{A}|^2(t, x) \le \frac{L}{t} r^{2-\epsilon}(F_0(x))$$

for some $L > 0$.

Then $F = \widetilde{F}$.

The precise definition in condition (1) and (2) is given in Section 2.

If we compare the above theorem to Theorems 1.1 and 1.3 in [5], we assume weaker curvature bounds, while in the expense of assuming the volume growth (1.2) of the initial immersions. We do not make any graphic/curvature assumptions on the initial immersion. We remark that condition (1) and (2) are both satisfied if $|A|, |\widetilde{A}|$ are uniformly bounded. Note that in Theorem 1.1, we assume that F_0 is smooth and the MCF F and $\widetilde{F}$ are both smooth up to time $t = 0$. In the next theorem, it is shown that under a better bound on the second fundamental forms and $\bar{R}$, one can relax these assumptions and drop the volume growth condition.

Theorem 1.2. *Let (M, h) be a non-compact complete Riemannian manifold with positive injectivity radius lower bound and uniformly bound on $\bar{\nabla}^i \bar{R}$ for $i = 0, 1, 2$. Let $F_0 : \Sigma \to M$ be a C^3 proper immersion and let $F, \widetilde{F} : (0, T] \times \Sigma \to M$ be solutions to the MCF so that $F, \widetilde{F}$ converges to F_0 locally in C^3 as $t \to 0$. If the curvatures satisfy $|A| + |\widetilde{A}| \le L/t^\alpha$ for some $\alpha < 1/2$ and $L > 0$. Then $F = \widetilde{F}$.*

The proof of the above theorems, like all other uniqueness results in MCF, use the parabolicity of the MCF equation. The technical issue is that the equation is not strictly parabolic - it's invariant under diffeomorphisms. In the previous approaches [5], [3], [4], [11], [24], [20], they use the well-known De Turck trick to construct a family of diffeomorphisms so that the resulting equation (Mean curvature De Turck flow) becomes strictly parabolic (note that the use of De Turck tricks are implicit in the graphical case, see for example p.548-549 in [8]).

Our proof of Theorem 1.1 and Theorem 1.2 use directly the parabolic equation satisfied by the second fundamental form. We employ an energy argument first performed by Kotschwar in [12], where he proves a uniqueness result for non-compact Ricci flow. The energy argument was then used again

for other geometric flows [13], [15], [19], [1], [2], [18], [23]. The main idea is to consider the quantity

$$E(t) = \int_\Sigma \mathcal{Q} d\mu_t,$$

where $\mathcal{Q}$ is a quantity so that $\mathcal{Q} = 0$ implies uniqueness. The goal is to show $E(t) = 0$ given that $E(0) = 0$. For example, in the Ricci flow situation [12], $\mathcal{Q}$ contains a term of the form $t^{-\beta}|g - \tilde{g}|^2$ for some β, where both $g, \tilde{g}$ are solutions to the Ricci flow.

In our situation, we choose our $\mathcal{Q}$ to contain the zeroth order term

(1.4) $$d_M(F(t,x), \widetilde{F}(t,x)).$$

As we will see later, first and second order terms should also be present in $\mathcal{Q}$ in order to obtain a nice differential inequality for $E(t)$. We do not need higher order quantity though: the parabolic nature of MCF gives a nice parabolic equation for the second fundamental form A, and an integration by part give a strictly **negative** term containing the third order quantities ∇A, which cancels all other third order quantities. As we will see later, cut-off functions are inserted in the energy E to deal with the non-compact situation.

Let us point out one key technical difference between our works and those in [12], [13], [19], [1], [2], [18]: In their energy arguments, the flows they consider are intrinsic, as opposed to MCF which is extrinsic. Not only that both the curvatures of Σ and M play a role, but also that the geometric quantities of two a-priori different MCFs live in different vector bundles on Σ. Thus one needs to use a bundle isomorphisms P to identity these bundles before estimating the difference. In our situation, we construct P using a parallel transport along the shortest paths between two MCFs. We remark that the same construction is also carried out in [23], [21] in the context of Schrödinger flow. We expect the same argument should work for other extrinsic geometric flows.

As a by-product, we obtain the following uniqueness result for Ricci flow, which generalizes results in [6], [12].

Theorem 1.3. *Let (M, g_0) be a smooth complete noncompact Riemannian manifold. Let $g(t), \tilde{g}(t), t \in [0, T]$ be two smooth complete solutions to Ricci flow with initial metric g_0. Suppose that $g(t), \tilde{g}(t)$ are uniformly equivalent to g_0 and*

$$|Rm| + |\widetilde{Rm}| \le \frac{L}{t}$$

for some constant $L > 0$. Then $g(t) = \tilde{g}(t)$ for all $t \in [0, T]$.

In contrast with the result in [18], we do not assume any growth rate on $|Rm(g_0)|_{g_0}$ and the size of L, while in the expense of assuming the uniform equivalence of metrics. The last main result is the following backward uniqueness theorem for MCF.

Theorem 1.4. *Let (M, h) be a complete non-compact Riemannian manifold with positive lower bound on injectivity radius and uniform upper bound on $|\bar\nabla^i \bar R|$ for $i \leq 4$. Let $F, \widetilde F : [0, T] \times \Sigma \to M$ be smooth MCFs with uniformly bounded second fundamental forms. If $F(T, \cdot) = \widetilde F(T, \cdot)$, then $F = \widetilde F$ for all t.*

The proof of Theorem 1.4 uses again that the second fundamental form A and its derivatives ∇A both satisfy strictly parabolic equations. While the lower order quantities do not, one can show that they satisfy an ordinary differential inequality. These coupled inequalities are sufficient to show Theorem 1.4 by a general backward uniqueness Theorem in [16]. The reader may find more historical remarks in the introduction of [16].

One slight technical issue is that the distance (1.4) is non-differentiable when it's zero and thus we need another zeroth order quantity. We treat $\widetilde F$ as a graph of F and represent $\widetilde F$ by a section on the pullback bundle $F^{-1}TM$. The assumptions on the fourth covariant derivatives of $\bar R$ in Theorem 1.4 is used in estimating the parabolic equation for ∇A, which we do not need in the proof of Theorem 1.1.

When the ambient space M in Theorem 1.4 is Euclidean, the result is proved in [10] in the co-dimension one case and recently in [25] for arbitrary co-dimensions.

In Section 2, we fix the notations and prove some elementary results. The parallel transport P will be studied in Sections 3 and 4. The main estimates are performed in Section 5. Theorem 1.1, Theorem 1.2 and Theorem 1.3 are proved in Section 6 and Theorem 1.4 is proved in Section 7.

2. Prelminary and notations

In this section, we review some definitions and results in basic submanifold theory and MCF. Let Σ be a smooth manifold and (M, h) be a smooth Riemannian manifold. Let

$$F, \widetilde{F} : \Sigma \times [0, T] \to M$$

be two families of smooth immersions.

Next we introduce several notations. We write only the notations for F. A tilde will be added to the corresponding notations for $\widetilde{F}$. We use $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^N)$ respectively to denote the local coordinates on Σ, M. We use $i, j, \ldots$ to denote the indices of Σ, $\alpha, \beta, \ldots$, and $\alpha', \beta', \ldots$ respectively to denote the indices on $F^{-1}TM$, $\widetilde{F}^{-1}TM$. For each t, let $g = g(t) = F(t, \cdot)^* h$ and ∇ be the Levi-Civita connection with respect to g. We use the same notation ∇ to denote the induced connection on all (p, q)-tensor bundle $T^{p,q}\Sigma$.

We say that the family of immersions F is uniformly continuous with respect to t, if for all $\delta > 0$ and $s \in [0, T)$, there is $s_\delta > 0$ so that

$$(2.1) \qquad d_M(F(t, x), F(s, x)) \le \delta, \quad \forall (t, x) \in [s, s + s_\delta] \times \Sigma.$$

We say that the induced metric $g(t)$ is uniformly equivalent to $g_0 = F_0^* h$, if there is $\lambda > 1$ so that

$$(2.2) \qquad \lambda^{-1} g_0 \le g(t) \le \lambda g_0, \quad \forall (t, x) \in [0, T] \times \Sigma.$$

On the pullback bundle $N := F^{-1}TM$ we have the connection induced from h, F:

$$(2.3) \qquad \nabla_i^F Y^\alpha = \partial_i Y^\alpha + \Gamma_{\beta\gamma}^\alpha F_i^\beta Y^\gamma, \quad Y \in \Gamma(M, N).$$

Here $\Gamma_{\beta\gamma}^\alpha$ denote the Christoffel symbols of the Levi-Civita connection $\bar{\nabla}$ on (M, h). Note that $\Gamma_{\beta\gamma}^\alpha$ is indeed $\Gamma_{\beta\gamma}^\alpha \circ F$, but we suppress F for simplicity. We also remark that

$$\nabla_i Y = \bar{\nabla}_{F_i} \widetilde{Y},$$

where $F_i = \partial_i F$ and $\widetilde{Y}$ is any extension of Y in M. We use the same notation ∇^F to denote the connection induced by ∇ and ∇^F on any N-valued tensor

bundle. Thus there could be six notations in total:

$$\nabla, \widetilde{\nabla}, \nabla^F, \nabla^{\widetilde{F}}, \widetilde{\nabla}^F, \widetilde{\nabla}^{\widetilde{F}}.$$

However, for simplicity we use only ∇ and $\widetilde{\nabla}$. It will be clear from the context which connection we are using.

Next we consider covariant derivatives with respect to time. Define the covariant time derivative D_t on $\Gamma(\Sigma, T^{0,p}\Sigma \otimes N)$ by

$$D_t Y^\alpha_{i_1 \cdots i_p} = \partial_t Y^\alpha_{i_1 \cdots i_p} + \Gamma^\alpha_{\beta\gamma} Y^\beta_{i_1 \cdots i_p} F^\gamma_t.$$

Note that when acts on vector fields along F, D_t is metric with respect to h. That is,

$$\partial_t h(Y, Z) = h(D_t Y, Z) + h(Y, D_t Z), \quad Y, Z \in \Gamma(\Sigma, N).$$

Next we introduce several standard geometric quantities from an immersion. For each fixed t, the differential of $F(t, \cdot) : \Sigma \to M$ is denoted F_*, thus

$$(F_*)^\alpha_i = F^\alpha_i = \frac{\partial F^\alpha}{\partial x^i} \in \Gamma(\Sigma, T^*\Sigma \otimes N).$$

The second fundamental form A is the covariant derivative of F_*:

$$\begin{aligned} A^\alpha_{ij} &= (\nabla F_*)^\alpha_{ij} \\ &= F^\alpha_{ij} - \Gamma^k_{ij} F^\alpha_k + \Gamma^\alpha_{\beta\delta} F^\beta_i F^\delta_j \in \Gamma(\Sigma, T^{0,2}\Sigma \otimes N) \end{aligned}$$

The mean curvature vector H is the trace of A given by

$$H^\alpha = g^{ij} A^\alpha_{ij} \in \Gamma(\Sigma, N).$$

We say that a family of immersions $F : [0, T] \times \Sigma \to M$ is a MCF starting from F_0 if

$$(2.4) \qquad \frac{\partial F}{\partial t} = H, \quad F(0, \cdot) = F_0(\cdot).$$

Remark 1. In this paper we use the following convention: We use B_k (resp. B_k^{loc}) to denote the (resp. local) bound on $|\bar{\nabla}^k \bar{R}|$. Unless otherwise specified, we use C to denote constants that depend only on the dimensions of Σ and M, constants λ in (2.2), L in the statement of Theorem 1.1 and the lower bound on injectivity radius i_0. Constants that depend also on $B_0, B_1, \ldots, B_k$ (resp. $B_0^{loc}, B_1^{loc}, \ldots, B_k^{loc}$) are denoted C_k (resp. C_k^{loc}). The explicit values

of the constants C, C_k, C_k^{loc} are not important and might change from line to line unless otherwise specified.

The following simple lemma is used a lot in this paper.

Lemma 2.1. *We have $|F_*|$, $|\widetilde{F}_*| \leq C$.*

Proof. Note that $|F_*|^2 = h_{\alpha\beta}g^{ij}F_i^{\alpha}F_j^{\beta} = g^{ij}g_{ij} = n$. Thus $|\widetilde{F}_*|^2 \leq C$ by (2.2). $\square$

Next we recall the differential equations of the following quantities along the MCF, the proof can be found in [22].

Lemma 2.2. *Under the MCF, we have*

$$(2.5) \qquad \partial_t g_{ij} = -2g^{pq}h(A_{pq}, A_{ij}),$$

$$(2.6) \qquad \partial_t g^{ij} = 2g^{pq}g^{ik}g^{jl}h(A_{pq}, A_{kl}),$$

$$(2.7) \qquad \partial_t d\mu = -|H|^2 d\mu,$$

$$(2.8) \qquad \partial_t \Gamma_{ij}^k = -g^{kl}\left(\nabla_i h(H, A_{jl}) + \nabla_j h(H, A_{il}) - \nabla_l h(H, A_{ij})\right).$$

We also need the equation for the higher covariant derivatives of A. Recall that in the notation of Chen and Yin [5], we have $F_* = \nabla F, A = \nabla^2 F, \nabla A = \nabla^3 F$ and so on. Proposition 2.3 in [5] together with Gauss equation give

$$(2.9) \quad (D_t - \Delta)\nabla^k F = \sum_{l=0}^{k-1} \nabla^l [h(A, A) * g^{-2} + \bar{R} * (\nabla F)^a * g^{-b}] * \nabla^{k-l} F,$$

where $a = 2$ or 4, $b = 1$ or 2, $*$ are any contraction of tensors and $\bar{R}$ include any contraction of the Riemann curvature tensor on M with h^{-1}.

In [5], the authors derive an apriori estimates for $|\nabla^k F|^2$ assuming that the second fundamental form $\nabla^2 F$ is uniformly bounded. When the second fundamental form $|A|^2(t, x)$ is bounded by $Lr^{2-\epsilon}(F_0(x))/t$, one can modify the proof of Theorem 3.2 in [5] to obtain the following lemma.

Lemma 2.3. *Let $L, \epsilon > 0$ and let (M, h) be a complete Riemannian manifold with*

$$|\bar{R}| \leq B_0, |\bar{\nabla}\bar{R}| \leq B_1, \quad |\bar{\nabla}^2\bar{R}|^2 \leq Lr^{2-\epsilon},$$

where $r(y) = d_M(y, y_0)$. Let F be a MCF so that $|A|^2(t, x) \leq Lr^{2-\epsilon}(F_0(x))/t$. Then

$$(2.10) \qquad |\nabla A(t, x)| \leq \frac{C_1 Lr^{2-\epsilon}(F_0(x))}{t}.$$

Sketch of proof. From Proposition 3.1 in [5], we have

$$(\partial_t - \Delta)|\nabla^2 F|^2 \leq -|\nabla^3 F|^2 + C_1|\nabla^2 F|^4 + C_1,$$
$$(\partial_t - \Delta)|\nabla^3 F|^2 \leq -|\nabla^4 F|^2 + C(C_1 + |\nabla^2 F|^2 + |\bar{\nabla}^2\bar{R}|^2)|\nabla^3 F|^2$$
$$+ C|\nabla^3 F|^3 + C_0|\nabla^2 F|^2 + C_0,$$

Write $G^2 = Lr^{2-\epsilon}$. Using the conditions on $|\bar{\nabla}^2\bar{R}|$ and $|A|$, we have

$$(\partial_t - \Delta)|\nabla^2 F|^2 \leq -|\nabla^3 F|^2 + C_1 G^4 t^{-2},$$
$$(\partial_t - \Delta)|\nabla^3 F|^2 \leq -|\nabla^4 F|^2 + C_1|\nabla^3 F|^3 + C_1 G^6 t^{-3}.$$

Thus one can proceed as in the proof of Theorem 3.2 in [5] to conclude.   □

3. Writing $\widetilde{F}$ as a graph of F: Basic estimates

In this section, we represent $\widetilde{F}$ as a graph of F and provide some basic estimates. Let

$$F, \widetilde{F} : [0, T] \times \Sigma \to M$$

be two families of immersions so that $F(0, \cdot) = \widetilde{F}(0, \cdot)$. Let $d : [0, T] \times \Sigma \to \mathbb{R}$ be the pointwise distance between F and $\widetilde{F}$. That is,

$$d = d(t, x) := d_M(F(t, x), \widetilde{F}(t, x)).$$

For each (t, x), write $p = F(t, x)$ and $\tilde{p} = \widetilde{F}(t, x)$. We assume that

$$(3.1) \qquad d < \min\left\{i_0, 1, \frac{1}{\sqrt{2B_0^{loc}}}\right\}, \quad \forall (t, x) \in [0, T] \times \Sigma.$$

Since $d < i_0$, for all (t, x) there is a unique shortest geodesic joining p and $\tilde{p}$. Write $\exp_p(sv)$, where $s \in [0, 1]$, to denote this geodesic. Then $\tilde{p} = \exp_p v$,

$|v| = d$ and v is a smooth section on N. The collection of these geodesics forms a smooth homotopy

$$\gamma : [0, T] \times \Sigma \times [0, 1] \to M, \quad \gamma(t, x, s) = \exp_p(sv)$$

connecting F and $\widetilde{F}$. We use $\dot{\gamma}, J_t$ and J_i to denote the derivative of γ with respect to s, t, and the coordinate x^i respectively (the notations are so chosen since J_t, J_i are Jacobi fields). Note also that v, F_t and F_i are the restrictions of $\dot{\gamma}, J_t, J_i$ to $s = 0$ respectively.

Let $P : T_{\tilde{p}}M \to T_pM$ be the parallel transport along the geodesic $-\gamma$. The inverse P^{-1} is the parallel transport along γ.

On the endomorphism bundle $\mathrm{End}(\widetilde{N}, N) = \widetilde{N}^* \otimes N$ over Σ, there is a connection induced from ∇^F and $\nabla^{\widetilde{F}}$. Together with the two connections ∇ and $\widetilde{\nabla}$ defined on $T^{p,q}\Sigma$ by g and $\tilde{g}$ respectively, there are two connections defined on any endomorphism valued (p, q)-tensor bundle $T^{p,q}\Sigma \otimes \mathrm{End}(\widetilde{N}, N)$, which again we denote by ∇ and $\widetilde{\nabla}$. Note that the connections satisfy the Leibniz rule:

$$\nabla(PZ) = \nabla P \cdot Z + P(\nabla Z), \quad Z \in \Gamma(\Sigma, T^{p,q}\Sigma \otimes \mathrm{End}(\widetilde{N}, N)).$$

In the following, we will derive estimates for v and d. Since the calculations might be useful for other geometric situations, we do not assume that $F, \widetilde{F}$ satisfy the MCF equation except for Theorem 3.1 in this section. We remark that all of the estimates follow from the Jacobi field equation (and its higher order derivatives).

First we prove a useful lemma.

Lemma 3.1. *Let $p = p(\tau), \tilde{p} = \tilde{p}(\tau)$ be two curves in M so that*

$$0 < d = d_M(p, \tilde{p}) < \min\{i_0, 1/\sqrt{2B_0^{loc}}\}$$

for all τ. Let $\gamma_\tau(s) = \exp_{p(\tau)}(sv_\tau)$, where $s \in [0, 1]$, be the unique geodesic joining $p(\tau)$ to $\tilde{p}(\tau)$. Let J be the Jacobi field given by the variation of geodesics γ_τ. Then

$$(3.2) \qquad |J| \leq |\partial_\tau p| + C_0^{loc}|\partial_\tau p|d^2 + 2|P\partial_\tau \tilde{p} - \partial_\tau p|,$$
$$(3.3) \qquad |\bar{\nabla}_{\dot{\gamma}} J| \leq C_0^{loc}|\partial_\tau p|d^2 + 2|P\partial_\tau \tilde{p} - \partial_\tau p|.$$

Proof. The first inequality (3.2) follows from the second one using $|J| \leq |\partial_\tau p| + \sup |\nabla_{\dot\gamma} J|$. To show (3.3), note that J satisfies the Jacobi field equation

$$(3.4) \qquad \bar\nabla_{\dot\gamma} \bar\nabla_{\dot\gamma} J + \bar R(\dot\gamma, J)\dot\gamma = 0.$$

Let $\{e_1, \ldots, e_n\}$ be a parallel orthonormal fields along γ. Write $J(s) = \sum J_i(s) e_i(s)$, then $\bar\nabla_{\dot\gamma} J = \sum J_i'(s) e_i(s)$. Also

$$(3.5) \qquad \partial_\tau p = \sum J_i(0) e_i(0) \quad \text{and} \quad P^{-1}\partial_\tau p = \sum J_i(0) e_i(1).$$

By the mean value theorem, there is $\zeta \in [0, 1]$ so that

$$|P\partial_\tau \tilde p - \partial_\tau p| = |\partial_\tau \tilde p - P^{-1}\partial_\tau p| = |\bar\nabla_{\dot\gamma} J(\zeta)|.$$

Thus by mean value theorem again, for any $s \in [0, 1]$, there is ζ_s between ζ and s so that

$$(3.6) \qquad \begin{aligned} |\bar\nabla_{\dot\gamma} J(s)| &\leq \left| \sum (J_i'(s) - J_i'(\zeta)) e_i(s) \right| + |\bar\nabla_{\dot\gamma} J(\zeta)| \\ &\leq \left| \sum J_i''(\zeta_s) e_i(s) \right| + |P\partial_\tau \tilde p - \partial_\tau p| \\ &= \left| \sum J_i''(\zeta_s) e_i(\zeta_s) \right| + |P\partial_\tau \tilde p - \partial_\tau p| \\ &= |\bar\nabla_{\dot\gamma} \bar\nabla_{\dot\gamma} J(\zeta_s)| + |P\partial_\tau \tilde p - \partial_\tau p| \\ &\leq B_0^{loc} d^2 |J(\zeta_s)| + |P\partial_\tau \tilde p - \partial_\tau p| \\ &\leq B_0^{loc} d^2 (|\partial_\tau p| + \sup |\bar\nabla_{\dot\gamma} J|) + |P\partial_\tau \tilde p - \partial_\tau p|, \end{aligned}$$

where we have used (3.4) and $|\bar R| \leq B_0^{loc}$. Since $B_0^{loc} d^2 < 1/2$ by assumption, (3.3) is shown with $C_0^{loc} = 2B_0^{loc}$. □

Remark 2. Note that in Lemma 3.1 we assume that $d > 0$. Indeed the Jacobi field is not even defined at points where $d = 0$. Thus in the MCF setting, the lemma cannot be applied directly when e.g. $t = 0$. To deal with this, we first consider the case $d > 0$ and conclude by letting $d \to 0$ (See the proof of Proposition 3.1 for an example).

The first application of Lemma 3.1 is the following proposition.

Proposition 3.1. *Let* $F, \widetilde{F}$ *be two families of immersions so that (3.1) holds. Then we have*

(3.7) $$|D_t v| \leq C_0^{loc} |F_t| d^2 + 2|P\widetilde{F}_t - F_t|,$$

(3.8) $$|\bar{\nabla} v| \leq C_0^{loc} d^2 + 2|P\widetilde{F}_* - F_*|.$$

Proof. Note that when $d > 0$, we have

$$D_t v = \bar{\nabla}_{F_t} \dot{\gamma}\big|_{s=0} = \bar{\nabla}_{\dot{\gamma}} J_t\big|_{s=0},$$
$$\bar{\nabla}_i v = \bar{\nabla}_{F_i} \dot{\gamma}\big|_{s=0} = \bar{\nabla}_{\dot{\gamma}} J_i\big|_{s=0}$$

Thus (3.7), (3.8) follows from Lemma 3.1 when $d > 0$. Assume $d = 0$ at some (t, x). Let (t_i, x_i) be a sequence so that $(t_i, x_i) \to (t, x)$ and $d(t_i, x_i) > 0$. Since $D_t v$, ∇v are smooth, the two inequalities can be shown by taking $i \to \infty$. If such a sequence does not exists, then v is identically zero in a space time neighbourhood and so $D_t v = \nabla v = 0$. $\qquad\square$

Since $|v| = d$, Proposition 3.1 gives an estimate for $\partial_t d^2$. We also use the following lemma which can also be proved easily (See Lemma 2.2 in [23]).

Lemma 3.2. *We have*

(3.9) $$\partial_t d^2 = 2h(P\widetilde{F}_t - F_t, v),$$

(3.10) $$|\nabla d| \leq |P\widetilde{F}_* - F_*|.$$

Next we need the following generalization of Lemma 3.1. First we need a definition.

Definition 3.1. We say that a polynomial Q of k-variables is universal if it has non-negative coefficients, $Q(0) = 0$ and the coefficients depend only on the dimensions of Σ and M.

Proposition 3.2. *Assume the same notations as in Lemma 3.1. Then for any* $k = 0, 1, 2, \ldots,$

(3.11) $$|\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^k J| \leq C_k^{loc} d^2 P_k^1\big(|\partial_\tau p|, |D_\tau \partial_\tau p|, \ldots, |D_\tau^k \partial_\tau p|\big)$$
$$+ Q_k^1\big(|P\partial_\tau \tilde{p} - \partial_\tau p|, |PD_\tau \partial_\tau \tilde{p} - D_\tau \partial_\tau p|, \ldots, |PD_\tau^k \partial_\tau \tilde{p} - D_\tau^k \partial_\tau p|\big),$$

where P_k^1, Q_k^1 *are universal polynomials in* $(k+1)$*-variables.*

Proof. We argue by induction. The case $k = 1$ is shown using (3.3) with

$$P_1^1(x_0) = \bar{x}_0, \quad Q_1^1(x_0) = 2x_0.$$

Assume that (3.11) holds for all integers strictly smaller than k . First we see that

$$
\begin{aligned}
(3.12) \qquad \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_{\dot{\gamma}} \nabla_J^k J &= -\bar{R}(\dot{\gamma}, \bar{\nabla}_J^k J)\dot{\gamma} \\
&+ \sum (\bar{\nabla}^{i_r} \bar{R})^{i'_r} * (\dot{\gamma})^j * (\bar{\nabla}_J^{k_p} J)^{k'_p} * (\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^{l_q} J)^{l'_q},
\end{aligned}
$$

with $i_r \leq k$ for all r and

$$j + \sum_q l'_q = 2, \quad \sum_p (k_p + 1)k'_p + \sum_q (l_q + 1)l'_q = k.$$

That is, each term has exactly two $\dot{\gamma}$'s and k- J's. When $k = 0$, (3.12) reduces to the Jacobi field equation (3.4) and the conditions on indices are satisfied trivially. In general, (3.12) can be proved by induction, using again the following consequences of commuting covariant derivatives (note that $[J, \dot{\gamma}] = 0$):

$$
\begin{aligned}
\bar{\nabla}_J \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_{\dot{\gamma}} \nabla_J^k J &= \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^{k+1} J + \bar{\nabla}_{\dot{\gamma}}(\bar{R}(\dot{\gamma}, J)\bar{\nabla}_J^k J) + \bar{R}(\dot{\gamma}, J)\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^k J, \\
\bar{\nabla}_J \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^{l_q} J &= \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^{l_q+1} J + \bar{R}(\dot{\gamma}, J)\bar{\nabla}_J^{l_q} J.
\end{aligned}
$$

Using (3.12), $|\dot{\gamma}| = d$ and $|\bar{\nabla}_J^{k_p} J| \leq |D_\tau^{k_p} \partial_\tau p| + \sup |\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^{k_p} J|$, we have

$$
\begin{aligned}
|\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^k J| &\leq C_k^{loc} \sum d^j (|D_\tau^{k_p} \partial_\tau p| + \sup |\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^{k_p} J|)^{k'_p} |\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^{l_q} J|^{l'_q} \\
&+ B_0 d^2 |\bar{\nabla}_J^k J|,
\end{aligned}
$$

Using the induction hypothesis (note $k_q, l_q \leq k - 1$) and Cauchy Schwarz inequality, one easily obtains

$$
\begin{aligned}
(3.13) \quad |\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^k J| &\leq C_k^{loc} d^2 \widehat{P}_{k-1}(|\partial_\tau p|, \ldots, |D_\tau^{k-1} \partial_\tau p|) \\
&+ \widehat{Q}_{k-1}(|P\partial_\tau \tilde{p} - \partial_\tau p|, \ldots, |PD_\tau^{k-1} \partial_\tau \tilde{p} - D_\tau^{k-1} \partial_\tau p|) \\
&+ B_0 d^2 |\bar{\nabla}_J^k J|,
\end{aligned}
$$

where $\widehat{P}_{k-1}, \widehat{Q}_{k-1}$ are universal polynomials in k-variables. Now, as in the proof of Lemma 3.1, we use mean value theorem twice to obtain

$$|\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^k J| \leq \sup |\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^k J| + |PD_\tau^k \partial_\tau \tilde{p} - D_\tau^k \partial_\tau p|$$

Finally we use (3.13), $|\bar{\nabla}_J^k J| \leq |D_\tau^k \partial_\tau p| + \sup |\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^k J|$ and $B_0^{loc} d^2 < 1/2$ to finish the induction step, with

$$P_k^1(x_0, \dots, x_k) = \widehat{P}_{k-1}(x_0, \dots, x_{k-1}) + x_k,$$
$$Q_k^1(x_0, \dots, x_k) = \widehat{Q}_{k-1}(x_0, \dots, x_{k-1}) + 2x_k.$$

This finishes the proof of the proposition. □

Proposition 3.3. *Assume the same notations as in Lemma 3.1. Then*

$$\begin{align}
(3.14) \qquad |\bar{\nabla}_J^k \dot{\gamma}| &\leq C_{k-1}^{loc} d P_{k-1}^2(|\partial_\tau p|, |D_\tau \partial_\tau p|, \dots, |D_\tau^{k-1} \partial_\tau p|) \\
&\quad + Q_{k-1}^2(|P\partial_\tau \tilde{p} - \partial_\tau p|, |PD_\tau \partial_\tau \tilde{p} - D_\tau \partial_\tau p|, \dots, \\
&\qquad |PD_\tau^{k-1} \partial_\tau \tilde{p} - D_\tau^{k-1} \partial_\tau p|),
\end{align}$$

where P_{k-1}^2, Q_{k-1}^2 *are universal polynomials of k-variables.*

Proof. By commuting covariant derivatives, we have

$$\begin{align}
\bar{\nabla}_J^k \dot{\gamma} &= \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^{k-1} J + \sum_{i=0}^{k-2} \bar{\nabla}_J^i (\bar{R}(\dot{\gamma}, J) \bar{\nabla}_J^{k-2-i} J) \\
&= \bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^{k-1} J + \sum (\bar{\nabla}^{i_r} \bar{R})^{i'_r} * (\dot{\gamma})^j * (\bar{\nabla}_J^{k_p} J)^{k'_p} * (\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^l J)^{l'},
\end{align}$$

where $l, k_p \leq k - 3$, $i_r \leq k - 2$ for all r,

$$j + l' = 1 \quad \text{and} \quad \sum_p (k_p + 1)k'_p + (l+1)l' = k.$$

By Proposition 3.2 and $|\bar{\nabla}_J^{k_p} J| \leq |D_\tau^{k_p} \partial_\tau p| + \sup |\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_J^{k_p} J|$, one obtains (3.14). □

An immediate consequence is the following theorem, which says that if F and $\widetilde{F}$ are both MCF starting at F_0, then they agree infinitesimally.

Theorem 3.1. *Let* $F, \widetilde{F}$ *be smooth MCFs starting at* F_0. *Then* $D_t^k v|_{t=0} = 0$ *for all* $k = 0, 1, 2, \dots$.

Proof. We use an approximation argument as in the proof of Proposition 3.1. Let $x \in \Sigma$. If v is identically zero in a space time neighbourhood of $(0, x)$, then $D_t^k v(x) = 0$ is clear. If not, then there are $(\tau_i, x_i) \to (0, x)$ so that

$v(\tau_i, x_i) \neq 0$. For each (τ_i, x_i), In general, whenever $v \neq 0$ at some (t, x), write $p(\tau) = F(\tau, x)$ and $\tilde{p}(\tau) = \tilde{F}(\tau, x)$, then

$$D_t^k v = \bar{\nabla}_J^k \dot{\gamma}|_{s=0},$$

where J is the variational vector field of the family of geodesics joining p to $\tilde{p}$. From Proposition 3.3, we have

$$|D_t^k v| \leq C_{k-1}^{loc} d^2 P_{k-1}^2(|H|, |D_t H|, \ldots, |D_t^k H|)$$
$$+ Q_{k-1}^2(|P\tilde{H} - H|, |PD_t\tilde{H} - D_t H|, \ldots, |PD_t^k\tilde{H} - D_t^k H|).$$

Thus it suffices to show that

$$(3.15) \qquad \lim_{i \to \infty} (PD_t^m \tilde{H} - D_t^m H)(\tau_i, x_i) \to 0,$$

for $m = 0, 1, 2, \ldots$. By using (2.6), $H = g^{ij}\nabla_{ij}^2 F$ and inductively applying (2.9), we see that for any $m \in \mathbb{N}$,

$$(3.16) \qquad D_t^m H = Q(g, h, F, m)$$

where Q is a quantity involving only the spatial derivatives of F, g, h and the derivatives of $\overline{Rm}_h$ evaluating at (x, t) and $F(x, t)$. Since F and $\tilde{F}$ are smooth up to $t = 0$, for $m = 0$, we have

$$\lim_{i \to \infty} D_t H(\tau_i, x_i) = \lim_{i \to \infty} D_t\tilde{H}(\tau_i, x_i) = Q(g_0, h, F_0, 0)$$

where the right hand side is the value of Q evaluating at x and $F_0(x)$. Since P is continuous and $P = \text{Id}$ at $t = 0$, we obtain (3.15) for $m = 0$. We can argue similarly for all $m \in \mathbb{N}$. $\qquad \square$

4. Estimates for the parallel transport P

Next we estimate the norm of the parallel transport P. We follow the same notations as in the previous section. First we prove the following lemma.

Lemma 4.1. *Let $F, \tilde{F}$ be two families of immersions which satisfy (2.2). Then*

$$(4.1) \qquad |g - \tilde{g}| \leq C|P\tilde{F}_* - F_*|,$$
$$(4.2) \qquad |P\tilde{H} - H| \leq C(|P\tilde{F}_* - F_*||\tilde{A}| + |P\tilde{A} - A|).$$

Proof. Since $g = h(F_*, F_*)$, together with Lemma 2.1,

$$
\begin{aligned}
|g - \tilde{g}| &= |h(F_*, F_*) - h(\widetilde{F}_*, \widetilde{F}_*)| \\
&= |h(F_*, F_*) - h(P\widetilde{F}_*, P\widetilde{F}_*)| \\
&= |h(P\widetilde{F}_* - F_*, P\widetilde{F}_*) + h(F_*, P\widetilde{F}_* - F_*)| \\
&\leq C(|F_*| + |P\widetilde{F}_*|)|P\widetilde{F}_* - F_*|.
\end{aligned}
$$

Using the same argument and $H = g^{ij} A_{ij}$,

$$
|P\widetilde{H} - H| \leq C(|g^{-1} - \tilde{g}^{-1}||\widetilde{A}| + |P\widetilde{A} - A|).
$$

Thus (4.2) follows from (4.1). □

Proposition 4.1. *Let $F, \widetilde{F}$ be two families of immersions so that (3.1) holds. Then for any p, q, the parallel transport*

$$
P : \Gamma(\Sigma, T^{p,q}\Sigma \otimes \widetilde{N}) \to \Gamma(\Sigma, T^{p,q}\Sigma \otimes N)
$$

satisfy

(4.3) $$|P| = |P^{-1}| = 1,$$

(4.4) $$|\nabla P|, |\nabla P^{-1}| \leq C_0^{loc} d$$

and

(4.5) $$|\Delta P| \leq C_1^{loc}\big(d \cdot \tilde{P}(|A|) + |P\widetilde{F}_* - F_*| + |\Gamma - \widetilde{\Gamma}| + |P\widetilde{A} - A|\big)$$

for some polynomial $\tilde{P}$. Here ΔP is the rough Laplacian of P with respect to the connection ∇: $\Delta P = g^{ij}\nabla_i \nabla_j P$.

Proof. Since P is given by $P(X \otimes Z) = X \otimes PZ$ for all $X \in \Gamma(\Sigma, T^{p,q}\Sigma)$ and $Z \in \Gamma(\Sigma, \widetilde{N})$, one can without loss of generality assume that $p = q = 0$.

Since parallel transport preserves length, $|PZ| = |Z|$ for all $Z \in T_{\tilde{p}}M$ and thus $|P| = |P^{-1}| = 1$. To show (4.4), note again it suffices to assume $d \neq 0$. Let $x = (x^1, \ldots, x^n) \in \Sigma$, $i \in \{1, \ldots, n\}$ and $Z \in T_{\tilde{p}}M$ be fixed. Parallel transport Z along the curve in M with tangent vector $\widetilde{F}_i$. Thus $\nabla_i Z = 0$ and by the Leibniz rule,

$$
(\nabla_i P)(Z) = \nabla_i(PZ) - P(\nabla_i Z) = \nabla_i(PZ).
$$

Let X be the parallel vector field along γ with $X(1) = Z$. Then $X(0) = PZ$ by definition of P. By the definition of the connection on N, we have

$\nabla_i(PZ) = \overline{\nabla}_{F_i}X(0)$. Lastly, since $F_i = J_i(0)$, we have $\nabla_i(PZ) = (\overline{\nabla}_{J_i}X)(0)$. Differentiating the parallel transport equation gives

$$\overline{\nabla}_{\dot\gamma}\overline{\nabla}_{J_i}X = -\bar{R}(\dot\gamma, J_i)X, \quad \nabla_{J_i}X(1) = \nabla_i Z = 0.$$

By (3.2), we have

$$(4.6) \qquad |\overline{\nabla}_{\dot\gamma}\overline{\nabla}_{J_i}X| \le C_0^{loc}d(|F_i| + |P\widetilde{F}_i - F_i|)|Z|.$$

Together with Lemma 2.1 this implies

$$|(\nabla_i P)Z| = |\nabla_i(PZ)| \le C_0^{loc}d|Z|.$$

Since this holds for all Z, we obtain (4.4). To show (4.5), we calculate under the normal coordinate at x in (Σ, g_t). Thus $\Delta P = \sum_i \nabla_i \nabla_i P$. Now for each fixed i and $Z \in T_{\tilde{p}}M$, we again parallel transport Z along the curve in M with tangent vector $\widetilde{F}_i$. Thus

$$\begin{aligned}
(\nabla_i\nabla_i P)(Z) &= \nabla_i[(\nabla_i P)(Z)] - \nabla P(\nabla_{e_i}e_i, Z) - (\nabla_i P)(\nabla_i Z) \\
&= \nabla_i(\nabla_i(PZ)) - \nabla_i(P(\nabla_i Z)) - \nabla P(\nabla_{e_i}e_i, Z) - (\nabla_i P)(\nabla_i Z) \\
&= \nabla_i(\nabla_i(PZ)) \\
&= \overline{\nabla}_{J_i}\overline{\nabla}_{J_i}X(0),
\end{aligned}$$

where again X is the parallel vector field along γ with $X(1) = Z$. Since

$$\overline{\nabla}_{\dot\gamma}\overline{\nabla}_{J_i}\overline{\nabla}_{J_i}X = -\bar{R}(\dot\gamma, J_i)\overline{\nabla}_{J_i}X - \overline{\nabla}_{J_i}(\bar{R}(\dot\gamma, J_i)X),$$

using (4.6), $\overline{\nabla}_{J_i}\overline{\nabla}_{J_i}X(1) = 0$, we obtain

$$|2\bar{R}(\dot\gamma, J_i)\overline{\nabla}_{J_i}X| \le C_0^{loc}d^2|Z|.$$

On the other hand, if we trace the induction proof of Proposition 3.2 and use (4.2), we have

$$\begin{aligned}
|\overline{\nabla}_{J_i}J_i| &\le |\overline{\nabla}_{F_i}F_i| + |\overline{\nabla}_{\dot\gamma}\overline{\nabla}_{J_i}J_i| \\
&\le |A_{ii}| + C_0^{loc}d^2(P(|F_i|, |A_{ii}|) + C(|P\widetilde{F}_* - F_*| + |P\overline{\nabla}_{\widetilde{F}_i}\widetilde{F}_i - \overline{\nabla}_{F_i}F_i|) \\
&\le |A| + C_0^{loc}d^2 P(1, |A|) + C(|P\widetilde{F}_* - F_*| + |P\widetilde{A}_{ii} - A_{ii}| + |\widetilde{\Gamma}_{ii}^k\widetilde{F}_k|) \\
&\le C_0^{loc}\big(\tilde{P}(|A|) + |P\widetilde{F}_* - F_*| + |\Gamma - \widetilde{\Gamma}| + |P\widetilde{A} - A|\big)
\end{aligned}$$

for some polynomial $\tilde{P}$. This implies

$$|\bar{R}(\dot{\gamma}, \bar{\nabla}_{J_i} J_i)X| \leq C_0^{loc} d|Z|\left(\tilde{P}(|A|) + |P\widetilde{F}_* - F_*| + |\Gamma - \widetilde{\Gamma}| + |P\widetilde{A} - A|\right).$$

Thus we have

$$|(\nabla_i \nabla_i P)(Z)| \leq C_1^{loc}\left(d\tilde{P}(|A|) + |P\widetilde{F}_* - F_*| + |\Gamma - \widetilde{\Gamma}| + |P\widetilde{A} - A|\right)|Z|$$

and this gives (4.5). $\hspace{8cm}\square$

The proof of the following proposition is similar to that of Proposition 4.1 and is skipped.

Proposition 4.2. *We have the estimates*

$$(4.7) \hspace{3cm} |D_t P| \leq C_0^{loc} d(|F_t| + |P\widetilde{F}_t - F_t|).$$

Next we derive an estimates for higher time covariant derivatives of P.

Proposition 4.3. *The k-th time derivatives of P satisfies*

$$(4.8) \hspace{1cm} |D_t^k P| \leq C_{k-1}^{loc} d P_{k-1}^3(|F_t|, |D_t F_t|, \ldots, |D_{k-1} F_t|) +$$
$$+ Q_{k-1}^3(|P\widetilde{F}_t - F_t|, \ldots, |PD_t^{k-1}\widetilde{F}_t - D_t^{k-1} F_t|),$$

where P_{k-1}^3, Q_{k-1}^3 are universal polynomials of k-variables.

In particular, we have

Theorem 4.1. *Let $F, \widetilde{F}$ be smooth MCFs starting at F_0, then*

$$(4.9) \hspace{3cm} D_t^k P\big|_{t=0} = 0$$

for $k = 1, 2, \ldots$.

Now we prove Proposition 4.3.

Proof. As in the proof of Proposition 4.1, let $Z \in T_{\tilde{p}}M$ and extend it to a parallel vector fields along the integral curve of $\widetilde{F}_t$. Then

$$(D_t^k P)(Z) = D_t^k(PZ).$$

Note that $D_t^k(PZ) = \bar{\nabla}_{J_t}^k X(0)$, where X is the parallel transport of Z along γ. We will prove by induction that

$$(4.10) \qquad |\bar{\nabla}_{J_t}^k X| \leq \bigg((C_{k-1}^{loc} d P_{k-1}^3(|F_t|, |D_t F_t|, \ldots, |D_t^{k-1} F_t|)$$

$$+ Q_{k-1}^3(|P\widetilde{F}_t - F_t|, \ldots, |PD_t^{k-1}\widetilde{F}_t - D_t^{k-1} F_t|) \bigg) |Z|,$$

where P_{k-1}^3, Q_{k-1}^3 are universal polynomials of k-variables.

When $k = 1$, one can show as in the proof of Proposition 4.1 the following estimates:

$$(4.11) \qquad\qquad |\bar{\nabla}_{J_t} X| \leq \big(C_0^{loc} d |F_t| + C|P\widetilde{F}_t - F_t| \big) |Z|.$$

Next we assume that (4.10) holds for all integers strictly smaller then k. by commuting covariant derivatives, we have

$$\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_{J_t}^k X = \sum_{i=0}^{k-1} \bar{\nabla}_{J_t}^i (\bar{R}(J_t, \dot{\gamma}) \bar{\nabla}_{J_t}^{k-1-i} X)$$

$$= \sum_{j+i+l+m=k-1} (\bar{\nabla}_{J_t}^j \bar{R}) * (\bar{\nabla}_{J_t}^i J_t) * (\bar{\nabla}_{J_t}^l \dot{\gamma}) * (\bar{\nabla}_{J_t}^m X).$$

Thus the induction step is proved using $|\bar{\nabla}_{J_t}^k J_t| \leq |D_t F_t| + \sup |\bar{\nabla}_{\dot{\gamma}} \bar{\nabla}_{J_t}^k J_t|$, Proposition 3.2, Proposition 3.3, the induction hypothesis and Cauchy Schwarz inequality. This finishes the proof of the proposition.   □

Next, we prove the following lemma which estimates the difference of the restriction of ambient tensors to F and $\widetilde{F}$. Let S be a (p, q)-tensor on M. Then $S|_F$ is a section of the bundle $N^{\otimes p} \otimes (N^*)^{\otimes q}$ over Σ. Let $P^*(S|_{\widetilde{F}})$ be given by

$$P^* S|_{\widetilde{F}}(a_1, \ldots, a_p, b^1, \ldots, b^q) = S|_{\widetilde{F}}(P^* a_1, \ldots, P^* a_p, P^{-1} b^1, \ldots, P^{-1} b^q)$$

for all $a_i \in N^*, b^j \in N$.

Lemma 4.2. *With the above definition,*

$$|P^*(S|_{\widetilde{F}}) - S|_F| \leq \sup|\bar{\nabla} S| \cdot d.$$

Proof. Let a_i, b^j be arbitrary and $a_i(s), b^j(s)$ be the respective parallel transport along $-\gamma$. Then by the fundamental theorem of calculus,

$$
\begin{aligned}
(P^* S|_{\widetilde{F}} &- S|_F)(a_1, \ldots, a_p, b^1, \ldots, b^q) \\
&= \int_0^1 \partial_s \big(S(a_1(s), \ldots, a_p(s), b^1(s), \ldots, b^q(s)) \big) ds \\
&= -\int_0^1 (\bar{\nabla}_{\dot{\gamma}} S)(a_1(s), \ldots, a_p(s), b^1(s), \ldots, b^q(s)) ds
\end{aligned}
$$

since a_i and b^j's are parallel along $-\gamma$. Thus

$$|(P^* S|_{\widetilde{F}} - S|_F)(a_1, \ldots, a_p, b^1, \ldots, b^q)| \leq \sup|\bar{\nabla} S| |a_1| \cdots |a_p| |b^1| \cdots |b^q| |\dot{\gamma}|.$$

Since $d = |\dot{\gamma}|$, the lemma is shown. $\qquad \square$

5. Main estimates

In this section we provide the necessary estimates for the next two sections. In this section, we assume that $F, \widetilde{F}$ are both solutions to the MCF starting at F_0 which satisfies (2.2) and (2.1). In particular, by choosing a small T, we assume that d satisfies (3.1).

First we estimate the time derivative of the quantities $P\widetilde{F}_* - F_*$ and $\Gamma - \widetilde{\Gamma}$.

Lemma 5.1. *We have*

$$
\begin{aligned}
(5.1) \qquad |D_t(P\widetilde{F}_* - F_*)| &\leq C_0(|A| + |\widetilde{A}|)d \\
&\quad + C|\widetilde{\nabla}\widetilde{A}||P\widetilde{F}_* - F_*| + C|P\widetilde{\nabla}\widetilde{A} - \nabla A|.
\end{aligned}
$$

Proof. Recall that

$$
\begin{aligned}
D_t(P\widetilde{F}_i - F_i) &= (D_t P)\widetilde{F}_i + P D_t \widetilde{F}_i - D_t F_i \\
&= (D_t P)\widetilde{F}_i + P\widetilde{\nabla}_i \widetilde{H} - \nabla_i H \\
&= (D_t P)\widetilde{F}_i + (\tilde{g}^{kl} - g^{kl})(P\widetilde{\nabla}_i \widetilde{A}_{kl}) + g^{kl}(P\widetilde{\nabla}_i \widetilde{A}_{kl} - \nabla_i A_{kl}).
\end{aligned}
$$

Using Proposition 4.2, Lemma 4.1 and (2.2), the result follows. $\qquad \square$

To estimate the time derivative of $\Gamma - \widetilde{\Gamma}$. From (2.8) we have

$$\partial_t \Gamma = g^{-2} * h(A, \nabla A),$$

Thus $|\partial_t(\Gamma - \widetilde{\Gamma})|$ can be estimated as in the proof of Lemma 5.1 and (4.1). We skip the proof of the following lemma:

Lemma 5.2.

$$(5.2) \qquad |\partial_t(\Gamma - \widetilde{\Gamma})| \leq C|A||\nabla A||P\widetilde{F}_* - F_*|$$
$$+ C|\nabla A||P\widetilde{A} - A| + C|\widetilde{A}||P\widetilde{\nabla}\widetilde{A} - \nabla A|.$$

Next let us consider the second order quantity

$$(5.3) \qquad |P\widetilde{A} - A|^2 = g^{ik}g^{jl}h(P\widetilde{A}_{ij} - A_{ij}, P\widetilde{A}_{kl} - A_{kl}).$$

Proposition 5.1. *We have the estimate*

$$(5.4) \qquad \partial_t|P\widetilde{A} - A|^2 - 2h(P\widetilde{\Delta}\widetilde{A} - \Delta A, P\widetilde{A} - A)$$
$$\leq \frac{1}{6}|P\widetilde{\nabla}\widetilde{A} - \nabla A|^2 + C((B_2^{loc})^2 + C_1^{loc} + |A|^2 + |\widetilde{A}|^2)d^2$$
$$+ C(|\widetilde{A}|^4 + |\widetilde{\nabla}\widetilde{A}|^2 + C_1^{loc})|P\widetilde{F}_* - F_*|^2$$
$$+ C(|A|^2 + |\widetilde{A}|^2 + C_0^{loc})|P\widetilde{A} - A|^2.$$

Proof. From (5.3) and (2.6) and Proposition 4.2,

$$\partial_t|P\widetilde{A} - A|^2 = \partial_t\big(g^{ik}g^{jl}h(P\widetilde{A}_{ij} - A_{ij}, P\widetilde{A}_{kl} - A_{kl})\big)$$
$$= 2(\partial_t g^{ik})g^{jl}h(P\widetilde{A}_{ij} - A_{ij}, P\widetilde{A}_{kl} - A_{kl}))$$
$$+ 2g^{ik}g^{jl}h(D_t(P\widetilde{A}_{ij} - A_{ij}), P\widetilde{A}_{kl} - A_{kl})$$
$$\leq C|A|^2|P\widetilde{A} - A|^2 + C_0^{loc}d(|A|^2 + |\widetilde{A}|^2)|P\widetilde{A} - A|$$
$$+ 2g^{ik}g^{jl}h(PD_t\widetilde{A}_{ij} - D_tA_{ij}, P\widetilde{A}_{kl} - A_{kl}).$$

Now use (2.9) with $k = 2$ to write

$$PD_t\widetilde{A} - D_tA = P\widetilde{\Delta}\widetilde{A} - \Delta A + (I) + (II) + (III) + (IV)$$

where

$$(I) = P(\tilde{g}^{-2} * \tilde{h}(\widetilde{A}, \widetilde{A}) * \widetilde{A}) - g^{-2} * h(A, A) * A;$$
$$(II) = P(\tilde{g}^{-2} * \tilde{h}(\widetilde{\nabla}\widetilde{A}, \widetilde{A}) * \widetilde{F}_*) - g^{-2} * h(\nabla A, A) * F_*;$$
$$(III) = P(\bar{\nabla}\bar{R}|_{\widetilde{F}} * (\widetilde{F}_*)^{a+2} * \tilde{g}^{-b}) - \bar{\nabla}\bar{R}|_F * (F_*)^{a+2} * g^{-b};$$
$$(IV) = P(\bar{R}|_{\widetilde{F}} * (\widetilde{F}_*)^a * \widetilde{A} * \tilde{g}^{-b}) - \bar{R}|_F * (F_*)^a * A * g^{-b}.$$

Using $\tilde{h}(\widetilde{A}, \widetilde{A}) = h(P\widetilde{A}, P\widetilde{A})$, one has

$$(I) = (g^{-1} - \tilde{g}^{-1}) * \tilde{g}^{-1} * h(P\widetilde{A}, P\widetilde{A}) * P\widetilde{A}$$
$$+ g^{-1} * (g^{-1} - \tilde{g}^{-1}) * h(P\widetilde{A}, P\widetilde{A}) * P\widetilde{A}$$
$$+ g^{-2} * \left(h(P\widetilde{A} - A, P\widetilde{A}) + h(A, P\widetilde{A} - A) \right) * P\widetilde{A}$$
$$+ g^{-2} * h(A, A) * (P\widetilde{A} - A)$$
$$\Rightarrow |(I)| \le C|\widetilde{A}|^3 |P\widetilde{F}_* - F_*| + C(|A|^2 + |\widetilde{A}|^2)|P\widetilde{A} - A|.$$

Similarly we have

$$|(II)| \le C|\widetilde{A}||\widetilde{\nabla}\widetilde{A}||P\widetilde{F}_* - F_*| + C|\widetilde{A}||P\widetilde{A} - A| + C|A||P\widetilde{\nabla}\widetilde{A} - \nabla A|.$$

For (III), note

$$P(\bar{\nabla}\bar{R}|_{\widetilde{F}} * (\widetilde{F}_*)^{a+2} * \tilde{g}^{-b}) = (P^*\bar{\nabla}\bar{R}|_{\widetilde{F}} * (P\widetilde{F}_*)^{a+2} * \tilde{g}^{-b}).$$

Thus a similar calculation and Lemma 4.2 give

$$|(III)| \le CB_2^{loc}d + C_1^{loc}|P\widetilde{F}_* - F_*|.$$

Similar for (IV) we have

$$|(IV)| \le C_1^{loc}|\widetilde{A}|d + C_0^{loc}|\widetilde{A}||P\widetilde{F}_* - F_*| + C_0^{loc}|P\widetilde{A} - A|.$$

Therefore

$$
\begin{aligned}
\partial_t |P\widetilde{A} - A|^2 \leq{}& 2h((P\widetilde{\Delta}\widetilde{A} - \Delta A), P\widetilde{A} - A) + C_0^{loc}(|A|^2 + |\widetilde{A}|^2)d^2 \\
&+ C(|A|^2 + |\widetilde{A}|^2)|P\widetilde{A} - A|^2 \\
&+ C\Big[(|\widetilde{A}||\widetilde{\nabla}\widetilde{A}| + |\widetilde{A}|^3 + C_1^{loc} + C_0^{loc}|\widetilde{A}|)|P\widetilde{F}_* - F_*| \\
&+ (|A|^2 + |\widetilde{A}|^2 + C_0^{loc})|P\widetilde{A} - A| + |A||P\widetilde{\nabla}\widetilde{A} - \nabla A| \\
&+ (CB_2^{loc} + C_1^{loc}|\widetilde{A}|)d\Big]|P\widetilde{A} - A|.
\end{aligned}
$$

Now (5.4) is obtained using Cauchy Schwarz inequalities. $\qquad\square$

6. Proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3

In this section, we use the energy argument to prove the Theorem 1.1, Theorem 1.2 and Theorem 1.3.

To prove Theorem 1.1 using the energy method, we introduce the following energy quantity. By [9], we can find $\rho \in C^\infty(M)$ such that $|\bar{\nabla}\rho| \leq 2$ and

$$
d_M(\cdot, y_0) \leq \rho(\cdot) \leq d_M(\cdot, y_0) + 1.
$$

Now define

$$
\mathcal{Q} = d^2 + |\Gamma - \widetilde{\Gamma}|^2 + \frac{\rho^{2-\epsilon}(F_0)}{t}|P\widetilde{F}_* - F_*|^2 + |P\widetilde{A} - A|^2, \quad t > 0.
$$

Lemma 6.1. *Under the assumption of the Theorem 1.1, there exists C_1 such that on $F_0^{-1}(B_M(p, r))$, where $r \gg 1$,*

$$
\partial_t \mathcal{Q} \leq \frac{C_1 r^{2-\epsilon}}{t}\mathcal{Q} + 2h(P\widetilde{\Delta}\widetilde{A} - \Delta A, P\widetilde{A} - A) + \frac{1}{2}|P\widetilde{\nabla}\widetilde{A} - \nabla A|^2.
$$

Proof. Recall that from Lemma 3.2, (4.2) and (1.3), we have

$$
\partial_t d^2 \leq Cd(|\widetilde{A}||P\widetilde{F}_* - F_*| + |P\widetilde{A} - A|) \leq \frac{Cr^{2-\epsilon}}{t}\mathcal{Q},
$$

On the other hand, using (5.1), (5.2) and (1.3), (2.10) give

$$
\begin{aligned}
\partial_t |\widetilde{\Gamma} - \Gamma|^2 &\leq C|\widetilde{\Gamma} - \Gamma| \Big[|A||\nabla A||P\widetilde{F}_* - F_*| + |\nabla A||P\widetilde{A} - A| \\
&\quad + |\widetilde{A}||P\widetilde{\nabla}\widetilde{A} - \nabla A| \Big] + C|A|^2 |\widetilde{\Gamma} - \Gamma|^2 \\
&\leq \frac{C_1 r^{2-\epsilon}}{t} \mathcal{Q} + \frac{1}{6} |P\widetilde{\nabla}\widetilde{A} - \nabla A|^2
\end{aligned}
$$

and

$$
\frac{\partial}{\partial t} \left(\frac{\rho^{2-\epsilon}}{t} |P\widetilde{F}_* - F_*|^2 \right) \leq \frac{C_1 r^{2-\epsilon}}{t} \mathcal{Q} + \frac{1}{6} |P\widetilde{\nabla}\widetilde{A} - \nabla A|^2.
$$

Lastly, using $(B_2^{loc})^2 \leq C(1 + r^{2-\epsilon})$ for $r \gg 1$ and (5.4), we have

$$
\partial_t |P\widetilde{A} - A|^2 \leq 2h(P\widetilde{\Delta}\widetilde{A} - \Delta A, P\widetilde{A} - A) + C_1 \frac{r^{2-\epsilon}}{t} \mathcal{Q} + \frac{1}{6} |P\widetilde{\nabla}\widetilde{A} - \nabla A|^2
$$

and the lemma is proved. $\qquad\square$

Proof of Theorem 1.1. For each $r \gg 1$, let $\phi(x) = \varphi^p(\rho(F_0)/r)$ where φ is smooth, equals 1 on $[0, 1/2]$, vanishes outside $[0, 1]$ and satisfies $0 \leq -\varphi' \leq 10$. Here p possibly depends on r. Let $\eta(x, t) = \frac{[\rho(F_0)]^2}{a - bt}$. For $t \in (0, a/(2b)]$, $\eta(x, t) \geq \rho^2/a$. Moreover,

$$
\partial_t \eta = \frac{b}{(a - bt)^2} \rho^2 \geq \frac{b}{8n\lambda} |\nabla \eta|^2.
$$

Here a and b are some constants to be fixed later and λ is a constant such that $g(t) \geq \lambda^{-1} g_0$. Now for $t \in (0, a/2b]$, define the energy $E_r(t)$ as

$$
E_r(t) = \int_\Sigma \mathcal{Q} e^{-\eta} \phi \, d\mu, \quad \text{for } t > 0.
$$

Note that the above is well defined, since ϕ is of compact support, while (2.1) and the properness of F_0 together imply that $F(t, \cdot)$ is also proper. From Lemma 6.1 and $\partial_t d\mu \leq 0$ by (2.7),

$$
\begin{aligned}
\frac{d}{dt} E_r(t) &\leq - \int_\Sigma \mathcal{Q} e^{-\eta} \phi \partial_t \eta \, d\mu + \frac{C_1 r^{2-\epsilon}}{t} E_r(t) + \int_\Sigma |P\widetilde{\nabla}\widetilde{A} - \nabla A|^2 e^{-\eta} \phi d\mu \\
&\quad + 2 \int_\Sigma h(P\widetilde{\Delta}\widetilde{A} - \Delta A, P\widetilde{A} - A) e^{-\eta} \phi \, d\mu.
\end{aligned}
$$

We focus on the term containing the Laplacians. Using (4.4), Lemma 2.3 and Cauchy Schwarz inequality,

$$
2\int_\Sigma h(P\widetilde{\Delta}\widetilde{A} - \Delta A, P\widetilde{A} - A)e^{-\eta}\phi\, d\mu
$$

$$
= 2\int_\Sigma h\left(P\tilde{g}^{ij}\widetilde{\nabla}_i\widetilde{\nabla}_j\widetilde{A} - g^{ij}\nabla_i\nabla_j A, P\widetilde{A} - A\right)e^{-\eta}\phi\, d\mu
$$

$$
= 2\int_\Sigma h\left(-\tilde{g}^{ij}(\nabla_i P)\widetilde{\nabla}_j\widetilde{A} + \widetilde{\nabla}_i(P\tilde{g}^{ij}\widetilde{\nabla}_j\widetilde{A}) - g^{ij}\nabla_i\nabla_j A, P\widetilde{A} - A\right)e^{-\eta}\phi\, d\mu
$$

$$
\leq C_0\int_\Sigma d|\widetilde{\nabla}\widetilde{A}||P\widetilde{A} - A|e^{-\eta}\phi\, d\mu
$$

$$
+ 2\int_\Sigma h\left(\nabla_i(P\tilde{g}^{ij}\widetilde{\nabla}_j\widetilde{A} - g^{ij}\nabla_j A), P\widetilde{A} - A\right)e^{-\eta}\phi\, d\mu
$$

$$
+ C\int_\Sigma |\Gamma - \widetilde{\Gamma}||\widetilde{\nabla}\widetilde{A}||P\widetilde{A} - A|e^{-\eta}\phi\, d\mu
$$

$$
\leq \frac{C_1 r^{2-\epsilon}}{t}E_r(t) + 2\int_\Sigma h\left(\nabla_i(P\tilde{g}^{ij}\widetilde{\nabla}_j\widetilde{A} - g^{ij}\nabla_j A), P\widetilde{A} - A\right)e^{-\eta}\phi\, d\mu.
$$

Now we use integration by part to the second term on the right hand side to obtain

$$
2\int_\Sigma h\left(\nabla_i(P\tilde{g}^{ij}\widetilde{\nabla}_j\widetilde{A} - g^{ij}\nabla_j A), P\widetilde{A} - A\right)e^{-\eta}\phi\, d\mu
$$

$$
= -2\int_\Sigma h\left(P\tilde{g}^{ij}\widetilde{\nabla}_j\widetilde{A} - g^{ij}\nabla_j A, (\nabla_i P)\widetilde{A}\right)e^{-\eta}\phi\, d\mu
$$

$$
- 2\int_\Sigma h\left(P\tilde{g}^{ij}\widetilde{\nabla}_j\widetilde{A} - g^{ij}\nabla_j A, P(\nabla_i\widetilde{A}) - \nabla_i A\right)e^{-\eta}\phi\, d\mu
$$

$$
- 2\int_\Sigma h\left(P\tilde{g}^{ij}\widetilde{\nabla}_j\widetilde{A} - g^{ij}\nabla_j A, P\widetilde{A} - A\right)\nabla_i(e^{-\eta}\phi)\, d\mu
$$

$$
= (A) + (B) + (C).
$$

For the first two terms, we use again

$$
\tilde{g}^{-1} = \tilde{g}^{-1} - g^{-1} + g^{-1}, \quad \nabla_i = \nabla_i - \widetilde{\nabla}_i + \widetilde{\nabla}_i,
$$

Proposition 4.1 and Cauchy Schwarz inequality to get

$$
(A) + (B) \leq \frac{C_1 r^{2-\epsilon}}{t}E_r(t) - \int_\Sigma |P\widetilde{\nabla}\widetilde{A} - \nabla A|^2 e^{-\eta}\phi\, d\mu.
$$

For (C) we have similarly

$$(C) \leq \frac{1}{2} \int_\Sigma |P\widetilde{\nabla}\widetilde{A} - \nabla A|^2 e^{-\eta}\phi d\mu + \frac{C_1 r^{2-\epsilon}}{t} E_r(t)$$
$$+ C \int_\Sigma |P\widetilde{A} - A|^2 e^{-\eta}\left[\frac{|\nabla\phi|^2}{\phi} + \phi|\nabla\eta|^2\right]d\mu.$$

Combine all these,

$$(6.1) \qquad \frac{d}{dt}E_r(t) \leq \frac{C_1 r^{2-\epsilon}}{t}E_r(t) + \int_\Sigma \mathcal{Q}e^{-\eta}\phi\left[-\frac{\partial}{\partial t}\eta + \tilde{C}|\nabla\eta|^2\right]d\mu$$
$$+ C\int_{F_0^{-1}(A_{y_0}(r/2,r))} |P\widetilde{A} - A|^2 e^{-\eta}\frac{|\nabla\phi|^2}{\phi}d\mu,$$

where $A_{y_0}(r/2,r) \subset M$ is the annulus centred at y_0 and $\tilde{C}$ is a fixed constants depending only on the dimensions of Σ, M, λ and L. To estimate the last term on the right hand side of (6.1), note

$$\frac{|\nabla\phi|^2}{\phi} \leq \frac{Cp^2}{r^2}\phi^{1-2/p}.$$

By Young's inequality, (1.3), $\partial_t d\mu \leq 0$ and (1.2), we have

$$\frac{Cp^2}{r^2}\int_{F_0^{-1}(A_{y_0}(r/2,r))} |P\widetilde{A} - A|^2 e^{-\eta}\phi^{1-2/p}d\mu$$
$$\leq \frac{Cp^2}{r^2}E_r^{1-2/p}\left(\int_{F_0^{-1}(A_{y_0}(r/2,r))} |P\widetilde{A} - A|^2 e^{-\eta}d\mu\right)^{2/p}$$
$$\leq \frac{r^{2-\epsilon}}{t}E_r + \frac{C^{p/2}p^p t^{p/2-1}}{r^{p+(2-\epsilon)(p/2-1)}}\cdot\left(\int_{F_0^{-1}(A_{y_0}(r/2,r))} (|A|^2 + |\widetilde{A}|^2)e^{-\eta}d\mu\right)$$
$$\leq \frac{r^{2-\epsilon}}{t}E_r + \frac{C^{p/2}p^p t^{p/2-2}}{r^{p+(2-\epsilon)(p/2-2)}}e^{-r^2/a}\cdot\int_{F_0^{-1}(A_{y_0}(r/2,r))} d\mu$$
$$\leq \frac{r^{2-\epsilon}}{t}E_r + \frac{C^{p/2}p^p t^{p/2-2}}{r^{p+(2-\epsilon)(p/2-2)}}e^{-r^2/a}\cdot\int_{F_0^{-1}(A_{y_0}(r/2,r))} d\mu_0$$
$$\leq \frac{r^{2-\epsilon}}{t}E_r + \frac{DC^{p/2}p^p t^{p/2-2}}{r^{p+(2-\epsilon)(p/2-2)}}e^{-r^2/a+Dr^2}.$$

Now we require that a, b satisfy $a^{-1} \geq 2D$ and $b/(8n\lambda) \geq \tilde{C}$. Therefore, the differential inequality for the energy quantity reduces to

$$\frac{d}{dt} E_r(t) \leq \frac{C_1 r^{2-\epsilon}}{t} E_r(t) + D(\sqrt{C}p)^p t^{p/2-2} e^{-r^2/(2a)}.$$

For each $r \gg 1$, from now on we consider C_1, C as fixed constants and write $\alpha = C_1 r^{2-\epsilon} > 0$. Solve the above ode on $0 < s < t < a/(2b)$:

$$\frac{E_r(t)}{t^\alpha} \leq \frac{E_r(s)}{s^\alpha} + D(\sqrt{C}p)^p e^{-r^2/(2a)} \int_s^t x^{p/2-2-\alpha}\, dx$$

At each $r \gg 1$, choose $p = 2(2+\alpha) = 2(2 + C_1 r^{2-\epsilon})$. For r large (depending only on C, D, ϵ, a) we have $e^{r^2/(4a)} > D(\sqrt{C}p)^p$. Hence,

$$(6.2) \qquad \frac{E_r(t)}{t^\alpha} \leq \frac{E_r(s)}{s^\alpha} + \frac{a}{2b} e^{-r^2/(4a)}$$

for large enough r. By Theorem 4.1 and the MCF equation, since the convergence $F(t, \cdot), \widetilde{F}(t, \cdot) \to F_0(\cdot)$ are smooth, $\mathcal{Q}^{(m)}(0) = 0$ for any $m \in \mathbb{N}$. Since F_0 is proper and M is complete, $F_0^{-1}(B_M(y_0, 2r))$ is a compact set and we may apply the dominated convergent theorem to conclude that

$$\lim_{s \to 0} \frac{E_r(s)}{s^\alpha} = 0.$$

Followed by letting $r \to \infty$ in (6.2), we have $\mathcal{Q} \equiv 0$ for all $t \in [0, a/(2b)]$, in particular $F = \widetilde{F}$ in $[0, a/2b]$. Extension to the whole interval $[0, T]$ follows from an open-closed argument and this finishes the proof of Theorem 1.1. $\square$

Next we prove Theorem 1.2.

Proof of Theorem 1.2. In this situation, we observe that conditions (1) and (2) in Theorem 1.1 hold since $t^{-2\alpha}$ is integrable in $[0, T]$. Thus all the calculations in the Sections 3, 4, and 5 can be applied. Let

$$\mathcal{Q}^\alpha = d^2 + |\Gamma - \widetilde{\Gamma}|^2 + t^{-2\alpha}|P\widetilde{F}_* - F_*|^2 + |P\widetilde{A} - A|^2.$$

In this situation, one uses an intrinsic cutoff function: Let $\rho = \rho_T \in C^\infty(\Sigma)$ be an exhaustion of (Σ, g_T) so that for some $x_0 \in \Sigma$,

$$d_{g_T}(x, x_0) \leq \rho(x) \leq d_{g_T}(x, x_0) + 1, \quad |\nabla \rho| \leq 2.$$

Let ϕ, η be defined as in the proof of Theorem 1.1 with this new ρ and let

$$E_r^\alpha(t) = \int_\Sigma \mathcal{Q}^\alpha e^{-\eta} \phi \, d\mu.$$

The assumption $|A| + |\widetilde{A}| \leq C/t^\alpha$ implies the estimates $|\nabla A| + |\widetilde{\nabla}\widetilde{A}| \leq C_1/t^{2\alpha}$. Arguing as in the proof of Theorem 1.1, we have for $r \gg 1$ and $b/(8n\lambda) \geq \widetilde{C}$,

$$\frac{d}{dt}E_r^\alpha(t) \leq \frac{C_2}{t^{2\alpha}}E_r^\alpha(t) + C\int_{F_0^{-1}A_{y_0}(r/2,r)} |P\widetilde{A} - A|^2 e^{-\eta}\frac{|\nabla\phi|^2}{\phi}\,d\mu.$$

Using the assumption on $|A|, |\widetilde{A}|$, (2.2) and pick $p = 2$, we have

$$\frac{d}{dt}E_r^\alpha(t) \leq \frac{C_2}{t^{2\alpha}}\left(E_r^\alpha(t) + e^{-ar^2}\mathrm{Vol}_{g_T}(B_r(x_0))\right).$$

From the Gauss equation and the assumptions on A, (Σ, g_T) has bounded curvature, thus the volume comparison theorem gives

$$\mathrm{Vol}_{g_T}(B_r(x_0)) \leq De^{Dr}$$

for some $D = D(n, m, T, B_0)$. Choosing $a^{-1} \geq 2D$,

$$\frac{d}{dt}E_r^\alpha(t) \leq \frac{C_2}{t^{2\alpha}}\left(E_r^\alpha(t) + e^{-r^2/2a}\right).$$

Since the convergence $F(t, \cdot), \widetilde{F}(t, \cdot) \to F_0(\cdot)$ is C^3, $E_r(t)$ is continuous at $t = 0$ and $E_r(0) = 0$. Integrating the above inequality (note $t^{-2\alpha}$ is integrable) gives

$$E_r(t) \leq \left(e^{\frac{C_2}{1-2\alpha}t^{1-2\alpha}} - 1\right)e^{-r^2/2a}.$$

Let $r \to \infty$ gives $\mathcal{Q}^\alpha = 0$ for all $t \in [0, a/2b]$. Thus $F = \widetilde{F}$ in $[0, a/2b]$ and the theorem follows from iterating the argument. $\qquad\square$

Using the above cutoff technique and the argument in the proof of Theorem 1.1, we sketch how one can prove Theorem 1.3.

Sketch of proof of Theorem 1.3. We argue using similar argument in [12]. Define the energy to be

$$E_R(t) = \int_M \left(t^{-2}|g - \tilde{g}|^2 + t^{-1}|\Gamma - \tilde{\Gamma}|^2 + |Rm - \widetilde{Rm}|^2 \right) \phi e^{-\eta} \, d\mu_{g(t)}.$$

Here we choose the cutoff function and exhaustion function as in the proof of Theorem 1.1: $\phi(x) = \phi(\rho(x)/R)$ and $\eta(x,t) = \frac{\rho(x)^2}{a-bt}$ where ρ is a smooth function on M such that

$$d_0(x, x_0) \leq \rho(x) \leq d_0(x, x_0) + 1 \quad \text{and} \quad |\nabla^{g_0}\rho| \leq 2$$

for some $x_0 \in M$. By volume comparison and equivalence of metrics, we know that

$$V_t(B_0(p, R)) \leq V_T(B_T(p, CR)) \leq C' e^{C'R}.$$

Using integration by part, we obtain a evolution inequality of E_R which is in the same form as before.

$$E_R'(t) \leq \frac{L}{t} E_R(t) + C_n \int_M \frac{|\nabla\phi|^2}{\phi} |Rm - \widetilde{Rm}|^2 e^{-\eta} \, d\mu$$

for some $L = L(n, \lambda)$. We can now employ the same trick in the proof of Theorem 1.1 to conclude that $g(t) = \tilde{g}(t)$ for all $t \in [0, T]$. $\square$

7. Backward uniqueness

In this section, we modify a general backward uniqueness result in [16] to prove Theorem 1.4. When the ambient space is Euclidean, similar results were obtained in [10] in co-dimension one case and [25] in arbitrary co-dimension. However, the issue of parallel transport is not addressed in [10], [25] when the ambient space is not Euclidean.

To start the proof, let $F, \widetilde{F} : [0, T] \times \Sigma \to M$ be two MCFs with uniformly bounded second fundamental forms $|A| + |\widetilde{A}| \leq C$ and $F = \widetilde{F}$ at time T. To show backward uniqueness, it suffices to show that $F = \widetilde{F}$ on $[1/l, T]$ for all $l \in \mathbb{N}$. Now consider l as fixed number. By Theorem 3.2 in [5], we have

$$|\nabla^k A| + |\widetilde{\nabla}^k \widetilde{A}| \leq C_{k+1}, \quad k = 0, 1, 2, \dots \text{ and } t \in [1/l, T].$$

Consider two (time-dependent) vector bundles over Σ:

$$\mathcal{X} = (T^{0,2}\Sigma \otimes N) \oplus (T^{0,3}\Sigma \otimes N), \quad \mathcal{Y} = N \oplus (T^{0,1}\Sigma \otimes N) \oplus T^{1,2}\Sigma \oplus T^{1,3}\Sigma.$$

We use the metric induced from g and h and the direct sums are orthogonal. Define the following time covariant derivatives on $\mathcal{X}$ and $\mathcal{Y}$ respectively:

$$D_t^{\mathcal{X}} = D_t \oplus D_t, \quad D_t^{\mathcal{Y}} = D_t \oplus D_t \oplus \partial_t \oplus \partial_t.$$

Consider the following two sections X, Y on $\mathcal{X}, \mathcal{Y}$ respectively:

$$X = (P\widetilde{A} - A) \oplus (P\widetilde{\nabla}\widetilde{A} - \nabla A),$$
$$Y = v \oplus (P\widetilde{F}_* - F_*) \oplus (\Gamma - \widetilde{\Gamma}) \oplus \nabla(\Gamma - \widetilde{\Gamma}),$$

where v is defined in Section 3. Theorem 1.4 follows from the following

Theorem 7.1. *There are constants C_4 so that*

(7.1) $$|(D_t^{\mathcal{X}} - \Delta)X| \leq C_4(|X| + |\nabla X| + |Y|),$$
(7.2) $$|D_t^{\mathcal{Y}}Y| \leq C_4(|X| + |\nabla X| + |Y|).$$

Proof. First we estimate $\partial_t(\nabla(\Gamma - \widetilde{\Gamma}))$. We remark that for any (p, q) tensors S on Σ, we have

$$(\partial_t \nabla - \nabla \partial_t)S = (\partial_t \Gamma) * S.$$

Then we have the estimates

$$
\begin{aligned}
|\partial_t \nabla(\Gamma - \widetilde{\Gamma})| &= |\nabla(\partial_t \Gamma - \partial_t \widetilde{\Gamma})| + C_2|\Gamma - \widetilde{\Gamma}| \\
&= \left|\nabla\left[g^{-2} * h(A, \nabla A) - \tilde{g}^{-2} * \tilde{h}(\widetilde{A}, \widetilde{\nabla}\widetilde{A})\right]\right| + C_2|\Gamma - \widetilde{\Gamma}| \\
&\leq |g^{-2} * (h(\nabla A, \nabla A) + h(A, \nabla^2 A)) - \tilde{g}^{-2} * (h(P\widetilde{\nabla}\widetilde{A}, P\widetilde{\nabla}\widetilde{A})| \\
&\quad + h(P\widetilde{A}, P\widetilde{\nabla}^2\widetilde{A})) + C_2|\Gamma - \widetilde{\Gamma}| \\
&\leq C_2(|\Gamma - \widetilde{\Gamma}| + |P\widetilde{F}_* - F_*| + |P\widetilde{A} - A| + |P\widetilde{\nabla}\widetilde{A} - \nabla A| \\
&\quad + |P\widetilde{\nabla}^2\widetilde{A} - \nabla^2 A|) \\
&\leq C_2(|\Gamma - \widetilde{\Gamma}| + |P\widetilde{F}_* - F_*| + |P\widetilde{A} - A| + |P\widetilde{\nabla}\widetilde{A} - \nabla A| \\
&\quad + |\nabla(P\widetilde{\nabla}\widetilde{A} - \nabla A)| + |v|) \\
&\leq C_2(|X| + |\nabla X| + |Y|).
\end{aligned}
$$

The above inequality together with (3.7), (5.1) and (5.2) give us (7.2). To derive (7.1), note that for any k,

$$
\begin{aligned}
(D_t - \Delta)(P\widetilde{\nabla}^k\widetilde{A} - \nabla^k A) = {}&((D_t - \Delta)P)\widetilde{\nabla}^k\widetilde{A} - 2g^{ij}(\nabla_i P)(\nabla_j\widetilde{\nabla}^k\widetilde{A}) \\
&+ P((D_t - \Delta)\widetilde{\nabla}^k\widetilde{A}) - (D_t - \Delta)\nabla^k A.
\end{aligned}
$$

The first two terms on the right hand side is estimated using (4.7), (4.5) and (4.4):

$$\begin{aligned}
|((D_t - \Delta)P)\widetilde{\nabla}^k\widetilde{A}| &\leq C_1|\widetilde{\nabla}^k\widetilde{A}|[(|A| + |\widetilde{A}|)|v| + \tilde{P}(|A|)|v| \\
&\quad + |P\widetilde{F}_* - F_*| + |\Gamma - \widetilde{\Gamma}| + |P\widetilde{A} - A|] \\
&\leq C_{k+1}(|X| + |Y|), \\
|2g^{ij}(\nabla_i P)(\nabla_j\widetilde{\nabla}^k\widetilde{A})| &\leq C_{k+2}|Y|.
\end{aligned}$$

To estimate the third term we use

$$\Delta - \widetilde{\Delta} = g^{-1} * \nabla(\Gamma - \widetilde{\Gamma}) + g^{-1} * (\Gamma - \widetilde{\Gamma}) * \widetilde{\nabla} + (g^{-1} - \tilde{g}^{-1}) * \widetilde{\nabla}^2$$

and get

$$\begin{aligned}
|(D_t &- \Delta)(P\widetilde{\nabla}^k\widetilde{A} - \nabla^k A)| \\
&\leq |P(D_t - \widetilde{\Delta})\widetilde{\nabla}^k\widetilde{A} - (D_t - \Delta)\nabla^k A| \\
&\quad + C\big[(|\widetilde{\nabla}^k\widetilde{A}| + |\widetilde{\nabla}^{k+1}\widetilde{A}|)|v| + |\widetilde{\nabla}^k\widetilde{A}||\nabla(\Gamma - \widetilde{\Gamma})| \\
&\quad + (|\widetilde{\nabla}^k\widetilde{A}| + |\widetilde{\nabla}^{k+1}\widetilde{A}|)|\Gamma - \widetilde{\Gamma}| + |\widetilde{\nabla}^{k+2}\widetilde{A}||P\widetilde{F}_* - F_*|\big] \\
&\leq |P(D_t - \widetilde{\Delta})\widetilde{\nabla}^k\widetilde{A} - (D_t - \Delta)\nabla^k A| + C_{k+3}(|X| + |Y|).
\end{aligned}$$

From (2.9), one can check that

$$\begin{aligned}
|P(D_t &- \widetilde{\Delta})\widetilde{\nabla}^k\widetilde{A} - (D_t - \Delta)\nabla^k A| \\
&\leq C_{k+2}\left(|P\widetilde{A} - A| + \sum_{i=0}^{k}|\nabla(P\widetilde{\nabla}^i\widetilde{A} - \nabla^i A)| + |\Gamma - \widetilde{\Gamma}| + |P\widetilde{F}_* - F_*| + |v|\right).
\end{aligned}$$

Using the above inequalities with the case $k = 0, 1$ give (7.1) and the theorem is proved. $\qquad\square$

Proof of Theorem 1.4. In [16], the author proves a general backward uniqueness theorem for two sections X, Y in two fixed vector bundles $\mathcal{X}$, $\mathcal{Y}$ on Σ respectively. We remark that their proof goes through if one assume that $\mathcal{X}$, $\mathcal{Y}$ are both time dependent vector bundle with ∂_t replaced by $D_t^{\mathcal{X}}, D_t^{\mathcal{Y}}$. In particular, to apply Theorem 3 in [16] to our situation, let

$$\tau = T - t, \ \Lambda^{ij} = g^{ij}.$$

Note $\nabla\Lambda = 0$ and $b = \partial_\tau g, \nabla b, \partial_\tau\Lambda, R_\Sigma$ are all uniformly bounded, so is $[D_t, \nabla]$ since

$$[D_t, \nabla] = \partial_t\Gamma + \bar{R} * H * F_*.$$

Thus Theorem 7.1 and Theorem 3 in [16] imply that $X = Y = 0$ on $[1/l, T]$. Thus $F = \widetilde{F}$ on $[1/l, T]$. □

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