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Quantitative stratification of F-subharmonic functions

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In this paper, we study the singular sets of F-subharmonic functions $u : B_2(0^n) \to \mathbf{R}$, where F is a subequation. The singular set $\mathcal{S}(u) \subset B_2(0^n)$ has a stratification $\mathcal{S}^0(u) \subset \mathcal{S}^1(u) \subset \cdots \subset \mathcal{S}^k(u) \subset \cdots \subset \mathcal{S}(u)$, where $x \in \mathcal{S}^k(u)$ if no tangent function to u at x is $(k+1)$-homogeneous. We define the quantitative stratifications $\mathcal{S}^k_\eta(u)$ and $\mathcal{S}^k_{\eta,r}(u)$ satisfying $\mathcal{S}^k(u) = \cup_\eta \mathcal{S}^k_\eta(u) = \cup_\eta \cap_r \mathcal{S}^k_{\eta,r}(u)$.

When homogeneity of tangents holds for F, we prove that $dim_H \mathcal{S}^k(u) \leq k$ and $\mathcal{S}(u) = \mathcal{S}^{n-p}(u)$, where p is the Riesz characteristic of F. And for the top quantitative stratification $\mathcal{S}^{n-p}_\eta(u)$, we have the Minkowski estimate $\mathrm{Vol}(B_r(\mathcal{S}^{n-p}_\eta(u) \cap B_1(0^n))) \leq C\eta^{-1}(\int_{B_{1+r}(0^n)} \Delta u) r^p$.

When uniqueness of tangents holds for F, we show that $S^k_\eta(u)$ is k-rectifiable, which implies $\mathcal{S}^k(u)$ is k-rectifiable.

When strong uniqueness of tangents holds for F, we introduce the monotonicity condition and the notion of F-energy. By using refined covering argument, we obtain a definite upper bound on the number of $\{\Theta(u,x) \geq c\}$ for $c > 0$, where $\Theta(u,x)$ is the density of F-subharmonic function u at x.

Geometrically determined subequations $F(\mathbb{G})$ are a very important type of subequation (when $p = 2$, homogeneity of tangents holds for $F(\mathbb{G})$; when $p > 2$, uniqueness of tangents holds for $F(\mathbb{G})$). By introducing the notion of $\mathbb{G}$-energy and using quantitative differentation argument, we obtain the Minkowski estimate of quantitative stratification $\mathrm{Vol}(B_r(\mathcal{S}^k_{\eta,r}(u)) \cap B_1(0^n)) \leq Cr^{n-k-\eta}$.

1. Introduction

1.1. Background

Recently, Harvey and Lawson [21, 22] (see also [10–20, 23]) established a theory of elliptic equations. The aim of this theory is to study the behavior of subsolutions in the viscosity sense. They introduced the definitions of Riesz characteristic, tangential p-flow, tangent and density function. And many interesting theorems, formulas and properties of subsolutions, tangents and density functions were established.

In this theory, there is a very important kind of examples called geometrically defined subequations (see [21, Example 4.4] and [22]). To be specific, let $\mathbb{G}$ be a compact subset of the Grassmannian manifold $G(p, \mathbf{R}^n)$ such that $\mathbb{G}$ is invariant under a subgroup $G \subset O(n)$ acting transitively on the sphere $S^{n-1} \subset \mathbf{R}^n$. The geometric subequation determined by $\mathbb{G}$ is defined by

$$F(\mathbb{G}) = \{A \in \mathrm{Sym}(n) \mid tr_W(A) \geq 0 \text{ for any } W \in \mathbb{G}\}, \tag{1.1}$$

where $\mathrm{Sym}(n)$ denotes the space of symmetric $n \times n$ matrices with real entries and $tr_W(A)$ denotes the trace of $A|_W$. Let u be a $F(\mathbb{G})$-subharmonic function, by the Restriction Theorem 3.2 in [20], we obtain $u|_W$ is subharmonic on W for any $W \in \mathbb{G}$. $F(\mathbb{G})$-subharmonic functions are usually called $\mathbb{G}$-plurisubharmonic functions. And as we can see, convex, $\mathbb{C}$-plurisubharmonic and $\mathbb{H}$-plurisubharmonic functions are all special cases of $\mathbb{G}$-plurisubharmonic functions.

In [21], Harvey and Lawson introduced the definitions of homogeneity, uniqueness and strong uniqueness of tangents. In [22], for geometrically defined subequations $F(\mathbb{G})$, it was proved that homogeneity of tangents holds when $p = 2$ and uniqueness of tangents holds when $p > 2$. They also proved strong uniqueness of tangents holds for many subequations (see [21, Theorem 13.1] and [22, Theorem 3.2, Theorem 3.12]). When the subequation F is convex, for any F-subharmonic function u, upper semicontinuity of density functions $\Theta^M(u,\cdot)$, $\Theta^S(u,\cdot)$ and $\Theta^V(u,\cdot)$ was proved (see [21, Theorem 7.4]), which implies that for any $c > 0$ and each density function as above, the set

$$E_c(u) := \{x \mid \Theta(u,x) \geq c\}$$

is closed (see [21, Corollary 7.5]). Furthermore, the discreteness of the set $E_c(u)$ was established when strong uniqueness of tangents holds for F and $p > 2$, where p is the Riesz characteristic of F (see [21, Theorem 14.1, Theorem 14.1']).

1.2. Definitions and notations

In this paper, many definitions in Harvey and Lawson's theory will be used. For the reader's convenience, we list some related definitions. For more details, we refer the reader to [21, 22]. We shall use the following notations, for any function u, point $x \in \mathbf{R}^n$ and $r > 0$,

$$M(u,x,r) = \sup_{y \in B_1(0^n)} u(x+ry),$$

$$S(u,x,r) = \frac{1}{n\omega_n} \int_{\partial B_1(0^n)} u(x+ry)dy,$$

$$V(u,x,r) = \frac{1}{\omega_n} \int_{B_1(0^n)} u(x+ry)dy,$$

where 0^n is the origin in $\mathbf{R}^n$ and ω_n is the volume of unit ball in $\mathbf{R}^n$.

Let F be a closed subset of $Sym^2(\mathbf{R}^n)$ (the set of $n \times n$ symmetric matrices). We always assume that the set F has the following properties:

(1) Positivity: $F + \mathcal{P} \subset F$, where $\mathcal{P} = \{A \in Sym^2(\mathbf{R}^n) \mid A \geq 0\}$;

(2) ST-Invariance: F is invariant under a subgroup $G \subset O(n)$ which acts transitively on the sphere S^{n-1};

(3) Cone Property: $tF \subset F$ for all $t \geq 0$;

(4) Convexity: F is convex.

A closed set F satisfying Positivity is called a subequation. For each subequation F, the viscosity F-subsolutions are called F-subharmonic functions.

Definition 1.1. ([8–10, 13]) Let Ω be a domain in $\mathbf{R}^n$. An upper semicontinuous function u on Ω is called a F-subharmonic function if for any $x_0 \in \Omega$ and any function $\varphi \in C^2(\Omega)$ such that

$$u(x_0) = \varphi(x_0) \text{ and } u - \varphi \leq 0 \text{ near } x_0,$$

then $D^2\varphi(x_0) \in F$.

First, let us recall the definition of the classical p^{th} Riesz kernel and the Riesz characteristic. The classical p^{th} Riesz kernel K_p is defined by

$$K_p(t) = \begin{cases} t^{2-p} & \text{if } 1 \leq p < 2 \\ \log t & \text{if } p = 2 \\ -\frac{1}{t^{p-2}} & \text{if } 2 < p < \infty. \end{cases}$$

Definition 1.2. ([21, Definition 3.2]) Suppose that F is an ST-invariant cone subequation. The Riesz characteristic p_F of F is defined to be

$$p_F := \sup\{\overline{p} \mid P_{e^\perp} - (\overline{p} - 1)P_e \in F \text{ for any unit vector } e \in \mathbf{R}^n\},$$

where P_e and $P_{e^\perp}$ denote orthogonal projection onto the line through e and the hyperplane with normal e respectively.

In this paper, we assume that $p_F = p$ for convenience. Next, let us recall the definitions of tangent and tangent set.

Definition 1.3. ([21, Definition 9.1]) Suppose that u is a F-subharmonic function. Let x be a point such that $B_\rho(x)$ is in the domain of u, where $\rho > 0$. For any $r > 0$, the tangential p-flow (or p-homothety) of u at x is defined as follows.

(1) If $p > 2$, $u_{x,r}(y) := r^{p-2}u(x + ry)$ in $B_{\frac{\rho}{r}}(0^n)$;

(2) If $2 > p \geq 1$, $u_{x,r}(y) := \frac{1}{r^{2-p}}(u(x + ry) - u(x))$ in $B_{\frac{\rho}{r}}(0^n)$;

(3) If $p = 2$, $u_{x,r}(y) := u(x + ry) - M(u, x, r)$ in $B_{\frac{\rho}{r}}(0^n)$.

Definition 1.4. ([21, Definition 9.3]) Suppose that u is a F-subharmonic function. Let x be an interior point in the domain of u. For each sequence

$r_j \searrow 0$ such that

$$\overline{U} = \lim_{j \to \infty} u_{x,r_j} \quad \text{in } L^1_{loc}(\mathbf{R}^n),$$

the point-wise defined function

$$U(y) := \lim_{r \to 0} \operatorname*{ess\,sup}_{B_r(y)} \overline{U}$$

is called a tangent to u at x. We let $T_x(u)$ denote the set of all such tangents U.

In [21], Harvey-Lawson proved that each tangent $U \in T_x(u)$ is F-subharmonic, and U is the unique F-subharmonic function in the L^1_{loc}-class $\overline{U} \in L^1_{loc}(\mathbf{R}^n)$ (see [21, Proposition 9.4]).

Definition 1.5. ([21, Definition 12.1]) Suppose that u is a F-subharmonic function. Let x be an interior point in the domain of u.

(1) For any u and x, if every tangent $\varphi \in T_x(u)$ satisfies $\varphi_{0^n,r} = \varphi$ for any $r > 0$, we say that homogeneity of tangents holds for F;

(2) For any u and x, if $T_x(u)$ is a singleton, we say that uniqueness of tangents holds for F;

(3) For any u and x, if $T_x(u) = \{\Theta K_p(|\cdot|)\}$, where $\Theta \geq 0$ is a constant, we say that strong uniqueness of tangents holds for F.

Remark 1.6. In Definition 1.5, it is clear that (3) implies (2) and (2) implies (1).

Next, in order to study the singular sets of F-subharmonic functions, we have the following definitions.

Definition 1.7. A function $h : \mathbf{R}^n \to \mathbf{R}$ is said to be k-homogeneous at $x \in \mathbf{R}^n$ with respect to k-plane $V^k \subset \mathbf{R}^n$ if h satisfies the following properties:

(1) h is subharmonic on $\mathbf{R}^n$;

(2) For any $r > 0$, $h_{x,r}(y) = h(y + x)$ for every $y \in \mathbf{R}^n$, where $h_{x,r}$ is the tangential p-flow of h at x;

(3) For any $y \in \mathbf{R}^n$ and $v \in V^k$, $h(y + v + x) = h(y + x)$.

If $x = 0^n$, we say h is k-homogeneous (or h is a k-homogeneous function) for convenience.

Definition 1.8. A function $u : B_r(x) \subset \mathbf{R}^n \to \mathbf{R}$ is said to be (k, ϵ, r, x)-homogeneous, if there exists a k-homogeneous function $h : \mathbf{R}^n \to \mathbf{R}$ such that

$$\|u_{x,r} - h\|_{L^1(B_1(0^n))} < \epsilon.$$

Definition 1.9. Suppose that homogeneity of tangents holds for F. Let u be a F-subharmonic function on $B_2(0^n)$. For any $\eta > 0$ and $r \in (0, 1)$, we have the following definitions:

(1) The singular set $\mathcal{S}(u)$ is defined by

$$\mathcal{S}(u) := \{x \in B_2(0^n) \mid \text{no tangent at } x \text{ is } n\text{-homogeneous}\}.$$

(2) The k^{th} stratification $\mathcal{S}^k(u)$ is defined by

$$\mathcal{S}^k(u) := \{x \in B_2(0^n) \mid \text{no tangent at } x \text{ is } (k+1)\text{-homogeneous}\}.$$

(3) The k^{th} η-stratification $\mathcal{S}^k_\eta(u)$ is defined by

$$\begin{aligned}\mathcal{S}^k_\eta(u) := \{x \in B_2(0^n) \mid u \text{ is not } (k+1, \eta, s, x)\text{-homogeneous} \\ \text{for any } s \in (0,1)\}.\end{aligned}$$

(4) The k^{th} (η, r)-stratification $\mathcal{S}^k_{\eta,r}(u)$ is defined by

$$\begin{aligned}\mathcal{S}^k_{\eta,r}(u) := \{x \in B_2(0^n) \mid u \text{ is not } (k+1, \eta, s, x)\text{-homogeneous} \\ \text{for any } s \in [r,1)\}.\end{aligned}$$

Remark 1.10. When homogeneity of tangents holds for F, we have the following relationships (see Proposition 8.2):

$$\mathcal{S}^0(u) \subset \mathcal{S}^1(u) \subset \cdots \subset \mathcal{S}^{n-1}(u) = \mathcal{S}(u)$$

and

$$\mathcal{S}^k(u) = \bigcup_\eta \mathcal{S}^k_\eta(u) = \bigcup_\eta \bigcap_r \mathcal{S}^k_{\eta,r}(u). \tag{1.2}$$

Remark 1.11. When strong uniqueness of tangents holds for F, three density functions $\Theta^M(u,\cdot)$, $\Theta^S(u,\cdot)$ and $\Theta^V(u,\cdot)$ are equivalent (see [21, Proposition 7.1, (12.3)]). And for each density function as above, we have

$$\mathcal{S}(u) = \mathcal{S}^0(u) = \bigcup_{c>0} E_c(u),$$

where $E_c(u) = \{x \in B_2(0^n) \mid \Theta(u, x) \geq c\}$.

1.3. Main results

In this paper, we assume that F is a subequation satisfies Positivity, ST-Invariance, Cone Property and Convexity. Let p be the Riesz characteristic of F. When $1 \leq p < 2$, the F-subharmonic function is Hölder continuous (see [21, Theorem 15.1]). Hence, we focus on the case $p \geq 2$ in this paper. When F satisfies different conditions, we obtain different results of the singular sets.

Theorem 1.12. *Suppose that F is a subequation such that homogeneity of tangents holds for F. Let u be a F-subharmonic function defined on $B_2(0^n)$ with $\|u\|_{L^1(B_2(0^n))} \leq \Lambda$. For any $\eta > 0$, we have*

(1) Vol$(B_r(S^{n-p}_\eta(u) \cap B_1(0^n))) \leq C(p,n)\eta^{-1} \left(\int_{B_{1+r}(0^n)} \Delta u\right) r^p$ for any $r \in (0, \frac{1}{5})$;

(2) $\mathcal{S}(u) = \mathcal{S}^{n-p}(u)$;

(3) $dim_H(\mathcal{S}^k(u)) \leq k$ for any $k = 1, 2, \ldots, n$, where $dim_H\mathcal{S}^k(u)$ is the Hausdorff dimension of $\mathcal{S}^k(u)$.

Theorem 1.13. *Suppose that F is a subequation such that uniqueness of tangents holds for F. Let u be a F-subharmonic function defined on $B_2(0^n)$. Then $\mathcal{S}^k(u)$ is k-rectifiable for any $k = 1, 2, \ldots, n$.*

Theorem 1.14. *Suppose that F is a subequation such that strong uniqueness of tangents holds for F and $p > 2$. Let u be a F-subharmonic function defined on $B_2(0^n)$ with $\|u\|_{L^1(B_2(0^n))} \leq \Lambda$. For any $c > 0$, there exists a constant $C(c, \Lambda, F)$ such that*

$$\#\left(E_c(u) \cap B_1(0^n)\right) \leq C(c, \Lambda, F) \tag{1.3}$$

where $\#\left(E_c(u) \cap B_1(0^n)\right)$ is the cardinality of $E_c(u) \cap B_1(0^n)$.

As alluded to above, there are many subequations satisfying the assumptions of Theorem 1.14 (see [21, 22]). Under the assumption of strong uniqueness of tangents, Harvey-Lawson proved that the set $E_c(u)$ is discrete (see [21, Theorem 14.1, Theorem 14.1']). Theorem 1.14 gives a quantitative estimate of the cardinality of this set.

In the proof of Theorem 1.14, we introduce the monotonicity condition and the notion of F-energy. And we prove every F-subharmonic function satisfies monotonicity condition after subtracting a constant. For F-subharmonic function satisfies monotonicity condition, we prove (6.6) by using refined covering arguments, which is introduced in [31]. Since the set $E_c(u)$ is invariant after subtracting a constant, we obtain Theorem 1.14.

For geometrically defined subequations $F(\mathbb{G})$ (i.e., $\mathbb{G}$-plurisubharmonic case), we have the following Minkowski estimate of quantitative stratification.

Theorem 1.15. *Let u be a $\mathbb{G}$-plurisubharmonic function on $B_2(0^n)$ with $\|u\|_{L^1(B_2(0^n))} \leq \Lambda$. For any $\eta > 0$, there exists constant $C(\eta, \Lambda, \mathbb{G})$ such that for any $r \in (0,1)$, we have*

$$Vol(B_r(\mathcal{S}^k_{\eta,r}(u)) \cap B_1(0^n)) \leq C(\eta, \Lambda, \mathbb{G}) r^{n-k-\eta}. \tag{1.4}$$

Remark 1.16. It suffices to prove Theorem 1.15 when $\mathbb{G}$ is a smooth submanifold of $G(p, \mathbf{R}^n)$. For general $\mathbb{G}$, since $\mathbb{G}$ is invariant under a subgroup $G \subset O(n)$ acting transitively on the sphere $S^{n-1} \subset \mathbf{R}^n$, we fix $W \in \mathbb{G}$ and consider $\mathbb{G}_0 = G \cdot W$. Then $\mathbb{G}_0$ is a smooth submanifold of $G(p, \mathbf{R}^n)$ and $F(\mathbb{G}) \subset F(\mathbb{G}_0)$. It follows that any $\mathbb{G}$-plurisubharmonic function is $\mathbb{G}_0$-plurisubharmonic function. Then Theorem 1.15 for smooth $\mathbb{G}_0$ implies Theorem 1.15 for general $\mathbb{G}$ (see [22, p.2198]). Therefore, without loss of generality, we assume that $\mathbb{G}$ is a smooth submanifold of $G(p, \mathbf{R}^n)$ in Section 7.

In the proof of Theorem 1.15, we introduce the notion of $\mathbb{G}$-energy, which is a monotone quantity. The key point is to establish the quantitative rigidity theorem (Theorem 7.5 and Theorem 7.7). Roughly speaking, we prove it by making use of the information of tangent at infinity, together with a contradiction argument. Next, combining quantitative rigidity theorem (Theorem 7.5 and Theorem 7.7) and cone-splitting lemma (Lemma 2.3), we obtain decomposition lemma (Lemma 7.14), which implies Theorem 1.15.

In general outline, we will follow a scheme introduced in [5], where quantitative differentation argument was established. By this method, Cheeger and Naber proved some new estimates on non-collapsed Riemannian manifolds with Ricci curvature bounded from below, especially Einstein manifolds. In fact, this method has already been applied to many areas. Analogous results were obtained in the study of mean curvature flows, elliptic equations, harmonic maps and so on (see [3–7]).

Recently, Naber and Valtorta [26] introduced new techniques for estimating the critical and singular set of elliptic PDEs. In [27, 28, 30], they also got some new results on stationary and minimizing harmonic maps. It was proved that the k^{th} stratification of singular set is k-rectifiable and obtained more stronger estimates of the quantitative stratification. And these techniques have also been applied to the study of stationary Yang Mills (see [29]) and L^2 curvature bounds on non-collapsed Riemannian manifolds with bounded Ricci curvature (see [25]).

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2. Cone-splitting lemma

In this section, we prove cone-splitting lemma (Lemma 2.3) for F-subharmonic functions. And we will use it throughout this paper.

Theorem 2.1 (Cone-splitting principle). *Let h be a function which is k-homogeneous at x_1 with respect to k-plane V^k. If there exists a point $x_2 \notin x_1 + V^k$ such that h is 0-homogeneous at x_2, then h is $(k+1)$-homogeneous at x_1 with respect to $(k+1)$-plane $V^{k+1} = span\{x_2 - x_1, V^k\}$.*

Proof. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbf{R}^n$. Without loss of generality, we assume that $x_1 = 0^n$, $x_2 = e_{k+1}$ and $V^k = span\{e_i\}_{i=1}^k$. Since h is k-homogeneous at x_1 respect to V^k, it suffices to prove

$$h(x + te_{k+1}) = h(x), \tag{2.1}$$

for all $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$. We split into different cases according to p (Riesz characteristic of F).

Case 1. $p > 2$.

Since h is k-homogeneous at 0^n and 0-homogeneous at e_{k+1}, By the definition of homogeneous function, we have

$$h(x) = h_{0^n, \frac{1}{|x|}}(x) = |x|^{2-p} h\left(\frac{x}{|x|}\right).$$

Let $g_1 = h|_{S^{n-1}}$, we obtain

$$h(x) = |x|^{2-p} g_1\left(\frac{x}{|x|}\right). \tag{2.2}$$

Similarly, there exists function g_2 on the unit sphere $S^{n-1} \subset \mathbf{R}^n$ such that

$$h(x) = |x - e_{k+1}|^{2-p} g_2\left(\frac{x - e_{k+1}}{|x - e_{k+1}|}\right). \tag{2.3}$$

We split up into different subcases.

Subcase 1.1. $x \in span\{e_{k+1}\}$.

By (2.2) and (2.3), we have

$$2^{2-p} g_1(e_{k+1}) = h(2e_{k+1}) = g_2(e_{k+1})$$

and

$$3^{2-p} g_1(e_{k+1}) = h(3e_{k+1}) = 2^{2-p} g_2(e_{k+1}).$$

Hence, we obtain $g_1(e_{k+1}) = g_2(e_{k+1}) = 0$ or $g_1(e_{k+1}) = g_2(e_{k+1}) = \infty$, which implies (2.1).

Subcase 1.2. $x \notin span\{e_{k+1}\}$ and $t < 1$.

By (2.2) and (2.3), we have

$$h\left(\frac{x}{1-t}\right) = \left|\frac{x}{1-t}\right|^{2-p} g_1\left(\frac{x}{|x|}\right) = \frac{1}{|1-t|^{2-p}} h(x)$$

and

$$h\left(\frac{x}{1-t}\right) = \left|\frac{x}{1-t} - e_{k+1}\right|^{2-p} g_2\left(\frac{\frac{x}{1-t} - e_{k+1}}{|\frac{x}{1-t} - e_{k+1}|}\right) = \frac{1}{|1-t|^{2-p}} h(x + te_{k+1}).$$

Then we obtain (2.1).

Subcase 1.3. $x \notin span\{e_{k+1}\}$ and $t \geq 1$.

If $x \notin span\{e_{k+1}\}$, then $x + te_{k+1} \notin span\{e_{k+1}\}$. By Subcase 1.2, we have $h(x) = h(x + te_{k+1} - te_{k+1}) = h(x + te_{k+1})$, which implies (2.1).

Case 2. $p = 2$.

By the property of homogeneous function (see [21, Section 9]), there exists two constants $\Theta_1, \Theta_2 \geq 0$ and two functions g_1, g_2 defined on the unit sphere $S^{n-1} \subset \mathbf{R}^n$ such that

$$\begin{aligned} h(x) &= \Theta_1 \log |x| + g_1 \left(\frac{x}{|x|} \right) \\ &= \Theta_2 \log |x - e_{k+1}| + g_2 \left(\frac{x - e_{k+1}}{|x - e_{k+1}|} \right). \end{aligned}$$

First, let us prove $\Theta_1 = \Theta_2$. For any point $y \notin span\{e_{k+1}\}$ such that $h(y) > -\infty$, by similar calculations in Subcase 1.2, for any $t < 1$, we obtain

$$h(y + te_{k+1}) = h(y) + (\Theta_2 - \Theta_1) \log(1 - t). \tag{2.4}$$

Since $h \not\equiv -\infty$, there exists a point $x_0 \notin span\{e_{k+1}\}$ such that $h(x_0) > -\infty$. By (2.4), we have

$$h\left(x_0 + \frac{1}{3} e_{k+1} \right) = h(x_0) + (\Theta_2 - \Theta_1) \log \frac{2}{3} \tag{2.5}$$

and

$$h\left(x_0 + \frac{2}{3} e_{k+1} \right) = h(x_0) + (\Theta_2 - \Theta_1) \log \frac{1}{3}. \tag{2.6}$$

By (2.5), we obtain $h(x_0 + \frac{1}{3} e_{k+1}) > -\infty$. Combining this and (2.4), it is clear that

$$h\left(x_0 + \frac{1}{3} e_{k+1} + \frac{1}{3} e_{k+1} \right) = h\left(x_0 + \frac{1}{3} e_{k+1} \right) + (\Theta_2 - \Theta_1) \log \frac{2}{3}. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), we get $\Theta_1 = \Theta_2$. Next, by the similar argument of Case 1, we obtain (2.1). □

Lemma 2.2. *Let u_i be a sequence of F-subharmonic functions on $B_2(0^n)$ with $\|u_i\|_{L^1(B_2(0^n))} \leq \Lambda$. Then there exists a subsequence u_{i_k} such that u_{i_k} converges to u in $L^1_{loc}(B_2(0^n))$, where u is a F-subharmonic function on $B_2(0^n)$.*

Proof. Every F-subharmonic function is subharmonic function (see [21, (6.3)]). By the compactness of subharmonic functions, there exists a subsequence u_{i_k} converges to u in $L^1_{loc}(B_2(0^n))$. On the other hand, F is a subequation satisfying ST-Invariance and Convexity, which implies that F is regular (see [18, Section 8]) and can not be defined using fewer of the independent variables (see [21, Proof of Proposition 9.4]). Since u_{i_k} is F-subharmonic, we obtain that u is a F-subharmonic distribution (see [18, Definition 2.3]). By [18, Theroem 1.1] or [21, Theorem 9.5], there exists a F-subharmonic function v in the L^1_{loc}-class u. It suffices to prove $u = v$ in $B_2(0^n)$. Since u and v are subharmonic, for any $x \in B_2(0^n)$, we obtain

$$u(x) = \lim_{s\to\infty} \frac{1}{\omega_n s^n}\int_{B_s(x)} u(y)dy = \lim_{s\to\infty} \frac{1}{\omega_n s^n}\int_{B_s(x)} v(y)dy = v(x),$$

as required. □

Lemma 2.3 (Cone-splitting lemma). *Let u be a F-subharmonic function on $B_2(0^n)$ with $\|u\|_{L^1(B_2(0^n))} \leq \Lambda$. For any $\epsilon, \tau > 0$, there exists constant $\delta(\epsilon, \tau, \Lambda, F)$ such that if*

(1) u is $(k, \delta, 1, 0^n)$-homogeneous with respect to k-plane V^k;

(2) u is $(0, \delta, 1, y)$-homogeneous, where $y \in B_1(0^n) \setminus B_\tau(V^k)$,

then u is $(k+1, \epsilon, 1, 0^n)$-homogeneous.

Proof. We argue by contradiction, assuming that there exists a sequence of F-subharmonic functions u_i with $\|u\|_{L^1(B_2(0^n))} \leq \Lambda$ and satisfy the following properties:

(1) u_i is $(k, i^{-1}, 1, 0^n)$-homogeneous with respect to k-plane V_i^k;

(2) u_i is $(0, i^{-1}, 1, y_i)$-homogeneous, where $y_i \in B_1(0^n) \setminus B_\tau(V_i^k)$;

(3) u_i is not $(k+1, \epsilon, 1, 0^n)$-homogeneous.

After passing to a subsequence, we assume that $\lim_{i\to\infty} V_i^k = V^k$, $\lim_{i\to\infty} y_i = y \in \overline{B_1(0^n)} \setminus B_{2\tau}(V^k)$ and u_i converges to u in $L^1_{loc}(B_2(0^n))$, where u is a F-subharmonic function (see Lemma 2.2). By (1), (2) and Lemma 8.1, there exists a function h such that

(a) h is k-homogeneous at 0^n with respect to V^k;

(b) h is 0-homogeneous at y;

(c) $h = u$ in $B_2(0^n)$.

Hence, by Theorem 2.1, we obtain that h is a $(k+1)$-homogeneous function. Combining this with u_i converges to u in $L^1_{loc}(B_2(0^n))$ and (c), it is clear that u_i is $(k+1, \epsilon, 1, 0^n)$-homogeneous when i is sufficiently large, which is a contradiction. $\square$

3. Top stratification of $\mathcal{S}(u)$

In this section, we give proofs of (1) and (2) in Theorem 1.12.

Proof of (1) in Theorem 1.12. For any $r \in (0, \frac{1}{5})$, $\{B_r(x)\}_{x \in E_\eta(u) \cap B_1(0^n)}$ is a covering of $B_r(E_\eta(u) \cap B_1(0^n))$, where $E_\eta(u) = \{x \in B_2(0^n) \mid \Theta^S(u, x) \geq \eta\}$. We take a Vitali covering $\{B_r(x_i)\}_{i=1}^M$ such that

(a) $B_r(x_i) \cap B_r(x_j) = \emptyset$ for any $i \neq j$;

(b) $B_r(E_\eta(u) \cap B_1(0^n)) \subset \bigcup_i B_{5r}(x_i)$;

(c) $x_i \in E_\eta(u) \cap B_1(0^n)$ for each i.

For each x_i, by the properties of $S(u, x_i, \cdot)$ (see [21, Corollary 5.3, Theorem 6.4]), we have

$$\lim_{t \to 0} \frac{S'_-(u, x_i, t)}{K'_p(t)} = \Theta^S(u, x_i)$$

and

$$\frac{S'_-(u, x_i, t)}{K'_p(t)} \quad \textit{is nondecreasing with respect to } t.$$

Since $x_i \in E_\eta(u) \cap B_1(0^n)$, it then follows that

$$\frac{S'_-(u, x_i, r)}{K'_p(r)} \geq \Theta^S(u, x_i) \geq \eta.$$

Using $S'_-(u, x_i, r) = C(n) K'_n(r) \int_{B_r(x_i)} \Delta u$ (see e.g. [24, Theorem 3.2.16] or [21, (7.8)]), it is clear that

$$\int_{B_r(x_i)} \Delta u \geq C(p, n) \eta r^{n-p}.$$

By (a), we obtain

$$\int_{B_{1+r}(0^n)} \Delta u \geq \sum_{i=1}^{M} \int_{B_r(x_i)} \Delta u \geq C(p, n) \eta M r^{n-p}. \tag{3.1}$$

Combining (b) and (3.1), we get

$$\begin{aligned}\operatorname{Vol}(B_r(E_\eta(u)\cap B_1(0^n))) &\leq \sum_{i=1}^{M}\operatorname{Vol}(B_{5r}(x_i)) \\ &\leq C(p,n)\eta^{-1}\left(\int_{B_{1+r}(0^n)}\Delta u\right)r^p.\end{aligned} \tag{3.2}$$

On the other hand, for every $y\in\mathcal{S}^{n-p}_\eta(u)\cap B_1(0^n)$, since 0 is a $(n-p+1)$-homogeneous function, by the definition of $\mathcal{S}^{n-p}_\eta(u)$, we have

$$\|u_{y,r}-0\|_{L^1(B_1(0^n))}\geq\eta, \tag{3.3}$$

for any $r\in(0,1)$. Now, we take $U\in T_y(u)$. Combining (3.3) and the definition of tangent, it is clear that $\|U\|_{L^1(B_1(0^n))}\geq\eta$. By [21, Theorem 10.1], there exists a constant $\tilde{C}(p,n)$ such that

$$\tilde{C}(p,n)\Theta^S(u,y)\geq-\int_{B_1(0^n)}U=\|U\|_{L^1(B_1(0^n))}\geq\eta,$$

which implies $\mathcal{S}^{n-p}_\eta(u)\cap B_1(0^n)\subset E_{\tilde{C}^{-1}\eta}(u)\cap B_1(0^n)$. Then by (3.2) (replace η by $\tilde{C}^{-1}\eta$), we obtain

$$\operatorname{Vol}(B_r(\mathcal{S}^{n-p}_\eta(u)\cap B_1(0^n)))\leq C(p,n)\tilde{C}(p,n)\eta^{-1}\left(\int_{B_{1+r}(0^n)}\Delta u\right)r^p,$$

as required. □

Proof of (2) in Theorem 1.12. We argue by contradiction, assuming that there exists a point $x\in\mathcal{S}(u)\setminus\mathcal{S}^{n-p}(u)$. By definition, there exists $\varphi\in T_x(u)$ such that φ is $(n-p+1)$-homogeneous but not n-homogeneous. It is clear that

$$dim_H(\mathcal{S}(\varphi))\geq n-p+1,$$

where $dim_H(\mathcal{S}(\varphi))$ denotes the Hausdorff dimension of $\mathcal{S}(\varphi)$. By (3.2) (replace u by φ), we get $dim_H(E_\eta(\varphi)\cap B_1(0^n))\leq n-p$. By the similar argument, it is clear that $dim_H(E_\eta(\varphi))\leq n-p$. Since $\mathcal{S}(\varphi)=\bigcup_\eta E_\eta(\varphi)$, we get

$$dim_H(\mathcal{S}(\varphi))\leq n-p,$$

which is a contradiction. □

4. Hausdorff dimension of $\mathcal{S}^k(u)$

In this section, we study the Hausdorff dimension of $\mathcal{S}^k(u)$. We use an iterated blow up argument as in [2] to prove (3) of Theorem 1.12. For convenience, we use $T_x(u)$ to denote the tangent set to u at x in the following argument.

Lemma 4.1. *Let h be a F-subharmonic function which is k-homogeneous at 0^n with respect to k-plane V^k. For any $x_0 \notin V^k$, if $\varphi \in T_{x_0}(h)$, then φ is $(k+1)$-homogeneous at 0^n with respect to $(k+1)$-plane $V^{k+1} = span\{x_0, V^k\}$.*

Proof. By the definition of tangent, there exists a sequence $\{r_i\}$ ($\lim_{i\to\infty} r_i = 0$) such that h_{x_0,r_i} converges to φ in $L^1_{loc}(\mathbf{R}^n)$. Since φ is subharmonic, in order to prove Lemma 4.1, it suffices to prove

$$\int_{B_r(y)} \varphi(x)dx = \int_{B_r(y+v)} \varphi(x)dx, \tag{4.1}$$

for any $y \in \mathbf{R}^n$, $v \in V^{k+1}$ and $r > 0$. First, we consider the case $p > 2$.

Case 1. $p > 2$.

We split up into different subcases.

Subcase 1.1. $v = \lambda x_0$ for some $\lambda \in \mathbf{R}$.

By direct calculations, we have

$$\begin{aligned}\int_{B_r(y+v)} \varphi(x)dx &= \int_{B_r(y+\lambda x_0)} \varphi(x)dx \\ &= \lim_{i\to\infty} \int_{B_r(y+\lambda x_0)} h_{x_0,r_i}(x)dx \\ &= \lim_{i\to\infty} \int_{B_r(y+\lambda x_0)} r_i^{p-2} h(x_0 + r_i x)dx \\ &= \lim_{i\to\infty} \int_{B_r(0^n)} r_i^{p-2} h(x_0 + r_i x + r_i y + \lambda r_i x_0)dx.\end{aligned} \tag{4.2}$$

Since h is homogeneous, it is clear that

$$
\begin{aligned}
(4.3)\qquad & \int_{B_r(0^n)} r_i^{p-2} h(x_0 + r_i x + r_i y + \lambda r_i x_0)dx \\
&= \int_{B_r(0^n)} (1+\lambda r_i)^{2-p} r_i^{p-2} h\left(x_0 + \frac{r_i x}{1+\lambda r_i} + \frac{r_i y}{1+\lambda r_i}\right)dx \\
&= \int_{B_{\frac{r}{1+\lambda r_i}}(\frac{y}{1+\lambda r_i})} (1+\lambda r_i)^{n+2-p} h_{x_0,r_i}(x)dx.
\end{aligned}
$$

On the other hand, since h_{x_0,r_i} converges to φ in $L^1_{loc}(\mathbf{R}^n)$, it then follows that

$$
\begin{aligned}
(4.4)\qquad & \lim_{i\to\infty} \int_{B_{\frac{r}{1+\lambda r_i}}(\frac{y}{1+\lambda r_i})} (1+\lambda r_i)^{n+2-p} |h_{x_0,r_i}(x) - \varphi(x)|dx \\
&\leq \lim_{i\to\infty} \int_{B_{r+1}(y)} 2|h_{x_0,r_i}(x) - \varphi(x)|dx \\
&= 0.
\end{aligned}
$$

Combining (4.2), (4.3) and (4.4), we obtain

$$
\begin{aligned}
& \left| \int_{B_r(y+v)} \varphi(x)dx - \int_{B_r(y)} \varphi(x)dx \right| \\
&= \left| \lim_{i\to\infty} \int_{B_{\frac{r}{1+\lambda r_i}}(\frac{y}{1+\lambda r_i})} (1+\lambda r_i)^{n+2-p} h_{x_0,r_i}(x)dx - \int_{B_r(y)} \varphi(x)dx \right| \\
&\leq \lim_{i\to\infty} \int_{B_{\frac{r}{1+\lambda r_i}}(\frac{y}{1+\lambda r_i})} (1+\lambda r_i)^{n+2-p} |h_{x_0,r_i}(x) - \varphi(x)|dx \\
&\quad + \left| \lim_{i\to\infty} \int_{B_{\frac{r}{1+\lambda r_i}}(\frac{y}{1+\lambda r_i})} (1+\lambda r_i)^{n+2-p} \varphi(x)dx - \int_{B_r(y)} \varphi(x)dx \right| \\
&\leq 0,
\end{aligned}
$$

where we used Lebesgue's dominated convergence theorem for the last inequality. This completes the proof of Subcase 1.1.

Subcase 1.2. $v \in V^k$.

By similar calculations in Subcase 1.1, we have

$$
(4.5)\qquad \int_{B_r(y+v)} \varphi(x)dx = \lim_{i\to\infty} \int_{B_r(0^n)} r_i^{p-2} h(x_0 + r_i x + r_i y + r_i v)dx.
$$

Since h is k-homogeneous with respect to k-plane V^k, it is clear that

$$\begin{aligned}(4.6)\qquad &\int_{B_r(0^n)} r_i^{p-2} h(x_0 + r_i x + r_i y + r_i v)dx \\ &= \int_{B_r(0^n)} r_i^{p-2} h(x_0 + r_i x + r_i y)dx \\ &= \int_{B_r(y)} h_{x_0,r_i}(x)dx.\end{aligned}$$

Combining (4.5), (4.6) and h_{x_0,r_i} converges to φ in $L^1_{loc}(\mathbf{R}^n)$, we get (4.1), which completes the proof of Subcase 1.2.

Next, we consider the case $p = 2$.

Case 2. $p = 2$.

Similarly, we split up into different subcases.

Subcase 2.1. $v = \lambda x_0$ for some $\lambda \in \mathbf{R}$.

By the definition of tangential 2-flow (see Definition 1.3), we have

$$\begin{aligned}\int_{B_r(y+v)} \varphi(x)dx &= \int_{B_r(y+\lambda x_0)} \varphi(x)dx \\ &= \lim_{i\to\infty} \int_{B_r(y+\lambda x_0)} h_{x_0,r_i}(x)dx \\ &= \lim_{i\to\infty} \int_{B_r(y+\lambda x_0)} \left(h(x_0 + r_i x) - M(h, x_0, r_i)\right) dx \\ &= \lim_{i\to\infty} \int_{B_r(0^n)} \left(h(x_0 + r_i x + r_i y + \lambda r_i x_0) - M(h, x_0, r_i)\right) dx.\end{aligned}$$

By the homogeneity of h, we obtain

$$\begin{aligned}&\int_{B_r(0^n)} \left(h(x_0 + r_i x + r_i y + \lambda r_i x_0) - M(h, x_0, r_i)\right) dx \\ &= \int_{B_r(0^n)} \Bigg(h\left(x_0 + \frac{r_i x}{1+\lambda r_i} + \frac{r_i y}{1+\lambda r_i}\right) \\ &\qquad + M(h, 0^n, 1+\lambda r_i) - M(h, x_0, r_i) \Bigg) dx \\ &= \int_{B_{\frac{r}{1+\lambda r_i}}\left(\frac{y}{1+\lambda r_i}\right)} h_{x_0,r_i}(x)dx + \int_{B_r(0^n)} M(h, 0^n, 1+\lambda r_i)dx.\end{aligned}$$

Since h is homogeneous, we get $M(h,0^n,1)=0$. By the continuity of $M(h,0^n,\cdot)$, it is clear that

$$\int_{B_r(y+v)}\varphi(x)dx=\lim_{i\to\infty}\int_{B_{\frac{r}{1+\lambda r_i}}(\frac{y}{1+\lambda r_i})}h_{x_0,r_i}(x)dx.$$

By the similar argument in Subcase 1.1, we complete the proof of Subcase 2.1.

Subcase 2.2. $v\in V^k$. The proof of Subcase 2.2 is similar to the proof of Subcase 1.2. □

Lemma 4.2. *Let u be a F-subharmonic function on $B_2(0^n)$. If $Haus^l(\mathcal{S}^k(u))>0$ for $l>k$, then $Haus^l(A)>0$, where*

$$A:=\{y\in B_2(0^n)\mid \textit{there exists a tangent } \varphi\in T_y(u) \textit{ such that } Haus^l(\mathcal{S}^k(\varphi))>0\}.$$

Proof. Combining $\mathrm{Haus}^l(\mathcal{S}^k(u))>0$ and $\mathcal{S}^k(u)=\bigcup_\eta\mathcal{S}^k_\eta(u)$ (see (1.2)), there exists a constant $\eta_0>0$ such that $\mathrm{Haus}^l(\mathcal{S}^k_{\eta_0}(u))>0$. By the property of Hausdorff measure, we have $\mathrm{Haus}^l(\mathcal{S}^k_{\eta_0}(u)\setminus D^l_{\eta_0}(u))=0$, where

$$D^l_{\eta_0}(u)=\left\{x\in\mathcal{S}^k_{\eta_0}(u)\mid \limsup_{r\to 0}\frac{\mathrm{Haus}^l_\infty(\mathcal{S}^k_{\eta_0}(u)\cap B_r(x))}{\omega_l r^l}\geq 2^{-l}\right\}.$$

Therefore, in order to prove Lemma 4.2, it suffices to prove that there exists a tangent $\varphi\in T_y(u)$ such that $\mathrm{Haus}^l(\mathcal{S}^k(\varphi))>0$ for any $y\in D^l_{\eta_0}(u)$. By the definition of $D^l_{\eta_0}(u)$, there exists a sequence of $\{r_j\}$ ($\lim_{i\to\infty}r_j=0$) such that

$$\lim_{j\to\infty}\frac{\mathrm{Haus}^l_\infty(\mathcal{S}^k_{\eta_0}(u)\cap B_{r_j}(y))}{\omega_l r_j^l}\geq 2^{-l}.$$

If $y+r_iz\in\mathcal{S}^k_{\eta_0}(u)\cap B_{r_j}(y)$, then $z\in\mathcal{S}^k_{\eta_0}(u_{y,r_j})\cap B_1(0^n)$. Combining this and the definition of Hausdorff measure, we have

$$\lim_{j\to\infty}\mathrm{Haus}^l_\infty(\mathcal{S}^k_{\eta_0}(u_{y,r_j})\cap B_1(0^n))\geq 2^{-l}.$$

After passing to a subsequence, we can assume that u_{y,r_j} converges to $\varphi\in T_y(u)$ in $L^1_{loc}(\mathbf{R}^n)$.

Claim. If $z_j\in\mathcal{S}^k_{\eta_0}(u_{y,r_j})$ and $\lim_{j\to\infty}z_j=z$, then $z\in\mathcal{S}^k_{\eta_0}(\varphi)$.

Proof of Claim. For any $r \in (0,1)$ and $(k+1)$-homogeneous function h, we have

$$\begin{aligned}\int_{B_1(0^n)} |\varphi_{z,r}(x) - h(x)|dx \geq & \int_{B_1(0^n)} |(u_{y,r_j})_{z_j,r}(x) - h(x)|dx \\ & - \int_{B_1(0^n)} |\varphi_{z_j,r}(x) - (u_{y,r_j})_{z_j,r}(x)|dx \\ & - \int_{B_1(0^n)} |\varphi_{z,r}(x) - \varphi_{z_j,r}(x)|dx.\end{aligned}$$

Letting $j \to \infty$, by Lemma 8.8, we obtain

$$\int_{B_1(0^n)} |\varphi_{z,r}(x) - h(x)|dx \geq \eta_0,$$

which implies $z \in \mathcal{S}^k_{\eta_0}(\varphi)$. We complete the proof of Claim. □

Combining Claim and the property of Hausdorff measure, it is clear that

$$\mathrm{Haus}^l(\mathcal{S}^k_{\eta_0}(\varphi) \cap B_1(0^n)) \geq \lim_{j\to\infty} \mathrm{Haus}^l_\infty(\mathcal{S}^k_{\eta_0}(u_{y,r_j}) \cap B_1(0^n)) \geq 2^{-l} > 0,$$

as desired. □

Theorem 4.3. *Let u be a F-subharmonic function on $B_2(0^n)$. Then for any $1 \leq k \leq n$, we have*

$$dim_H(\mathcal{S}^k(u)) \leq k.$$

Proof. We argue by contradiction. Suppose that $\mathrm{Haus}^l(\mathcal{S}^k(u)) > 0$ for some $l > k$. By Lemma 4.2, there exists $y_0 \in \mathcal{S}^k(u)$ and $\varphi_0 \in T_{y_0}(u)$ such that $\mathrm{Haus}^l(\mathcal{S}^k(\varphi_0)) > 0$. We assume that φ_0 is m-homogeneous with respect to m-plane V_0^m, where $m \leq k$. By Lemma 4.2, $\mathrm{Haus}^l(\mathcal{S}^k(\varphi_0)) > 0$ and $m \leq k < l$, there exists $y_1 \in \mathcal{S}^k(\varphi_0) \setminus V_0^m$ and $\varphi_1 \in T_{y_1}(\varphi_0)$ such that $\mathrm{Haus}^l(\mathcal{S}^k(\varphi_1)) > 0$. By Lemma 4.1, we obtain that φ_1 is $(m+1)$-homogeneous with respect to $(m+1)$-plane $V_1^{m+1} = span\{V_0^m, y_1\}$. Repeating this process, there exist $y_{k-m+1} \in \mathcal{S}^k(\varphi_{k-m}) \setminus V^k_{k-m}$ and $\varphi_{k-m+1} \in T_{y_{k-m+1}}(\varphi_{k-m})$ such that φ_{k-m+1} is $(k+1)$-homogeneous, which contradicts with the definition of $\mathcal{S}^k(\varphi_{k-m})$. □

5. Rectifiability of $\mathcal{S}^k(u)$

In this section, we prove the k^{th} stratification $\mathcal{S}^k(u)$ is k-rectifiable when uniqueness of tangents holds for F (i.e., Theorem 1.13). Let u be a F-subharmonic function on $B_2(0^n)$ with $\|u\|_{L^1(B_1(0^n))} \leq \Lambda$. First, we define

$$F_{\delta,\eta}(u) = \{x \in B_2(0^n) \mid u \text{ is } (0,\delta,r,x)\text{-homogeneous for any } r \in (0,\eta)\}.$$

For any $x \in \big(F_{\delta,\eta}(u) \cap \mathcal{S}^k_\epsilon(u)\big) \setminus \mathcal{S}^{k-1}(u)$, where $\epsilon > 0$, let φ be the unique tangent to u at x. We assume φ is k-homogeneous with respect to k-plane V^k_φ. It then follows that $\|\varphi\|_{L^1(B_2(0^n))} \leq \Lambda_1(\Lambda, F)$.

Lemma 5.1. *For any $\tau \in (0,1)$, there exists r_x such that for any $r < r_x$, we have*

$$F_{\delta,1}(u_{x,r}) \subset B_{2\tau}(V^k_\varphi),$$

where $\delta = \delta(\epsilon, 2\tau, \Lambda_1, F)$ is the constant in Lemma 2.3.

Proof. We argue by contradiction, assuming that there exist $\{r_i\}$ and $\{z_i\}$ such that $\lim_{i\to\infty} r_i = 0$ and $z_i \in F_{\delta,1}(u_{x,r_i}) \setminus B_{2\tau}(V^k_\varphi)$. It then follows that there exists homogeneous function h_i such that

$$\int_{B_1(0^n)} |(u_{x,r_i})_{z_i,r}(y) - h_i(y)|dy \leq \delta$$

for any $r \in (0,1)$. Since u_{x,r_i} converges to φ in $L^1_{loc}(\mathbf{R}^n)$, by Lemma 8.8, after passing to a subsequence, we can assume that $\lim_{i\to\infty} z_i = z$ and h_i converges to h in $L^1_{loc}(B_2(0^n))$. For any $r \in (0,1)$, by Lemma 8.8, we have

$$\begin{aligned}
\int_{B_1(0^n)} |\varphi_{z,r}(y) - h(y)|dy \leq{}& \lim_{i\to\infty} \int_{B_1(0^n)} |\varphi_{z,r}(y) - (u_{x,r_i})_{z_i,r}(y)|dy \\
&+ \lim_{i\to\infty} \int_{B_1(0^n)} |(u_{x,r_i})_{z_i,r}(y) - h_i(y)|dy \\
&+ \lim_{i\to\infty} \int_{B_1(0^n)} |h_i(y) - h(y)|dy \\
\leq{}& \delta
\end{aligned}$$

which implies $z \in F_{\delta,1}(\varphi) \setminus B_{2\tau}(V^k_\varphi)$. However, by Lemma 2.3 and $x \in S^k_\epsilon(u)$, we get $F_{\delta,1}(\varphi) \subset B_\tau(V^k_\varphi)$, which is a contradiction. $\square$

Lemma 5.2. *For any $r \le r_x$, we have*

$$F_{\delta,r}(u) \cap B_r(x) \subset B_{2\tau r}(V_\varphi^k + x)$$

Proof. For any $x + rz \in F_{\delta,r}(u) \cap B_r(x)$, where $z \in B_1(0^n)$, there exists homogeneous function h such that for any $s \in (0, r)$, we have

$$\int_{B_1(0^n)} |u_{x+rz,s}(y) - h(y)| dy \le \delta$$

It then follows that

$$\int_{B_1(0^n)} |(u_{x,r})_{z,\frac{s}{r}}(y) - h(y)| dy \le \delta,$$

which implies $z \in F_{\delta,1}(u_{x,r})$. Combining this with Lemma 5.1, we have $x + rz \in B_{2\tau r}(V_\varphi^k + x)$. □

Now, we are in a position to prove Theorem 1.13.

Proof of Theorem 1.13. For any $\eta > 0$ and $x \in \big(F_{\delta,\eta}(u) \cap \mathcal{S}_\epsilon^k(u)\big) \setminus \mathcal{S}^{k-1}(u)$, by Lemma 5.2, there exists $r_x \le \eta$ such that for any $r < r_x$, we have $F_{\delta,r}(u) \cap B_r(x) \subset B_{2\tau r}(V_\varphi^k + x)$, which implies

$$\left(\left(F_{\delta,\eta}(u) \cap \mathcal{S}_\epsilon^k(u)\right) \setminus \mathcal{S}^{k-1}(u)\right) \cap B_r(x) \subset B_{2\tau r}(V_\varphi^k + x).$$

Hence, $\big(F_{\delta,\eta}(u) \cap \mathcal{S}_\epsilon^k(u)\big) \setminus \mathcal{S}^{k-1}(u)$ is k-rectifiable (see e.g. [32, p.61, Lemma 1]). Since uniqueness of tangents holds for F, we have $\mathcal{S}_\epsilon^k(u) = \cup_\eta (F_{\delta,\eta}(u) \cap \mathcal{S}_\epsilon^k(u))$. By (1.2), it then follows that

$$\begin{aligned} \mathcal{S}^k(u) \setminus \mathcal{S}^{k-1}(u) &= \bigcup_\epsilon \left(\mathcal{S}_\epsilon^k(u) \setminus \mathcal{S}^{k-1}(u)\right) \\ &= \bigcup_\epsilon \bigcup_\eta \left(\left(F_{\delta,\eta}(u) \cap \mathcal{S}_\epsilon^k(u)\right) \setminus \mathcal{S}^{k-1}(u)\right), \end{aligned}$$

which implies $\mathcal{S}^k(u) \setminus \mathcal{S}^{k-1}(u)$ is k-rectifiable. On the other hand, since uniqueness of tangents holds for F implies homogeneity of tangents holds for F, by (3) of Theorem 1.12, we have $\mathrm{Haus}^k(\mathcal{S}^{k-1}(u)) = 0$. It then follows that $\mathcal{S}^{k-1}(u)$ is k-rectifiable. Hence, $\mathcal{S}^k(u) = \big(\mathcal{S}^k(u) \setminus \mathcal{S}^{k-1}(u)\big) \cup \mathcal{S}^{k-1}(u)$ is k-rectifiable. □

6. F-subharmonic functions

In this section, we consider the singular sets of F-subharmonic functions and give the proof of Theorem 1.14. We assume that strong uniqueness of tangents holds for F and $p > 2$, where p is the Riesz characteristic of F. By [21, Proposition 7.1 and (12.3)], all density functions are equivalent, i.e., $\Theta^M = \Theta^S = \frac{n-p+2}{n}\Theta^V$. For convenience, if u is a F-subharmonic function on $B_2(0^n)$, we use $E_c(u)$ to denote the set $\{x \in B_2(0^n) \mid \Theta^V(u,x) \geq c\}$ in this section.

6.1. Monotonicity condition and F-energy

In this subsection, we introduce the monotonicity condition and F-energies of F-subharmonic functions. And then we prove every F-subharmonic function satisfies monotonicity condition after subtracting a constant.

Definition 6.1. Let u be a F-subharmonic function on $B_2(0^n)$. We say that u satisfies monotonicity condition if F-energy defined by

$$\theta_F(u,x,r) := \frac{S(u,x,r)}{K_p(r)} + \frac{M(u,x,r)}{K_p(r)}$$

is nondecreasing in $r \in (0, \frac{1}{2})$ for any $x \in B_1(0^n)$. And we define $\theta_F(u,x,0) = \lim_{r\to 0}\theta_F(u,x,r)$

Lemma 6.2. *Let u be a F-subharmonic function on $B_2(0^n)$ with $\|u\|_{L^1(B_2(0^n))} \leq \Lambda$. Then there exists constant $N(\Lambda,p,n)$ such that $u - N$ satisfies monotonicity condition.*

Proof. For any $x \in B_1(0^n)$, since $S(u,x,\cdot)$ is K_p-convex on $(0,1)$ (see [21, Theorem 6.4]). Then we have

$$S(u,x,r) = f(K_p(r)),$$

where f is a convex function on $(-\infty,-1)$. It follows that

$$f'_+(K_p(\frac{1}{2})) \leq \frac{f(K_p(\frac{2}{3})) - f(K_p(\frac{1}{2}))}{K_p(\frac{2}{3}) - K_p(\frac{1}{2})} = \frac{S(u,x,\frac{2}{3}) - S(u,x,\frac{1}{2})}{K_p(\frac{2}{3}) - K_p(\frac{1}{2})}. \tag{6.1}$$

Since F-subharmonic function is subharmonic (see [21, (6.3)]), by $\|u\|_{L^1(B_2(0^n))} \leq \Lambda$, Lemma 8.5 and the submean value property, we see that

$$-C(\Lambda, n) \leq S(u, x, r) \leq C(\Lambda, n) \quad \text{for } r \in \left(0, \frac{4}{5}\right), \tag{6.2}$$

where $C(\Lambda, n)$ is a constant. Substituting (6.2) into (6.1), it is clear that

$$f'_+\left(K_p\left(\frac{1}{2}\right)\right) \leq \tilde{N}(\Lambda, p, n), \tag{6.3}$$

where $\tilde{N}(\Lambda, p, n)$ is a constant. Using (6.2), (6.3) and $K_p(\frac{1}{2}) < 0$, there exists a constant $N(\tilde{N}, \Lambda, p)$ such that

$$\begin{aligned} f\left(\frac{1}{2}\right) - f'_+\left(K_p\left(\frac{1}{2}\right)\right) K_p\left(\frac{1}{2}\right) &= S\left(u, x, \frac{1}{2}\right) - f'_+\left(K_p\left(\frac{1}{2}\right)\right) K_p\left(\frac{1}{2}\right) \\ &\leq N(\tilde{N}, \Lambda, p). \end{aligned}$$

Figure 1 is the graph of f on $(-\infty, K_p(\frac{1}{2}))$. The red line is tangent line of f at $K_p(\frac{1}{2})$.

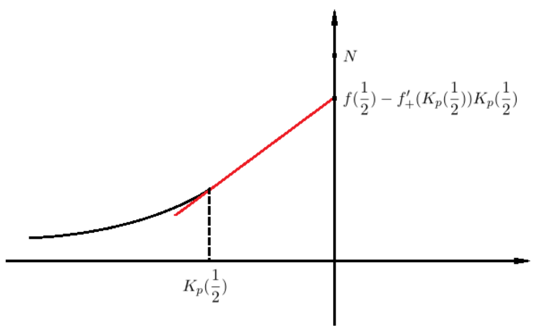

Figure 1.

In Figure 2, by the convexity of f, the slope of line 1 is larger than that of line 2. It follows that

$$\frac{f(K_p(r)) - N}{K_p(r) - 0} \geq \frac{f(K_p(s)) - N}{K_p(s) - 0} \quad \text{for } 0 < s < r < \frac{1}{2}.$$

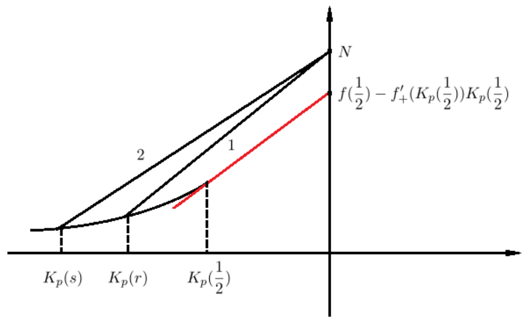

Figure 2.

Hence,

$$\frac{S(u-N,x,r)}{K_p(r)} = \frac{S(u,x,r)-N}{K_p(r)}$$

is nondecreasing in $r \in (0, \frac{1}{2})$. Similarly, by increasing the value of N (if necessary), we can prove $\frac{M(u-N,x,r)}{K_p(r)}$ is also nondecreasing. □

Remark 6.3. In [21], Harvey-Lawson proved the same result except for the dependence of the constants on L^1 norm (see [21, Lemma 5.4]).

6.2. Quantitative rigidity results

In this subsection, we prove some quantitative rigidity results of F-subharmonic functions.

Lemma 6.4. *Let u_i and u be F-subharmonic functions on $B_2(0^n)$. For $c > 0$, if u_i converges to u in $L^1_{loc}(B_2(0^n))$ and x_i converges to x, where $x_i \in E_c(u_i) \cap \overline{B_1(0^n)}$, then*

$$x \in E_c(u) \cap \overline{B_1(0^n)}.$$

Proof. For any $t > 0$, we compute

$$\begin{aligned}
&|V(u,x,t) - V(u_i,x_i,t)| \\
&\quad \le \frac{1}{\omega_n t^n}\int_{B_t(x_i)} |u(y) - u_i(y)|dy \\
&\qquad + \frac{1}{\omega_n t^n}\left|\int_{B_t(x)} u(y)dy - \int_{B_t(x_i)} u(y)dy\right| \\
&\quad \le \frac{1}{\omega_n t^n}\int_{B_{1+t}(0^n)} |u(y) - u_i(y)|dy \\
&\qquad + \frac{1}{\omega_n t^n}\left|\int_{B_t(x)} u(y)dy - \int_{B_t(x_i)} u(y)dy\right|.
\end{aligned}$$

which implies

$$\lim_{i\to\infty} V(u_i,x_i,t) = V(u,x,t).$$

Therefore, for any $0 < s < r < \frac{1}{2}$, we obtain

$$\frac{V(u,x,r) - V(u,x,s)}{K_p(r) - K_p(s)} = \lim_{i\to\infty} \frac{V(u_i,x_i,r) - V(u_i,x_i,s)}{K_p(r) - K_p(s)} \ge c,$$

where we used the condition $x_i \in E_c(u_i) \cap \overline{B_1(0^n)}$. By the definition of density function Θ^V (see [21, Corollary 5.3]), we obtain $\Theta(u,x) \ge c$. This completes the proof. $\square$

Lemma 6.5. *Let u be a F-subharmonic function on $B_2(0^n)$ with $\|u\|_{L^1(B_2(0^n))} \le \Lambda$ and satisfies monotonicity condition. For any $\epsilon > 0$, there exists constant $\delta_0(\epsilon, \Lambda, F)$ such that if*

$$\theta_F\left(u, 0^n, \frac{1}{2}\right) - \theta_F(u, 0^n, \delta_0) < \delta_0,$$

then u is $(0, \epsilon, 2, 0^n)$-homogeneous.

Proof. We argue by contradiction. Assuming that there exists a sequence of F-subharmonic function u_i on $B_2(0^n)$ such that

(1) $\|u_i\|_{L^1(B_2(0^n))} \le \Lambda$;

(2) u_i satisfies monotonicity condition;

(3) $\theta_F(u_i, 0^n, \frac{1}{2}) - \theta_F(u_i, 0^n, i^{-1}) < i^{-1}$;

(4) u_i is not $(0, \epsilon, 2, 0^n)$-homogeneous.

By Lemma 2.2, after passing to a subsequence, we can assume u_i converges to u in $L^1_{loc}(B_2(0^n))$, where u is a F-subharmonic function. By [21, Lemma 6.5] (or Lemma 8.6), it is clear that u also satisfies monotonicity condition. For each $t \in (0, \frac{1}{2})$, we obtain

$$\begin{aligned}
\theta_F(u, 0^n, \frac{1}{2}) - \theta_F(u, 0^n, t) &= \frac{S(u, 0^n, \frac{1}{2})}{K_p(\frac{1}{2})} - \frac{S(u, 0^n, t)}{K_p(t)} \\
&\quad + \frac{M(u, 0^n, \frac{1}{2})}{K_p(\frac{1}{2})} - \frac{M(u, 0^n, t)}{K_p(t)} \\
&= \lim_{i\to\infty} \left(\theta_F(u_i, 0^n, \frac{1}{2}) - \theta_F(u_i, 0^n, t) \right) \\
&\leq 0,
\end{aligned}$$

which implies

$$S(u, 0^n, r) = \Theta^S(u, 0^n) K_p(r) \ \text{ and } \ M(u, 0^n, r) = \Theta^M(u, 0^n) K_p(r)$$

for any $r \in (0, \frac{1}{2})$, where Θ^S and Θ^M are S-density and M-density (see [21, Section 6]). Since strong uniqueness holds for u, then $\Theta^S = \Theta^M$ (see [21, (12.3)]). It follows that

$$S(u, 0^n, r) = M(u, 0^n, r).$$

Hence, for each sphere $\partial B_r(0^n)$, its average is equal to its maximum. Then we have

$$u(x) = \Theta^S(u, 0^n) K_p(r) \ \text{ for } x \in \partial B_r(0^n),$$

which implies

$$u(x) = \Theta^S(u, 0^n) K_p(|x|) \ \text{ for } x \in B_{\frac{1}{2}}(0^n).$$

However, u_i converges to u in $L^1_{loc}(B_2(0^n))$. Thus, u_i are $(0, \epsilon, 2, 0^n)$-homogenous when i is sufficiently large, which is a contradiction. □

Lemma 6.6. *Let u be a F-subharmonic function on $B_2(0^n)$ with $\|u\|_{L^1(B_2(0^n))} \leq \Lambda$. For any $c > 0$, there exists constant $\epsilon(c, \Lambda, F)$ such that* *if u is $(0,\epsilon,1,0^n)$-homogenous, then*

$$E_c(u)\cap A_{\frac{1}{16},\frac{1}{4}}=\emptyset,$$

where $A_{\frac{1}{16},\frac{1}{4}}=\{x\in\mathbf{R}^n \mid \frac{1}{16}\le|x|\le\frac{1}{4}\}$.

Proof. We argue by contradiction, assuming that there exists a sequence of F-subharmonic functions u_i on $B_2(0^n)$ such that

(1) $\|u_i\|_{L^1(B_2(0^n))}\le\Lambda$;

(2) u_i is $(0,i^{-1},1,0^n)$-homogeneous;

(3) there exists point $x_i\in E_c(u_i)\cap A_{\frac{1}{16},\frac{1}{4}}$.

By Lemma 2.2), after passing to a subsequence, we can assume u_i converges to u in $L^1_{loc}(B_2(0^n))$ and x_i converges to x, where u is a F-subharmonic function. By (2), Lemma 8.1 and strong uniqueness holds for F, then there exists a constant $\Theta\ge 0$ such that

$$u(x)=\Theta K_p(|x|)\ \text{ in } B_1(0^n). \tag{6.4}$$

By (3) and Lemma 6.4, we have $x\in E_c(u)\cap A_{\frac{1}{16},\frac{1}{4}}$, which contradicts with (6.4). □

Remark 6.7. In [21], Harvey and Lawson proved the discreteness of $E_c(u)$ (see [21, Theorem 14.1, Theorem 14.1']). As an immediate corollary of Lemma 6.5, Lemma 6.6 and scaling argument, we also prove that every point in $E_c(u)$ is isolated, which gives another proof of discreteness of $E_c(u)$.

6.3. Proof of Theorem 1.14

First, we have the following lemma.

Lemma 6.8. *Let u be a F-subharmonic function on $B_2(0^n)$ with $\|u\|_{L^1(B_2(0^n))}\le\Lambda$. For any $x\in B_1(0^n)$, $r\in(0,\frac{1}{2})$, there exists constant $N(\Lambda,F)$ such that*

$$\int_{B_1(0^n)}|u_{x,r}(y)|dy\le N.$$

Proof. Without loss of generality, we assume $u \leq 0$ on $B_{\frac{3}{2}}(0^n)$. Since $V(u,x,\cdot)$ is K_p-convex, we have

$$\frac{V(u,x,1)-V(u,x,r)}{K_p(1)-K_p(r)} \leq \frac{V(u,x,1)-V(u,x,\frac{1}{2})}{K_p(1)-K_p(\frac{1}{2})} \leq C(\Lambda,p,n),$$

which implies

$$\frac{V(u,x,r)}{K_p(r)} \leq \frac{V(u,x,1)}{K_p(r)} + C(\Lambda,p,n)\frac{K_p(r)-K_p(1)}{K_p(r)} \leq N(\Lambda,n,p).$$

Since $u \leq 0$ on $B_{\frac{3}{2}}(0^n)$, it then follows that

$$\int_{B_1(0^n)} |u_{x,r}(y)|dy = -\int_{B_1(0^n)} u_{x,r}(y)dy = \omega_n\frac{V(u,x,r)}{K_p(r)} \leq N(\Lambda,n,p),$$

as desired. □

Now, we are in the position to prove Theorem 1.14.

Proof of Theorem 1.14. We split up into two cases.

Case 1. u satisfies monotonicity condition.

For convenience, we let S_0 denote $\#\left(E_c(u)\cap B_1(0^n)\right)$. And we will obtain an upper bound of S_0 by induction argument.

For $i=1$, we consider the covering $\{B_{2^{-1}}(x_j)\}$ of $E_c(u)\cap B_1(0^n)$ such that

(1) $x_j \in E_c(u)\cap B_1(0^n)$;

(2) $B_{2^{-2}}(x_j)$ are disjoint.

In this covering, there exists a ball containing the largest number of points in $E_c(u)\cap B_1(0^n)$ (say $B_{2^{-1}}(x_1)$, contains S_1 points in $E_c(u)\cap B_1(0^n)$).

If $S_1 = S_0$, we put $T_1 = 0$, otherwise put $T_1 = 1$. If $T_1 = 1$, by (2) and the definition of S_1, it is clear that

$$2^{-2n}S_0 \leq S_1 < S_0.$$

Furthermore, in this case, we have

$$(E_c(u)\cap B_1(0^n))\cap\left(B_2(x_1)\setminus B_{\frac{1}{2}}(x_1)\right) \neq \emptyset.$$

We repeat this process by covering $E_c(u)\cap B_{2^{-i}}(x_i)$ with balls of radius 2^{-i-1}. Since $E_c(u)\cap B_1(0^n)$ is discrete, there exists $i_0 \in \mathbb{Z}_+$ such that $S_{i_0} =$

1. We define

$$I := \{0 \le i \le i_0 \mid T_i = 1\}.$$

Then we obtain

$$S_0 \le (2^{2n})^{|I|}. \tag{6.5}$$

In order to get an upper bound of $|I|$, we consider the point x_{i_0}. For any $i \in I$, by construction, we have

$$(E_c(u) \cap B_1(0^n)) \cap (B_{2^{-i+1}}(x_{i_0}) \setminus B_{2^{-i-1}}(x_{i_0})) \ne \emptyset,$$

which implies

$$E_c(u_{x_{i_0},2^{-i+3}}) \cap A_{\frac{1}{16},\frac{1}{4}} \ne \emptyset. \tag{6.6}$$

We claim that

$$\theta\left(u_{x_{i_0},2^{-i+2}}, 0^n, \frac{1}{2}\right) - \theta(u_{x_{i_0},2^{-i+2}}, 0^n, \delta_0) \ge \delta_0, \tag{6.7}$$

where $\delta_0(\epsilon, c, N, F)$, $\epsilon(c, N, F)$ and $N(\Lambda, F)$ are the constants in Lemma 6.5, Lemma 6.6 and Lemma 6.8, respectively.

If (6.7) is false, then we have

$$\theta\left(u_{x_{i_0},2^{-i+2}}, 0^n, \frac{1}{2}\right) - \theta(u_{x_{i_0},2^{-i+2}}, 0^n, \delta_0) < \delta_0.$$

By Lemma 6.5, $u_{x_{i_0},2^{-i+2}}$ is $(0, \epsilon, 2, 0^n)$-homogenous. It follows that $u_{x_{i_0},2^{-i+3}}$ is $(0, \epsilon, 1, 0^n)$-homogenous. Combining this with Lemma 6.6, we obtain

$$E_c(u_{x_{i_0},2^{-i+3}}) \cap A_{\frac{1}{16},\frac{1}{4}} = \emptyset,$$

which contradicts with (6.6).

By (6.7), for any $i \in I$, we have

$$\theta(u, x_{i_0}, 2^{-i+1}) - \theta(u, x_{i_0}, 2^{-i+2}\delta_0) \ge \delta_0.$$

Since F-subharmonic function is subharmonic (see [21, (6.3)]), by Lemma 8.5, it is clear that

$$\theta\left(u, x_{i_0}, \frac{1}{2}\right) - \theta(u, x_{i_0}, 0) \le L(\Lambda, p, n),$$

which implies

$$|I| \leq C(L, \delta_0). \tag{6.8}$$

Combining (6.5) and (6.8), we get the desired estimate.

Case 2. u does not satisfies monotonicity condition.

By Lemma 6.2, we obtain $u - N$ satisfies monotonicity condition. By Case 1, we have

$$\# \left(E_c(u) \cap B_1(0^n)\right) \leq C(c, \Lambda, F).$$

By the definition of $E_c(u)$, it is clear that $E_c(u) = E_c(u - N)$. This completes the proof. □

7. $\mathbb{G}$-plurisubharmonic functions

In this section, we study the singular sets of $\mathbb{G}$-plurisubharmonic functions and give the proof of Theorem 1.15. For $\mathbb{G}$, we use $F(\mathbb{G})$ to denote the associated subequation (see (1.1)). We note that the Riesz characteristic of $F(\mathbb{G})$ is p (see [21, (4.8)]). Without loss of generality, we assume that $\mathbb{G}$ is a smooth submanifold of $G(p, \mathbf{R}^n)$ (see Remark 1.16).

7.1. $\mathbb{G}$-energy

In this subsection, we introduce the $\mathbb{G}$-energies of $\mathbb{G}$-plurisubharmonic functions. And then we prove a property of $\mathbb{G}$-energy.

Definition 7.1. Let u be a $\mathbb{G}$-plurisubharmonic function on $B_{2R}(0^n)$. For any $x \in B_R(0^n)$ and $r \in (0, R)$, the $\mathbb{G}$ energy of u is defined by

$$\begin{aligned}\theta_{\mathbb{G}}(u, x, r) = \int_{\mathbb{G}} \frac{S'_-(u|_{W+x}, x, r)}{K'_p(r)} dW + \int_{\mathbb{G}} \frac{M'_-(u|_{W+x}, x, r)}{K'_p(r)} dW \\ + \frac{M'_-(u, x, r)}{K'_p(r)}.\end{aligned}$$

where K_p is the Riesz kernel. We define $\theta_{\mathbb{G}}(u, x, 0) = \lim_{r \to 0} \theta_{\mathbb{G}}(u, x, r)$.

Since u is $\mathbb{G}$-plurisubharmonic, $S(u|_{W+x}, x, \cdot)$, $M(u|_{W+x}, x, \cdot)$ are K_p-convex for any $W \in \mathbb{G}$ and $x \in B_1(0^n)$. It is clear that $\theta_{\mathbb{G}}(u, x, r)$ is nondecreasing function in r.

Lemma 7.2. *Let u be a $\mathbb{G}$-plurisubharmonic function on $B_R(0^n)$. Then for any $0 < a < b < R$, there exists constant $C(a, b, \mathbb{G})$ such that*

$$\int_{\mathbb{G}} \|u|_W\|_{L^1(A_{a,b}\cap W)} dW \leq C\|u\|_{L^1(A_{a,b})},$$

where $A_{a,b} = \{x \in \mathbf{R}^n | a \leq |x| \leq b\}$.

Proof. For any $0 < a < b < R$, we define

$$E_{a,b} := \{(W, x) \in \mathbb{G} \times A_{a,b} \mid x \in W\}.$$

Thus, $E_{a,b} \xrightarrow{\sigma} \mathbb{G}$ and $E_{a,b} \xrightarrow{\pi} A_{a,b}$ are fiber bundles, where σ and π are projections onto the first and second factor (see [22, p.2196]). We consider the pull back function $\pi^\star u$ on $E_{a,b}$. Since the fiber bundle is locally a product space, then there exists constants $C_\sigma(a, b, \mathbb{G})$ and $C_\pi(a, b, \mathbb{G})$ such that

$$\begin{aligned}
\int_{\mathbb{G}} \|u|_W\|_{L^1(A_{a,b}\cap W)} dW &= \int_{\mathbb{G}} \int_{A_{a,b}\cap W} |u|_W(x)| dx dW \\
&\leq C_\sigma \int_{E_{a,b}} |u|_W(x)| dV_{E_{a,b}} \\
&= C_\sigma \int_{E_{a,b}} |(\pi^\star u)(W, x)| dV_{E_{a,b}} \\
&\leq C_\sigma C_\pi \int_{A_{a,b}} |u(x)| dx,
\end{aligned}$$

where $dV_{E_{a,b}}$ is the volume form on $E_{a,b}$. □

Lemma 7.3. *Let u be a $\mathbb{G}$-plurisubharmonic function on $B_2(0^n)$ with $\|u\|_{L^1(B_2(0^n))} \leq \Lambda$. Then for any $x \in B_1(0^n)$, there exists constant $C(\mathbb{G})$ such that*

$$\theta_{\mathbb{G}}\left(u, x, \frac{1}{2}\right) \leq C(\mathbb{G})\Lambda.$$

Proof. Since $S(u|_{W+x},x,\cdot)$ and $M(u|_{W+x},x,\cdot)$ are K_p-convex, we have

$$\begin{aligned}
\theta_{\mathbb{G}}\left(u,x,\frac{1}{2}\right) = {} & \int_{\mathbb{G}} \frac{S'_{-}(u|_{W+x},x,\frac{1}{2})}{K'_p(\frac{1}{2})}dW \\
& + \int_{\mathbb{G}} \frac{M'_{-}(u|_{W+x},x,\frac{1}{2})}{K'_p(\frac{1}{2})}dW + \frac{M'_{-}(u,x,\frac{1}{2})}{K'_p(\frac{1}{2})} \\
\leq {} & \int_{\mathbb{G}} \frac{S(u|_{W+x},x,\frac{2}{3}) - S(u|_{W+x},x,\frac{1}{2})}{K_p(\frac{2}{3})-K_p(\frac{1}{2})}dW \\
& + \int_{\mathbb{G}} \frac{M(u|_{W+x},x,\frac{2}{3}) - M(u|_{W+x},x,\frac{1}{2})}{K_p(\frac{2}{3})-K_p(\frac{1}{2})}dW \\
& + \frac{M(u,x,\frac{2}{3}) - M(u,x,\frac{1}{2})}{K_p(\frac{2}{3})-K_p(\frac{1}{2})}.
\end{aligned} \tag{7.1}$$

By the submean value property of subharmonic functions, it is clear that

$$S\left(u|_{W+x},x,\frac{2}{3}\right) \leq M\left(u|_{W+x},x,\frac{2}{3}\right) \leq \frac{3^p}{\omega_p}\|u|_{W+x}\|_{L^1((A_{\frac{1}{3},1}\cap W)+x)}, \tag{7.2}$$

where ω_p is the volume of unit ball in $\mathbb{R}^p$. Combining (7.1), (7.2), Lemma 7.2 and Lemma 8.5, we obtain

$$\begin{aligned}
\theta_{\mathbb{G}}\left(u,x,\frac{1}{2}\right) \leq {} & \int_{\mathbb{G}} \frac{S(u|_{W+x},x,\frac{2}{3}) - S(u|_{W+x},x,\frac{1}{2})}{K_p(\frac{2}{3})-K_p(\frac{1}{2})}dW \\
& + \int_{\mathbb{G}} \frac{M(u|_{W+x},x,\frac{2}{3}) - M(u|_{W+x},x,\frac{1}{2})}{K_p(\frac{2}{3})-K_p(\frac{1}{2})}dW \\
& + \frac{M(u,x,\frac{2}{3}) - M(u,x,\frac{1}{2})}{K_p(\frac{2}{3})-K_p(\frac{1}{2})} \\
\leq {} & C\int_{\mathbb{G}} \|u|_{W+x}\|_{L^1((A_{\frac{1}{3},1}\cap W)+x)}dW + C\Lambda \\
\leq {} & C\|u\|_{L^1(A_{\frac{1}{3},1}+x)} + C\Lambda \\
\leq {} & C\Lambda,
\end{aligned}$$

where C depends only on $\mathbb{G}$. □

7.2. Quantitative rigidity theorem

In this subsection, we prove quantitative rigidity theorem of $\mathbb{G}$-plurisubharmonic functions.

Lemma 7.4. *Let $\{u_i\}$ be a sequence of $\mathbb{G}$-plurisubharmonic functions on $B_R(0^n)$ with $\|u_i\|_{L^1(B_R(0^n))} \leq \Lambda$. Then there exists a subsequence $\{u_{i_k}\}$ such that u_{i_k} converges to u in $L^1_{loc}(B_R(0^n))$, where u is a $\mathbb{G}$-plurisubharmonic function. And for almost every $W \in \mathbb{G}$, u_{i_k} converges to u in $L^1(A_{a,b} \cap W)$ for any $0 < a < b < R$. In particular, for every $r \in (0, R)$, we have*

$$\lim_{k\to\infty} S(u_{i_k}|_W, 0^p, r) = S(u|_W, 0^p, r)$$

and

$$\lim_{k\to\infty} M(u_{i_k}|_W, 0^p, r) = M(u|_W, 0^p, r)$$

for almost every $W \in \mathbb{G}$, where 0^p is the origin in $\mathbf{R}^p$.

Proof. By Lemma 2.2, there exists a subsequence $\{u_{i_k}\}$ such that u_{i_k} converges to u in $L^1_{loc}(B_R(0^n))$, where u is a $\mathbb{G}$-plurisubharmonic function.

For any $0 < a < b < R$, recalling $E_{a,b} \xrightarrow{\pi} A_{a,b}$ is a fiber bundle, we consider the pull back function $\pi^\star u_{i_k}$ and $\pi^\star u$ on $E_{a,b}$. Since u_{i_k} converges to u in $L^1(A_{a,b})$, we have $\pi^\star u_{i_k}$ converges to $\pi^\star u$ in $L^1(E_{a,b})$, i.e.,

$$\lim_{k\to\infty} \int_{E_{a,b}} |\pi^\star u_{i_k} - \pi^\star u| = 0,$$

which implies

$$\lim_{k\to\infty} \int_{\mathbb{G}} \int_{A_{a,b}\cap W} |u_{i_k}(x) - u(x)| dx dW = 0.$$

By Fatou's Lemma, we have

$$\begin{aligned}\int_{\mathbb{G}} \lim_{k\to\infty} \int_{A_{a,b}\cap W} &|u_{i_k}(x) - u(x)| dx dW \\ &\leq \lim_{k\to\infty} \int_{\mathbb{G}} \int_{A_{a,b}\cap W} |u_{i_k}(x) - u(x)| dx dW = 0.\end{aligned}$$

Thus, for almost every $W \in \mathbb{G}$, we obtain

$$\lim_{k\to\infty} \int_{A_{a,b}\cap W} |u_{i_k}(x) - u(x)| dx = 0,$$

which implies $u_{i_k}|_W$ converges to $u|_W$ in $L^1(A_{a,b} \cap W)$. Since $u_{i_k}|_W$ and $u|_W$ are subharmonic functions on $A_{a,b} \cap W$, for any $r \in (a, b)$, by Lemma 8.6, we obtain

$$\lim_{k\to\infty} S(u_{i_k}|_W, 0^p, r) = S(u|_W, 0^p, r)$$

and

$$\lim_{k\to\infty} M(u_{i_k}|_W, 0^p, r) = M(u|_W, 0^p, r)$$

for almost every $W \in \mathbb{G}$. $\square$

In order to prove quantitative rigidity theorem, we split up into different cases. First, we consider the case $p > 2$.

Theorem 7.5 (Quantitative rigidity theorem, $p > 2$). *For any $\epsilon, \lambda > 0$, there exists constant $\delta_0(\epsilon, \lambda, \mathbb{G})$ such that if u is a $\mathbb{G}$-plurisubharmonic function on $B_{\delta_0^{-1}}(0^n)$ and satisfies*

(1) $\|u\|_{L^1(B_r(0^n))} \leq \lambda r^{n-p+2}$, for any $r \in (0, \delta_0^{-1})$;

(2) $\theta_{\mathbb{G}}(u, 0^n, \delta_0^{-1}) - \theta_{\mathbb{G}}(u, 0^n, \delta_0) \leq \delta_0$,

then u is $(0, \epsilon, 1, 0^n)$-homogeneous.

Proof. We argue by contradiction, assuming that there exists a sequence of $\mathbb{G}$-plurisubharmonic functions u_i on $B_i(0^n)$ such that

(1) $\|u_i\|_{L^1(B_r(0^n))} \leq \lambda r^{n-p+2}$, for any $r \in (0, i)$;

(2) $\theta_{\mathbb{G}}(u_i, 0^n, i) - \theta_{\mathbb{G}}(u_i, 0^n, i^{-1}) \leq i^{-1}$;

(3) u_i is not $(0, \epsilon, 1, 0^n)$-homogeneous.

By Lemma 7.4, there exists a subsequence $\{u_{i_k}\}$ such that u_{i_k} converges to u in $L^1_{loc}(\mathbf{R}^n)$, where u is a $\mathbb{G}$-plurisubharmonic function on $\mathbf{R}^n$. And for any $r > 0$, we have

$$\lim_{k\to\infty} S(u_{i_k}|_W, 0^p, r) = S(u|_W, 0^p, r)$$

and

$$\lim_{k\to\infty} M(u_{i_k}|_W, 0^p, r) = M(u|_W, 0^p, r)$$

for almost every $W \in \mathbb{G}$.

Since $S(u|_W, 0^p, \cdot)$ and $M(u|_W, 0^p, \cdot)$ are K_p-convex functions, combining this with Fatou's Lemma, Lemma 7.4 and Lemma 8.4, for almost any $r >$

$t > 0$, we obtain

$$\begin{aligned}
&\theta_{\mathbb{G}}(u, 0^n, r) - \theta_{\mathbb{G}}(u, 0^n, t) \\
&\quad = \int_{\mathbb{G}} \lim_{k\to\infty} \left(\frac{S'_-(u_{i_k}|_W, 0^p, r)}{K'_p(r)} - \frac{S'_-(u_{i_k}|_W, 0^p, t)}{K'_p(t)} \right) dW \\
&\qquad + \int_{\mathbb{G}} \lim_{k\to\infty} \left(\frac{M'_-(u_{i_k}|_W, 0^p, r)}{K'_p(r)} - \frac{M'_-(u_{i_k}|_W, 0^p, t)}{K'_p(t)} \right) dW \\
&\qquad + \lim_{k\to\infty} \left(\frac{M'_-(u_{i_k}, 0^p, r)}{k'_p(r)} - \frac{M'_-(u_{i_k}, 0^p, t)}{k'_p(t)} \right) \\
&\quad \le \int_{\mathbb{G}} \lim_{k\to\infty} \left(\frac{S'_-(u_{i_k}|_W, 0^p, i_k)}{K'_p(i_k)} - \frac{S'_-(u_{i_k}|_W, 0^p, i_k^{-1})}{K'_p(i_k^{-1})} \right) dW \\
&\qquad + \int_{\mathbb{G}} \lim_{k\to\infty} \left(\frac{M'_-(u_{i_k}|_W, 0^p, i_k)}{K'_p(i_k)} - \frac{M'_-(u_{i_k}|_W, 0^p, i_k^{-1})}{K'_p(i_k^{-1})} \right) dW \\
&\qquad + \lim_{k\to\infty} \left(\frac{M'_-(u_{i_k}, 0^p, i_k)}{k'_p(i_k)} - \frac{M'_-(u_{i_k}, 0^p, i_k^{-1})}{k'_p(i_k^{-1})} \right) \\
&\quad \le \lim_{k\to\infty} \left(\theta_{\mathbb{G}}(u_{i_k}, 0^n, i_k) - \theta_{\mathbb{G}}(u_{i_k}, 0^n, i_k^{-1}) \right) \\
&\quad \le 0.
\end{aligned}$$

By the monotonicity of $\theta_{\mathbb{G}}(u, 0^n, \cdot)$, we have

$$\theta_{\mathbb{G}}(u, 0^n, r) = \theta_{\mathbb{G}}(u, 0^n, 0),$$

for any $r > 0$. It then follows that

$$S(u|_W, 0^p, r) = \Theta(u|_W, 0^p) K_p(r) + C_S(W) \tag{7.3}$$

and

$$M(u|_W, 0^p, r) = \Theta(u|_W, 0^p) K_p(r) + C_M(W) \tag{7.4}$$

for almost every $W \in \mathbb{G}$, where

$$\Theta(u|_W, 0^p) = \Theta^S(u|_W, 0^p) = \Theta^M(u|_W, 0^p) \quad \text{(see [21, (12.3)]).}$$

By (7.3), for any $b > a > 0$, we obtain

$$\begin{aligned}\int_{(B_b(0^n)\setminus B_a(0^n))\cap W} \Delta(u|_W) &= \int_{B_b(0^n)\cap W} \Delta(u|_W) - \int_{B_a(0^n)\cap W} \Delta(u|_W) \\ &= C(p)\left(\frac{S'_-(u|_W, 0^p, b)}{K'_p(b)} - \frac{S'_-(u|_W, 0^p, a)}{K'_p(a)}\right) \\ &= 0,\end{aligned}$$

where we used $S'_-(u|_W, 0^n, r) = C(p)K'_p(r)\int_{B_r(0^n)} \Delta(u|_W)$ for any $r > 0$ (see e.g. [24, Theorem 3.2.16] or [21, (7.8)]). It then follows that $u|_W$ is harmonic on $W \setminus \{0^p\}$. By Harnack's inequality and (7.4), it is clear that

$$\limsup_{x\to 0^p} |x|^{p-2}|u|_W(x)| < +\infty. \tag{7.5}$$

Combining Theorem 10.5 in [1] and (7.5), we get

$$u|_W(x) = \Theta(u|_W, 0^p)K_p(|x|) + h_W(x) \tag{7.6}$$

on W, where h_W is a harmonic function on W. By (7.4) and (7.6), we have

$$M(h_W, 0^p, r) = C_M(W),$$

for any $r > 0$. By Strong Maximum Principle, we conclude that $h_W = C_M(W)$. It then follows that $u|_W = \Theta(u|_W, 0^p)K_p + C_M(W)$ for almost every $W \in \mathbb{G}$. Combining Lemma 7.2 and (1), by scaling, we obtain

$$\int_{\mathbb{G}}\int_{A_{r,2r}\cap W} |u|_W|dW \leq C(\mathbb{G})r^{p-n}\int_{A_{r,2r}} |u(x)|dx \leq C(\mathbb{G})\lambda r^2,$$

which implies

$$\int_{\mathbb{G}}\int_{A_{r,2r}\cap W} |-\Theta(u|_W, 0^p)|x|^{2-p} + C_M(W)|dW \leq C(\mathbb{G})\lambda r^2.$$

It then follows that

$$\left(\int_{\mathbb{G}} |C_M(W)|dW\right) r^p \leq C(\mathbb{G})\left(\int_{\mathbb{G}} \Theta(u|_W)dW + \lambda\right) r^2.$$

Since $p > 2$ and r is arbitrary, we have

$$\int_{\mathbb{G}} |C_M(W)|dW = 0.$$

Therefore, it is clear that $u|_W = \Theta(u|_W, 0^p)K_p$ for almost every $W \in \mathbb{G}$. Recalling u is a subharmonic function on $\mathbf{R}^n$, we get u is 0-homogeneous. However, u_{i_k} converges to u in $L^1_{loc}(B_2(0^n))$. Then u_{i_k} are $(0, \epsilon, 1, 0^n)$-homogeneous when k is sufficiently large, which is a contradiction. □

Next, we prove quantitative rigidity theorem for the case $p = 2$. First, we need the following lemma.

Lemma 7.6. *Let u be a $\mathbb{G}$-subharmonic function on $B_2(0^n)$. If $p = 2$, then*

$$\Theta^M(u|_W, 0^2) = \Theta^M(u, 0^n),$$

for almost every $W \in \mathbb{G}$.

Proof. Let φ be a tangent to u at 0^n. Then there exists a sequence $\{r_i\}$ such that $\lim_{i\to\infty} r_i = 0$ and u_{0^n,r_i} converges to φ in $L^1_{loc}(\mathbf{R}^n)$. For almost every $W \in \mathbb{G}$. By Lemma 7.4, we obtain that $u_{0^n,r_i}|_W$ converges to $\varphi|_W$ in $L^1(A_{1,2} \cap W)$. On the other hand, for any non-polar plane $W \in \mathbb{G}$ (for definition of non-polar plane, see [22, p.2194]), by passing to a subsequence, we can assume $(u|_W)_{0^2,r_i}$ converges to ψ in $L^1_{loc}(\mathbf{R}^2)$, where $\psi \in T_{0^2}(u|_W)$. By the definition of the tangential 2-flow, it is clear that

$$(u_{0^n,r_i})|_W(x) - (u|_W)_{0^2,r_i}(x) = M(u|_W, 0^2, r_i) - M(u, 0^n, r_i),$$

for almost every $x \in A_{1,2} \cap W$. Since the left hand side converges to $(\varphi|_W - \psi)$ in $L^1(A_{1,2} \cap W)$ and the right hand side is independent of x, then we obtain

$$\lim_{i\to\infty} \big(M(u|_W, 0^2, r_i) - M(u, 0^n, r_i)\big) = C,$$

where C is a constant. It then follows that

$$\Theta^M(u|_W, 0^2) - \Theta^M(u, 0^n) = \lim_{i\to\infty} \left(\frac{M(u|_W, 0^2, r_i)}{K_2(r_i)} - \frac{M(u|_W, 0^n, r_i)}{K_2(r_i)}\right) = 0,$$

as required. □

Theorem 7.7 (Quantitative rigidity theorem, $p = 2$). *For any $\epsilon, \lambda > 0$, there exists constant $\delta_0(\epsilon, \lambda, \mathbb{G})$ such that if u is a $\mathbb{G}$-plurisubharmonic function on $B_{\delta_0^{-1}}(0^n)$ and satisfies*

(1) $\|u\|_{L^1(B_r(0^n))} \leq \lambda r^n(|\log r| + 1)$, *for any $r \in (0, \delta_0^{-1})$;*

(2) $M(u, 0^n, 1) = 0$;

(3) $\theta_{\mathbb{G}}(u, 0^n, \delta_0^{-1}) - \theta_{\mathbb{G}}(u, 0^n, \delta_0) \leq \delta_0$,

then u *is* $(0, \epsilon, 1, 0^n)$*-homogeneous.*

Proof. We argue by contradiction, assuming that there exists a sequence of $\mathbb{G}$-plurisubharmonic functions u_i on $B_i(0^n)$ such that

(1) $\|u_i\|_{L^1(B_r(0^n))} \leq \lambda r^n(|\log r| + 1)$, for any $r \in (0, i)$;

(2) $M(u_i, 0^n, 1) = 0$;

(3) $\theta_{\mathbb{G}}(u_i, 0^n, i) - \theta_{\mathbb{G}}(u_i, 0^n, i^{-1}) \leq i^{-1}$;

(4) u_i is not $(0, \epsilon, 1, 0^n)$-homogeneous.

By Lemma 2.2, there exists a subsequence $\{u_{i_k}\}$ such that u_{i_k} converges to u in $L^1_{loc}(\mathbf{R}^n)$, where u is a $\mathbb{G}$-plurisubharmonic function on $\mathbf{R}^n$. By (2) and Lemma 8.6, we obtain $M(u, 0^n, 1) = 0$. Combining this and the similar argument in Theorem 7.5, for any $r > 0$, we have

$$\theta_{\mathbb{G}}(u, 0^n, r) = \theta_{\mathbb{G}}(u, 0^n, 0),$$

which implies

$$M(u, 0^n, r) = \Theta^M(u, 0^n)K_2(r)$$

and

$$u|_W = \Theta^M(u|_W, 0^2)K_2 + C_W,$$

for almost every $W \in \mathbb{G}$, where C_W is a constant on W. By Lemma 7.6, we obtain

$$u|_W = \Theta^M(u, 0^n)K_2 + C_W.$$

For $x \in W$, by definition of tangential 2-flow, it is clear that

$$\begin{aligned} u_{0^n,r}(x) &= u(rx) - M(u, 0^n, r) \\ &= \Theta^M(u, 0^n)K_2(rx) + C_W - \Theta^M(u, 0^n)K_2(r) \\ &= u(x). \end{aligned}$$

It then follows that $u_{0^n,r}(x) = u(x)$ for almost every $x \in \mathbf{R}^n$. Since $u_{0^n,r}$ and u are subharmonic functions. We obtain that $u_{0^n,r} = u$ for any $r > 0$. Then u is 0-homogeneous. When k is sufficiently large, u_{i_k} is $(0, \epsilon, 1, 0^n)$-homogeneous, which contradicts with (4). $\square$

7.3. Covering lemma and decomposition lemma

Let u be a $\mathbb{G}$-plurisubharmonic function on $B_2(0^n)$ with $\|u\|_{L^1(B_2(0^n))} \leq \Lambda$. First, we introduce the following definitions.

Definition 7.8. For any $\epsilon > 0$, $t \geq 1$ and $0 < r < 1$, we define

$$\mathcal{H}_{t,r,\epsilon} = \{x \in B_1(0^n) \mid \mathcal{N}_t(u, B_r(x)) > \epsilon\}$$

and

$$\mathcal{L}_{t,r,\epsilon} = \{x \in B_1(0^n) \mid \mathcal{N}_t(u, B_r(x)) \leq \epsilon\},$$

where

$$\mathcal{N}_t(u, B_r(x)) = \inf\{\delta > 0 \mid u \text{ is } (0, \delta, tr, x)\text{-homogeneous}\}.$$

Definition 7.9. For any $x \in B_1(0^n)$ and $\gamma \in (0,1)$, we define j-tuple $T^j(x) = (T_1^j(x), T_2^j(x), \ldots, T_j^j(x))$ by

$$T_i^j(x) = \begin{cases} 1 & \text{if } x \in \mathcal{H}_{\gamma^{-1},\gamma^i,\epsilon} \\ 0 & \text{if } x \in \mathcal{L}_{\gamma^{-1},\gamma^i,\epsilon} \end{cases}$$

for all $1 \leq i \leq j$, where $\epsilon = \epsilon(\eta, \gamma, \Lambda, \mathbb{G})$ is the constant in Lemma 7.13 and $\gamma > 0$ is a constant to be determined later.

Definition 7.10. For any j-tuple T^j, we define

$$E_{T^j} = \{x \in B_1(0^n) \mid T^j(x) = T^j\}.$$

Next, for each $E_{T^j} \neq \emptyset$, we define a collection of sets $\{\mathcal{C}^k_{\eta,\gamma^j}(T^j)\}$ by induction, where $\mathcal{C}^k_{\eta,\gamma^j}(T^j)$ is the union of balls of radius γ^j. For $j = 0$, we put $\mathcal{C}^k_{\eta,\gamma^0}(T^j) = B_1(0^n)$. Assume that $\mathcal{C}^k_{\eta,\gamma^{j-1}}(T^{j-1})$ has been defined and satisfies $\mathcal{S}^k_{\eta,\gamma^j}(u) \cap E_{T^j} \subset \mathcal{C}^k_{\eta,\gamma^{j-1}}(T^{j-1})$, where T^{j-1} is the $(j-1)$-tuple obtained from T^j by dropping the last entry. For each ball $B_{\gamma^{j-1}}(x)$ of $\mathcal{C}^k_{\eta,\gamma^{j-1}}(T^{j-1})$, take a minimal covering of $B_{\gamma^{j-1}}(x) \cap \mathcal{S}^k_{\eta,\gamma^j}(u) \cap E_{T^j}$ by balls of radius γ^j with centers in $B_{\gamma^{j-1}}(x) \cap \mathcal{S}^k_{\eta,\gamma^j}(u) \cap E_{T^j}$. Define the union of all balls so obtained to be $\mathcal{C}^k_{\eta,\gamma^j}(T^j)$.

Lemma 7.11. *For any $x \in B_1(0^n)$, $s \in (0, \frac{1}{2})$ and $r \in (0, \frac{1}{2}s^{-1})$, there exists constant $N(\Lambda, p, n)$ such that*

$$\int_{B_r(0^n)} |u_{x,s}(y)| dy \leq \begin{cases} N r^{n-p+2} & \textit{when } p > 2 \\ N r^n (|\log r| + 1) & \textit{when } p = 2. \end{cases}$$

Proof. Without loss of generality, we assume $u \leq 0$ on $B_{\frac{3}{2}}(0^n)$. When $p > 2$, since $V(u, x, \cdot)$ is K_p-convex, we have

$$0 \leq \frac{V(u,x,1) - V(u,x,rs)}{K_p(1) - K_p(rs)} \leq \frac{V(u,x,1) - V(u,x,\frac{1}{2})}{K_p(1) - K_p(\frac{1}{2})} \leq C(\Lambda, p, n),$$

which implies

$$\frac{V(u,x,rs)}{K_p(rs)} \leq \frac{V(u,x,1)}{K_p(rs)} + C(\Lambda,p,n) \frac{K_p(rs) - K_p(1)}{K_p(rs)} \leq N(\Lambda, p, n).$$

Since $u \leq 0$ on $B_{\frac{3}{2}}(0^n)$, it then follows that

$$\int_{B_r(0^n)} |u_{x,s}(y)| dy = - \int_{B_r(0^n)} u_{x,s}(y) dy = \omega_p \frac{V(u,x,rs)}{K_p(rs)} r^{n-p+2} \leq N r^{n-p+2}.$$

When $p = 2$, by similar calculations, we have

$$|M(u_{x,s}, 0^n, r)| = \frac{M(u,x,sr) - M(u,x,s)}{K_2(sr) - K_2(s)} |\log r| \leq C(\Lambda, n) |\log r|. \tag{7.7}$$

By Harnack's inequality (see [21, (7.10)]), we obtain

$$\begin{aligned} S(u_{x,s}, 0^n, r) &\geq C \left(M\left(u_{x,s}, 0^n, \frac{r}{2}\right) - M(u_{x,s}, 0^n, r) \right) + M(u_{x,s}, 0^n, r) \\ &\geq -C |\log r|, \end{aligned}$$

which implies

$$V(u_{x,s}, 0^n, r) = n \int_0^1 S(u_{x,s}, 0^n, rt) t^{n-1} dt \geq -C(|\log r| + 1). \tag{7.8}$$

Combining (7.7) and (7.8), it is clear that

$$\begin{aligned}\int_{B_r(0^n)} |u_{x,s}(y)|dy &= \int_{B_r(0^n)} (M(u_{x,s}, 0^n, r) - u_{x,s}(y))dy \\ &\quad + \int_{B_r(0^n)} |M(u_{x,s}, 0^n, r)|dy \\ &\leq Cr^n(|\log r| + 1),\end{aligned}$$

as desired. □

Lemma 7.12. *For all ϵ, τ, $\gamma > 0$, there exists constant $\delta(\epsilon, \tau, \gamma, \Lambda, \mathbb{G})$ with the following property. For any $r \leq 1$, if $x \in \mathcal{L}_{\gamma^{-1},\gamma r,\delta}(u)$, then there exists nonnegative integer $l \leq n$ such that*

(1) u is (l, ϵ, r, x)-homogeneous with respect to k-plane $V^k_{u,x}$;

(2) $\mathcal{L}_{\gamma^{-1},\gamma r,\delta} \cap B_r(x) \subset B_{\tau r}(V^k_{u,x})$.

Proof. First, we define $\delta^{[l]}$ by induction. We put $\delta^{[n]} = \frac{\epsilon}{2}$. Then we define $\delta^{[l]} = \delta(\tau, \delta^{[l+1]}, N(\Lambda, \mathbb{G}), \mathbb{G})$, where δ and N are the constants in Lemma 2.3 and Lemma 7.11, respectively. Let us put $\delta < \delta^{[0]}$. Then $\delta < \delta^{[0]} \leq \delta^{[1]} \leq \cdots \leq \delta^{[n]} = \frac{\epsilon}{2}$. Since $x \in \mathcal{L}_{\gamma^{-1},\gamma r,\delta}(u)$, we have u is $(0, \delta^{[0]}, r, x)$-homogeneous. Then there exists a largest l such that u is $(l, \delta^{[l]}, r, x)$-homogeneous, which implies $u_{x,r}$ is $(l, \delta^{[l]}, 1, 0^n)$-homogeneous at 0^n.

If there exists $y \in \big(\mathcal{L}_{\gamma^{-1},\gamma r,\delta} \cap B_r(x)\big) \setminus B_{\tau r}(V^l_{u,x})$, then $\tilde{y} = \frac{1}{r}(y - x) \in B_1(0^n) \setminus B_\tau(V^l_{u_{x,r},0^n})$ and $u_{x,r}$ is $(l, \delta^{[l]}, 1, \tilde{y})$-homogeneous. By Lemma 2.3, we obtain $u_{x,r}$ is $(l+1, \delta^{[l+1]}, 1, 0^n)$-homogeneous, which implies u is $(l+1, \delta^{[l+1]}, r, x)$-homogeneous, which contradicts with our assumption that l is the largest one. □

Lemma 7.13 (Covering lemma). *There exists constant $\epsilon(\eta, \gamma, \Lambda, \mathbb{G})$ such that if $x \in \mathcal{L}_{\gamma^{-1},\gamma^j,\epsilon}$ and $B_{\gamma^{j-1}}(x)$ is a ball of $\mathcal{C}^k_{\eta,\gamma^{j-1}}(T^{j-1})$, then the number of balls in the minimal covering of $B_{\gamma^{j-1}}(x) \cap \mathcal{S}^k_{\eta,\gamma^j}(u) \cap \mathcal{L}_{\gamma^{-1},\gamma^j,\epsilon}$ is less than $C(n)\gamma^{-k}$.*

Proof. We put $\epsilon = \delta(\eta, \tau, \gamma, \Lambda, \mathbb{G})$, where δ is the constant in Lemma 7.12. Since $x \in \mathcal{L}_{\gamma^{-1},\gamma^j,\epsilon}$, by Lemma 7.12, there exists nonnegative integer $l \leq n$ such that

(1) u is $(l, \eta, \gamma^{j-1}, x)$-homogeneous with respect to k-plane $V^k_{u,x}$;

(2) $\mathcal{L}_{\gamma^{-1},\gamma^j,\eta} \cap B_{\gamma^{j-1}}(x) \subset B_{\tau\gamma^{j-1}}(V^k_{u,x})$.

Since $x \in \mathcal{S}^k_{\eta,\gamma^j}(u)$, we obtain that u is not $(k+1, \eta, \gamma^{j-1}, x)$-homogeneous, which implies $l \leq k$. Hence, by choosing $\tau = \frac{\gamma}{10}$, we have

$$B_{\gamma^{j-1}}(x) \cap \mathcal{S}^k_{\eta,\gamma^j}(u) \cap \mathcal{L}_{\gamma^{-1},\gamma^j,\epsilon} \subset B_{\gamma^{j-1}}(x) \cap B_{\frac{\gamma^j}{10}}(V^k_{u,x}).$$

This completes the proof. $\square$

Lemma 7.14 (Decomposition lemma). *There exists constants $C_0(n)$, $C_1(n)$, $K(\eta, \gamma, \Lambda, \mathbb{G})$, $Q(\eta, \gamma, \Lambda, \mathbb{G})$ and $\gamma_0(\eta, \Lambda, \mathbb{G})$ such that for any $\gamma < \gamma_0$ and $j \in \mathbb{Z}_+$, we have*

(1) The set $\mathcal{S}^k_{\eta,\gamma^j}(u) \cap B_1(0^n)$ can be covered by at most j^K nonempty sets $\mathcal{C}^k_{\eta,\gamma^j}$.

(2) Each set $\mathcal{C}^k_{\eta,\gamma^j}$ is the union of at most $(C_1\gamma^{-n})^Q \cdot (C_0\gamma^{-k})^{j-Q}$ balls of radius γ^j.

Proof. First, we prove (1). We need to prove $|T^j| \leq K(\eta, \gamma, \Lambda, \mathbb{G})$ if $E_{T^j} \neq \emptyset$. For any $0 < s < t < 1$ and $x \in B_1(0^n)$, we define

$$\mathcal{W}_{s,t}(x) = \theta_{\mathbb{G}}(u, x, t) - \theta_{\mathbb{G}}(u, x, s) \geq 0.$$

Fixing a point $x_0 \in E_{T^j}$, we consider the set

$$I = \{i \in \mathbb{Z}_+ \mid \mathcal{W}_{\gamma^i,\gamma^{i-2}}(x_0) \geq \delta_0\},$$

where δ_0 is the constant in Theorem 7.5 ($p > 2$) or Theorem 7.7 ($p = 2$). It is clear that

$$\sum_{i \in I} \mathcal{W}_{\gamma^i,\gamma^{i-2}}(x_0) \leq 3\mathcal{W}_{0,1}(x_0).$$

By Lemma 7.3, we have

$$|I| \cdot \delta_0 \leq 3C(\mathbb{G})\Lambda.$$

For any $i \notin I$, by $\mathcal{W}_{\gamma^i,\gamma^{i-2}}(x_0) \leq \delta_0$, we have

$$\theta(u_{x_0,\gamma^{i-1}}, 0^n, \gamma^{-1}) - \theta(u_{x_0,\gamma^{i-1}}, 0^n, \gamma) = \mathcal{W}_{\gamma^i,\gamma^{i-2}}(x_0) < \delta_0. \tag{7.9}$$

Now, we put $\gamma_0 = \delta_0$. Thus, if $\gamma < \gamma_0$, combining (7.9), Theorem 7.5 ($p > 2$), Theorem 7.7 ($p = 2$), Lemma 7.11 and $M(u_{x_0,\gamma^{i-1}}, 0^n, 1) = 0$ when $p = 2$, we obtain $u_{x_0,\gamma^{i-1}}$ is $(0, \epsilon, 1, 0^n)$-homogeneous, which implies u is $(0, \epsilon, \gamma^{i-1}, x_0)$-homogeneous. Hence, we have $x_0 \in \mathcal{L}_{\gamma^{-1},\gamma^i,\epsilon}$, which implies $T^j_i(x_0) = 0$. It then follows that there exists constant K depending only on $\mathbb{G}$ and Λ such that

$$|T^j| := \sum_{i=1}^{j} T_i^j \leq |I| \leq K,$$

which implies the cardinality of $\{\mathcal{C}^k_{\eta,\gamma^j}(T^j)\}$ is at most

$$\binom{j}{K} \leq j^K.$$

This proves (1).

Next, we prove (2). Clearly, by an induction argument, (2) is an immediate corollary of Lemma 7.13. □

7.4. Proof of Theorem 1.15

In this subsection, we give the proof of Theorem 1.15.

Proof of Theorem 1.15. First, we put $\gamma = \min(\gamma_0, C_0^{-\frac{2}{\eta}})$, where γ_0 and C_0 are the constants in Lemma 7.14. Clearly, it suffices to prove (1.4) when $r < \gamma$. There exists $j \in \mathbb{Z}_+$ such that $\gamma^{j+1} \leq r < \gamma^j$. By Lemma 7.14, $\mathcal{S}^k_{\eta,\gamma^j}(u) \cap B_1(0^n)$ can be covered by $j^K(C_1\gamma^{-n})^Q(C_0\gamma^{-k})^{j-Q}$ balls of radius γ^j, which implies

$$\begin{aligned}\mathrm{Vol}(B_{\gamma^j}(\mathcal{S}^k_{\eta,\gamma^j}(u)) \cap B_1(0^n)) &\leq j^K(C_1\gamma^{-n})^Q(C_0\gamma^{-k})^{j-Q}(2\gamma^j)^n \\ &\leq C(n,Q,K)(\gamma^j)^{n-k-\eta}.\end{aligned}$$

Since $\gamma^{j+1} \leq r < \gamma^j$, we have $\mathcal{S}^k_{\eta,r}(u) \subset \mathcal{S}^k_{\eta,\gamma^j}(u)$, which implies

$$\begin{aligned}\mathrm{Vol}(B_r(\mathcal{S}^k_{\eta,r}(u)) \cap B_1(0^n)) &\leq \mathrm{Vol}(B_{\gamma^j}(\mathcal{S}^k_{\eta,\gamma^j}(u)) \cap B_1(0^n)) \\ &\leq C(n,Q,K)(\gamma^j)^{n-k-\eta} \\ &\leq C(\eta,\Lambda,\mathbb{G})r^{n-k-\eta},\end{aligned}$$

as desired. □

8. Appendix

8.1. Homogeneous functions

In this subsection, we assume that homogeneity of tangents holds for F and Riesz characteristic $p \geq 2$. In Lemma 8.1, we prove a basic property of homogeneous functions. By using this property, we give the proof of (1.2).

Lemma 8.1. *Let h_i be a sequence of functions on $\mathbf{R}^n$. Suppose that h_i is k-homogeneous at y_i with respect to k-plane V_i^k. If $\lim_{i\to\infty} y_i = y$, $\lim_{i\to\infty} V_i^k = V^k$ and h_i converges to u in $L^1(B_r(0^n))$. Then there exists a function h such that*

(1) h is defined on $\mathbf{R}^n$;

(2) h is k-homogeneous at y with respect to k-plane V^k;

(3) $h = u$ in $B_r(0^n)$.

Proof. Without loss of generality, we assume $y = 0^n$ and $r = 1$. We split up in to different cases.

Case 1. For any i, we have $y_i = 0^n$ and $V_i^k = V^k$.

When $p = 2$, for any $R > 1$, we have

$$\begin{aligned}
\int_{B_R(0^n)} |h_i(x) - h_j(x)|dx &= \int_{B_R(0^n)} \left|(h_i)_{0^n,\frac{1}{R}}(x) - (h_j)_{0^n,\frac{1}{R}}(x)\right| dx \\
&\leq \int_{B_R(0^n)} \left|h_i\left(\frac{x}{R}\right) - h_j\left(\frac{x}{R}\right)\right| dx \\
&\quad + \omega_n R^n \left|M\left(h_i, 0^n, \frac{1}{R}\right) - M\left(h_j, 0^n, \frac{1}{R}\right)\right| \\
&\leq R^n \|h_i - h_j\|_{L^1(B_1(0^n))} + \omega_n R^n \left|M\left(h_i, 0^n, \frac{1}{R}\right) - M\left(h_j, 0^n, \frac{1}{R}\right)\right|.
\end{aligned}$$

By Lemma 8.6, we obtain

$$\lim_{i,j\to\infty} \|h_i - h_j\|_{L^1(B_R(0^n))} = 0. \tag{8.1}$$

On the other hand, when $p > 2$, by the similar argument, we still have (8.1).

Next, by (8.1), h_i is a Cauchy sequence in $L^1_{loc}(\mathbf{R}^n)$. There exists a function h on $\mathbf{R}^n$ such that h_i converges to h in $L^1_{loc}(\mathbf{R}^n)$. It is clear that $h = u$ in $B_1(0^n)$. Now, it suffices to prove h is k-homogeneous at 0^n with respect to V^k. For any $r > 0$, we have $(h_i)_{0^n,r} = h_i$. Letting $i \to \infty$, we obtain h is k-homogeneous. Since h_i is homogeneous at 0^n with respect to V^k, then for any $x \in \mathbf{R}^n$ and $v \in V^k$, by the property of subharmonic functions, we have

$$\begin{aligned} h(x+v) - h(x) &= \lim_{s\to 0} \frac{1}{\omega_n s^n}\left(\int_{B_s(x+v)} h(y)dy - \int_{B_s(x)} h(y)dy\right) \\ &= \lim_{s\to 0}\lim_{i\to\infty} \frac{1}{\omega_n s^n}\left(\int_{B_s(x+v)} h_i(y)dy - \int_{B_s(x)} h_i(y)dy\right) \\ &= 0, \end{aligned}$$

as desired.

Case 2. General case.

Since $\lim_{i\to\infty} V_i^k = V^k$, there exists a sequence of $n \times n$ orthogonal matrices A_i such that $V_i^k = A_i V^k$ and $\lim_{i\to\infty} A_i = I_n$, where I_n is the $n \times n$ identity matrix. We define $\tilde{h}_i(x) = h_i(A_i x + y_i)$, which implies that $\tilde{h}_i$ is k-homogeneous at 0^n with respect to V^k. For any $r \in [\frac{1}{2}, 1)$, we compute

$$\begin{aligned} &\int_{B_r(0^n)} |\tilde{h}_i(x) - u(x)|dx \\ &\quad\le \int_{B_r(0^n)} |h_i(A_i x + y_i) - u(A_i x + y_i)|dx \\ &\qquad + \int_{B_r(0^n)} |u(A_i x + y_i) - u(x)|dx \\ &\quad\le \int_{B_r(y_i)} |h_i(x) - u(x)|dx + \int_{B_r(0^n)} |u(A_i x + y_i) - u(x)|dx \\ &\quad\to 0, \end{aligned} \tag{8.2}$$

where we used h_i converges to u in $L^1(B_1(0^n))$ and Lemma 8.7. By Case 1, (8.2) and scaling argument, for each $r \in [\frac{1}{2}, 1)$, there exists a function h^r such that

(1) h^r is defined on $\mathbf{R}^n$;

(2) h^r is k-homogeneous at 0^n with respect to k-plane V^k;

(3) $h^r = u$ in $B_r(0^n)$.

By (2) and (3), we have

$$h^r = h^{\frac{1}{2}} \quad \text{in } \mathbf{R}^n.$$

Hence, $h^{\frac{1}{2}}$ is the desired function. $\square$

Proposition 8.2. *If homogeneity of tangents holds for F, then for any F-subharmonic function u on $B_2(0^n)$, we have*

$$\mathcal{S}^k(u) = \bigcup_\eta \mathcal{S}^k_\eta(u) = \bigcup_\eta \bigcap_r \mathcal{S}^k_{\eta,r}(u).$$

Proof. For any $\eta > 0$, by definition, we have $\mathcal{S}^k_\eta(u) = \bigcap_r \mathcal{S}^k_{\eta,r}(u)$ and $\mathcal{S}^k_\eta(u) \subset \mathcal{S}^k(u)$. It suffices to prove $\mathcal{S}^k(u) \subset \bigcup_\eta \mathcal{S}^k_\eta(u)$. We argue by contradiction, assuming that $\mathcal{S}^k(u) \not\subseteq \bigcup_\eta \mathcal{S}^k_\eta(u)$. Then there exists a point $x \in B_2(0^n)$ such that

(1) $x \in \mathcal{S}^k(u)$;

(2) For each $i \in \mathbf{Z}_+$, there exists a $(k+1)$-homogeneous function h_i and $r_i \in (0,1)$ such that

$$\int_{B_1(0^n)} |u_{x,r_i}(y) - h_i(y)| dy < i^{-1}.$$

By the compactness of subharmonic functions, after passing to a subsequence, we assume

$$\begin{aligned} &\lim_{i\to\infty} r_i = r, \quad \lim_{i\to\infty} \|h_i - h\|_{L^1(B_{\frac{1}{2}}(0^n))} = 0 \\ &\textit{and} \quad \lim_{i\to\infty} \|u_{x,r_i} - h\|_{L^1(B_{\frac{1}{2}}(0^n))} = 0. \end{aligned} \tag{8.3}$$

If $r = 0$, by the definition of tangent (see [21, Definition 9.3, Proposition 9.4]), there exists $U \in T_x(u)$ such that u_{x,r_i} converges to U in $L^1_{loc}(\mathbf{R}^n)$. Combining this and (8.3), we have $U = h$ in $B_{\frac{1}{2}}(0^n)$. On the other hand, since homogeneity of tangents holds for F, U is 0-homogeneous. By Lemma 8.1, there exists a $(k+1)$-plane V^{k+1} such that h is $(k+1)$-homogeneous with respect to V^{k+1}. By Definition 1.7, we get $U = h$ is $(k+1)$-homogeneous, which contradicts with $x \in \mathcal{S}^k(u)$.

If $r > 0$, by Lemma 8.7, $h = u_{x,r}$ in $B_{\frac{1}{2}}(0^n)$. By the definition of tangent set, we have $T_x(u) = T_{0^n}(u_{x,r}) = T_{0^n}(h)$. By Lemma 8.1, h is a $(k+1)$-homogeneous function, which implies $T_{0^n}(h) = \{h\}$, which contradicts with $x \in \mathcal{S}^k(u)$. $\square$

8.2. K_p-convex functions

In this subsection, we recall some properties of K_p-convex functions, where K_p is the Riesz kernel.

Lemma 8.3. *Let $\{f_i\}$ be a sequence of K_p-convex functions on $(0, R)$. If $\lim_{i\to\infty} f_i(r) = f(r)$ for almost every $r \in (0, R)$, then we have $\lim_{i\to\infty} f_i(r) = f(r)$ for every $r \in (0, R)$.*

Proof. For any $\epsilon > 0$ and $r \in (0, R)$, by assumption, there exists $0 < s_1 < s_2 < r < s_3 < s_4 < R$ such that

$$\lim_{i\to\infty} f_i(s_j) = f(s_j) \text{ for } j = 1, 2, 3, 4. \tag{8.4}$$

By the definition of K_p-convex functions, for any $r_1, r_2 \in (s_2, s_3)$, we have

$$\frac{f_i(s_2) - f_i(s_1)}{K_p(s_2) - K_p(s_1)} \le \frac{f_i(r_2) - f_i(r_1)}{K_p(r_2) - K_p(r_1)} \le \frac{f_i(s_4) - f_i(s_3)}{K_p(s_4) - K_p(s_3)}. \tag{8.5}$$

Combining (8.4) and (8.5), we obtain that f_i and f are Lipschitz functions on $[s_2, s_3]$ with uniform Lipschitz constant $L(s_1, s_2, s_3, s_4, f, p)$. We can choose $\tilde{r} \in (s_2, s_3)$ such that $|\tilde{r} - r| \le \epsilon$ and $\lim_{i\to\infty} f_i(\tilde{r}) = f(\tilde{r})$. It then follows that for i sufficiently large, we have

$$\begin{aligned} |f_i(r) - f(r)| &\le |f_i(r) - f_i(\tilde{r})| + |f_i(\tilde{r}) - f(\tilde{r})| + |f(\tilde{r}) - f(r)| \\ &\le 2L\epsilon + |f_i(\tilde{r}) - f(\tilde{r})| \\ &\le (2L + 1)\epsilon, \end{aligned}$$

which implies $\lim_{i\to\infty} f_i(r) = f(r)$. Since r is arbitrary, we complete the proof. □

Lemma 8.4. *Let $\{f_i\}$ be a sequence of K_p-convex functions on $(0, R)$. If $\lim_{i\to\infty} f_i(r) = f(r)$ for every $r \in (0, R)$, then we have*

$$\lim_{i\to\infty} \frac{(f_i)'_{\pm}(r)}{K_p'(r)} = \frac{f'(r)}{K_p'(r)},$$

for almost every $r \in (0, R)$.

Proof. Since f_i are K_p-convex functions and $f = \lim_{i\to\infty} f_i$, it is clear that f is also K_p-convex function. As a result, we obtain f is differentiable almost everywhere in $(0,R)$. For any $r_0 \in (0,R)$ at which f is differentiable and for any $\epsilon > 0$, there exists $r > 0$ such that

$$\frac{f'(r_0)}{K_p'(r_0)} - \epsilon \leq \frac{f(r_0)-f(r_0-r)}{K_p(r_0)-K_p(r_0-r)} \leq \frac{f(r_0+r)-f(r_0)}{K_p(r_0+r)-K_p(r_0)} \leq \frac{f'(r_0)}{K_p'(r_0)} + \epsilon.$$

Then there exists $N > 0$ such that for any $i \geq N$, we have

$$\frac{f'(r_0)}{K_p(r_0)} - 2\epsilon \leq \frac{f(r_0)-f(r_0-r)}{K_p(r_0)-K_p(r_0-r)} - \epsilon \leq \frac{f_i(r_0)-f_i(r_0-r)}{K_p(r_0)-K_p(r_0-r)} \leq \frac{(f_i)'_-(r_0)}{K_p'(r_0)}$$

and

$$\begin{aligned}\frac{(f_i)'_+(r_0)}{K_p'(r_0)} &\leq \frac{f_i(r_0+r)-f_i(r_0)}{K_p(r_0+r)-K_p(r_0)} \\ &\leq \frac{f(r_0+r)-f(r_0)}{K_p(r_0+r)-K_p(r_0)} + \epsilon \leq \frac{f'(r_0)}{K_p(r_0)} + 2\epsilon.\end{aligned}$$

Combining with

$$\frac{(f_i)'_-(r_0)}{K_p'(r_0)} \leq \frac{(f_i)'_+(r_0)}{K_p'(r_0)},$$

we complete the proof. □

8.3. Subharmonic function in $\mathbf{R}^p$

In this subsection, we recall some properties of subharmonic functions.

Lemma 8.5. *Let v be a subharmonic function on $B_R(0^p) \subset \mathbf{R}^p$ with $\|v\|_{L^1(B_b(0^p)\setminus(B_a(0^p))} \leq \Lambda$, where $0 < a < b < R$. Then for any $t \in (a+d, b-d)$, where $d > 0$, there exists a constant $C(t,a,d)$ such that*

$$M(v,0^p,t) \geq S(v,0^p,t) \geq -C(t,a,d)\Lambda,$$

where 0^p is the origin in $\mathbf{R}^p$.

Proof. It suffices to prove $S(v,0^p,t) \geq -C(t,a,d)\Lambda$. First, by the submean value property of subharmonic functions, there exists a constant $\tilde{C}(d,\Lambda)$

such that

$$\sup_{B_{b-d}(0^p)\setminus B_{a+d}(0^p)} v \leq \tilde{C}(d,\Lambda).$$

Thus, we compute

$$\begin{aligned}\int_{B_t(0^p)\setminus B_{a+d}(0^p)} |\tilde{C}-v(x)|dx &= \int_{B_t(0^p)\setminus B_{a+d}(0^p)} \left(\tilde{C}-v(x)\right) dx \\ &= \int_{a+d}^{t} p\omega_p s^{p-1}\left(\tilde{C}-S(v,0^p,s)\right) ds \\ &\geq \left(\tilde{C}-S(v,0^p,t)\right)\omega_p\left(t^p-(a+d)^p\right),\end{aligned}$$

where ω_p is the volume of unit ball in $\mathbf{R}^p$. It is clear that

$$\begin{aligned}\left(\tilde{C}-S(v,0^p,t)\right)\omega_p\left(t^p-(a+d)^p\right) &\leq \|\tilde{C}-v\|_{L^1(B_t(0^p)\setminus B_{a+d}(0^p))} \\ &\leq \tilde{C}\omega_p\left(t^p-(a+d)^p\right)+\Lambda.\end{aligned}$$

Hence, we obtain

$$S(v,0^p,t) \geq -\frac{\Lambda}{\omega_p\left(t^p-(a+d)^p\right)}.$$

□

Lemma 8.6. *Let v_i and v be subharmonic functions on $B_R(0^p)\subset\mathbf{R}^p$. If v_i converges to v in $L^1(B_b(0^p)\setminus B_a(0^p))$, where $0<a<b<R$, then for any $r\in(a,b)$, we have*

$$\lim_{i\to\infty} M(v_i,0^p,r) = M(v,0^p,r) \tag{8.6}$$

and

$$\lim_{i\to\infty} S(v_i,0^p,r) = S(v,0^p,r). \tag{8.7}$$

Proof. First, by the property of subharmonic functions, for any $x\in B_b(0^p)\setminus B_a(0^p)$, we have

$$v_i(x)\leq v_i*\phi_\delta(x) \quad \text{and} \quad \lim_{i\to\infty} v_i*\phi_\delta(x) = v*\phi_\delta(x),$$

where ϕ is a mollifier. It then follows that

$$\limsup_{i\to\infty} v_i(x) \leq \lim_{\delta\to 0} v*\phi_\delta(x) = v(x),$$

which implies

$$\limsup_{i\to\infty} M(v_i, 0^p, r) \le M(v, 0^p, r). \tag{8.8}$$

Suppose we have

$$\liminf_{i\to\infty} M(v_i, 0^p, r) < M(v, 0^p, r),$$

then there exists a subsequence $\{v_{i_k}\}$ and a number d such that

$$\lim_{k\to\infty} M(v_{i_k}, 0^p, r) = \liminf_{i\to\infty} M(v_i, 0^p, r) < d < M(v, 0^p, r). \tag{8.9}$$

Then we get $v_{i_k} \le d$ on $B_r(0^p)$ when k is sufficiently large. By the convergence in $L^1(B_b(0^p) \setminus B_a(0^p))$, we obtain $v \le d$ on $B_r(0^p) \setminus B_a(0^p)$. Since v is subharmonic function, we have

$$M(v, 0^p, r) \le d,$$

which contradicts with (8.9). Therefore, we conclude that

$$\liminf_{i\to\infty} M(v_i, 0^p, r) \ge M(v, 0^p, r). \tag{8.10}$$

Combining (8.8) and (8.10), we prove (8.6).

For the proof of (8.7), by Fatou's lemma, it is clear that

$$\int_a^b \left(\lim_{i\to\infty} \int_{\partial B_r(0^p)} |v_i - v| \right) dr \le \lim_{i\to\infty} \int_{B_b(0^p)\setminus B_a(0^p)} |v_i(x) - v(x)| dx \to 0,$$

which implies

$$\lim_{i\to\infty} S(v_i, 0^p, r) = S(v, 0^p, r)$$

for almost every $r \in (a, b)$. Since $S(v_i, 0^p, \cdot)$ and $S(v, 0^p, \cdot)$ are K_p-convex functions, by Lemma 8.3, we obtain (8.7). $\square$

Lemma 8.7. *Suppose that A_i is a sequence of $p \times p$ orthogonal matrices and z_i is a sequence of points. Let v be a subharmonic function on $B_R(0^n)$. If $\lim_{i\to\infty} z_i = 0^n$ and $\lim_{i\to\infty} A_i = I_p$ (I_p is the $p \times p$ identity matrix), then for any $r \in (0, R)$, we have*

$$\lim_{i\to\infty} \int_{B_r(0^n)} |v(A_i x + z_i) - v(x)| dx = 0.$$

Proof. For convenience, we use v_δ to denote $v * \phi_\delta$, where ϕ_δ is a mollifier. By the property of smooth approximation, it is clear that v_δ converges to v in $L^1_{loc}(B_R(0^n))$. On the other hand, since v_δ is smooth, we have

$$\lim_{i\to\infty} \int_{B_r(0^n)} |v_\delta(A_i x + z_i) - v_\delta(x)| dx = 0.$$

Therefore, we obtain

$$\begin{aligned}
&\int_{B_r(0^n)} |v(A_i x + z_i) - v(x)| dx \\
&\quad \le \int_{B_r(0^n)} |v(A_i x + z_i) - v_\delta(A_i x + z_i)| dx \\
&\qquad + \int_{B_r(0^n)} |v_\delta(A_i x + z_i) - v_\delta(x)| dx + \int_{B_r(0^n)} |v_\delta(x) - v(x)| dx \\
&\quad \le \int_{B_r(z_i)} |v(x) - v_\delta(x)| dx + \int_{B_r(0^n)} |v_\delta(A_i x + z_i) - v_\delta(x)| dx \\
&\qquad + \int_{B_r(0^n)} |v_\delta(x) - v(x)| dx \\
&\quad \to 0,
\end{aligned}$$

as desired. □

Lemma 8.8. *Let v_i and v be subharmonic functions on $B_2(0^n)$, and suppose that v_i converges to v in $L^1_{loc}(B_2(0^n))$. For any sequence of point $\{z_i\} \subset B_1(0^n)$, if z_i converges to z, then we have*

$$\lim_{i\to\infty} \int_{B_1(0^n)} |(v_i)_{z_i,r}(x) - v_{z,r}(x)| dx,$$

for any $r \in (0, 1)$.

Proof. We split up into different cases.

Case 1. $p > 2$.

For any $r \in (0,1)$, by Lemma 8.7, we have

$$\begin{aligned}
&\int_{B_1(0^n)} |(v_i)_{z_i,r}(x) - v_{z,r}(x)|dx \\
&\quad \leq \int_{B_1(0^n)} |(v_i)_{z_i,r}(x) - v_{z_i,r}(x)|dx + \int_{B_1(0^n)} |v_{z_i,r}(x) - v_{z,r}(x)|dx \\
&\quad = \int_{B_r(z_i)} r^{p-2-n}|v_i(x) - v(x)|dx \\
&\quad\quad + \int_{B_r(0^n)} r^{p-2-n}|v(x+z_i) - v(x+z)|dx \\
&\quad \to 0,
\end{aligned}$$

as desired.

Case 2. $p = 2$.

By the definition of tangential 2-flow, we have

$$\begin{aligned}
&\int_{B_1(0^n)} |(v_i)_{z_i,r}(x) - v_{z,r}(x)|dx \\
&\quad \leq \int_{B_1(0^n)} |v_i(rx+z_i) - v(rx+z)|dx \\
&\quad\quad + \int_{B_1(0^n)} |M(v_i,z_i,r) - M(v,z,r)|dx.
\end{aligned}$$

By the similar argument in Case 1, we obtain

$$\lim_{i\to\infty} \int_{B_1(0^n)} |v_i(rx+z_i) - v(rx+z)|dx = 0$$

Hence, it suffices to prove $\lim_{i\to\infty} M(v_i,z_i,r) = M(v,z,r)$. Next, we define $\tilde{v}_i(x) = v_i(x+z_i-z)$ for every $x \in B_1(0^n)$. It then follows that $M(\tilde{v}_i,z,r) = M(v_i,z_i,r)$. It is clear that

$$\begin{aligned}
&\int_{B_1(0^n)} |\tilde{v}_i(x) - v(x)|dx \\
&\quad \leq \int_{B_1(0^n)} |v_i(x+z_i-z) - v(x+z_i-z)|dx \\
&\quad\quad + \int_{B_1(0^n)} |v(x+z_i-z) - v(x)|dx \\
&\quad = \int_{B_1(z_i-z)} |v_i(x) - v(x)|dx + \int_{B_1(0^n)} |v(x+z_i-z) - v(x)|dx \\
&\quad \to 0,
\end{aligned}$$

where we used Lemma 8.7. Hence, by Lemma 8.6, we obtain

$$\lim_{i\to\infty} M(v_i, z_i, r) = \lim_{i\to\infty} M(\tilde{v}_i, z, r) = M(v, z, r).$$

□

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