

On the fundamental group of semi-Riemannian manifolds with positive curvature tensor

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This paper presents an investigation of the relation between some positivity of the curvature and the finiteness of fundamental groups in semi-Riemannian geometry. We consider semi-Riemannian submersions $\pi : (E, g) \rightarrow (B, -g_B)$ under the condition with (B, g_B) Riemannian, the fiber closed Riemannian, and the horizontal distribution integrable. Then we prove that, if the lightlike geodesically complete or timelike geodesically complete semi-Riemannian manifold E has some positivity of curvature, then the fundamental group of the fiber is finite. Moreover we construct an example of semi-Riemannian submersions with some positivity of curvature, non-integrable horizontal distribution, and the finiteness of the fundamental group of the fiber.

1. Introduction

This paper discusses our study of the fundamental group of semi-Riemannian manifolds with positive curvature tensor. In the case of positive constant curvature, Calabi–Markus and Wolf proved the following theorem:

Theorem 1.1 (Calabi–Markus [CM62] ($q = 1$), Wolf [Wol62]). *Let M be an n -dimensional geodesically complete semi-Riemannian manifold of index q with constant positive curvature, where $n \geq 2q > 0$. Then the fundamental group $\pi_1(M)$ is finite.*

Kobayashi considered whether the finiteness of the fundamental group would continue to hold if we perturb the metric of positive constant curvature. Kobayashi proposed the following conjecture:

Conjecture (Kobayashi [Kob01]). *Let n and q be positive integers with $n \geq 2q$. Assume that (M, g) is an n -dimensional geodesically complete semi-Riemannian manifold of index q . Suppose that we have a positive lower bound on the sectional curvature of (M, g) . Then,*

- 1) M is never compact;
- 2) if $n \geq 3$, the fundamental group of M is always finite.

In the previous paper [Muk15] we remarked that the conjecture is true by using Kulkarni's theorem [Kul79] that states that the one-sided bound on the sectional curvature leads to constant curvature. Therefore, we replace the curvature condition of the conjecture by another condition.

We study the following curvature condition of Andersson and Howard [AH98]:

$$(1.1) \quad g(R(u, v)v, u) \geq k(g(u, u)g(v, v) - g(u, v)^2)$$

for any tangent vectors u, v , where R is the curvature tensor. Following Andersson–Howard [AH98], we denote this condition by $R \geq k$. Reversing the inequality (1.1), we write the condition as $R \leq k$. These conditions are extensions of the curvature conditions of Riemannian geometry that the sectional curvature is bounded above or below. Alexander–Bishop [AB08] found that these conditions geometrically means a signed local triangle comparison condition. Under the curvature conditions of Andersson–Howard, several analogues of Riemannian comparison theory have been obtained, for instance, gap rigidity theorem (Andersson–Howard [AH98]), volume comparison theorem (Díaz-Ramos–García-Río–Hervella [DRGRH05]), and local triangle comparison theorem (Alexander–Bishop [AB08]).

Our previous paper [Muk15] presents our study of an analogy of the Myers theorem in Lorentzian geometry. In the current paper, we investigate an analogy of the Myers theorem in semi-Riemannian geometry. A semi-Riemannian manifold E is said to be *lightlike geodesically complete* (resp. *timelike geodesically complete*) if any inextensible lightlike (resp. timelike) geodesic is defined on the real line. We obtain the following theorem:

Theorem 1.2. *Let (E, g) be an either lightlike geodesically complete or timelike geodesically complete semi-Riemannian manifold with $R \geq k > 0$, and (B, g_B) a complete Riemannian manifold of dimension greater than or equal to 2. Suppose that there exists a semi-Riemannian submersion $\pi : (E, g) \rightarrow (B, -g_B)$ such that the fibers are closed Riemannian manifolds, that the dimension of the fibers is greater than or equal to 2, and that*

the horizontal distribution is integrable. Then the induced homomorphism $\pi_ : \pi_1(E) \rightarrow \pi_1(B)$ is surjective and has a finite kernel. Moreover, the fundamental group of the fibers is finite.*

We should remark that the theorem includes not only warped products but also non-warped products. In fact, the semi-Riemannian product $(\mathbb{H}^l \times S^m, -g_{\mathbb{H}^l} + g_{S^m})$ satisfies the assumptions of the theorem, where $(\mathbb{H}^l, g_{\mathbb{H}^l})$ is the l -dimensional hyperbolic space, and (S^m, g_{S^m}) is the l -dimensional sphere. By the stability of the lightlike geodesic completeness by Beem–Ehrlich [BE87], we can perturb the fiber metric (S^m, g_{S^m}) of the product $(\mathbb{H}^l \times S^m, -g_{\mathbb{H}^l} + g_{S^m})$ with the assumption satisfied.

Note that the base manifold (B, g_B) has negative curvature bounded above by $-k$ by Lemma 3.1 of the paper. Hence, the theorem leads us to the following corollary:

Corollary 1.1. *Assume that, in addition to the hypotheses of the theorem, E is a closed semi-Riemannian manifold. Then the fundamental group $\pi_1(E)$ has exponential growth.*

The theorem gives a sufficient condition of geodesic incompleteness by its contraposition:

Theorem 1.2. *Let (E, g) be semi-Riemannian manifold with $R \geq k > 0$, and (B, g_B) a complete Riemannian manifold of dimension greater than or equal to 2. Assume that there exists a semi-Riemannian submersion $\pi : (E, g) \rightarrow (B, -g_B)$ such that the fibers are closed Riemannian manifolds, that the dimension of the fibers is greater than or equal to 2, that the fundamental group of the fibers is infinite, and that the horizontal distribution is integrable. Then (E, g) is neither lightlike geodesically complete nor timelike geodesically complete.*

Let (B, g_B) be a Riemannian manifold with negative curvature bounded above by $-k$, F a closed manifold of which the fundamental group is infinite, and let $g_F^* = \{g_F^b\}_{b \in B}$ be a smooth family of Riemannian metrics of F . By the theorem we see that $(B \times F, -g_B + g_F^*)$ is never geodesically complete, satisfying $R \geq k > 0$. In fact, $(\mathbb{H}^l \times \mathbb{T}^m, -g_{\mathbb{H}^l} + e^{2b}g_{\mathbb{T}^m})$, which is a special case of Alexander–Bishop [AB08, Example 7.5 (c)], satisfies $R \geq 1 > 0$, but is not geodesically complete, where b is a Busemann function of \mathbb{H}^l .

Note that there exists a semi-Riemannian submersion with $R \geq k > 0$ and without the integrability of horizontal distribution. In fact, we construct

a semi-Riemannian example $\pi : SU(2, 1)/S^1 \rightarrow SU(2, 1)/U(2)$, which satisfies $R \geq k > 0$ and of which the horizontal distribution is not integrable and fiber $U(2)/S^1$ has a finite fundamental group. This construction is motivated by the previously reported construction of the positively curved Riemannian manifold $SU(3)/S^1$ [AW75]. It would be interesting to determine whether we can construct new semi-Riemannian manifolds with $R \geq k > 0$ by using the construction methods of curved Riemannian manifolds. In respect of Theorem 1.2, in general we do not know whether the theorem can be extended to semi-Riemannian submersions without the integrability of the horizontal distribution. We conjecture that we expect it to be possible to remove the integrability of the horizontal distribution from the theorem.

Organization of the paper

In Section 2, we introduce some notions and propositions on semi-Riemannian submersions needed for proving Theorem 1.2. In Section 3, we prove Theorem 1.2. In Section 4, we construct a semi-Riemannian submersion with $R \geq k > 0$ and non-integrable horizontal distribution.

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2. Preliminary

A *semi-Riemannian metric* g of a manifold M is a symmetric non-degenerate $(0, 2)$ tensor. If a manifold M is equipped with a semi-Riemannian metric g , (M, g) is said to be a *semi-Riemannian manifold*. In this section, we recall some terminology and results of semi-Riemannian geometry, which can mainly be found in the papers by O'Neill [O'N66, O'N67, O'N83]. Note that, although O'Neill [O'N66, O'N67] considered the Riemannian case, many results are generalized to semi-Riemannian cases.

Definition 2.1 (O'Neill [O'N83, Definition 44]). Let (E, g) and (B, g_B) be semi-Riemannian manifolds. A map $\pi : E \rightarrow B$ is a *semi-Riemannian submersion* if π is a submersion satisfying the following conditions:

- The fibers $\pi^{-1}(x)$ are semi-Riemannian manifolds for any $x \in B$;

- differential map $\pi_{*p} : (T_p\pi^{-1}(\pi(p)))^\perp \rightarrow T_{\pi(p)}B$ is isometric for any $p \in E$.

Especially, this research only considers the case in which the fiber and $(B, -g_B)$ are Riemannian. Tangent vectors normal (resp. tangent) to fibers are known as *horizontal* (resp. *vertical*) vectors. A distribution \mathcal{H} (resp. \mathcal{V}) over E is *horizontal* (resp. *vertical*) if \mathcal{H}_p (resp. \mathcal{V}_p) is a tangent subspace in E of which the elements are horizontal (resp. vertical). Note that $T_pE = \mathcal{H}_p \oplus \mathcal{V}_p$. Then for any tangent vector $v \in T_pE$, $v^{\mathcal{H}} \in \mathcal{H}_p$ and $v^{\mathcal{V}} \in \mathcal{V}_p$ are given by $v = v^{\mathcal{H}} + v^{\mathcal{V}}$. For a tangent vector X of the base B , a tangent vector \hat{X} on E is termed a *lift* of X if \hat{X} is horizontal and $\pi_*\hat{X} = X$. We often identify vectors or vector fields on B with their lifts. A vector field X is said to be *basic* if X is horizontal and $\pi_*\hat{X}$ is independent of any points of fibers. Every vector field on B has a unique horizontal and basic lift on E .

We denote by $\hat{\nabla}$ (resp. ∇^B) the Levi-Civita connection of (E, g) (resp. (B, g_B)). O'Neill [O’N66] defined the following important tensors:

Definition 2.2 (O’Neill [O’N66]). (2,1) tensor fields T and A on E are defined by satisfying that, for any tangent vectors v, w of E ,

$$T_vw = (\hat{\nabla}_{v^{\mathcal{V}}}w^{\mathcal{V}})^{\mathcal{H}} + (\hat{\nabla}_{v^{\mathcal{V}}}w^{\mathcal{H}})^{\mathcal{V}};$$

$$A_vw = (\hat{\nabla}_{v^{\mathcal{H}}}w^{\mathcal{H}})^{\mathcal{V}} + (\hat{\nabla}_{v^{\mathcal{H}}}w^{\mathcal{V}})^{\mathcal{H}}.$$

Note that for vertical tangent vectors V, W the tensor T_VW is the second fundamental form of the fiber. It is well known that the horizontal distribution is integrable if and only if $A = 0$. Therefore, the case we consider in this work is $A = 0$. The following formulas of the curvature by the tensors T and A hold:

Proposition 2.1 (O’Neill [O’N66, Corollary 1]). Let \hat{K}, K_* and K^\perp be the sectional curvatures of E and B , a fiber, respectively. Let X, Y be horizontal vectors, and let V, W be vertical vectors. Then we have

$$(2.1) \quad \hat{K}(X, Y) = K_*(\pi_*X, \pi_*Y) - \frac{3g(A_XY, A_XY)}{g(X, X)g(Y, Y) - g(X, Y)^2};$$

$$(2.2) \quad \hat{K}(V, W) = K^\perp(V, W) - \frac{g(T_VV, T_WW) - g(T_VW, T_VW)}{g(V, V)g(W, W) - g(V, W)^2}.$$

Let $c(t)$ be a curve in E and $Z(t)$ a vector field along $c(t)$. We consider the covariant derivative of $Z(t)$. Throughout this paper, the covariant derivative of the vector field along a curve is denoted by ι . Then we have

Proposition 2.2 (O’Neill [O’N67, Theorem 1]). *Let $c(t)$ and $Z(t)$ be as above. The following equation holds:*

$$(2.3) \quad \begin{aligned} (Z'(t))^{\mathcal{H}} &= (\widehat{\pi_* (Z)})'(t) + A_{Z(t)^{\mathcal{H}}}(c'(t)^{\mathcal{V}}) \\ &\quad + A_{c'(t)^{\mathcal{H}}}(Z(t)^{\mathcal{V}}) + T_{c'(t)^{\mathcal{V}}}(Z^{\mathcal{V}}(t)). \end{aligned}$$

O’Neill proved that Proposition 2.2 implies the following proposition, which means that any geodesic in B lifts to a unique horizontal geodesic in E :

Proposition 2.3 (O’Neill [O’N67, Corollary 2]). *Let $\pi : E \rightarrow B$ be a semi-Riemannian submersion. If the initial velocity of a geodesic is horizontal, any velocity of the geodesic is also horizontal at any time.*

Recall warped products and their generalization, i.e., twisted products, as important examples of semi-Riemannian submersions. Let (B, g_B) and (F, g_F) be Riemannian manifolds and α a smooth function of $B \times F$. A semi-Riemannian manifold $(B \times F, -g_B + e^{2\alpha}g_F)$, is known as a *semi-Riemannian twisted product*, and especially, if α does not depend on B , it is known as a *semi-Riemannian warped product*. Note that the natural projection $\pi : (B \times F, -g_B + e^{2\alpha}g_F) \rightarrow (B, -g_B)$ is a semi-Riemannian submersion. We denote by $\widehat{\nabla}$, ∇^B , and ∇^F the Levi-Civita connections of $(B \times F, -g_B + e^{2\alpha}g_F)$, (B, g_B) , and (F, g_F) , respectively. The natural projection from $B \times F$ to F is denoted by π_F . For a vector field Z on B (resp. F), a vector field \widehat{Z} on the product manifold $B \times F$ is a *lift* of Z if $\pi_{F*}(\widehat{Z}) = 0$ and $\pi_*(\widehat{Z}) = Z$ (resp. $\pi_*(\widehat{Z}) = 0$ and $\pi_{F*}(\widehat{Z}) = Z$). We have the following formula of the mean curvature vector of the fibers in warped products:

Proposition 2.4 (O’Neill [O’N83, Chapter 7, Prop. 35] (warped products case), Chen [Che81, Chapter VII, Prop. 1.2] (twisted product cases)). *For any lifts \widehat{U} , \widehat{V} of vector fields U, V on F in $(B \times F, -g_B + e^{2\alpha}g_F)$,*

$$(2.4) \quad T_{\widehat{U}}\widehat{V} = e^{2\alpha}g_F(U, V)\nabla^B \log \alpha.$$

We use the following equations on geodesics in warped products later:

Proposition 2.5 (O’Neill [O’N83, Chapter 7, Prop. 38]). *Any geodesic $\gamma(t) = (\gamma_B(t), \gamma_F(t))$ in the warped product $(B \times F, -g_B + e^{2\alpha}g_F)$ satisfies*

the following two conditions:

$$\begin{aligned} \nabla_{\partial/\partial t}^B \gamma'_B(t) &= -e^{2\alpha} g_F(\gamma'_F(t), \gamma'_F(t)) \nabla^B \alpha; \\ \nabla_{\partial/\partial t}^F \gamma'_F(t) &= -2 \frac{d\alpha(\gamma_B(t))}{dt} \gamma'_F(t). \end{aligned}$$

3. Proof of Theorem 1.2

Proof of Theorem 1.2. First, we prove the following lemma:

Lemma 3.1. *(B, g_B) has negative curvature bounded above by -k.*

Proof. We assume that A = 0. By using the equation (2.1) of Proposition 2.1, we have $K(X, Y) = K_*(\pi_*X, \pi_*Y)$. The curvature condition $R \geq k > 0$ implies $K_*(\pi_*X, \pi_*Y)$. Let K_B be the curvature of (B, g_B) . Since $K_*(\pi_*X, \pi_*Y) = -K_B(\pi_*X, \pi_*Y)$, the curvature of (B, g_B) is bounded above by $-k$. □

Let b_0 be a fixed point of B , and F the fiber $\pi^{-1}(b_0)$. It follows that the universal covering space of B is contractible by the Hadamard–Cartan theorem. By the following exact sequence of the homotopy group

$$1 \rightarrow \pi_1(F) \rightarrow \pi_1(E) \xrightarrow{\pi_*} \pi_1(B) \rightarrow 1,$$

we see that π_* is surjective and that the kernel of π_* is isomorphic to $\pi_1(F)$. Therefore, it is sufficient to prove that the fundamental group of F is finite. Moreover, we assume that B is simply connected and contractible throughout the proof. In fact, this is because, for the universal covering $\text{cov} : \tilde{B} \rightarrow B$, the fiber of the induced submersion $\text{cov}^*E \rightarrow \tilde{B}$ is the same as the original fiber of $\pi : E \rightarrow B$.

Here we investigate the metric structure of the entire space E . We can define the projective map $\pi_F : E \rightarrow F$ by the following: For any $p \in E$, we have a unique geodesic γ in B from $\pi(p)$ to b_0 . Let $\hat{\gamma}$ be the horizontal lift of γ . Then $\pi_F(p)$ is given by the end point of $\hat{\gamma}$. Note that π_F is surjective and smooth since geodesics smoothly depend on initial points. As the horizontal distribution is integrable, a horizontal manifold $\pi^{-1}(f)$ is diffeomorphic to B under π . Then we see that $\phi = (\pi, \pi_F) : E \rightarrow B \times F$ is a diffeomorphism. By the construction of π_F , the fiber $\pi_F^{-1}(f)$ diffeomorphic to $B \times \{f\}$ is a horizontal manifold. The definition of semi-Riemannian submersion determines that $\pi : \pi_F^{-1}(f) \rightarrow B$ is isometric. Therefore, we obtain the following lemma:

Lemma 3.2. *Set $g_F^b = g|_{\pi^{-1}(b)}$ and a smooth family of the Riemannian metric $g_F^* = \{g_F^b\}_{b \in B}$ of F with respect to the points of B . Then (E, g) is isometric to $(B \times F, -g_B + g_F^*)$.*

Throughout the proof, (E, g) is regarded as $(B \times F, -g_B + g_F^*)$. This metric structure leads us to define lifts of any vector field on fibers in E as well as warped products and twisted products. Any horizontal lift of vector fields on the base space does not depend on the fibers.

Next, we investigate the curvature of the fiber. Parallel vector fields along horizontal geodesics preserve verticality.

Lemma 3.3. *Let $c : \mathbb{R} \rightarrow E$ be a horizontal curve and $V(t)$ a parallel vector field along $c(t)$ with $V(0)$ vertical. Then $V(t)$ is a vertical vector field.*

Proof. Since $A = 0$ and $c(t)$ is a horizontal curve, the equation (2.3) gives $(V'(t))^{\mathcal{H}} = (\pi_*(\widehat{V(t)}))'$. As $V(t)$ is parallel, $V'(t) = 0$. It follows that $(\pi_*(V(t)))' = 0$. Since $\pi_*V(0) = 0$, we have $\pi_*V(t) = 0$. Therefore, $V(t)$ is vertical. □

Place any point $p_0 \in E$ and unit vertical vector $V_0 \in \mathcal{V}_{p_0}$. Set $x_0 = \pi(p_0)$. Let u_0 be any unit tangent vector at x_0 and γ the geodesic in B starting from x_0 with initial velocity u_0 . Further, let N be the C^1 gradient vector field of the Busemann function b_γ of B associated to γ . Note that the flow of N is a geodesic in B . Then the Busemann function b_γ is extended to the entire space $E = B \times F$ by $B \times F \ni (b, f) \mapsto b_\gamma(b) \in \mathbb{R}$. This extended Busemann function is denoted by \widehat{b}_γ . We write the gradient vector of \widehat{b}_γ as \widehat{N} , which is the horizontal lift of N . Then for any $p \in E$ we define a map $S : \mathcal{V}_p \rightarrow T_pE$ by $S(V_0) = \widehat{V}_{V_0}\widehat{N}$ for any vertical vector $V_0 \in \mathcal{V}_p$. Then we have

Lemma 3.4. *$S(V_0)$ is vertical. Moreover, $S(V_0)$ has no term of differentials of N .*

Proof. For $p = (b, f)$, take a coordinate $(x_1, x_2, \dots, x_i, \dots, x_{\dim B})$ of $B \times \{f\}$ and a coordinate $(y_1, y_2, \dots, y_k, \dots, y_{\dim E - \dim B})$ of $\{b\} \times F$ around p . Then we have

$$V_0 = \sum_k V_0^k \frac{\partial}{\partial y_k}, \quad \widehat{N} = \sum_i \widehat{N}^i \frac{\partial}{\partial x_i}.$$

Note that $\partial \widehat{N}^i / \partial y_k = 0$ since \widehat{N} is a horizontal lift and E is the product manifold $B \times F$. Therefore,

$$S(V_0) = \widehat{\nabla}_{V_0} \widehat{N} = \sum_k V_0^k \widehat{\nabla}_{\frac{\partial}{\partial y_k}} \left(\sum_i \widehat{N}^i \frac{\partial}{\partial x_i} \right) = \sum_{k,i} \widehat{N}^i V_0^k \widehat{\nabla}_{\frac{\partial}{\partial y_k}} \frac{\partial}{\partial x_i}.$$

Moreover, we have

$$\widehat{\nabla}_{\frac{\partial}{\partial y_k}} \frac{\partial}{\partial x_i} = \sum_{l,m} \frac{g^{lm}}{2} \frac{\partial g_{kl}}{\partial x_i} \frac{\partial}{\partial y_m},$$

where l, m are the index of the coordinate of the fiber. Hence, $S(V_0)$ is vertical and $S(V_0)$ has no term of differentials of N . □

We can extend the vertical vector V_0 to a vertical vector field on E by using partitions of unity, and restrict this vector field to the C^2 -submanifold $\widehat{b}_\gamma^{-1}(\widehat{b}_\gamma(p_0))$. The restricted vector field stands for V_0 by abuse of notation. We consider the differential equation $\widehat{\nabla}_{\widehat{N}} V = 0$ with the initial value $V(p) = V_0(p)$ for any $p \in \widehat{b}_\gamma^{-1}(\widehat{b}_\gamma(p_0))$. Take any flow $\tau(t)$ of \widehat{N} with $\tau(0) \in \widehat{b}_\gamma^{-1}(\widehat{b}_\gamma(p_0))$ and $\tau(0) = V_0(\tau(0))$. Then $\widehat{\nabla}_{\widehat{N}} V = \widehat{\nabla}_{\tau'(t)} V(\tau(t)) = 0$ on the geodesic $\tau(t)$. Then $V(\tau(t))$ is solved and vertical by Lemma 3.3. By collecting $V(\tau(t))$, we see that V is C^2 vertical vector fields on E . Let $\widehat{\gamma}(t)$ be the horizontal lift of $\gamma(t)$ starting from p_0 . We denote the tangent vector $S(V)(\widehat{\gamma}(t))$ at $\widehat{\gamma}(t)$ by $S_t(V)$. Set $h(t) = g(S_t(V), V)$. We should remark that

$$g(S_t(V), V) = g(\widehat{\nabla}_V \widehat{N}, V) = -g(\widehat{N}, \widehat{\nabla}_V V).$$

We have

$$\begin{aligned} \frac{d}{dt} h(t) &= -\widehat{N} g(\widehat{N}, \widehat{\nabla}_V V) \\ &= -g(\widehat{\nabla}_{\widehat{N}} \widehat{N}, \widehat{\nabla}_V V) - g(\widehat{N}, \widehat{\nabla}_{\widehat{N}} \widehat{\nabla}_V V) \\ &= -g(\widehat{N}, \widehat{\nabla}_{\widehat{N}} \widehat{\nabla}_V V) \\ &= -g(\widehat{N}, R(\widehat{N}, V)V) - g(\widehat{N}, \widehat{\nabla}_V \widehat{\nabla}_{\widehat{N}} V) - g(\widehat{N}, \widehat{\nabla}_{[\widehat{N}, V]} V) \\ &= -g(\widehat{N}, R(\widehat{N}, V)V) + g(\widehat{N}, \widehat{\nabla}_{\widehat{\nabla}_V \widehat{N}} V) \\ &= -g(\widehat{N}, R(\widehat{N}, V)V) + g(\widehat{N}, [\widehat{\nabla}_V \widehat{N}, V] + \widehat{\nabla}_V \widehat{\nabla}_V \widehat{N}) \end{aligned}$$

$$\begin{aligned}
 &= -g(\widehat{N}, R(\widehat{N}, V)V) + g(\widehat{N}, \widehat{\nabla}_V \widehat{\nabla}_V \widehat{N}) \\
 &= -g(R(V, \widehat{N})\widehat{N}, V) - g(\widehat{\nabla}_V \widehat{N}, \widehat{\nabla}_V \widehat{N}) \\
 &\leq -kg(V, V)g(\widehat{N}, \widehat{N}) - g(\widehat{\nabla}_V \widehat{N}, V)^2 \\
 &\leq k - h(t)^2.
 \end{aligned}$$

By Ricatti’s argument, we obtain $|h(t)| \leq \sqrt{k}$. We should remark that

$$|h(t)| = |g(\widehat{N}, \widehat{\nabla}_V V)| = |g(\widehat{N}, T_V V)| = |g(\widehat{N}, T_V V)|$$

as $h(t)$ is defined on \mathbb{R} . Since $|h(0)| = |g(u_0, T_{V_0} V_0)|$. Here $|V|$ represents $\sqrt{|g(V, V)|}$. If $|T_{V_0} V_0| \neq 0$, we take u_0 as $\frac{T_{V_0} V_0}{|T_{V_0} V_0|}$. Then we obtain $|T_{V_0} V_0| \leq \sqrt{k}$. Therefore, we have confirmed the following lemma:

Lemma 3.5. $|T_V V| \leq \sqrt{k}$ for any vertical unit tangent vector V .

Hereafter, we calculate the curvature of the fiber.

Proposition 3.1. *The curvature of fibers is non-negative.*

Proof. Equation (2.2) of Proposition 2.1 implies that for any unit vertical vector U and V ,

$$\begin{aligned}
 \widehat{K}(U, V) &= K^\perp(U, V) - \frac{g(T_U U, T_V V) - g(T_U V, T_U V)}{g(U, U)g(V, V) - g(U, V)^2} \\
 &\leq K^\perp(U, V) - \frac{g(T_U U, T_V V)}{g(U, U)g(V, V) - g(U, V)^2} \\
 &\leq K^\perp(U, V) + \frac{|T_U U||T_V V|}{g(U, U)g(V, V) - g(U, V)^2}.
 \end{aligned}$$

Since $\widehat{K}(U, V) \geq k$, we have

$$\begin{aligned}
 K^\perp(U, V) &\geq k - \frac{|T_U U||T_V V|}{g(U, U)g(V, V) - g(U, V)^2} \\
 &\geq k - (\sqrt{k})^2 = 0.
 \end{aligned}$$

□

Therefore, the following structural theorem for the fundamental group of a closed Riemannian manifold of non-negative curvature leads us to the restriction of the topology of the fibers:

Theorem 3.1 (Toponogov [Top59], Cheeger–Gromoll [CG72, Theorem 3]). *Let M be a closed Riemannian manifold of non-negative sectional curvature. Then, the universal covering Riemannian manifold \tilde{M} of M can be split isometrically as $\mathbb{R}^l \times \tilde{N}$, where \tilde{N} is a closed Riemannian manifold. Moreover, the fundamental group $\pi_1(M)$ includes a free abelian subgroup \mathbb{Z}^l of finite index that acts properly discontinuously and co-compactly as a deck transformation on the Euclidean factor.*

From Theorem 3.1, it follows that the universal covering Riemannian manifold of the fiber is the Riemannian product manifold of the Euclidean space and some closed Riemannian manifold. It is sufficient to prove that the dimension of the Euclidean factor is zero. Suppose, by way of contradiction, that this dimension is not zero. Then, the fundamental group of the fiber has a free abelian normal subgroup \mathbb{Z}^l of finite index for some $l > 0$. In the previous paper [Muk15], which considered the Lorentzian case, we used the Penrose singularity theorem. However, we are aware of few semi-Riemannian analogies of the Penrose singularity theorem. Therefore, we need another strategy. First we show the following proposition:

Proposition 3.2. *If $l > 0$, the entire space E admits a structure of warped products.*

Proof of Proposition 3.2. To prove the proposition, we prove several lemmas. First, we check that the fibers are totally umbilical.

Lemma 3.6. *For any $b \in B$, the fiber $\pi^{-1}(b)$ is a totally umbilical submanifold with constant mean curvature.*

Proof. Let ∇^\perp be the Levi-Civita connection of the fibers $\pi^{-1}(b)$ with the induced metric and K^\perp the sectional curvature of ∇^\perp . Theorem 3.1 implies that there exists a closed Riemannian manifold S such that the universal covering of $\pi^{-1}(b)$ is isometric to $S \times \mathbb{R}^l$. Take $p \in \pi^{-1}(x)$. We have $T_p\pi^{-1}(x) \simeq T_{\tilde{p}}S + T_{\tilde{p}}\mathbb{R}^l$, where \tilde{p} is a lift of p . Hence, we identify $T_p\pi^{-1}(b)$ with $T_{\tilde{p}}S + T_{\tilde{p}}\mathbb{R}^l$. For any unit tangent vector $U_1 \in T_{\tilde{p}}S$, $U_2 \in T_{\tilde{p}}\mathbb{R}^l$ with $g(U_1, U_2) = 0$, we have

$$K^\perp(U_1, U_2) = 0.$$

By the equation (2.2), we obtain

$$\begin{aligned} 0 &= K^\perp(U_1, U_2) \\ &= \widehat{K}(U_1, U_2) + g(T_{U_1}U_1, T_{U_2}U_2) - g(T_{U_1}U_2, T_{U_1}U_2) \\ &\geq k + g(T_{U_1}U_1, T_{U_2}U_2) \\ &\geq k - |T_{U_1}U_1||T_{U_2}U_2|. \end{aligned}$$

Lemma 3.5 implies that the right-hand side is non-negative. Therefore, we have $k - |T_{U_1}U_1||T_{U_2}U_2| = 0$. Since the equality of the Cauchy–Schwartz inequality holds, $T_{U_1}U_1 = T_{U_2}U_2 = \sqrt{k}\nu(U_1, U_2)$, where $\nu(U_1, U_2)$ is a horizontal unit vector. Moreover, $T_{U_1}U_2 = 0$ holds. We see that $\nu = \nu(U_1, U_2)$ is independent of the choice of $U_1 \in T_{\bar{p}}S$, $U_2 \in T_{\bar{p}}\mathbb{R}^l$. For any $U = U_1 + U_2 \in T_p\pi^{-1}(b)$,

$$\begin{aligned} T_UU &= T_{U_1}U_1 + 2T_{U_1}U_2 + T_{U_2}U_2 \\ &= \sqrt{k}(g(U_1, U_1) + g(U_2, U_2))\nu = \sqrt{k}(g(U, U))\nu. \end{aligned}$$

For any $U, V \in T_p\pi^{-1}(b)$ with $g(U, V) = 0$,

$$\begin{aligned} T_UV &= \frac{T_{U+V}(U + V) - T_UU - T_VV}{2} \\ &= \frac{\sqrt{k}}{2}(g(U + V, U + V) - g(U, U) - g(V, V))\nu = 0. \end{aligned}$$

Therefore, we obtain $T_UV = \sqrt{k}g(U, V)\nu$ such that $g(\nu, \nu) = -1$. □

We prove the following lemma:

Lemma 3.7. *E has the structure of twisted product $(B \times F, -g_B + e^{2\alpha}g_F)$, where α is a function on E and g_F is some Riemannian metric of F .*

Proof. Note that ν is a unit horizontal vector field. Recall that $E = (B \times F, -g_B + g_F^*)$. For any point $p = (b, f) \in B \times F = E$, let U be any tangent vector in T_fF and let \widehat{U} be the vertical lift of U to E . Note that \widehat{U} is the vertical vector field along the horizontal submanifold $B \times \{f\}$. For fixed $f \in F$, we can define $h_U(b) = g_F^b(f)(\widehat{U}, \widehat{U})$. Let X be any vector field on B

and let \widehat{X} be its lift. Then we have

$$\begin{aligned} Xh_U(b) &= Xg_F^b(\widehat{U}, \widehat{U}) = \widehat{X}g(\widehat{U}, \widehat{U}) = 2g(\widehat{U}, \widehat{\nabla}_{\widehat{X}}\widehat{U}) = 2g(\widehat{U}, \widehat{\nabla}_{\widehat{U}}\widehat{X}) \\ &= 2(\widehat{U}g(\widehat{U}, \widehat{X}) - g(\widehat{\nabla}_{\widehat{U}}\widehat{U}, \widehat{X})) = -2g(\widehat{\nabla}_{\widehat{U}}\widehat{U}, \widehat{X}) \\ &= -2g(T_{\widehat{U}}\widehat{U}, \widehat{X}) = -2\sqrt{k}g(\widehat{U}, \widehat{U})g(\nu, \widehat{X}) = -2\sqrt{k}g(\nu, \widehat{X})h_U(b). \end{aligned}$$

Set

$$H_U(b) = \frac{\log h_U(b)}{2\sqrt{k}}.$$

Then we obtain

$$g_B(X, \nabla^B H_U) = g_B(X, \nu).$$

Therefore, on any $B \times \{f\}$, we have $\nabla^B H_U = \nu$. It follows that $\nabla^B H_U$ does not depend on U .

Take a fixed point $b_0 \in B$. Since the fibers have torus factors, there exists a unit vector field along $\pi^{-1}(b_0)$, denoted by U' . Let \widehat{U}' be the vertical lift of U' to E . Note that \widehat{U}' is the vertical vector field on the entire space E . We know that $\nabla^B(H_U - H_{U'}) = 0$. Hence, $H_U - H_{U'}$ does not depend on B . It follows that $H_U(b) - H_{U'}(b) = H_U(b_0) - H_{U'}(b_0)$ for any $b \in B$. Then we have

$$g_F^b(f)(\widehat{U}, \widehat{U}) = h_U(b) = \frac{h_{U'}(b)}{h_{U'}(b_0)} h_U(b_0) = \frac{h_{U'}(b)}{h_{U'}(b_0)} g_F^{b_0}(f)(\widehat{U}, \widehat{U}).$$

We denote by $\alpha(b, f)$ the function $\frac{1}{2} \log \frac{h_{U'}(b)}{h_{U'}(b_0)}$, by F the fiber $\pi^{-1}(b_0)$, and by g_F the metric $g_F^{b_0}(f)$ of F . We obtain $g_F^b(f)(\widehat{U}, \widehat{U}) = e^{2\alpha} g_F(\widehat{U}, \widehat{U})$. We have proved that E is a twisted product. \square

Note that $F = \mathbb{T}^l \times S$ by the splitting theorem, where S is a closed Riemannian manifold. It follows that $E = (B \times \mathbb{T}^l \times S, -g_B + e^{2\alpha} g_{\mathbb{T}^l} + e^{2\alpha} g_S)$, where S is some closed manifold without torus factors and $g_{\mathbb{T}^l}$ is the flat metric of the torus. First, we consider the case $l \geq 2$. Then we have

Lemma 3.8. *The submanifold $(\mathbb{T}^l, e^{2\alpha} g_{\mathbb{T}^l})$ has non-negative curvature.*

Proof. Let Π be the fundamental form of $(\mathbb{T}^l, e^{2\alpha} g_{\mathbb{T}^l})$ in the fiber $(\mathbb{T}^l \times S, e^{2\alpha} g_{\mathbb{T}^l} + e^{2\alpha} g_S)$. Note that the fiber has non-negative curvature. Take any

tangent vectors U, V of \mathbb{T}^l . Then

$$(3.1) \quad \Pi(U, V) = -g_{\mathbb{T}^l}(U, V)(\nabla^F \alpha)^\perp,$$

where ∇^F is the Levi-Civita connection of (F, g_F) and $(\nabla^F \alpha)^\perp$ is the component of $\nabla^F \alpha$ perpendicular to the torus. By using the Gaussian equation of the torus in the fiber, we see that the sectional curvature of $(\mathbb{T}^l, e^{2\alpha} g_{\mathbb{T}^l})$ is non-negative. \square

Lemma implies the following lemma:

Lemma 3.9. *α does not depend on the torus.*

Proof. Note that the scalar curvatures $\text{Scal}_{(\mathbb{T}^l, g_{\mathbb{T}^l})}$ and $\text{Scal}_{(\mathbb{T}^l, e^{2\alpha} g_{\mathbb{T}^l})}$ of $(\mathbb{T}^l, g_{\mathbb{T}^l})$ and $(\mathbb{T}^l, e^{2\alpha} g_{\mathbb{T}^l})$ are zero and non-negative, respectively. Moreover, we have the following formula on scalar curvature:

$$(3.2) \quad \begin{aligned} \text{Scal}_{(\mathbb{T}^l, e^{2\alpha} g_{\mathbb{T}^l})} &= e^{-2\alpha} (\text{Scal}_{(\mathbb{T}^l, g_{\mathbb{T}^l})} + 2(l-1)\Delta_{(\mathbb{T}^l, g_{\mathbb{T}^l})}\alpha \\ &\quad - (l-2)(l-1)|d\alpha|^2), \end{aligned}$$

where $\Delta_{(\mathbb{T}^l, g_{\mathbb{T}^l})}$ is the Laplacian of $(\mathbb{T}^l, g_{\mathbb{T}^l})$. First, we consider the case $l = 2$. From the equation (3.2) it follows that

$$\text{Scal}_{(\mathbb{T}^l, e^{2\alpha} g_{\mathbb{T}^l})} = 2e^{-2\alpha} \Delta_{(\mathbb{T}^l, g_{\mathbb{T}^l})}\alpha.$$

Since $\text{Scal}_{(\mathbb{T}^l, e^{2\alpha} g_{\mathbb{T}^l})}$ is non-negative, $\Delta_{(\mathbb{T}^l, g_{\mathbb{T}^l})}\alpha \geq 0$ on the torus. By the maximal principle, α is constant on the torus. Next, we consider the remaining case $l \geq 3$. Then, by equation (3.2) we obtain the following inequality

$$2\Delta_{(\mathbb{T}^l, g_{\mathbb{T}^l})}\alpha \geq (l-2)|d\alpha|^2 \geq 0.$$

The maximal principle implies that α is constant on the torus. The proof is complete. \square

Therefore, we have

Lemma 3.10. *α does not depend on the fiber.*

Proof. Lemma 3.9 implies that the sectional curvature of $(\mathbb{T}^l, e^{2\alpha} g_{\mathbb{T}^l})$ is zero. Therefore, by equation (3.1) and the Gaussian equation, we obtain $(\nabla^F \alpha)^\perp = 0$. Since α does not depend on the torus by Lemma 3.9, we have $\nabla^F \alpha = 0$. The lemma has been proved. \square

Next, we consider the case of $l = 1$, namely $F_0 = \mathbb{T}^1 \times S$, where S is a closed Riemannian manifold. Since S is closed, there exists a closed geodesic S^1 in (S, g_S) . Therefore, we have the natural immersion $\iota : \mathbb{T}^1 \times S^1 = \mathbb{T}^2 \rightarrow \mathbb{T}^1 \times S$. Let the curvature of $(\mathbb{T}^2, e^{2\alpha \circ \iota} g_{\mathbb{T}^2})$ be \bar{S} . Then

$$\bar{S} = 2e^{-2\alpha} \Delta_{(\mathbb{T}^2, g_{\mathbb{T}^2})}(\alpha \circ \iota).$$

Since the fiber has non-negative curvature, $\bar{S} \geq 0$ by using the Gauss equation. The maximal principle implies that α does not depend on the image of $(\mathbb{T}^1 \times S^1)$ under the immersion ι . We denote by U the unit vector field on $(\mathbb{T}^1, g_{\mathbb{T}^1})$ in the fiber. Let $\text{Ric}_{(F, e^{2\alpha} g_F)}$ and $\text{Ric}_{(F, g_F)}$ be the Ricci curvatures of $(F, e^{2\alpha} g_F)$ and (F, g_F) , respectively. Then we have

$$\begin{aligned} \text{Ric}_{(F, e^{2\alpha} g_F)}(U, U) &= \text{Ric}_{(F, g_F)}(U, U) - (\dim F - 2)(\text{Hess } \alpha(U, U) - |d\alpha(U)|) \\ &\quad + \Delta_{(F, g_F)} \alpha - (\dim F - 2)|d\alpha|^2 \\ &= \Delta_{(F, g_F)} \alpha - (\dim F - 2)|d\alpha|^2, \end{aligned}$$

where $\Delta_{(F, g_F)}$ is the Laplacian of (F, g_F) . Since the left-hand side is positive,

$$\Delta_{(F, g_F)} \alpha \geq (\dim F - 2)|d\alpha|^2 \geq 0.$$

Therefore, by the maximal principle, α is constant on the fiber. We have proved the case of $l = 1$. Therefore, Proposition 3.2 has been proved. \square

It is sufficient to prove Theorem 1.2 for only warped products. From equation (2.4) it follows that for any lifts \hat{U}, \hat{V} of vector fields U, V on F ,

$$T_{\hat{U}} \hat{V} = e^{2\alpha} g_F(U, V) \nabla^B \alpha.$$

It follows that

$$\nu = \frac{1}{\sqrt{k}} \nabla^B \alpha.$$

Since $g(\nu, \nu) = -1$, we have

$$g_B(\nabla^B \alpha, \nabla^B \alpha) = k.$$

Consider a function

$$H = \frac{1}{\sqrt{k}} \alpha : B \rightarrow \mathbb{R}.$$

Then H is a signed distance function as $g_B(\nabla^B H, \nabla^B H) = 1$. It follows that B is diffeomorphic to $\mathbb{R} \times H^{-1}(0)$ since (B, g_B) is complete. Let γ_0 be the

geodesic with $\gamma'_0(s) = -\nabla^B H(\gamma_0(s))$ in (B, g_B) . We can construct incomplete lightlike and timelike geodesics $\gamma(t) = (\gamma_B(t), \gamma_F(t))$ in E by using the geodesic γ_0 in B . First we consider lightlike geodesics. By using Proposition 2.5, lightlike geodesics satisfy the following equations

$$\begin{aligned} \nabla_{\partial/\partial t}^B \gamma'_B(t) &= -e^{2\alpha} g_F(\gamma'_F(t), \gamma'_F(t)) \nabla^B \alpha \\ &= -g_B(\gamma'_B(t), \gamma'_B(t)) \nabla^B \alpha; \\ \nabla_{\partial/\partial t}^F \gamma'_F(t) &= -2 \frac{d\alpha(\gamma_B(t))}{dt} \gamma'_F(t), \end{aligned}$$

since $g_B(\gamma'_B(t), \gamma'_B(t)) = e^{2\alpha} g_F(\gamma'_F(t), \gamma'_F(t))$. Take a solution $s(t)$ of

$$(3.3) \quad \frac{d^2 s(t)}{dt^2} = \sqrt{k} \left(\frac{ds(t)}{dt} \right)^2.$$

Set $\gamma_B(t) = \gamma_0(s(t))$. Since $\gamma'_B(t) = \gamma'_0(s(t))s'(t)$, we have $g_B(\gamma'_B(t), \gamma'_B(t)) = s'(t)^2$. Therefore, we obtain

$$\begin{aligned} \nabla_{\partial/\partial t} \gamma'_B(t) &= s''(t)\gamma'_0(s(t)) + \nabla_{\partial/\partial s} \gamma'_0(s(t))(s'(t))^2 \\ &= s''(t)\gamma'_0(s(t)) \\ &= -\sqrt{k}(s'(t))^2 \nabla^B H \\ &= -g_B(\gamma'_B(s), \gamma'_B(s)) \nabla^B \alpha \end{aligned}$$

Thus, we have $\gamma_B(t)$, which means we can also obtain $\gamma_F(t)$. We have constructed a lightlike geodesic $(\gamma_B(t), \gamma_F(t))$.

We solve a solution $s(t)$ of the differential equation (3.3). The solution is

$$s(t) = -\frac{1}{\sqrt{k}} \log |\sqrt{k}t + C_1| + C_2,$$

where C_1 and C_2 are integral constants. Then $s((-\frac{C_1}{\sqrt{k}}, \infty)) = \mathbb{R}$, and

$$\lim_{t \rightarrow -\frac{C_1}{\sqrt{k}} + 0} s(t) = \infty.$$

Therefore, the geodesic $(\gamma_B(t), \gamma_F(t))$ is not defined on \mathbb{R} . We see that the entire semi-Riemannian manifold is not lightlike geodesically complete.

Next we consider timelike geodesics. By using Proposition 2.5, timelike geodesics $\gamma(t) = (\gamma_B(t), \gamma_F(t))$ with $g(\gamma'(t), \gamma'(t)) = -1$ satisfy the following

equations

$$\begin{aligned} \nabla_{\partial/\partial t}^B \gamma'_B(t) &= -e^{2\alpha} g_F(\gamma'_F(t), \gamma'_F(t)) \nabla^B \alpha \\ &= (1 - g_B(\gamma'_B(t), \gamma'_B(t))) \nabla^B \alpha; \\ \nabla_{\partial/\partial t}^F \gamma'_F(t) &= -2 \frac{d\alpha(\gamma_B(t))}{dt} \gamma'_F(t), \end{aligned}$$

since

$$-g_B(\gamma'_B(t), \gamma'_B(t)) + e^{2\alpha} g_F(\gamma'_F(t), \gamma'_F(t)) = -1.$$

Assume that $g_B(\gamma'_B(t), \gamma'_B(t)) \neq 1$, that is $\gamma'_F(t) \neq 0$. Take a solution $s(t)$ of

$$(3.4) \quad \frac{d^2 s(t)}{dt^2} = \sqrt{k} \left\{ \left(\frac{ds(t)}{dt} \right)^2 - 1 \right\},$$

where $\frac{ds(t)}{dt} > 1$. Set $\gamma_B(t) = \gamma_0(s(t))$. In the same manner as for the case of lightlike geodesics, we have

$$\nabla_{\partial/\partial t} \gamma'_B(t) = (1 - g_B(\gamma'_B(s), \gamma'_B(s))) \nabla^B \alpha.$$

Thus, we obtain $\gamma_B(t)$ and $\gamma_F(t)$, which means we have constructed a time-like geodesic $(\gamma_B(t), \gamma_F(t))$. We obtain the solution $s(t)$ of the differential equation (3.4). The solution is

$$s(t) = -\frac{1}{\sqrt{k}} \log |C_1 e^{2\sqrt{k}t} - 1| + t + C_2,$$

where $C_1 > 0$ and C_2 are integral constants. Note that $C_1 > 0$ follows from $\frac{ds(t)}{dt} > 1$. Then $s((-\infty, \frac{1}{2\sqrt{k}} \log \frac{1}{C_1})) = \mathbb{R}$, and

$$\lim_{t \rightarrow \frac{1}{2\sqrt{k}} \log \frac{1}{C_1} - 0} s(t) = \infty.$$

Therefore, the geodesic $(\gamma_B(t), \gamma_F(t))$ is not defined on \mathbb{R} . We see that the entire semi-Riemannian manifold is not timelike geodesically complete. This is a contradiction.

Hence, we have proved that the fiber never includes tori. It follows that the fundamental group of the fiber is finite. The proof of Theorem 1.2 is complete. □

4. Semi-Riemannian example with non-integrable horizontal distribution

In this section, we construct a semi-Riemannian submersion such that the entire space satisfies the curvature condition $R \geq k > 0$ and that the horizontal distribution is not integrable. Set

$$G = SU(2, 1) = \{g \in SL(3, \mathbb{C}) : {}^t \bar{g} I_{2,1} g = I_{2,1}\},$$

where $I_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Let \mathfrak{g} be the Lie algebra of G . The indefinite inner product $B(-, -)$ on \mathfrak{g} is given by

$$B(X, Y) = -\Re \operatorname{Tr}(XY).$$

This inner product is invariant under the adjoint action $Ad(g)$ for $g \in G$. We write K for a maximal compact subgroup

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix} : A \in U(2) \right\}.$$

Let \mathfrak{k} be the Lie algebra of K . Then B is positive definite on \mathfrak{k} and negative definite on \mathfrak{k}^\perp . Note that the homogeneous space G/K with the induced metric from $-B$ is a complex hyperbolic plane $\mathbb{C}\mathbb{H}^2$. Write

$$H = \left\{ \begin{pmatrix} e^{2\pi ti} & 0 & 0 \\ 0 & e^{2\pi ti} & 0 \\ 0 & 0 & e^{-4\pi ti} \end{pmatrix} : t \in \mathbb{R} \right\} \subset K.$$

Note that H is the circle S^1 . Let \mathfrak{h}_0 be the Lie algebra of H . We denote $\mathfrak{h}_0^\perp \cap \mathfrak{k}$ and \mathfrak{k}^\perp by \mathfrak{h}_1 and \mathfrak{h}_2 , respectively. Then we have $\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$. We write the projection of a vector $X \in \mathfrak{g}$ onto the subspace \mathfrak{h}_j as X_j for each j . Let us take the following basis of \mathfrak{g} :

$$e_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, e_2 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$f_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, f_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}.$$

Then \mathfrak{h}_0 is spanned by e_1 , \mathfrak{h}_1 is spanned by e_2, e_3, e_4 , and \mathfrak{h}_2 is spanned by f_1, f_2, f_3, f_4 . Let M be a homogeneous space G/H . We define the indefinite inner metric $(-, -)$ on $\mathfrak{g}/\mathfrak{h}_0$ by $(X, Y) = (1 + t)B(X_1, Y_1) + B(X_2, Y_2)$ for $X, Y \in \mathfrak{g}/\mathfrak{h}_0$ and can also define the semi-Riemannian metric $(-, -)$ on G/H , which is left-invariant under G . We see that the projective map $\pi : M \rightarrow \mathbb{C}\mathbb{H}^2$ is a semi-Riemannian submersion.

We will show that M satisfies the curvature condition $R \geq k > 0$. We denote the curvature tensor of $(M, (-, -))$ by R . Note that $[\mathfrak{h}_1, \mathfrak{h}_1] \subset \mathfrak{h}_1$, $[\mathfrak{h}_2, \mathfrak{h}_2] \subset \mathfrak{h}_0 \oplus \mathfrak{h}_1$, and $[\mathfrak{h}_1, \mathfrak{h}_2] \subset \mathfrak{h}_2$. Then we have

$$\begin{aligned} (R(X, Y)Y, X) &= \frac{1 - 3t}{4}B([X, Y]_1, [X, Y]_1) + (t - t^2)B([X_1, Y_1], [X, Y]) \\ &\quad + t^2B([X_1, Y_1], [X_1, Y_1]) + \frac{(1 + t)^2}{4}B([X, Y]_2, [X, Y]_2) \\ &\quad + B([X, Y]_0, [X, Y]_0) \\ &= \frac{1 + t}{4}B([X_1, Y_1], [X_1, Y_1]) + \frac{1 - 3t}{4}B([X_2, Y_2]_1, [X_2, Y_2]_1) \\ &\quad + \frac{1 - t - 2t^2}{2}B([X_1, Y_1], [X_2, Y_2]_1) \\ &\quad + \frac{(1 + t)^2}{4}B([X, Y]_2, [X, Y]_2) \\ &\quad + B([X, Y]_0, [X, Y]_0). \end{aligned}$$

Set $X = a_2e_2 + a_3e_3 + a_4e_4 + b_1f_1 + b_2f_2 + b_3f_3 + b_4f_4$, $Y = c_2e_2 + c_3e_3 + c_4e_4 + d_1f_1 + d_2f_2 + d_3f_3 + d_4f_4$. Then, straightforward computation implies

$$\begin{aligned} B([X, Y]_2, [X, Y]_2) &= -2 \left\{ \left(\det \begin{pmatrix} a_3 & c_3 \\ b_2 & d_2 \end{pmatrix} - \det \begin{pmatrix} a_2 & c_2 \\ b_3 & d_3 \end{pmatrix} - \det \begin{pmatrix} a_4 & c_4 \\ b_4 & d_4 \end{pmatrix} \right)^2 \right. \\ &\quad + \left(\det \begin{pmatrix} a_2 & c_2 \\ b_4 & d_4 \end{pmatrix} - \det \begin{pmatrix} a_3 & c_3 \\ b_1 & d_1 \end{pmatrix} - \det \begin{pmatrix} a_4 & c_4 \\ b_3 & d_3 \end{pmatrix} \right)^2 \\ &\quad + \left(\det \begin{pmatrix} a_2 & c_2 \\ b_1 & d_1 \end{pmatrix} + \det \begin{pmatrix} a_3 & c_3 \\ b_4 & d_4 \end{pmatrix} + \det \begin{pmatrix} a_4 & c_4 \\ b_2 & d_2 \end{pmatrix} \right)^2 \\ &\quad \left. + \left(\det \begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix} + \det \begin{pmatrix} a_3 & c_3 \\ b_3 & d_3 \end{pmatrix} - \det \begin{pmatrix} a_4 & c_4 \\ b_1 & d_1 \end{pmatrix} \right)^2 \right\} \\ &= -2 \sum_{i,j} \left(\det \begin{pmatrix} a_i & c_i \\ b_j & d_j \end{pmatrix} \right)^2 + B([X_1, Y_1], [X_2, Y_2]_1). \end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$B([X_1, Y_1], [X_2, Y_2]_1) \leq \sqrt{B([X_1, Y_1], [X_1, Y_1])B([X_2, Y_2]_1, [X_2, Y_2]_1)}.$$

Set

$$x = \sqrt{\sum_{i < j} \left(\det \begin{pmatrix} a_i & c_i \\ a_j & c_j \end{pmatrix} \right)^2}, \quad y = \sqrt{\sum_{i < j} \left(\det \begin{pmatrix} b_i & d_i \\ b_j & d_j \end{pmatrix} \right)^2},$$

$$z = \sqrt{\sum_{i, j} \left(\det \begin{pmatrix} a_i & c_i \\ b_j & d_j \end{pmatrix} \right)^2}.$$

We should remark that

$$B([X_1, Y_1], [X_1, Y_1]) = 8x^2, \quad B([X_2, Y_2]_1, [X_2, Y_2]_1) = 2y^2.$$

Then we have

$$\begin{aligned} (R(X, Y)Y, X) &\geq 2(1+t)x^2 + \frac{1-3t}{2}y^2 - \frac{(1+t)^2}{2}z^2 \\ &\quad - \left| \frac{1-t-2t^2}{2} + \frac{(1+t)^2}{4} \right| \sqrt{16x^2y^2} \\ &= 2(1+t)x^2 - 3|1-t^2|xy + \frac{1-3t}{2}y^2 - \frac{(1+t)^2}{2}z^2. \end{aligned}$$

Note that

$$(X, X)(Y, Y) - (X, Y)^2 = 4(1+t)^2x^2 + 4y^2 - 4(1+t)z^2.$$

Therefore, we obtain

$$\begin{aligned} &(R(X, Y)Y, X) - k((X, X)(Y, Y) - (X, Y)^2) \\ &\geq \{2(1+t) - 4k(1+t)^2\}x^2 - 3|1-t^2|xy \\ &\quad + \left\{ \frac{1-3t}{2} - 4k \right\} y^2 + \left(4k(1+t) - \frac{(1+t)^2}{2} \right) z^2. \end{aligned}$$

Hence, it is sufficient to prove that there exists $k > 0$ and $t > -1$ such that

$$\begin{aligned} &\{2(1+t) - 4k(1+t)^2\}x^2 - 3|1-t^2|xy \\ &\quad + \left\{ \frac{1-3t}{2} - 4k \right\} y^2 + \left\{ 4k(1+t) - \frac{(1+t)^2}{2} \right\} z^2 > 0. \end{aligned}$$

Therefore, our search is reduced to finding $k > 0$ and $t > -1$ satisfying the following four inequalities:

$$(4.1) \quad k > \frac{1+t}{8},$$

$$(4.2) \quad k < \frac{1}{2(1+t)},$$

$$(4.3) \quad k < \frac{1-3t}{8},$$

$$(4.4) \quad \{2(1+t) - 4k(1+t)^2\} \left(\frac{1-3t}{2} - 4k \right) - \frac{9}{4}(1-t^2)^2 > 0.$$

From the inequalities (4.1), (4.2), and (4.3), we see that

$$\frac{1+t}{8} < k < \frac{1-3t}{8} < \frac{1}{2(1+t)},$$

for $-1 < t < 0$. The inequality (4.4) is

$$(t+1) \left\{ 16(1+t)k^2 + 2(3t^2 + 2t - 5)k - \frac{1}{4}(9t^3 - 9t^2 + 3t + 5) \right\} > 0.$$

Set

$$\eta(t) = \frac{-3t^2 - 2t + 5 - \sqrt{45t^4 + 12t^3 - 50t^2 + 12t + 45}}{16(t+1)}.$$

We note that $\eta(t)$ is a solution of the equation with respect to k that the left-hand side of the inequality (4.4) is zero. Then the inequalities (4.1), (4.2), (4.3), and (4.4) hold for $-1 < t < -\frac{3}{5}$ and $\frac{1+t}{8} < k < \eta(t)$.

We will prove that M is geodesically complete. Since a natural projection $G \rightarrow M$ is a semi-Riemannian submersion, we consider only geodesics $\gamma(u)$ in G from the identity element of which the initial velocity belongs to $\mathfrak{h}_1 \oplus \mathfrak{h}_2$. Let $\Gamma(u)$ be a curve $\gamma(u)^{-1}\gamma'(u)$ in $\mathfrak{h}_1 \oplus \mathfrak{h}_2$. We denote by $\Gamma_j(u)$ the \mathfrak{h}_j -component of $\Gamma(u)$. Set a linear operator $\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\phi(X_0 + X_1 + X_2) = X_0 + (1+t)X_1 + X_2$ for any $X_j \in \mathfrak{h}_j$. Note that we have $(X, Y) = B(\phi(X), Y)$. Then $\gamma(u)$ satisfies the following Euler–Arnold equation:

$$\phi(\Gamma'(u)) = [\phi(\Gamma(u)), \Gamma(u)]$$

(for instance see [GL95]). Therefore, we have

$$(4.5) \quad (1+t)\Gamma'_1(u) = 0,$$

$$(4.6) \quad \Gamma'_2(u) = t[\Gamma_1(u), \Gamma_2(u)].$$

The equation (4.5) implies $\Gamma_1(u)$ is constant, denoted by v_1 . Then, the equation (4.6) means $\Gamma_2'(u) = t[v_1, \Gamma_2(u)]$, which is a system of first-order linear differential equations with a constant coefficient. It follows that $\gamma(u)$ is defined in the real line \mathbb{R} . Hence, the semi-Riemannian manifold M we have constructed is the desired solution.

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