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Conformal hypersurface geometry via a boundary Loewner–Nirenberg–Yamabe problem

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We develop a new approach to the conformal geometry of embedded hypersurfaces by treating them as conformal infinities of conformally compact manifolds. This involves the Loewner–Nirenberg-type problem of finding on the interior a metric that is both conformally compact and of constant scalar curvature. Our first result is an asymptotic solution to all orders. This involves log terms. We show that the coefficient of the first of these is a new hypersurface conformal invariant which generalises to higher dimensions the important Willmore invariant of embedded surfaces. We call this the obstruction density. For even dimensional hypersurfaces it is a fundamental curvature invariant. We make the latter notion precise and show that the obstruction density and the trace-free second fundamental form are, in a suitable sense, the only such invariants. We also show that this obstruction to smoothness is a scalar density analog of the Fefferman–Graham obstruction tensor for Poincaré–Einstein metrics; in part this is achieved by exploiting Bernstein–Gel'fand–Gel'fand machinery. The solution to the constant scalar curvature problem provides a smooth hypersurface defining density determined canonically by the embedding up to the order of the obstruction. We give two key applications: the construction of conformal hypersurface invariants and the construction of conformal differential operators. In particular we present an infinite family of conformal powers of the Laplacian determined canonically by the conformal embedding. In general these depend non-trivially on the embedding and, in contrast to Graham–Jennes–Mason–Sparling operators intrinsic to even dimensional hypersurfaces, exist to all orders. These extrinsic conformal Laplacian powers determine an explicit holographic formula for the obstruction density.

1. Introduction

A smoothly embedded codimension-1 submanifold Σ of a smooth, d dimensional manifold M, is termed a *hypersurface*. These are critically important in geometry and analysis, not least because manifold and domain boundaries are examples; almost any boundary problem calls on some aspect of hypersurface geometry. Despite this, a general theory of natural invariants, differential operators and functionals for conformal hypersurface geometries is lacking. We show here that a certain boundary problem provides a unified approach to such problems and a number of new results.

For any continuous geometry, understanding the existence and construction of local invariants is a critical first step. Given a hypersurface Σ in a Riemannian manifold (M, g), local conformal invariants are the natural scalar or tensor-valued fields determined by the data (M, g, Σ), which have the additional property that they are (as densities) unchanged when the metric g is replaced by a conformally related metric, that is $\widehat{g}$ where $\widehat{g} = \Omega^2 g$ for some positive function Ω. The precise definitions are given in Sections 2.4 and 4.1.

For surfaces in Riemannian 3-manifolds an extremely interesting conformal invariant appears as the left-hand-side of the Willmore equation,

$$\bar{\Delta} H + 2H(H^2 - \mathrm{K}) = 0, \tag{1.1}$$

here given for an embedded surface Σ in Euclidean 3-space $\mathbb{E}^3$ [61]. Here H and K are, respectively, the mean and Gauß curvatures, while $\bar{\Delta}$ is the Laplacian induced on Σ. We shall call this quantity the *Willmore invariant*; it is invariant under Möbius transformations of the ambient $\mathbb{E}^3$. A key feature is the linearity of the highest order term, $\bar{\Delta} H$. This is important for PDE problems, but also means that the Willmore invariant should be viewed as a fundamental conformal curvature quantity. The Willmore equation and its corresponding Willmore energy functional play an important *rôle* in both mathematics and physics (see *e.g.* [1, 53, 54]). Recently the celebrated Willmore conjecture [61] concerning absolute minimizers of this energy was settled in [50]. The Willmore energy is also linked to a holographic notion of physical observables [43] and in particular entanglement entropy [3, 34, 55, 56]. It is clearly important to understand whether the existence of the Willmore invariant is peculiar to surfaces, or if there also exist higher dimensional analogs. We provide the answer to this question: via a natural PDE problem, we show the Willmore invariant is the first member of a family of fundamental curvatures existing in each even hypersurface dimension. The nature of these, and the way that they arise, strongly suggests that they will also play an important *rôle* in both mathematics and physics. In addition, we shall find related objects for odd dimensional hypersurfaces, and there is evidence that these should also be of interest [23].

A powerful approach to the study of conformal geometry is provided by the Poincaré metric of Fefferman–Graham [21], (which in part follows the approach to CR geometry developed in [20] and [14]). This is a construction to study the conformal geometry of a $(d-1)$-manifold Σ by recovering it as the boundary of a conformally compact Einstein d-manifold. The Poincaré metric of d-dimensions is not directly suitable for capturing general extrinsic geometry of $(d-1)$-dimensional submanifolds, because the Einstein equation significantly constrains the allowable conformal geometry of the manifold hosting the hypersurface. For the boundary, only minimal regularity forces its embedding to be *totally umbilic* [28, 47], *i.e.*, everywhere vanishing trace-free second fundamental form. On the other hand, certain interesting conformal submanifold geometry is revealed by embedding suitable submanifolds in the boundary of a Poincaré manifold [43].

Our approach is to replace the Poincaré metric problem with one directly adapted to the situation of a conformally embedded hypersurface. We show that this replacement is given by the Loewner–Nirenberg-type problem of finding, conformally, a negative scalar curvature, conformally compact metric. For our purposes this non-compact analog of the well-known Yamabe problem will be formulated as follows. Given a smooth compact Riemannian manifold (M^d, g), with boundary $\partial M = \Sigma$, find a (non-negative) defining function u for Σ so that the scalar curvature of the metric $\bar{g} = u^{-2}g$ on M satisfies

$$\mathrm{Sc}^{\bar{g}} = -d(d-1).$$

Our use of this problem is inspired by the approach in the article [2] of Andersson, Chruściel and Friedrich (ACF): It follows from results in [49], [4, 5] and [2] that for $d \geq 3$ there exists a unique solution u. Moreover in [2, Theorem 1.3] the authors studied the boundary asymptotics in detail, and from this one can conclude that u has an asymptotic expansion involving powers of r and $\log r$, where r is a smooth defining function for Σ. The expansion below order $d+1$ is locally determined. Furthermore they explain (see also [2, Lemma 2.1]) that there always exists a smooth defining function r so that $\mathrm{Sc}^{r^{-2}g} = -d(d-1) + r^d R_d$ with $R_d \in C^\infty(M)$. Moreover, $R_d|_\Sigma = 0$ is a condition on the conformal class of g equivalent to C^∞ smoothness of u to the boundary. It is pointed out in [2, Remark 2.1] that $R_d|_\Sigma$ behaves as a density under conformal rescaling, and it is clear from their construction (in [2, Section 5]) that it is locally determined. These results cover our Theorems 1.2, 4.5 and 4.8. However we include these here to make the treatment self-contained and because the coordinate free approach we use is rather different and should be of independent interest.

Our main point is that this set-up is ideal for the development of a holographic approach to the study of conformal hypersurface invariants. Especially interesting in this direction is the obstruction term $R_d|_\Sigma$. Indeed, for the $d = 3$ case, $R_3|_\Sigma$ was explicitly identified in [2, Theorem 1.3] as a surface conformal invariant and given in terms of derivatives of the trace-free second fundamental (of Σ embedded in M), and ambient curvature. In fact, this invariant found by ACF is the same as that arising from the variation of the Willmore energy; in particular its specialisation to surfaces in $\mathbb{E}^3$ agrees with the Willmore invariant appearing in (1.1). This suggests the obstructions $R_d|_\sigma$ could provide higher dimensional Willmore invariants, and we show that when d is odd, this is indeed the case.

For later reference, let us now set up the problem in the terms we will use.

1.1. A scalar curvature boundary problem

Given a d-dimensional Riemannian manifold (M, g) with boundary $\Sigma := \partial M$, one may ask whether there is a smooth real-valued function u on M satisfying the following conditions:

1) u is a defining function for Σ (*i.e.*, Σ is the zero set of u, and $\boldsymbol{d}u_x \neq 0\ \forall x \in \Sigma$);

2) $\bar{g} := u^{-2}g$ has scalar curvature $\mathrm{Sc}^{\bar{g}} = -d(d-1)$.

Here $\boldsymbol{d}$ is the exterior derivative (on functions). We assume dimension $d \geq 3$ and all structures are C^∞.

Assuming $u > 0$ and setting $u = \rho^{-2/(d-2)}$, part (2) of this problem is governed by the Yamabe equation

$$\left[-4\,\frac{d-1}{d-2}\,\Delta + \mathrm{Sc}\right]\rho + d\,(d-1)\,\rho^{\frac{d+2}{d-2}} = 0. \tag{1.2}$$

The above problem fits nicely into the framework of conformal geometry as follows: Recall that a conformal structure $\boldsymbol{c}$ on a manifold is an equivalence class of metrics where the equivalence relation $\widehat{g} \sim g$ means that $\widehat{g} = \Omega^2 g$ for some positive function Ω. The line bundle $(\Lambda^d TM)^2$ is oriented and, for $w \in \mathbb{R}$, the bundle of *conformal densities* of weight w is denoted $\mathcal{E}M[w]$ (or simply $\mathcal{E}[w]$ if the underlying manifold is clear from context), and is defined to be the oriented $\frac{w}{2d}$-root of this. Locally each $g \in \boldsymbol{c}$ determines a volume form and, squaring this, globally a section of $(\Lambda^d T^*M)^2$. So, on a conformal manifold $(M, \boldsymbol{c})$ there is a canonical section $\boldsymbol{g}$ of $\odot^2 T^*M \otimes \mathcal{E}[2]$ called the conformal metric. Thus each metric $g \in \boldsymbol{c}$ is naturally in $1:1$ correspondence with a (strictly) positive section τ of $\mathcal{E}[1]$ via $g = \tau^{-2}\boldsymbol{g}$. Also, the Levi-Civita connection ∇ of g preserves τ, and hence $\boldsymbol{g}$. Thus we are led from Equation (1.2) to the conformally invariant equation on a weight 1 density $\sigma \in \Gamma(\mathcal{E}[1])$

$$S(\sigma) := \big(\nabla\sigma\big)^2 - \frac{2}{d}\sigma\left(\Delta + \frac{\mathrm{Sc}}{2(d-1)}\right)\sigma = 1, \tag{1.3}$$

where $\boldsymbol{g}$ and its inverse are used to raise and lower indices, $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b$ and Sc means $\boldsymbol{g}^{bd}R_{ab}{}^a{}_d$, with R the Riemann tensor. Now, choosing $\boldsymbol{c} \ni g = \tau^{-2}\boldsymbol{g}$, Equation (1.3) is the PDE governing the function $u = \sigma/\tau$ solving part (2) of the problem. Setting aside boundary aspects, this is exactly the Yamabe equation (1.2) above.

The critical point is that, in contrast to (1.2), Equation (1.3) is well-adapted to the boundary problem since u (or equivalently σ) is the variable defining the boundary. Let us write $\Sigma := \mathcal{Z}(\sigma)$ for the zero locus of σ. Since u is a defining function, this implies that σ is a *defining density* for Σ, meaning that it is a section of $\mathcal{E}[1]$ with zero locus Σ and $\nabla \sigma_x \neq 0 \ \forall x \in \Sigma$. For our purposes, we only need to treat the problem formally (so it applies to any hypersurface) and it may thus be stated as follows:

Problem 1.1. Let Σ be an embedded hypersurface in a conformal manifold $(M, \boldsymbol{c})$ with dimension $d \geq 3$. Find a smooth defining density $\bar{\sigma}$ such that

$$S(\bar{\sigma}) = 1 + \bar{\sigma}^{\ell} A_{\ell},$$

for some $A_\ell \in \Gamma(\mathcal{E}[-\ell])$, where $\ell \in \mathbb{N} \cup \infty$ is as high as possible.

Here and elsewhere $\Gamma(\mathcal{V})$ indicates the space of smooth, meaning C^∞, sections of a given bundle $\mathcal{V}$.

1.2. The main results

Our first main result is an asymptotic solution to the singular Yamabe Problem 1.1. This is given in Theorem 4.5, which states that in solving the equivalent Equation(1.3) we can smoothly achieve

$$S(\sigma) = 1 + \sigma^d B \quad \text{equivalently} \quad \mathrm{Sc}^{\sigma^{-2}\boldsymbol{g}} = -d(d-1)\left(1 + \sigma^d B\right), \tag{1.4}$$

for a certain smooth conformal density B. Proposition 4.9 and Proposition 4.11 provide the explicit recursive formulæ that solve the problem, including log terms. These simple formulæ follow after recasting Problem 1.1 in terms of tractor calculus on $(M, \boldsymbol{c})$; in particular as the almost scalar constant (ASC) problem stated in [28]. The equivalent tractor problem is given in Problem 4.3, and provides an effective simplification. The approach demonstrates that surprisngly, although the problem is non-linear, a fundamental $\mathfrak{sl}(2)$ structure inherent in the geometry [33] (see Section 3.5) may be applied.

The next main result concerns the obstruction to smoothly solving Problem 1.1 beyond order $\ell = d$:

Theorem 1.2. *There is an obstruction to solving Problem 1.1 smoothly to all orders and this is a conformal density* $\mathcal{B} \in \Gamma(\mathcal{E}\Sigma[-d])$ *determined entirely by the data* $(M, \boldsymbol{c}, \Sigma)$*, of the conformal embedding of the boundary. That is* *a smooth formal solution to the singular Yamabe Problem 1.1 exists if and only if the hypersurface invariant* $\mathcal{B}$ *is zero.*

This theorem is proved as the equivalent statement Theorem 4.8. From that we see that the *(ASC) obstruction density* $\mathcal{B}$ arises as $B|_\Sigma$, where B is as in (1.4). We later prove that the density $\mathcal{B}$ is natural in the sense that, in a scale, it may be given by an expression polynomial in the hypersurface conormal, ambient Riemann curvature and Levi-Civita covariant derivatives thereof, see Theorem 6.5. So it is conformal invariant of the hypersurface; hypersurface invariants are defined in Definitions 2.6 and 4.1 of Sections 2.4 and 4. If $\mathcal{B}$ is zero for a given hypersurface, then any smooth formal solution to all orders depends on the choice of a smooth conformal density in $\Gamma(\mathcal{E}\Sigma[-d])$.

Theorem 5.1 establishes that for hypersurfaces of even dimension $d-1$ the obstruction density takes the form

$$\bar{\Delta}^{\frac{d-1}{2}} H + \text{lower order terms},$$

generalising the Willmore invariant $\bar{\Delta} H + \text{lower order terms}$, *cf.* Equation (1.1). Here and throughout we use "lower order terms" to mean terms of lower order in the jets of the hypersurface conormal (or equivalently in the jets of the defining function or defining density). Then we prove the following (see the end of Section 5):

Theorem 1.3. *The obstruction density is a fundamental curvature invariant of even dimensional conformal hypersurfaces. For hypersurfaces of dimension 3 or greater, the trace-free second fundamental form* $\mathring{\mathrm{II}}$ *and the obstruction density* $\mathcal{B}$ *are the only fundamental conformal invariants of a hypersurface (up to the addition of lower order terms) taking values in an irreducible bundle.*

The notion of *fundamental curvature quantity* used in the Theorem here is defined in Section 5.1. Informally it means a conformal invariant that has a non-trivial linearisation with respect to embedding variation of flat structures, and in general cannot be constructed in a standard way from simpler invariants, see Remark 5.6. This property and the leading order behaviour show that on even dimensional hypersurfaces, $\mathcal{B}$ generalises the Willmore invariant. We also show in Section 5.1 that the obstruction density is a scalar density analog of the Fefferman–Graham obstruction tensor [22]. This uses the classification of linear conformally invariant operators and the appropriate Bernstein–Gel'fand–Gel'fand (BGG) complexes. Proposition 4.14 gives explicit formulæ for the obstruction density for hypersurfaces of dimensions two and three (with $(M, \boldsymbol{c})$ conformally flat for the latter case). In two dimensions this gives exactly the Willmore invariant.

The uniqueness of the solution $\bar{\sigma}$ to Problem 1.1 means that, in a conformal manifold $(M, \boldsymbol{c})$, a hypersurface Σ determines a corresponding distinguished *conformal unit defining density*, as defined in Definition 4.6; this is uniquely determined up to the order asserted in Theorem 4.5. This gives an interesting way to construct conformal hypersurface invariants: Consider any conformal invariant U on M, that couples the data of the jets of the conformal structure $(M, \boldsymbol{c})$ to the jets of the section $\bar{\sigma}$. Along Σ, this determines an invariant of $(M, \boldsymbol{c}, \Sigma)$ whenever U involves there no more than the d-jet of $\bar{\sigma}$. This is treated in Section 6.1.

Importantly, the stated uniqueness of $\bar{\sigma}$ also means that we can construct the basic invariant differential operators determined naturally by the data $(M, \boldsymbol{c}, \Sigma)$. In particular we give a family of conformally invariant, hypersurface Laplacian-power type operators depending on the extrinsic geometry and termed *extrinsic conformal Laplacians*:

Theorem 1.4. *Consider a hypersurface Σ embedded in a conformal manifold $(M, \boldsymbol{c})$. The data of this embedding determines canonically, for each $k \in 2\mathbb{Z}_{>0}$, a natural conformally invariant differential operator*

$$\mathsf{P}_k : \Gamma\left(\mathcal{T}^{\Phi} M\left[\frac{k-d+1}{2}\right]\right)\Big|_{\Sigma} \to \Gamma\left(\mathcal{T}^{\Phi} M\left[\frac{-k-d+1}{2}\right]\right)\Big|_{\Sigma},$$

with leading term $\bar{\Delta}^{\frac{k}{2}}$.

Here, the notation $\mathcal{T}M$ denotes the standard tractor bundle introduced in 3.1, the superscript Φ labels a higher tensor bundle built from tensoring these, while $\mathcal{T}^{\Phi}M[w]$ denotes its tensor product with $\mathcal{E}M[w]$. A simple example of $\mathcal{T}^{\Phi}M[w]$ is the conformal density bundle $\mathcal{E}M[w]$ itself. The above theorem is proved in Theorems 7.1 and 7.5. In fact, as we indicate in the first of these, certain linear operators also arise for k odd. In one sense the P_k are analogs of the Graham–Jennes–Mason–Sparling (GJMS) operators [41], but in contrast exist to all orders in both dimension parities and depend non-trivially on the conformal embedding. These operators are based on the tangential operators found in [33], and admit simple holographic formulæ–see Theorem 7.1. An important application of the extrinsic conformal Laplacians is to the provision of a (holographic) formula for the obstruction density. This is given in Theorem 7.7.

The first draft of this article also studied energy functions for the obstruction density and, based on evidence gathered at linear order, posed the question of whether such action functionals exist in all dimensions (a detailed discussion of integrated invariants may now be found in [35, 37], explicit formulæ in ambient dimensions 3 and 4 and a novel variational calculus are given in [23]). Since then, Graham has answered this question by showing that in all dimensions the obstruction density we define arises as the gradient, with respect to variations of the conformal embedding, of an integral that gives the anomaly term in a renormalised volume expansion for singular Yamabe metrics solving Problem 1.1 below, see [39]. This conclusion was subsequently also recovered as a special case of a broad framework for renormalised volume problems developed in [35, 36], where it is moreover shown that the given anomaly/energy is an integral of an extrinsically coupled Q-curvature. Moreover, very recently, Graham and Reichert [42] have employed earlier work by Graham and Witten [43] to study Willmore energies for even dimensional submanifolds, including detailed results for embedded four-manifolds. Aspects of the work presented in the current article were originally announced in [34].

1.3. Structure of the article

In Section 2 below, we show that the classical problem of constructing Riemannian hypersurface invariants can be treated holographically. This provides the idea to be generalised in the more difficult conformal case, and also a tool for calculating in the treatment of the latter. A review of basic conformal geometry and tractor calculus is given in Section 3, with key identities derived in Section 3.6. Conformal hypersurfaces are treated in Section 4. Section 5 deals with the linearised obstruction density.

Theorem 4.5 allows us proliferate natural invariants of conformal hypersurfaces; this is discussed in Section 6.1 where various explicit examples are also given. A recursive procedure for computing conformal hypersurface invariants is given in Theorem 6.2.

1.4. Notation

We will primarily employ an abstract index notation in the spirit of [52]. Occasionally a mixed notation such as $R(\hat{n}, b, c, \hat{n}) = R_{abcd}\hat{n}^a\hat{n}^d$, where $\hat{n}$ is a vector field and b, c are indices, is propitious.

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2. Riemannian hypersurfaces

Throughout this section we shall consider a smooth (C^∞) *hypersurface* Σ in a Riemannian manifold (M^d, g), meaning a smoothly embedded codimension 1 submanifold. Treating Riemannian hypersurfaces has several purposes. It enables us to set up the basic structures for later use. We also use this opportunity to show that such a hypersurface determines a canonical defining function. This provides a method for constructing Riemannian hypersurface invariants and thus gives a simple analog of the conformal ideas studied in the subsequent sections.

2.1. Riemannian notation and conventions

We work on manifolds M of dimension $d \geq 3$, unless stated otherwise. For simplicity we assume that this is connected and orientable. Also for simplicity we will consider only metrics of Riemannian signature, however nearly all results can be trivially extended to metrics of any signature (p, q). Given a metric g we write ∇_a to denote the corresponding Levi-Civita connection. Then the Riemann curvature tensor R is given by

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w,$$

where u, v, and w are arbitrary vector fields. In an abstract index notation (*cf.* [52]) R is denoted by $R_{ab}{}^c{}_d$, and $R(u, v)w$ is $u^a v^b w^d R_{ab}{}^c{}_d$. This can be decomposed into the totally trace-free *Weyl curvature* W_{abcd} and the symmetric *Schouten tensor* P_{ab} according to

$$R_{abcd} = W_{abcd} + 2g_{c[a}\mathrm{P}_{b]d} + 2g_{d[b}\mathrm{P}_{a]c}, \tag{2.1}$$

where $[\cdots]$ indicates antisymmetrisation over the enclosed indices. Thus P_{ab} is a trace modification of the Ricci tensor $\mathrm{Ric}_{ab} = R_{ca}{}^{c}{}_{b}$:

$$\mathrm{Ric}_{ab} = (d-2)\mathrm{P}_{ab} + \mathrm{J}g_{ab}, \qquad \mathrm{J} := \mathrm{P}^{a}_{a}.$$

The scalar curvature is $\mathrm{Sc} = g^{ab}\,\mathrm{Ric}_{ab}$ so $\mathrm{J} = \mathrm{Sc}\,/(2(d-1))$; in two dimensions we define $\mathrm{J} := \frac{1}{2}\,\mathrm{Sc}$. To simplify notation, we will often write $u.v$ to denote $g(u,v)$, for vectors u and v.

2.2. Hypersurfaces in Riemannian manifolds

Given a hypersurface Σ, a function $s \in C^\infty M$ will be called a *defining function* for Σ if $\Sigma = \mathcal{Z}(s)$ (the zero locus of s) and the exterior derivative $\boldsymbol{d}s$ is nowhere vanishing along Σ. Defining functions always exist, at least locally. We shall use the notation n_s (or simply n if s is understood) for $\boldsymbol{d}s$.

Definition 2.1. A defining function s for Σ is said to be *normal* if

$$|n_s|^2 = 1 + sA, \tag{2.2}$$

where $A \in C^\infty(M)$.

To see that normal defining functions always exist locally, let s be a defining function for a smooth hypersurface Σ in a Riemannian manifold (M^d, g). Then there is a neighbourhood U of Σ on which the function $\bar{s} := s/|\boldsymbol{d}s|_g$ is a defining function, and this is normal.

Now we consider further "improving" the defining function:

Problem 2.2. Given Σ, a smooth hypersurface in a Riemannian manifold (M, g), find a defining function $\bar{s}$ such that $n_{\bar{s}}$ obeys

$$|n_{\bar{s}}|^2 = 1 + s^{\ell+1}A, \tag{2.3}$$

for some $A \in C^\infty(M)$ and $\ell \in \mathbb{N} \cup \infty$ as high as possible.

Remark 2.3. In fact the equation $|n_{\bar{s}}| = 1$ may be solved exactly in a neighbourhood of Σ (this is part of the construction of Gaussian normal coordinates [60] in which $\bar{s}$ is the distance to the hypersurface) but we wish to illustrate an inductive approach to the problem as stated. The point being that an adaptation of this idea then treats Problem 1.1. Furthermore, the canonical defining function obtained by this process is very useful for calculations, *cf.* [37].

Next some points of notation. Equalities such as $E = F + s^k G$ for G smooth, will be denoted $E = F + \mathcal{O}(s^k)$, and we may write $E \stackrel{\Sigma}{=} F$ if $k \in \mathbb{Z}_{\geq 1}$. We will also use this notation when either E or F is defined only along Σ and agrees with the restriction of the other side to Σ. Norms squared of vector or 1-form fields will often be denoted simply by squares, *i.e.*, $v^2 := |v|^2$. Hence our problem is given s such that $(\boldsymbol{d}s)^2 = 1 + \mathcal{O}(s)$, find $\bar{s}(s)$ such that $(\boldsymbol{d}\bar{s})^2 = 1 + \mathcal{O}(s^{\ell+1})$.

Now the key point here is that Problem 2.2 can be solved, to arbitrarily high order by an explicit recursive formula. This is based on the following Lemma.

Lemma 2.4. *Suppose the defining function $s \in C^\infty(M)$ satisfies*

$$n^2 = 1 + \mathcal{O}(s^\ell), \quad \ell \in \mathbb{Z}_{\geq 1},$$

where $n := \boldsymbol{d}s$. Then

$$\bar{s} = s\left[1 - \frac{1}{2}\frac{n^2 - 1}{\ell + 1}\right] \tag{2.4}$$

solves (2.3).

Proof. Since $n^2 = 1 + s^\ell A$, with $\ell \geq 1$, for some $A \in C^\infty(M)$, then

$$\nabla\bar{s} = n - \frac{1}{2(\ell+1)}\big((\ell+1)s^\ell A\, n + s^{\ell+1}\nabla A\big) = \left(1 - \frac{1}{2}s^\ell A\right) n + \mathcal{O}(s^{\ell+1}).$$

Hence

$$(\nabla\bar{s})^2 = \big(1 - s^\ell A\big)\, n^2 + \mathcal{O}(s^{\ell+1}) = 1 + \mathcal{O}(s^{\ell+1}),$$

where the last equality used the fact that $n^2 = 1 + s^\ell A$. Note that, at the order treated, Equation (2.4) uniquely specifies the adjustment of $\bar{s}$ required to solve (2.3). □

An immediate consequence of this Lemma is a recursive formula solving Problem 2.2.

Proposition 2.5. *Problem 2.2 can be solved uniquely to order $\ell = \infty$. The solution is given recursively by*

$$\bar{s} := \bar{s}_\ell,$$

where

$$\bar{s}_\ell = s \prod_{k=0}^{\ell-1} \left[1 - \frac{1}{2} \frac{n_{\bar{s}_k}^2 - 1}{k+2} \right],$$

and $\bar{s}_0 := s$ is a normal defining function.

We will term a defining function s obeying $(\nabla s)^2 = 1$ everywhere a *unit defining function.* In the setting of formal asymptotics, we will slightly abuse this language by using it also to refer to solutions to Problem 2.2 (whose existence is guaranteed by the previous proposition).

2.3. The standard Riemannian hypersurface objects and identities

Let Σ be a smooth hypersurface in a Riemannian manifold (M^d, g), and write ∇ for the Levi-Civita connection of g. Suppose that $\hat{n}^a \in \Gamma(TM)$ is such that its restriction to Σ is a unit normal vector field. The tangent bundle $T\Sigma$ and the subbundle of $TM|_\Sigma$ orthogonal to $\hat{n}^a$ along Σ (denoted $TM^\top$) may be identified; this will be assumed in the following discussion. Thus we will use the same abstract indices for $T\Sigma = TM^\top$ as for TM.

We will also denote the projection of tensors to submanifold tensors by a superscript $\top$. In particular, for a vector $v \in \Gamma(TM)$, we have $v^\top := v - \hat{n}\,\hat{n}.v$. We will generally use a bar to distinguish intrinsic hypersurface quantities from the corresponding ambient objects. In particular, for a vector field $v^a \in \Gamma(T\Sigma)$, the intrinsic Levi–Civita connection $\bar{\nabla}$ is given in terms of the ambient connection restricted to Σ by the Gauß formula,

$$\bar{\nabla}_a v^b = \nabla_a^\top v^b + \hat{n}^b \mathrm{I\!I}_{ac} v^c, \tag{2.5}$$

where $\nabla_a^\top$ means $(\delta_a^b - \hat{n}^b \hat{n}_a)\nabla_b$ and the second fundamental form $\mathrm{I\!I}_{ab} \in \Gamma\big(\odot^2 T^*\Sigma\big)$ is given by

$$\mathrm{I\!I}_{ab} = \nabla_a^\top \hat{n}_b\big|_\Sigma. \tag{2.6}$$

Its trace yields the mean curvature

$$H := \frac{1}{d-1}\,\mathrm{II}_a^a,$$

so that the trace-free second fundamental form $\mathring{\mathrm{II}}_{ab}$ can be written

$$\mathring{\mathrm{II}}_{ab} = \mathrm{II}_{ab} - H\bar{g}_{ab},$$

where (along Σ) $g_{ab} - \hat{n}_a\hat{n}_b$ is called first fundamental form or the induced metric $\bar{g}_{ab}$. The intrinsic curvature is related to the (projected) ambient curvature along Σ by the Gauß equation

$$\bar{R}_{abcd} = R^{\top}_{abcd} + \mathrm{II}_{ac}\mathrm{II}_{bd} - \mathrm{II}_{ad}\mathrm{II}_{bc}.$$

When $d \geq 4$, writing the trace of this expression in terms of the trace-free second fundamental form, as well as Schouten and Weyl tensors, leads to what we shall call the Fialkow–Gauß equation:

$$\begin{aligned}(2.7)\qquad \mathring{\mathrm{II}}^2_{ab} - \frac{1}{2}\,\bar{g}_{ab}\frac{\mathring{\mathrm{II}}_{cd}\mathring{\mathrm{II}}^{cd}}{d-2} - W(\hat{n},a,b,\hat{n})&\\ = (d-3)\left(\mathrm{P}^{\top}_{ab} - \bar{\mathrm{P}}_{ab} + H\mathring{\mathrm{II}}_{ab} + \frac{1}{2}\,\bar{g}_{ab}H^2\right) &=: (d-3)\mathcal{F}_{ab}.\end{aligned}$$

Since $W(\hat{n},a,b,\hat{n})$ ($=: W_{cabd}\hat{n}^c\hat{n}^d$) and $\mathring{\mathrm{II}}_{ab}$ are both known to be conformally invariant, the same applies to the expression on the right hand side of the above display. This was coined the Fialkow tensor in [59].

Note that in Equation (2.7), some terms such as $W(\hat{n},a,b,\hat{n})$ may be defined off Σ, but it is implicitly clear that this expression as a whole only makes sense along Σ. In such situations, we avoid cumbersome notations such as $W(\hat{n},a,b,\hat{n})\big|_\Sigma$ wherever clarity allows.

The covariant curl of the second fundamental form is governed by the Codazzi–Mainardi equation

$$(2.8)\qquad \bar{\nabla}_a\mathrm{II}_{bc} - \bar{\nabla}_b\mathrm{II}_{ac} = \big(R_{abcd}\hat{n}^d\big)^{\top};$$

tracing this yields (in $d \geq 3$)

$$(2.9)\qquad \bar{\nabla}.\mathring{\mathrm{II}}_b - (d-2)\bar{\nabla}_b H = (d-2)\mathrm{P}(b,n)^{\top}.$$

Conversely, the trace-free part of the Codazzi–Mainardi equation gives

$$(2.10)\qquad \bar{\nabla}_a\mathring{\mathrm{II}}_{bc} - \bar{\nabla}_b\mathring{\mathrm{II}}_{ac} + \frac{1}{d-2}\big(\bar{\nabla}.\mathring{\mathrm{II}}_a\,\bar{g}_{bc} - \bar{\nabla}.\mathring{\mathrm{II}}_b\,\bar{g}_{ac}\big) = \big(W_{abcd}\hat{n}^d\big)^{\top}.$$

Since the Weyl tensor is conformally invariant, this shows that the trace-free curl of the trace-free second fundamental form is also invariant. Indeed the mapping of weight 1, trace-free, rank two symmetric tensors

$$\mathring{K}_{bc} \stackrel{\mathsf{Cod}}{\longmapsto} \bar{\nabla}_a \mathring{K}_{bc} - \bar{\nabla}_b \mathring{K}_{ac} + \frac{1}{d-2}\big(\bar{\nabla}.\mathring{K}_a \, \bar{g}_{bc} - \bar{\nabla}.\mathring{K}_b \, \bar{g}_{ac}\big), \tag{2.11}$$

is itself a conformally invariant operator. The operator Cod will be called the *conformal Codazzi operator* in the following.

The link between the intrinsic and ambient scalar curvatures is given by the formula

$$\mathrm{II}_{ab}\mathrm{II}^{ab} - (d-1)^2 H^2 = \mathrm{Sc} - 2\,\mathrm{Ric}(\hat{n},\hat{n}) - \overline{\mathrm{Sc}}.$$

This recovers Gauß' *Theorema Egregium* for a three dimensional Euclidean ambient space. In dimension $d > 2$, the above display may be written in terms of the Schouten tensor and trace-free second fundamental form and gives

$$\mathrm{J} - \mathrm{P}(\hat{n},\hat{n}) = \bar{\mathrm{J}} - \frac{d-1}{2}\, H^2 + \frac{\mathring{\mathrm{II}}_{ab}\mathring{\mathrm{II}}^{ab}}{2(d-2)}. \tag{2.12}$$

This result shows that $\mathrm{J} - \mathrm{P}(\hat{n},\hat{n}) - \bar{\mathrm{J}} + \frac{d-1}{2}\, H^2$ is conformally invariant.

Finally, we record the relation between the Laplacian of the mean curvature and divergence of the second fundamental form; this is a simple consequence of the Codazzi–Mainardi equation:

$$\begin{aligned} \bar{\Delta} H &= \frac{1}{d-1}\bar{\nabla}^a\big(\bar{\nabla}.\mathrm{II}_a - \mathrm{Ric}(\hat{n},a)^\top\big) \\ &= \frac{1}{d-2}\bar{\nabla}^a\big(\bar{\nabla}.\mathring{\mathrm{II}}_a - \mathrm{Ric}(\hat{n},a)^\top\big). \end{aligned} \tag{2.13}$$

2.4. Riemannian hypersurface invariants

Since locally any hypersurface is the zero set of some defining function, there is no loss of generality in restricting to those hypersurfaces Σ which are the zero locus $\mathcal{Z}(s)$ of some defining function s. To further simplify our discussion we also assume that M is oriented with volume form ω. Given a hypersurface in M, it has an orientation determined by s and ω, as $\boldsymbol{ds}$ is a conormal field. Different defining functions are *compatibly oriented* if they determine the same orientation on Σ.

Definition 2.6. For hypersurfaces, a *scalar Riemannian pre-invariant* is a function P which assigns to each pair consisting of a Riemannian d-manifold (M,g) and hypersurface defining function s, a function $P(s;g)$ such that:

(i) $P(s;g)$ is natural, in the sense that for any diffeomorphism $\phi : M \to M$ we have $P(\phi^* s; \phi^* g) = \phi^* P(s;g)$.

(ii) The restriction of $P(s;g)$ is independent of the choice of oriented defining functions, meaning that if s and s' are two compatibly oriented defining functions such that $\mathcal{Z}(s) = \mathcal{Z}(s') =: \Sigma$ then, $P(s;g)|_\Sigma = P(s';g)|_\Sigma$.

(iii) P is given by a universal polynomial expression such that, given a local coordinate system (x^a) on (M,g), $P(s;g)$ is given by a polynomial in the variables

$$g_{ab},\ \partial_{a_1} g_{bc},\ \ldots,\ \partial_{a_1}\partial_{a_2}\cdots\partial_{a_k} g_{bc},\ (\det g)^{-1},$$
$$s,\ \partial_{b_1} s,\ \ldots,\ \partial_{b_1}\partial_{b_2}\cdots\partial_{b_\ell} s,\ ||\boldsymbol{d}s||_g^{-1},\ \omega_{a_1\ldots a_d},$$

for some positive integers k, ℓ.

A *scalar Riemannian invariant* of a hypersurface Σ is the restriction $P(\Sigma;g) := P(s;g)|_\Sigma$ of a pre-invariant $P(s;g)$ to $\Sigma := \mathcal{Z}(s)$.

In (iii) ∂_a means $\partial/\partial x^a$, $g_{ab} = g(\partial_a, \partial_b)$, $\det g = \det(g_{ab})$ and $\omega_{a_1\ldots a_d} = \omega(\partial_{a_1}, \ldots, \partial_{a_d})$. Also, in the context of treating hypersurface invariants, there is no loss of generality studying defining functions such that $||\boldsymbol{d}s||_g^{-1} \neq 0$ everywhere in M, since we may, if necessary replace M by a local neighbourhood of Σ. For (i) note that if $\Sigma = \mathcal{Z}(s)$, then $\phi^{-1}(\Sigma)$ is a hypersurface with defining function $\phi^* s$. The conditions (i),(ii) and (iii) mean that any Riemannian invariant $P(s;g)|_\Sigma$ of Σ, is entirely determined by the data (M,g,Σ), and this justifies the notation $P(\Sigma;g)$. Then in this notation the naturality condition of (i) implies $\phi^*(P(\Sigma,g)) = P(\phi^{-1}(\Sigma), \phi^* g)$.

While scalar invariants are our main focus, the above definition is easily extended to define *tensor valued hypersurface pre-invariants* and *invariants.* In that case one considers instead tensor valued functions P and requires that the coordinate components of the image satisfy the conditions (ii), (iii), and the obvious adjustment of (i). We shall use the term *invariant* to mean either tensor or scalar valued hypersurface invariants.

A simple example of such a tensor valued invariant is the unit conormal defined by a defining function s, that is $\hat{n} := \boldsymbol{d}s/||\boldsymbol{d}s||_g$. Then the second fundamental form $\mathrm{I\!I}$ and its intrinsic covariant derivatives $\bar\nabla \mathrm{I\!I}$, $\bar\nabla\bar\nabla \mathrm{I\!I}$ and so forth are easily seen to arise from the restriction to Σ of tensor-valued pre-invariants. So, for example, $\mathrm{II}^{ab}\mathrm{II}_{ab}$ is a scalar hypersurface invariant.

It follows that, in the spirit of Weyl's clasical invariant theory, one may generate Riemannian hypersurface invariants by expressions of the form

$$\text{Partial-contraction}\big(n\cdots n\,(\bar\nabla\cdots\bar\nabla\mathrm{II})\dots (\bar\nabla\cdots\bar\nabla\mathrm{II})(\nabla\cdots\nabla R)\dots(\nabla\cdots\nabla R)\big)\big|_\Sigma, \tag{2.14}$$

where "Partial-contraction" is some use of the inverse metric (and the volume form if M is oriented) to contract some number of indices (and as usual we view $T\Sigma$ and $T^*\Sigma$ as subbundles of, respectively, $TM|_\Sigma$ and $T^*M|_\Sigma$).

2.5. Invariants via the unit defining function

Proposition 2.5 asserts that a Riemiannian hypersurface embedding $\Sigma \hookrightarrow M$, (formally) determines a unit defining function $\bar s$ that is unique to $\mathcal{O}(\bar s^\infty)$ along Σ. In fact is straightforward to re-express the derivatives of $\bar s$, along Σ, in terms of the objects introduced above:

Proposition 2.7. *If $\bar s$ is a unit defining function for a Riemannian hypersurface Σ then, for any integer $k \geq 1$, the quantity*

$$\nabla^k \bar s|_\Sigma$$

may be expressed as $\bar\nabla^{k-2}\mathrm{II}$ plus a linear combination of partial contractions involving $\bar\nabla^\ell \mathrm{II}$ for $0 \leq \ell \leq k-3$, and the Riemannian curvature R and its covariant derivatives (to order at most $k-3$) and the undifferentiated conormal n,.

Thus via Proposition 2.5 every partial contraction

$$\text{Partial-contraction}\big((\nabla\cdots\nabla\bar s)\dots(\nabla\cdots\nabla\bar s)(\nabla\cdots\nabla R)\dots(\nabla\cdots\nabla R)\big)\big|_\Sigma,$$

encodes a Riemannian hypersurface invariant and may be written in terms of expressions of the form (2.14) using Proposition 2.7. In the following we establish an analogous approach for conformal hypersurface invariants.

3. Conformal geometry and the ASC problem

A conformal structure $\boldsymbol{c}$ is an equivalence class of Riemannian metrics where any two metrics $g, g' \in \boldsymbol{c}$ are related by a *conformal rescaling*; that is $g' = fg$

with $C^\infty M \ni f > 0$. On a conformal manifold $(M, \boldsymbol{c})$, there is no distinguished connection on TM. However there is a canonical metric h and linear connection $\nabla^{\mathcal{T}}$ (preserving h) on a related higher rank vector bundle known as the tractor bundle. This enables us to greatly simplify the treatment of Problem 1.1.

3.1. Basic conformal tractor calculus

We follow here the development of [7], see also [31]. The standard tractor bundle and its connection are linked and equivalent to the normal conformal Cartan connection [10, 11], and are also equivalent to objects developed by Thomas [58].

On a conformal d-manifold $(M, \boldsymbol{c})$, the standard tractor bundle $\mathcal{T}M$ (or simply $\mathcal{T}$ when M is understood or $\mathcal{T}^A$ as its abstract index notation) is a rank $d+2$ vector bundle equipped with a canonical tractor connection $\nabla^{\mathcal{T}}$. The bundle $\mathcal{T}$ is not irreducible but has a composition series summarised via a semi-direct sum notation

$$\mathcal{T}^A M = \mathcal{E}M[1] \oplus\!\!\!\!+ \, \mathcal{E}_a M[1] \oplus\!\!\!\!+ \, \mathcal{E}M[-1].$$

Here $\mathcal{E}M[w]$ (or $\mathcal{E}[w]$ when M is understood), for $w \in \mathbb{R}$, is the conformal density bundle which recall, is the natural (oriented) line bundle equivalent, via the conformal structure $\boldsymbol{c}$, to $\big[(\wedge^d TM)^2\big]^{\frac{w}{2d}}$. Then while $\mathcal{E}_a M[w]$ denotes $\mathcal{E}M[w] \otimes T^*M =: T^*M[w]$. This means that there canonically is a bundle inclusion $\mathcal{E}[-1] \to \mathcal{T}^A$, with $\mathcal{E}_a[1]$ a subbundle of the quotient by this, and there is a surjective bundle map $\mathcal{T} \to \mathcal{E}[1]$. We denote by X^A the canonical section of $\mathcal{T}^A[1] := \mathcal{T}^A \otimes \mathcal{E}[1]$ giving the first of these:

$$X^A : \mathcal{E}[-1] \to \mathcal{T}^A; \tag{3.1}$$

we refer to X as the *canonical tractor*.

A choice of metric $g \in \boldsymbol{c}$ determines an isomorphism

$$\mathcal{T} \stackrel{g}{\cong} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1]. \tag{3.2}$$

We may write, for example, $U \stackrel{g}{=} (\sigma,\ \mu_a,\ \rho)$, or alternatively $[U^A]_g = (\sigma,\ \mu_a,\ \rho)$, to mean that U is a section of $\mathcal{T}$ and $(\sigma,\ \mu_a,\ \rho)$ is its image under the isomorphism (3.2). Sometimes we will use (3.2) without emphasis on the metric g, if this is understood by context. Changing to a conformally related metric $\widehat{g} = \Omega^2 g$ gives a different isomorphism, which is related to the previous one by the transformation formula

$$(3.3)\qquad \widehat{(\sigma, \mu_b, \rho)} = (\sigma, \mu_b + \sigma\Upsilon_b, \rho - \boldsymbol{g}^{cd}\Upsilon_c\mu_d - \tfrac{1}{2}\sigma\boldsymbol{g}^{cd}\Upsilon_c\Upsilon_d),$$

where Υ_b is the one-form $\Omega^{-1}\boldsymbol{d}\Omega$.

In terms of the above *splitting*, the tractor connection is given by

$$(3.4)\qquad \nabla_a^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + \boldsymbol{g}_{ab}\rho + \mathrm{P}_{ab}\sigma \\ \nabla_a\rho - \mathrm{P}_{ac}\mu^c \end{pmatrix}.$$

It is straightforward to verify that the right-hand-side of (3.4) transforms according to (3.3), and this verifies the conformal invariance of $\nabla^{\mathcal{T}}$. In the following we will usually write simply ∇ for the tractor connection. Since it is the only connection we shall use on $\mathcal{T}$, its dual, and tensor powers, this should not cause any confusion; in particular the ∇'s on the the right side of the above display denote the Levi–Civita connection. The curvature of the tractor connection is the tractor-endomorphism valued two-form $\mathcal{R}^{\sharp}$; in the above splitting this acts as

$$(3.5)\qquad \mathcal{R}^{\sharp}_{ab} \begin{pmatrix} \sigma \\ \mu_c \\ \rho \end{pmatrix} = [\nabla_a, \nabla_b] \begin{pmatrix} \sigma \\ \mu_c \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ W_{abc}{}^d\mu_d + C_{abc}\sigma \\ -C_{abc}\mu^c \end{pmatrix}.$$

Here $C_{abc} := \nabla_a\mathrm{P}_{bc} - \nabla_b\mathrm{P}_{ac}$ denotes the Cotton tensor.

For $[U^A] = (\sigma, \mu_a, \rho)$ and $[V^A] = (\tau, \nu_a, \kappa)$, the conformally invariant *tractor metric* h on $\mathcal{T}$ is given by

$$(3.6)\qquad h(U, V) = h_{AB}U^AV^B = \sigma\kappa + \boldsymbol{g}_{ab}\mu^a\nu^b + \rho\tau =: U{\cdot}V.$$

Note that this has signature $(d+1, 1)$ and, as mentioned, is preserved by the tractor connection, *i.e.*, $\nabla^{\mathcal{T}}h = 0$. It follows from this formula that $X_A = h_{AB}X^B$ provides the surjection $X_A : \mathcal{T}^A \to \mathcal{E}[1]$. The tractor metric h_{AB} and its inverse h^{AB} are used to identify $\mathcal{T}$ with its dual in the obvious way, equivalently, it is used to raise and lower tractor indices. We will often employ the shorthand notation V^2 for $V_AV^A = h(V, V)$.

Tensor powers of the standard tractor bundle $\mathcal{T}$, and tensor parts thereof, are vector bundles that are also termed tractor bundles. We shall denote an arbitrary tractor bundle by $\mathcal{T}^\Phi$ and write $\mathcal{T}^\Phi[w]$ to mean $\mathcal{T}^\Phi \otimes \mathcal{E}[w]$; w is then said to be the weight of $\mathcal{T}^\Phi[w]$.

Closely linked to $\nabla^{\mathcal{T}}$ is an important, second order, conformally invariant differential operator

$$D^A : \Gamma(\mathcal{T}^\Phi[w]) \to \Gamma(\mathcal{T}^{\Phi'}[w-1]),$$

known as the Thomas-D (or tractor D-) operator. Here $\mathcal{T}^{\Phi'}[w-1] := \mathcal{T}^A \otimes \mathcal{T}^\Phi[w-1]$. In a scale g,

$$[D^A]_g = \begin{pmatrix} (d+2w-2)\,w \\ (d+2w-2)\nabla_a \\ -(\Delta + \mathrm{J}w) \end{pmatrix}, \tag{3.7}$$

where $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b$, and ∇ is the coupled Levi-Civita-tractor connection [7, 58]. When $w = 1-\frac{d}{2}$ we have $D^A \stackrel{g}{=} -X^A(\Delta + [1-d/2]\mathrm{J})$ where $\Delta + [1-d/2]\mathrm{J}$ is conformally invariant; on densities this is the well-known Yamabe operator, so we term $w = 1-\frac{d}{2}$ the *Yamabe weight*. The following variant of the Thomas D-operator is also useful.

Definition 3.1. Suppose that $w \neq 1-\frac{d}{2}$. The operator

$$\widehat{D}^A : \Gamma(\mathcal{T}^\Phi[w]) \longrightarrow \Gamma\big(\mathcal{T}^{\Phi'}[w-1]\big)$$

is defined by

$$\widehat{D}^A T := \frac{1}{d+2w-2}\, D^A T.$$

Remark 3.2. Given a weight w' tractor $V^A \in \Gamma(\mathcal{T}^A M \otimes \mathcal{T}^{\Phi'} M[w'])$ subject to

$$X_A V^A = 0,$$

where $[V^A] \stackrel{g}{=} (0, v_a, v)$ for some $g \in \boldsymbol{c}$, then we may extend the above definition to the projection along V^A of the operator $\widehat{D}_A$ at the Yamabe weight $w = 1-\frac{d}{2}$, by defining

$$\begin{array}{rccc} V^A\widehat{D}_A: & \Gamma(\mathcal{T}^\Phi M[1-\frac{d}{2}]) & \longrightarrow & \Gamma(\mathcal{T}^{\Phi'}M[w'] \otimes \mathcal{T}^\Phi M[-\frac{d}{2}]) \\ & \Cup & & \Cup \\ & T & \stackrel{g}{\longmapsto} & \big(v^a\nabla_a + [1-\frac{d}{2}]\,v\big)\,T. \end{array}$$

[In [25] this invariant operator was denoted $V^A\widetilde{D}_A$.]

Finally, the following Lemma is easily verified by direct application of Equation (3.7):

Lemma 3.3. *Let $T \in \Gamma(\mathcal{T}^\Phi M[w])$. Then*

$$D_A(X^A T) = (d+w)(d+2w+2)T. \tag{3.8}$$

3.2. Conformal hypersurfaces

We now consider a hypersurface Σ smoothly embedded in a conformal manifold $(M, \boldsymbol{c})$, and term this a *conformal embedding of* Σ. Then the conformal structure $\boldsymbol{c}$ on M induces a conformal structure $\bar{\boldsymbol{c}}$ on Σ. Moreover, working locally, we may assume that there is a section $\hat{n}_a$ of $T^*M[1]$ such that $\boldsymbol{g}_{ab}\hat{n}^a\hat{n}^b = 1$ along Σ; so $\hat{n}_a$ is the conformal analog of a Riemannian unit conormal field.

There is also a corresponding unit tractor object (from [7]) called the *normal tractor* N^A which is defined along Σ, in some choice of scale, by

$$[N^A] \stackrel{\Sigma}{=} \begin{pmatrix} 0 \\ \hat{n}_a \\ -H \end{pmatrix}. \tag{3.9}$$

This is the basis for a conformal hypersurface calculus [28, 44, 57, 59] which is further developed in the article [37, Section 4]. The most basic ingredient of this is the conformal tractor analog of the Riemannian isomorphism between the intrinsic tangent bundle $T\Sigma$ and the subbundle $TM^\top$ of $TM|_\Sigma$ orthogonal to $\hat{n}^a$ along Σ; this relates the hypersurface tractor bundle $\mathcal{T}\Sigma$ and the ambient one $\mathcal{T}M$. In fact (see [9] and [28, Section 4.1]), the subbundle $N^\perp$ orthogonal to the normal tractor (with respect to the tractor metric h) along Σ is canonically isomorphic to the intrinsic hypersurface tractor bundle $\mathcal{T}\Sigma$. In the same spirit as our treatment of the Riemannian case (see Section 2.3), we will use this isomorphism to identify these bundles and use the same abstract index for $\mathcal{T}M$ and $\mathcal{T}\Sigma$. To employ this isomorphism for explicit computations in a given choice of scale, the detailed relationship between sections as given in corresponding splittings of the ambient and hypersurface tractor bundles is needed. In terms of sections expressed in a scale $g \in \boldsymbol{c}$ (determining $\bar{g} \in \bar{\boldsymbol{c}}$), this isomorphism is given by the map

$$\left[V^A\right]_g := \begin{pmatrix} v^+ \\ v_a \\ v^- \end{pmatrix} \mapsto \begin{pmatrix} v^+ \\ v_a - \hat{n}_a H v^+ \\ v^- + \frac{1}{2}H^2 v^+ \end{pmatrix} = \left[U^A{}_B\right]^g_{\bar{g}} \left[V^B\right]_g =: \left[\bar{V}^A\right]_{\bar{g}}, \tag{3.10}$$

where $V^A \in \Gamma(N^\perp)$ and $\bar{V}^A \in \Gamma(\mathcal{T}\Sigma)$. Here the $SO(d+1,1)$-valued matrix

$$\left[U^A{}_B\right]^{g}_{\bar{g}} := \begin{pmatrix} 1 & 0 & 0 \\ -\hat{n}_a H & \delta_a^b & 0 \\ -\frac{1}{2}H^2 & \hat{n}^b H & 1 \end{pmatrix},$$

and we have used the canonical isomorphism between $T^*M\big|_\Sigma$ and $T^*\Sigma$ defined by the (unit) normal vector $\hat{n}_a$ to identify sections of these. Note that for tractors along Σ in the joint kernel of $X_\llcorner$ and $N_\llcorner$ (contraction by the canonical and normal tractors), the map (3.10) is the identity.

3.3. Defining densities

The notion of a defining function adapts naturally to densities, as follows.

Given a hypersurface Σ, a section $\sigma \in \Gamma(\mathcal{E}[1])$ is said to be a *defining density* for Σ if $\Sigma = \mathcal{Z}(\sigma)$ and $\nabla\sigma$ is nowhere vanishing along Σ where ∇ is the Levi-Civita connection for some, equivalently any, $g \in \boldsymbol{c}$. For a defining density σ, we define a corresponding tractor field

$$I^A_\sigma := \widehat{D}^A\sigma. \tag{3.11}$$

For later use we introduce some notation for the components of this in a scale:

$$[\widehat{D}^A\sigma] \stackrel{g}{=} \left(\sigma, \nabla_a\sigma, -\frac{1}{d}(\Delta + \mathrm{J})\sigma\right) =: (\sigma, n_a, \rho).$$

Since I^A_σ includes the full 1-jet of σ it follows at once that it is nowhere vanishing. In fact since we assume Riemannian signature for any defining density σ we have that

$$I^2_\sigma > 0 \tag{3.12}$$

holds in a neighbourhood of Σ.

3.4. Scale and almost Riemannian structure

A choice of positive section $\tau \in \Gamma(\mathcal{E}_+[1])$ is equivalent to a choice of metric from the conformal class $\boldsymbol{c}$ via the relation $g^\tau := \tau^{-2}\boldsymbol{g}$. Traditionally any such section is thus termed a scale. However conformally compact manifolds, conformal hypersurfaces, and related structures are naturally treated by working with densities that may have a zero locus (and change sign). Thus we generalise our notion of scale as follows.

Definition 3.4. On a conformal manifold $(M, \boldsymbol{c})$ any section $\sigma \in \Gamma(\mathcal{E}[1])$ such that

$$I_\sigma^A := \widehat{D}^A \sigma$$

is nowhere vanishing is called a *scale* and I_σ^A is the corresponding *scale tractor*. We will describe such data $(M, \boldsymbol{c}, \sigma)$ (equivalently $(M, \boldsymbol{c}, I_\sigma^A)$) as an *almost Riemannian structure* or almost Riemannian geometry.

On an almost Riemannian structure $\sigma = X_A I^A$ (denoting $I := I_\sigma$) is non-vanishing on an open dense set; so σ determines a metric on such an open set. On the other hand, because it is nonzero, I^A provides a structure on the manifold M which connects the geometry of these metrics to that of the zero locus of σ. This idea was developed in [28] (see also [16]). Note that if I^2 has a fixed strict sign then I^A determines a structure group reduction of the conformal Cartan geometry, *cf.* [13] where holonomy reductions are treated.

Note that, in particular, if σ is a defining density for a hypersurface then σ is a scale away from its zero locus Σ. Thus a conformally compact manifold $\overline{M}$ is the same as an almost Riemannian geometry $(\overline{M}, \boldsymbol{c}, \sigma)$, where $\sigma \in \Gamma(\mathcal{E}[1])$ is a defining density for the boundary $\partial M = \mathcal{Z}(\sigma)$.

On conformally compact manifolds one may consider the natural curvature quantities on the bulk. In general these are not expected to extend smoothly (or in some reasonable way) to the boundary since the metric is singular there. It is therefore important to determine the extent to which such curvatures may have meaningful limiting values at the conformal infinity. In fact such questions can be treated quite mechanically by treating the structure as an almost Riemannian geometry, and this is a main motivation for this notion.

This idea applies in particular to the scalar curvature, which lies at the heart of our considerations here. The following refers to the quantity $S(\sigma)$ in Equation (1.3):

Proposition 3.5 (see [28]). *For $\sigma \in \Gamma(\mathcal{E}[1])$ the quantity $S(\sigma)$ is the squared length of the corresponding scale tractor:*

$$S(\sigma) = I_\sigma^2 := h_{AB} I_\sigma^A I_\sigma^B. \tag{3.13}$$

Thus the key Equation (1.3) has a simplifying and geometrically useful tractor fomulation

$$I_\sigma^2 = 1. \tag{3.14}$$

This observation is critical for our later developments. Meanwhile observe that although the Equation (1.3) is second order, it is a natural conformal analogue of the Riemannian equation $n_s^2 = 1$ of Problem 2.2: in place of $n_s = \boldsymbol{d}s$ we have $I_\sigma = \widehat{D}\sigma$. Moreover, if $I^2 = 1 + \mathcal{O}(\sigma^2)$ then $I_\sigma|_\Sigma = N$, the normal tractor of (3.9), see [28].

If I_σ is any scale tractor for $\sigma \in \Gamma(\mathcal{E}[1])$, then away from the zero set of σ we have $I_\sigma^2 = -\operatorname{Sc}^{g^o}/\big(d(d-1)\big)$, where $g^o = \sigma^{-2}\boldsymbol{g}$. Thus Proposition 3.5 shows that on smooth conformally compact manifolds, the scalar curvature extends smoothly to the boundary. Moreover, on M, the condition $I_\sigma^2 =$ *constant* generalises the condition of constant scalar curvature. If this holds, we term the manifold *almost scalar constant* (ASC) and there are severe restrictions on the possible zero loci $\mathcal{Z}(\sigma)$ of σ, see [28]. In particular, if $I_\sigma^2 = 1$ it is either the case that $\mathcal{Z}(\sigma)$ is empty, or else a smoothly embedded hypersurface. If not a boundary, the latter is separating. Thus the problem treated here fits very nicely into this theory.

Since our interests here concern conformally compact manifolds, and more generally conformal hypersurfaces (in the case of Riemannian signature), we will henceforth assume (3.12) holds on M, without loss of generality.

3.5. Laplace–Robin operator, $\mathfrak{sl}(2)$ and tangential operators

Combining the scale tractor $I := I_\sigma$ and the Thomas D-operator gives the *Laplace–Robin* operator [26, 27]

$$I{\cdot}D : \Gamma(\mathcal{T}^\Phi[w]) \longrightarrow \Gamma(\mathcal{T}^\Phi[w-1]).$$

This is a canonical degenerate Laplace operator with degeneracy precisely along $\Sigma = \mathcal{Z}(\sigma)$. Calculated in some scale $g \in \boldsymbol{c}$

$$\begin{aligned} I{\cdot}D &\stackrel{g}{=} (d+2w-2)(\nabla_n + w\rho) - \sigma(\Delta + w\mathrm{J}) \\ &= -\sigma\Delta + (d+2w-2)\Big[\nabla_n - \frac{w}{d}(\Delta\sigma)\Big] - \frac{2w}{d}(d+w-1)\sigma\mathrm{J}, \end{aligned} \tag{3.15}$$

where $n = \nabla\sigma$. Away from Σ, we may specialise this to the scale $g^o = \sigma^{-2}\boldsymbol{g}$; trivialising density bundles accordingly gives a tractor-coupled Laplace-type operator

$$I\cdot D \stackrel{g^o}{=} -\left(\Delta - \frac{2w(d+w-1)}{d}\mathrm{J}\right).$$

Conversely, along Σ, the Laplace–Robin operator becomes first order. In particular, when the scale tractor I_σ obeys the ASC condition (3.14) along Σ,

$$I\cdot D = (d+2w-2)\delta_n, \tag{3.16}$$

where the first order operator

$$\delta_n :\stackrel{g}{=} \nabla_n - wH.$$

The above is precisely the conformally invariant Robin operator of [9, 15].

In addition to unifying boundary dynamics and interior Laplace problems, the Laplace–Robin operator $I\cdot D$ and the scale σ are generators of a solution-generating $\mathfrak{sl}(2)$ algebra [33]. To display this, we first define a triplet of canonical operators.

Definition 3.6. Let $\sigma \in \Gamma(\mathcal{E}M[1])$ be a defining density with nowhere vanishing I^2. Then we define the triplet of operators x, h, and $y_\sigma =: y$, mapping

$$\Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}^\Phi M[w+\varepsilon]),$$

where $\varepsilon = 1, 0$, and -1, respectively, and for $f \in \Gamma(\mathcal{T}^\Phi M[w])$

$$xf := \sigma f, \qquad hf := (d+2w)f, \qquad yf := -\frac{1}{I_\sigma^2}\, I_\sigma\cdot Df.$$

Proposition 3.7 ([33], Proposition 3.4). *The operators $\{x, h, y\}$ obey the $\mathfrak{sl}(2)$ algebra*

$$[h,x] = 2x, \qquad [x,y] = h, \qquad [h,y] = -2y. \tag{3.17}$$

This operator algebra was called a *solution generating algebra* in the setting of linear extension problems in [33], because it generates formal solutions via an expansion in x. Here, it plays a similar *rôle* for the non-linear setting we treat. In fact, obstructions to smooth solutions of the extension problem $yf = 0$ (with appropriate conditions on f), can be recovered from a related tangential operator problem [33] underpinned by the following, standard $\mathfrak{sl}(2)$ identities

$$\begin{aligned}[x,y^k] &= y^{k-1}k(h-k+1),\\ [x^k,y] &= x^{k-1}k(h+k-1), \qquad \text{where } k\in\mathbb{Z}_{\geq 1}.\end{aligned} \tag{3.18}$$

Tangential operators provide a link between ambient and hypersurface geometry:

Definition 3.8. Given a hypersurface Σ and a defining density σ, an operator P, acting between smooth sections of vector bundles over M, is called *tangential* if

$$P\circ x = x\circ \widetilde{P},$$

and $\widetilde{P}$ is some smooth operator.

Observe that for smooth sections f and P tangential, we have that $Pf|_\Sigma$ depends only on $f|_\Sigma$. Thus tangential operators canonically define hypersurface operators, and hence our interest in them. The first identity in Equation (3.18) implies that the operator y^k is tangential acting on $\Gamma(\mathcal{T}^\Phi M\big[\frac{k-d+1}{2}\big])$. This allows us to construct extrinsic conformal Laplacians along the hypersurface Σ in Section 7.1, and in turn write holographic formulæ for the obstruction density in Section 7.2.

3.6. Tractor calculus identities

We list here some identities that are required for the subsequent discussion. The key result is a characterization of how the Thomas D-operator violates the Leibniz rule.

A powerful computational tool is a Leibniz-type rule for the Thomas D-operator acting on products of tractors[1]:

Proposition 3.9 (Leibniz's failure). *Let $T_i\in\Gamma(\mathcal{T}^{\Phi_i}M[w_i])$ for $i=1,2$, and $h_i := d+2w_i-2$, $h_{12} := d+2w_1+2w_2-2$ with $h_i\neq 0\neq h_{12}$. Then*

$$\begin{aligned}&\widehat{D}^A(T_1T_2) - (\widehat{D}^AT_1)\,T_2 - T_1(\widehat{D}^AT_2)\\ &\qquad = -\frac{2}{d+2w_1+2w_2-2}\,X^A\,(\widehat{D}_BT_1)(\widehat{D}^BT_2).\end{aligned} \tag{3.19}$$

Proof. The result is a simple consequence of Lemma A.1 given in Appendix A. □

[1] This rule also appeared, in a rather different physics context, in [46].

From this we obtain the following:

Corollary 3.10. *Let* $T_i \in \Gamma(\mathcal{T}^{\Phi_i}M[w_i])$ *for* $i = 1, \ldots, k$, *and* $h_i := d + 2w_i - 2$, $h_{1\ldots k} := d + 2\sum_{i=1}^k w_i - 2$ *with* $h_i \neq 0 \neq h_{1\ldots k}$. *Then*

$$(3.20)\quad \widehat{D}^A(T_1T_2\ldots T_k) - \sum_{i=1}^{k} T_1\ldots(\widehat{D}^AT_i)\ldots T_k = -\frac{2X^A}{d+2\sum_{i=1}^k w_i - 2}\sum_{1\leq i<j\leq k} T_1\ldots(\widehat{D}^BT_i)\ldots(\widehat{D}_BT_j)\ldots T_k.$$

Proof. A simple induction based on repeated use of Equation (3.19) gives the result up to a discrete set of weights determined by the statement of Proposition 3.9 (and the choice of grouping of terms used in the induction). Most of these excluded weights are artefacts of the induction procedure and correspond to removable singularities in the rational coefficients that arise. Standard rational continuation then yields the result for the set of weights listed which correspond precisely to the values where the $\widehat{D}$ operators on the left hand side of the display are ill-defined. □

Among the many identities that follow from the above proposition, two are key for our purposes:

Lemma 3.11. *Let* $T \in \Gamma(\mathcal{T}^{\Phi}M[w])$ *and* $k \in \mathbb{Z}_{\geq 0}$ *with* $d + 2k + 2w - 2 \neq 0 \neq d + 2w - 2$. *Then*

$$(3.21)\quad \widehat{D}^A(\sigma^kT) - \sigma^k\widehat{D}^AT = k\,\sigma^{k-1}I^AT - \frac{2k\,X^A\sigma^{k-1}I{\cdot}D\,T}{(d+2k+2w-2)(d+2w-2)} - \frac{k(k-1)X^A\sigma^{k-2}I^2T}{d+2k+2w-2}.$$

Proof. This result is a direct application of Equation (3.20). □

An immediate corollary of this Lemma, for symmetrized tensor products, will be particularly useful:

Corollary 3.12. *Let* $T \in \Gamma(\mathcal{T}^{\Phi}M[w])$ *and* $k \in \mathbb{Z}_{\geq 0}$ *with* $d + 2k + 2w - 2 \neq 0 \neq d + 2w - 2$. *Then*

$$(3.22)\quad \Big(\widehat{D}_A(\sigma^kT)\Big)\odot\Big(\widehat{D}^A(\sigma^kT)\Big) = \sigma^{2k}\big(\widehat{D}_A\,T\big)\odot\big(\widehat{D}^A\,T\big) + \frac{2k\,(d-2)}{d+2k+2w-2}\,T\odot\left[\frac{\sigma^{2k-1}\,I{\cdot}D\,T}{d+2w-2} + \frac{(k\,d+2w)\,\sigma^{2k-2}I^2T}{2(d-2)}\right].$$

Contracting Equation (3.21) with the scale tractor I^A and stripping off the arbitrary tractor T gives the operator identity,

$$[I{\cdot}D, \sigma^{k+1}] = I^2\sigma^k\,(k+1)(d+2\mathbf{w}+k), \tag{3.23}$$

up to apparent weight restrictions which can be removed by a standard polynomial continuation argument. Here, $\mathbf{w} = h - \frac{d}{2}$ is the weight operator

$$\mathbf{w} : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}^\Phi M[w]) \ni T,$$

with $\mathbf{w}T = wT$. Identity (3.23) is also a direct consequence of the $\mathfrak{sl}(2)$ algebra of Proposition 3.7 (*cf.* Equation (3.18)), which also shows its validity at all weights. The ambient space method used to establish Lemma A.1 in Appendix A can also be employed, for positive σ, to extend the validity of Lemma 3.11, its corollary and Display (3.23) to k replaced by any $\alpha \in \mathbb{R}$ (see [33, Equation 5.9]).

4. Conformal hypersurfaces and the extension problem

Throughout the following sections we shall assume that M is oriented, $\dim(M) \geq 3$, and that any hypersurface Σ is the zero locus of some smooth defining function.

4.1. Conformal hypersurface invariants

In Section 2.4 we defined (scalar and tensor-valued) invariants of a Riemannian hypersurface Σ. We now require the analogous concept for hypersurfaces in a conformal manifold.

Definition 4.1. A *weight w conformal covariant* of a hypersurface Σ is a Riemannian hypersurface invariant $P(\Sigma, g)$ with the property that $P(\Sigma, \Omega^2 g) = \Omega^w P(\Sigma, g)$, for any smooth positive function Ω. Any such covariant determines an invariant section of $\mathcal{E}\Sigma[w]$ that we shall denote $P(\Sigma; \boldsymbol{g})$, where $\boldsymbol{g}$ is the conformal metric of the conformal manifold $(M, [g])$. We shall say that $P(\Sigma; \boldsymbol{g})$ is a *conformal invariant* of Σ. When Σ is understood by context, the term *hypersurface conformal invariant* will refer to densities or weighted tensor fields which arise this way.

In particular one may naïvely attempt to construct conformal hypersurface invariants by seeking linear combinations of Riemannian hypersurface invariants of the form (2.14) that have the required conformal behaviour. But this method is intractible except at the lowest order.

4.2. The extension problem

We are now ready to treat the main problem. In the conformal setting it is efficient to use defining densities rather than defining functions to describe hypersurfaces analytically.

If σ is a defining density for some hypersurface Σ, and a background metric ($g \in \boldsymbol{c}$ on M) is chosen, we shall write $n_\sigma := \nabla\sigma$ (or simply n if σ is understood), where ∇ is the Levi-Civita connection.

Definition 4.2. A defining density σ for Σ is said to be *normal* if

$$n_\sigma^2 := \boldsymbol{g}^{-1}(n_\sigma, n_\sigma) = 1 + \mathcal{O}(\sigma).$$

Note that if s is a normal defining function in the scale $g = \tau^{-2}\boldsymbol{g}$, where $\tau \in \Gamma(\mathcal{E}[1])$ is nowhere zero, then $\sigma := s\tau$ is a normal defining density for Σ. In particular Σ has a normal defining density.

Observe now that if σ is a defining density for Σ then, computing in terms of any background metric, we have $I_\sigma^2 = n^2 + 2\sigma\rho$ for some density $\rho \in \Gamma(\mathcal{E}[-1])$. This follows from (3.6) and (3.11). So σ is a normal defining density if and only if

$$I_\sigma^2 = 1 + \mathcal{O}(\sigma),$$

and this statement is conformally invariant.

Thus we may always find σ such that $I_\sigma^2 = 1 + \sigma A_1$ for some smooth density $A_1 \in \Gamma(\mathcal{E}[-1])$. Now, by Proposition 3.5, Problem 1.1 is equivalent to the following.

Problem 4.3. Find a smooth defining density $\bar\sigma$ such that

$$I_{\bar\sigma}^2 = 1 + \bar\sigma^\ell A_\ell, \tag{4.1}$$

for some smooth $A_\ell \in \Gamma(\mathcal{E}[-\ell])$, where $\ell \in \mathbb{N} \cup \infty$ is as high as possible.

To treat this it is natural to set up a recursive approach similar to that used to solve Problem (2.2).

Lemma 4.4. *Suppose $\sigma \in \Gamma(\mathcal{E}[1])$ defines the hypersurface Σ in $(M, \boldsymbol{c})$, and*

$$I_\sigma^2 = 1 + \sigma^k A_k, \qquad A_k \in \Gamma(\mathcal{E}[-k]), \quad k \geq 1.$$

Then, if $k \neq d$, there exists $f_k \in \Gamma(\mathcal{E}[-k])$, unique to $\mathcal{O}(\sigma)$, such that the scale tractor $I_{\sigma'}$ of the new defining density $\sigma' := \sigma + \sigma^{k+1} f_k$ obeys

$$I_{\sigma'}^2 = 1 + \mathcal{O}(\sigma^{k+1}).$$

If $k = d$, then for any $f \in \Gamma(\mathcal{E}[-d])$ and $\sigma' := \sigma + \sigma^{d+1} f$, we have

$$I_{\sigma'}^2 = I_\sigma^2 + \mathcal{O}(\sigma^{d+1}).$$

Proof. Let $I := I_\sigma$ and first observe

$$\big(\widehat{D}\sigma'\big)^2 = I^2 + \frac{2}{d}\, I \cdot D\big(\sigma^{k+1} f_k\big) + \Big[\widehat{D}\big(\sigma^{k+1} f_k\big)\Big]^2.$$

Then using Eq. (3.22) and (3.23) we have

$$\big(\widehat{D}\sigma'\big)^2 = 1 + \sigma^k A_k + \frac{2}{d}\,\sigma^k\,(k+1)(d-k) f_k + \mathcal{O}(\sigma^{k+1}). \tag{4.2}$$

Thus, when $k \neq d$, setting

$$f_k = -\frac{d}{2(d-k)(k+1)}\, A_k, \tag{4.3}$$

gives the first result. The result at $k = d$ follows directly from Eq. (4.2). □

The first main result now follows:

Theorem 4.5. *There is a distinguished defining density $\bar\sigma \in \Gamma(\mathcal{E}[1])$, unique to $\mathcal{O}(\sigma^{d+1})$ where σ is any given defining density, such that*

$$I_{\bar\sigma}^2 = 1 + \bar\sigma^d B_{\bar\sigma}, \tag{4.4}$$

for some smooth density $B_{\bar\sigma} \in \Gamma(\mathcal{E}[-d])$. Moreover $\bar\sigma$ may be given by a canonical formula $\bar\sigma(\sigma)$ depending (smoothly) only on $(M, \boldsymbol{c}, \sigma)$.

Proof. Existence follows from the lemma, by induction, while uniqueness follows from Eq. (4.3) determining the improvement term of the form $\sigma^{k+1} f_k$ where $f_k = c_k A_k + \mathcal{O}(\sigma)$. The constant c_k is determined and non-zero (when $k \neq d$) at each order. The last statement is immediate by the construction given in the induction. □

Given the above theorem, we make the following definition:

Definition 4.6. We say that a smooth defining density $\bar{\sigma} \in \Gamma(\mathcal{E}[1])$, is a *conformal unit defining density* if it obeys

$$I^2_{\bar{\sigma}} = 1 + \mathcal{O}(\bar{\sigma}^d). \tag{4.5}$$

Remark 4.7. It follows from Theorem 4.5 that a conformal hypersurface embedding determines an ambient metric (singular along the hypersurface) from the conformal class, up to the order given. This has a host of applications. For example, as we shall later show, given a conformal unit defining density, there then exist distinguished extensions of hypersurface quantities.

Theorem 4.8. *Given two normal defining densities σ and σ', and $B_{\bar{\sigma}}$ and $\bar{\sigma}(\cdot)$ as determined in Theorem 4.5 above, then*

$$B_{\bar{\sigma}(\sigma)} = B_{\bar{\sigma}(\sigma')} + \mathcal{O}(\sigma)\,.$$

Thus $\mathcal{B} := B_{\bar{\sigma}(\cdot)}\big|_{\Sigma}$ is independent of σ and determined by $(M, \boldsymbol{c}, \Sigma)$.

Proof. If σ and σ' are two defining densities then, by the uniqueness statement for the f_k in the first part of Lemma 4.4, given a defining density σ we have

$$\bar{\sigma}(\sigma') - \bar{\sigma}(\sigma) = \bar{\sigma}^{d+1} f,$$

where $f \in \Gamma(\mathcal{E}[-d])$. The result now follows from the last statement of Lemma 4.4. □

We shall call the hypersurface conformal invariant $\mathcal{B}$ of Theorem 4.8 the *ASC obstruction density.*

If $\bar{\sigma}_{k-1}(\sigma)$ obeys $I^2_{\bar{\sigma}_{k-1}} = 1 + \mathcal{O}(\bar{\sigma}^k_{k-1})$, then

$$\frac{I^2_{\bar{\sigma}_{k-1}} - 1}{\bar{\sigma}^k_{k-1}}$$

is smooth along Σ. Thus, using Equations (4.2) and (4.3) we set

$$\bar{\sigma}_k := \bar{\sigma}_{k-1}\left[1 - \frac{d}{2}\frac{I^2_{\bar{\sigma}_{k-1}} - 1}{(d-k)(k+1)}\right], \qquad k \neq d.$$

Iterating this gives an explicit formula for the order σ^d solution to the boundary Problem 4.3, as well as its obstruction, which we summarise below.

Proposition 4.9. *Let $\sigma =: \bar\sigma_0$ be a normal defining density for Σ. Then Problem 4.3 is solved to order $\ell = d$ by recursive use of the formula*

$$\bar\sigma_k := \sigma \prod_{i=0}^{k-1} \left[1 - \frac{d}{2} \frac{I^2_{\bar\sigma_i} - 1}{(d-i-1)(i+2)}\right],$$

with also

$$I^2_{\bar\sigma_i} = (\widehat{D}^A \bar\sigma_i)\widehat{D}_A \bar\sigma_i, \qquad 1 \le k < d-1.$$

The ASC obstruction density is the obstruction to solving Problem 4.3 smoothly, and is given by

$$\mathcal{B} = \left[\frac{I^2_{\bar\sigma_{d-1}} - 1}{\sigma^d}\right]\Bigg|_\Sigma . \tag{4.6}$$

4.3. The higher order expansion and log terms

To continue the solution beyond order $\ell = d$, we relax our smoothness requirement to allow log terms. In particular we consider $\log\sigma$ which is defined as a section of a log density bundle; see Section 2.1 of [33]. Such a section is well defined away from Σ and obeys the following generalization of the algebra (3.23)

$$[I{\cdot}D, \log\sigma] = \frac{I^2}{\sigma}\,(d + 2\mathbf{w} - 1). \tag{4.7}$$

Thus we refine the notion of an order $\mathcal{O}(\sigma^\ell)$ solution to allow for any finite power of $\log\sigma$. Since we are solving for a density σ, following [33], we must also introduce a second, *true scale* $\tau \in \Gamma(\mathcal{E}[1])$, *i.e.* a scale τ that is nowhere vanishing on M. Thus, while $\log\tau$ is a log density, $\log(\sigma/\tau) = \log\sigma - \log\tau$ is a section of $\mathcal{E}M[0]$. We will also need a notion of log-polyhomogeneity in the following sense (*cf.* [51] and [2]):

Definition 4.10. Let σ be a smooth defining density for Σ and τ be some smooth, true scale. Then if densities f and g obey

$$f = g + \sigma^\ell \sum_{j=0}^{k} \big(\log(\sigma/\tau)\big)^j C_j,$$

away from Σ, where C_j are smooth densities, k is any non-negative integer and τ is any true scale we write

$$f = g + \tilde{\mathcal{O}}(\sigma^\ell).$$

When g is a smooth density, we say, for any non-negative integer ℓ, that f is a *polyhomogeneous density.*

Let us now deal with the case where log terms are first needed.

Proposition 4.11. *Let σ be a smooth conformal unit defining density for Σ. Then*

$$\sigma' := \sigma \left[1 + \frac{d}{2} \log(\sigma/\tau) \, \frac{I^2 - 1}{d+1}\right], \tag{4.8}$$

obeys

$$I^2_{\sigma'} = 1 + \sigma^{d+1}\big[\log(\sigma/\tau)\, A + B'\big] + \tilde{\mathcal{O}}(\sigma^{d+2}) = 1 + \tilde{\mathcal{O}}(\sigma^{d+1}),$$

for some smooth $A, B' \in \Gamma(\mathcal{E}[-d-1])$.

Proof. Because σ is a conformal unit density we have $I^2_\sigma = 1 + \sigma^d B$. The proof now mimics that of Lemma 4.4, save that when calculating the analog of Equation (4.2), one needs to apply the algebra (3.23) in conjunction with (4.7) to compute ($1/(d+1)$ times)

$$\begin{aligned} I{\cdot}D\, \sigma^{d+1} \log(\sigma/\tau) B &= \sigma^{d+1} I{\cdot}D\big((\log\sigma - \log\tau)B\big) \\ &= -\sigma^d (d+1) I^2 B + \sigma^{d+1}\big(\log(\sigma/\tau)\, I{\cdot}DB - [I{\cdot}D, \log\tau]B\big), \end{aligned}$$

which takes the quoted form since σ is a conformal unit density. Note that the first equality relied on the fact that the density $\log(\sigma/\tau)B$ has weight $-d$. □

Remark 4.12. Proposition 4.11 shows that the density B—determining the obstruction density $\mathcal{B}$—appears (up to a constant multiple) as the coefficient of the first logarithm term in a polyhomogeneous solution $\bar{\sigma}$ to a modified version of Problem 4.3

$$I^2_{\bar{\sigma}} = 1 + \tilde{\mathcal{O}}(\sigma^{d+1}),$$

where σ is a smooth defining density for Σ. In fact for this problem, there is a further freedom to add an additional term $\sigma^{d+1} f$ to the solution in

Equation (4.8) which gives

$$\bar\sigma = \sigma + \sigma^{d+1}\left[f + \frac{d}{2(d+1)}\log(\sigma/\tau)B\right] + \tilde{\mathcal{O}}(\sigma^{d+2}), \tag{4.9}$$

where σ is a smooth conformal unit density and f is any smooth, weight $-d$ density. Notice that when the obstruction B is nowhere vanishing along Σ, by suitably choosing the true scale τ, we may absorb away the density f. Moreover, because $\log(\sigma/\tau)$ is a weight zero density, it follows that $\bar\sigma$ itself is also a weight $w = 1$ density. However, the presence of terms involving the logarithm of σ means that it is no longer smooth, but instead polyhomogeneous in the sense of Definition 4.10.

A polyhomogeneous version of the recursion of Proposition 4.9 now solves Problem 4.3 to all orders beyond the obstruction. Non-linearity of the singular Yamabe problem then leads to products of logarithms in the higher order terms $\tilde{\mathcal{O}}(\sigma^{d+2})$ of the polyhomogeneous solution $\bar\sigma$. Assuming the obstruction density is nowhere vanishing, the failure for this solution to be unique is parameterised by the choice of the true scale τ restricted to Σ.

Proposition 4.13. *Let σ be a smooth conformal unit defining density for Σ and $k \geq d+1$. Suppose the polyhomogeneous density $\bar\sigma$ satisfies*

$$I^2_{\bar\sigma} = 1 + \bar\sigma^k\big[\log(\sigma/\tau)\,A + B' + \tilde{\mathcal{O}}(\sigma)\big],$$

for smooth $A, B' \in \Gamma(\mathcal{E}[-k])$. Then

$$\bar\sigma' = \bar\sigma\left[1 - \frac{d}{2}\frac{I^2_{\bar\sigma} - 1}{(d-k)(k+1)}\right] + \frac{d}{2}\frac{(d-2k-1)A}{(d-k)^2(k+1)^2}\,\bar\sigma^{k+1} \tag{4.10}$$

obeys

$$I^2_{\bar\sigma'} = 1 + \bar\sigma^{k+1}\big[\log(\sigma/\tau)\,A' + B'' + \tilde{\mathcal{O}}(\sigma)\big],$$

for smooth $A', B'' \in \Gamma(\mathcal{E}[-k-1])$.

Proof. The proof again closely mimics that of Lemma 4.4. We begin by computing the square of the scale tractor for an ansatz $\bar\sigma' = \bar\sigma + \bar\sigma^{k+1}F$

where $F = \big[f + \log(\sigma/\tau)\, g\big]$ with f and g smooth, and find (calling $I := I_{\bar\sigma}$)

$$I^2_{\bar\sigma'} = \big(\widehat{D}\bar\sigma'\big)^2 = 1 + \bar\sigma^k \Big[\log(\sigma/\tau)\, A + B' + \tilde{\mathcal{O}}(\sigma)\Big] + \frac{2}{d}\, I{\cdot}D\Big(\bar\sigma^{k+1}F\Big) + \tilde{\mathcal{O}}(\sigma^{k+2}).$$

The $\mathfrak{sl}(2)$ algebra of Propostion 3.7 still holds away from Σ because $\bar\sigma$ is a smooth weight one density there, whose scale tractor is non-vanishing. Thus we may use Equation (3.23) which implies the operator identity on densities of weight $-k$

$$I{\cdot}D \circ \bar\sigma^{k+1} = \bar\sigma^k \circ \big[(d-k)(k+1) + \bar\sigma I{\cdot}D\big] + \tilde{\mathcal{O}}(\sigma^{2k}),$$

and in turn, since $I_{\bar\sigma} = I_\sigma + X\tilde{\mathcal{O}}(\sigma^{d-1}) + \tilde{\mathcal{O}}(\sigma^d)$, Equation (4.7) yields

$$\big[(d-k)(k+1) + \bar\sigma I{\cdot}D\big]F = (d-k)(k+1)F + (d-2k-1)g + \tilde{\mathcal{O}}(\sigma).$$

Noting that $\log(\sigma/\tau)\, A + B' = \frac{I^2-1}{\bar\sigma^k} + \tilde{\mathcal{O}}(\sigma)$, altogether we have

$$I^2_{\bar\sigma'} - 1 = I^2 - 1 + \frac{2}{d}\,\bar\sigma^k \Big[(d-k)(k+1)F + (d-2k-1)g\Big] + \tilde{\mathcal{O}}(\sigma^{k+1}).$$

Cancelling the leading $\bar\sigma^k \log(\sigma/\tau)$ terms on the right-hand-side above requires $g = -\frac{d}{2}\frac{A}{(d-k)(k+1)}$ which gives the solution for F

$$\bar\sigma^k F = -\frac{d}{2}\frac{I^2-1}{(d-k)(k+1)} + \frac{d}{2}\frac{(d-2k-1)A}{(d-k)^2(k+1)^2}\,\bar\sigma^k.$$

Inserting the above in $\bar\sigma' = \bar\sigma(1 + \bar\sigma^k F)$ gives the quoted result. □

Observe that when $A \neq 0$, the improvement factor in square brackets in Equation (4.10) introduces one further logarithm in the improved, polyhomogeneous solution $\bar\sigma'$. Also, if the obstruction $\mathcal{B}$ is absent, so that $I^2_\sigma - 1 = \mathcal{O}(\sigma^{d+1})$, then the log term of the solution (4.9) can be omitted and Proposition 4.9 yields a *smooth* order $\ell = \infty$ solution to Problem 4.3. In that case, there remains the freedom to modify the solution by the term $\sigma^{d+1}f$ in Equation (4.9), for any smooth weight $-d$ density f.

4.4. Examples

It is not difficult to compute explicit formulæ for the obstruction density by using the recursion defined in Proposition 4.9 and the tools of Section 2.5.

However with increasing dimension (and hence order) the computations and expressions rapidly become complicated, see [23, 37] for details. We give two examples. For simplicity the second is given in conformally flat ambient spaces, in terms of a flat ambient metric.

Proposition 4.14. *For surfaces in dimension $d = 3$, the ASC obstruction density is given by*

$$\mathcal{B} = -\frac{1}{3}\Big(\bar{\nabla}_a\bar{\nabla}_b + H\mathring{\mathrm{II}}_{ab} + \mathrm{P}^{\top}_{ab}\Big)\mathring{\mathrm{II}}^{ab}. \tag{4.11}$$

For hypersurfaces in conformally flat four-manifolds, the ASC obstruction density is expressed, using a flat scale, by

$$\mathcal{B} = \frac{1}{6}\bigg((\bar{\nabla}_c\mathring{\mathrm{II}}_{ab})^2 + 2\mathring{\mathrm{II}}^{ab}\bar{\Delta}\mathring{\mathrm{II}}_{ab} + \frac{3}{2}\,(\bar{\nabla}^b\mathring{\mathrm{II}}_{ab})\bar{\nabla}^c\mathring{\mathrm{II}}^a{}_c - 2\bar{\mathrm{J}}\,\mathring{\mathrm{II}}_{ab}\mathring{\mathrm{II}}^{ab} + (\mathring{\mathrm{II}}_{ab}\mathring{\mathrm{II}}^{ab})^2\bigg). \tag{4.12}$$

5. Related linear problems and the form of the obstruction density

By construction the obstruction density $\mathcal{B}$ found in Theorem 4.5 above is conformally invariant. We will establish in Theorem 6.5 below, that it is moreover a hypersurface conformal invariant in the sense of Definition 4.1. Here we establish its structure at leading differential order.

Theorem 5.1. *Up to a non-zero constant multiple, the ASC obstruction density $\mathcal{B}$ takes the form:*

$$\bar{\Delta}^{\frac{d-1}{2}}H + \text{lower order terms}, \quad \textit{if } d-1 \textit{ is even};$$

and

$$\text{a fully non-linear expression}, \quad \textit{if } d-1 \textit{ is odd.}$$

For the case where $d-1$ is is odd, the "fully non-linear expression" means that there is an expression for $\mathcal{B}$ as a linear combination of terms, none of which is linear in the jets of the ambient curvature R and the conormal n. In particular this means that in this case the linearisation is zero in a flat background space.

In Lemma 6.4 and Lemma 6.2 below, we give a general algorithm for computing a formula for the obstruction density. To calculate its leading term (leading in terms of the number of derivatives on the conormal) we linearise by computing its infinitesimal variation. It is easily seen, using the algorithm there, that every term in the expression will involve jets of the conormal n; that is every term will be of polynomial degree at least 1 in the jets of the conormal n. In fact to establish Theorem 5.1 we need less than the full information of the leading term. The point is this: The obstruction density has weight $-d$. The algorithm shows that there is an expression for ASC obstruction density as a linear combination of terms, each of which is of weight $-d$ and is a contraction involving covariant derivatives of the conormal and curvatures. But in fact all we wish to establish at this point is whether there is a term of the maximal possible order (namely d derivatives) on the conormal or not. Such a leading term, if it exists, cannot (by elementary weight considerations) have curvature in its coefficients and must be linear in the conormal. By the same reasoning, if there is no such term, then each term must be either non-linear in the conormal or involve jets of curvature as well as jets of the conormal. Thus it here suffices to consider an $\mathbb{R}$-parametrised family of embeddings of $\mathbb{R}^{d-1}$ in $\mathbb{E}^d$, with corresponding defining densities σ_t such that the zero locus $\mathcal{Z}(\sigma_0)$ is the $x^d = 0$ hyperplane (where x^a are the standard coordinates on $\mathbb{E}^d = \mathbb{R}^d$) and so that $\mathcal{B}|_{t=0} = 0$. Then applying $\delta := \frac{d}{dt}(\,\cdot\,)\,|_{t=0}$ (which we will often denote by a dot) to Expression (5.1) we obtain the following:

Proposition 5.2. *The variation of the obstruction density is given by*

$$\dot{\mathcal{B}} = \begin{cases} a\,\bar{\Delta}^{\frac{d+1}{2}}\dot{\sigma} + \text{lower order terms}, & d-1 \textit{ even, with } a \neq 0 \textit{ a constant,} \\ \text{non-linear terms}, & d-1 \textit{ odd.} \end{cases}$$

This establishes Theorem 5.1 because, in the case of $d-1$ even—remembering that $\mathcal{B}$ depends polynomially on σ and its derivatives—the highest order term (in the jets of σ) in the variation of mean curvature is $\frac{1}{d-1}\bar{\Delta}\dot{\sigma}$. It also shows that when $d-1$ is odd the formula for $\mathcal{B}$, determined by Theorem 6.5 below, has no linear term. The argument here is that if any term in that formula is linear in the conormal and of degree 0 in curvature then it will survive upon linearisation in flat backgrounds. But evidently the collection of such terms add to zero in flat backgrounds (according to the Proposition, or by (5.3) below). Thus in the curved setting they may also be cancelled up to the addition of terms that involve the commutation of covariant derivatives on the conormal (or defining density). But (replacing the commutator with the appropriate curvature action) the latter can be written as terms that are each of degree at least 1 in both the curvature and the conormal jets, and hence non-linear overall.

Proof of Proposition 5.2. First, via Theorem 4.5, for each $t \in \mathbb{R}$ we can replace σ_t with the corresponding normalised defining density $\bar{\sigma}_t$ which solves

$$I^2_{\bar{\sigma}_t} = 1 + \bar{\sigma}_t^d \, B_{\bar{\sigma}_t}. \tag{5.1}$$

We can assume the family $\bar{\sigma}_t$ depends smoothly on t since, according to Proposition 4.9, we may take $\bar{\sigma}_t$ to depend polynomially on σ_t.

Next observe that $B_{\bar{\sigma}_0}|_{\mathcal{Z}(\bar{\sigma}_{t=0})} = 0$. This follows from Theorem 4.5 since, in this Euclidean hyperplane setting, there is a parallel standard tractor I such that $I^2 = 1$, and with $\sigma := X^A I_A$ also a defining density for $\mathcal{Z}(\bar{\sigma}_{t=0})$. (This follows, using stereographic projection, from the results in [28, Section 5].) In fact given a hypersurface defined by some defining density σ, the freedom to change σ is just that of multiplying by a nowhere vanishing function. Thus there is no loss of generality in assuming that our initial parametrised family σ_t obeys $\sigma_{t=0} = \sigma$, and we shall henceforth take this to be the case. Then, according to Proposition 4.9, $\bar{\sigma}_{t=0} = \sigma$.

Now we consider the variation at $t = 0$, through the family of embeddings $\bar{\sigma}_t$. Viewing the space $\mathbb{E}^d$ as the $t = 0$ hypersurface in $\mathbb{E}^d \times \mathbb{R}$ and applying δ to Expression 5.1 we have

$$2I{\cdot}\widehat{D}\,\dot{\sigma} = \sigma^d \, \dot{B}_\sigma, \tag{5.2}$$

where now $I := I_\sigma$ is parallel, $\dot{\sigma} = \delta\bar{\sigma}$, and $\dot{B}_\sigma = \delta B_{\bar{\sigma}_t}$. Given that (for each t) the solution $\bar{\sigma}_t$ is polynomial in the jets of the original σ_t, it is clear that the linearisation of (5.1) provides a solution of the linear problem $I{\cdot}\widehat{D}\dot{\sigma} = 0$, to the given order, and the linearisation $\dot{B}_\sigma$ of B is an obstruction to the linear problem. But the linear problem $I{\cdot}\widehat{D}\dot{\sigma} = 0$, for extending conformal weight-1 densities off a hypersurface, is treated in [33]. From [33, Proposition 5.4] this implies that when $d - 1$ is odd

$$\dot{B}_\sigma\,|_{\mathcal{Z}(\sigma)} = 0, \tag{5.3}$$

as the linear problem is unobstructed in this case, while when $d - 1$ is even

$$\dot{B}_\sigma\,|_{\mathcal{Z}(\sigma)} = a\,\mathsf{P}_{d+1}\dot{\sigma}, \quad \text{with } a \neq 0 \text{ a constant,}$$

where P_{d+1} is the order $d + 1$, conformally invariant, Laplacian power operator [17, 41, 45] along $\mathcal{Z}(\bar{\sigma}_0)$. In the flat Euclidean metric on $\mathcal{Z}(\sigma)$, we have simply $\mathsf{P}_{d+1} = \bar{\Delta}^{\frac{d+1}{2}}$. So we are done. □

Remark 5.3. Note that conformal Laplacian operator P_{d+1} is beyond the order for which such operators exist in general curved backgrounds [29, 38, 41]. But above the proof is in the conformally flat setting where the existence of P_{d+1} is well known. In fact, for an even dimensional hypersurface (with nowhere null conormal) in any conformal manifold the tangential operators $(I{\cdot}D)^\ell$ (on suitably weighted densities or tractor fields) give conformal Laplacian type operators on Σ, for all even orders ℓ. This does not contradict the non-existence of higher order conformal Laplacian operators, as these use the data of the conformal embedding.

5.1. Bernstein–Gel'fand–Gel'fand complexes

It is useful to see how the linearised operator of Proposition 5.2 fits into the standard theory of conformally invariant differential operators on conformally flat manifolds.

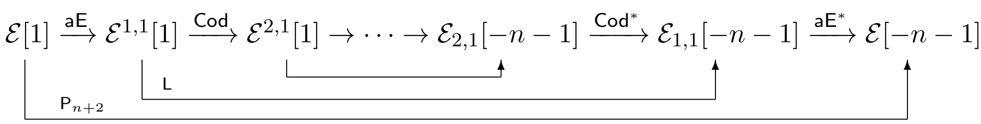
We work on a manifold of dimension n (which one should view as $d-1=\dim(\Sigma)$ for comparison with the above). For this section, it will be convenient to introduce an alternative notation for the bundles and corresponding smooth section spaces of certain tensor bundles. We will use $\mathcal{E}^k$ as a convenient alternative notation for $\Omega^k M=\Gamma(\wedge^k T^*M)$. The tensor product of $\mathcal{E}^k\otimes\mathcal{E}^\ell$, $\ell\le n/2$, $k\le\lceil n/2\rceil$, decomposes (as bundles) into irreducibles. We denote the highest weight component by $\mathcal{E}^{k,\ell}$. (Here "weight" does not refer to conformal weight, but rather the weight of the inducing $\mathrm{O}(n)$-representation.) We realise the tensors in $\mathcal{E}^{k,\ell}$ as trace-free covariant $(k+\ell)$-tensors $T_{a_1\cdots a_k b_1\cdots b_\ell}$ which are skew on the indices $a_1\cdots a_k$ and also on the set $b_1\cdots b_\ell$. Skewing over more than k indices annihilates T, as does symmetrising over any 3 indices. Then as usual, we write $\mathcal{E}^{k,\ell}[w]$ as a shorthand for the tensor product $\mathcal{E}^{k,\ell}\otimes\mathcal{E}[w]$. For example, $\mathcal{E}^{2,2}[2]$ is the bundle of *algebraic Weyl tensors* while the trace-free second fundamental form $\mathring{\mathrm{II}}$ is a section of $\mathcal{E}^{1,1}[1]$. Then we use the notation $\mathcal{E}_{k,\ell}[w]$ to stand for $\mathcal{E}^{k,\ell}[w+2k+2\ell-n]$. This notation is suggested by the global duality between $\mathcal{E}^{k,\ell}[w]$ and $\mathcal{E}_{k,\ell}[-w]$ given by contraction and integration over the manifold (in the case of compact manifolds, or else by using compactly supported sections).

The round sphere is conformally flat, and the sphere equipped with the conformal class of the round metric provides the basic "flat model" for conformal geometry in each dimension. This is acted upon by the conformal group $SO(n+1,1)$; the linear differential operators (between irreducible bundles) that intertwine this action are classified via corresponding dual Verma module homorphisms [8, 19]. Most of the existing operators arise as differentials in complexes known as BGG complexes [48]. The BGG complex of interest to us takes (for dimensions $n \geq 8$) the form

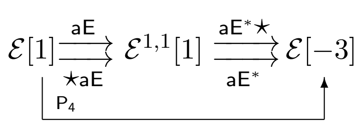
$$\mathcal{E}[1] \xrightarrow{\mathsf{aE}} \mathcal{E}^{1,1}[1] \xrightarrow{\mathsf{Cod}} \mathcal{E}^{2,1}[1] \to \cdots \to \mathcal{E}_{2,1}[-n-1] \xrightarrow{\mathsf{Cod}^*} \mathcal{E}_{1,1}[-n-1] \xrightarrow{\mathsf{aE}^*} \mathcal{E}[-n-1]\ .$$

The details of most of the operators will not be important for us. For the named operators: Cod is the conformal Codazzi operator and is given in a scale in Equation (2.11) while $\mathsf{aE} : \mathcal{E}[1] \to \mathcal{E}^{1,1}[1]$ is the so-called almost Einstein operator, which (in a scale) is given by $\sigma \mapsto \nabla_{(a}\nabla_{b)\circ}\sigma + P_{(ab)\circ}\sigma$, while Cod^* and aE^* are formal adjoints of these.

In odd dimensions the differentials of the BGG complex exhaust the list of all conformally invariant differential operators between the bundles concerned. However in even dimensions there are also *long operators* L : $\mathcal{E}^{k,\ell}[1] \to \mathcal{E}_{k,\ell}[-1]$, and an additional pair of operators about the centre of the diagram [8]. Altogether this gives the operator diagram



for dimensions 8 or greater. The operators in this diagram are unique (up to multiplying by a constant), and the diagram indicates by arrows all the operators between the bundles explicitly presented. In particular, all compositions shown vanish. The same diagram applies in dimensions 6 and 4 with minor adjustments: In dimension 6 there are two "short" operators with domain $\mathcal{E}^{2,1}[1]$ and two with range $\mathcal{E}_{2,1}[-1]$. Built from these there is one non-trivial composition $\mathcal{E}^{2,1}[1] \to \mathcal{E}_{2,1}[-1]$. Similiarly in dimension 4 we have $\star \circ \mathsf{Cod} : \mathcal{E}^{1,1}[1] \to \mathcal{E}^{2,1}[1]$ and $\mathsf{Cod}^* \circ \star : \mathcal{E}^{2,1}[1] \to \mathcal{E}_{1,1}[-1]$, as well as the operators indicated. Here $\star$ is a bundle involution related to the Hodge star operator on middle degree forms. Then L is the composition $\mathsf{Cod}^* \circ \mathsf{Cod}$. In dimension 2 the corresponding diagram is



and in this case

$$\mathsf{P}_4 := \mathsf{aE}^* \circ \mathsf{aE}. \tag{5.4}$$

We see here that in dimension 4 the operator P_4 factors through aE. This aspect generalises.

Lemma 5.4. *On any conformally flat manifold of even dimension n, the operator $\mathsf{P}_{n+2} : \mathcal{E}[1] \to \mathcal{E}[-n-1]$ can be written as a composition*

$$\mathsf{P}_{n+2} = \mathsf{H} \circ \mathsf{aE},$$

where $\mathsf{H} : \mathcal{E}^{1,1}[1] \to \mathcal{E}[-n-1]$ is a differential operator.

The operator $\mathsf{H} : \mathcal{E}^{1,1}[1] \to \mathcal{E}[-n-1]$ is not conformally invariant (except in dimension 2), but must be conformal on the range of aE. In fact, analogous factorisations as here hold for all the long operators (and in fact stronger results are available); a general approach to establishing this is discussed in [32]. Here, there is a simple proof:

Proof of Lemma 5.4. Any conformally flat manifold is locally conformally isomorphic to the sphere and such local maps give an injection between conformally invariant operators and differential operators on the sphere that intertwine the conformal group action.

In dimension 2 it is straightforward to verify (5.4). For higher dimensions, from the construction of the conformal Laplacians on conformally flat manifolds given in [24, Section 4.2] (see also [31]) one has

$$X_A \mathsf{P}_{n+2} = -\mathsf{P}_n \circ D_A,$$

where P_n is the order n conformal Laplacian. On the other hand from the same sources, or as also discussed below in Section 7.1, P_n, as an operator on tractor fields (of weight 0), factors through the tractor connection:

$$\mathsf{P}_n = \mathcal{G} \circ \nabla,$$

for some differential operator $\mathcal{G}$. So

$$X_A \mathsf{P}_{n+2} = -\mathcal{G} \circ \nabla \circ D_A.$$

But a straightforward calculation verifies that $\nabla \circ D_A$ factors through the operator aE (with this on the right). In fact this underlies the *construction* of the tractor connection in [7]. □

Let us write $\mathcal{K}^{1,1}[1]$ for the space of smooth sections in the kernel of

$$\mathsf{Cod} : \mathcal{E}^{1,1}[1] \to \mathcal{E}^{2,1}[1]$$

in even dimensions $n \geq 4$, and for the kernel of $\mathsf{aE}^* \circ \star$ in dimension 2. Then we have the following:

Proposition 5.5. *On any even dimensional conformally flat Riemannian n-manifold there is a differential operator $\mathsf{H} : \mathcal{E}^{1,1}[1] \to \mathcal{E}[-n-1]$ that upon restriction to $\mathcal{K}^{1,1}[1] \subset \mathcal{E}^{1,1}[1]$ is conformally invariant.*

Proof. Since the underlying BGG complex is locally exact, in each case, it follows that locally $\mathcal{K}^{1,1}[1]$ is the image of aE. Thus the result follows from the lemma. □

Now recall that on any conformal hypersurface $\mathring{\rm I\!I}$ is a section of $\mathcal{E}^{1,1}[1]$. For a hypersurface in Euclidean $(n+1)$-space this is in the kernel of the operator Cod (see Equation (2.10)). On the other hand on a conformally flat n-manifold with n even, given any section $\mathring{K}$ of $\mathcal{K}^{1,1}[1]$ we may form

$$\mathsf{B}_{\mathring{K}} := \mathsf{H}(\mathring{K}).$$

From Proposition 5.5 it follows that this is a conformally invariant section of $\mathcal{E}[-n-1]$. This is in general nontrivial; the last claim following from the fact that P_{n+2} is elliptic (and so has finite dimensional kernel). Thus we see that in this sense the existence of the obstruction density in even dimensions, and the absence of its linearisation in odd dimensions (*i.e.*, as in Proposition 5.2) is nicely compatible with linear conformal differential operator theory. Note that for comparison with Proposition 5.2, at leading order $\mathsf{aE}(\dot{\sigma})$ is the linearisation of $\mathring{\rm I\!I}$. So, for embedding variations of the standard sphere in Euclidean space, the linearisation of the obstruction density $\dot{B}_\sigma\big|_{S^n}$ is a non-zero multiple of $\mathsf{B}_{\mathsf{aE}(\dot{\sigma})}$, in concordance with (5.3).

Let us say that a conformally invariant curvature invariant of a hypersurface is a *(hypersurface) fundamental curvature quantity* if it has a non-trivial linearisation, with respect to variation of the hypersurface embedding, when evaluated on the conformal class of the round sphere embedded in Euclidean space. We have established above that the obstruction density is a fundamental curvature quantity in this sense. The trace free second fundamental form is also; its linearisation being the BGG operator $\mathsf{aE} : \mathcal{E}[1] \to \mathcal{E}^{1,1}[1]$. In fact these are effectively the only such invariants. In the following statement we ignore the possibility of multiplying an invariant by a non-zero constant.

We are ready to prove Theorem 1.3 mentioned in the introduction.

Proof of Theorem 1.3. Defining hypersurface invariants as we do, it is easily verified that any linearisation of a hypersurface invariant is a conformally invariant differential operator between irreducible bundles, with domain bundle $\mathcal{E}[1]$. Thus the result is immediate from the known classification of such operators mentioned above. □

For hypersurfaces of dimension 2 there is $\mathcal{B}$, $\mathring{\rm II}$, and also the invariant $\star\mathring{\rm II}$, and these give the full set fundamental curvature invariants.

Remark 5.6. It follows from the classification of conformally invariant operators on the sphere and the above discussion that, except for hypersurface dimension 2, the obstruction density cannot be written, even at leading order, as a conformally invariant operator acting on the trace-free second fundamental form.

Finally we point out that the discussion here gives a precise sense in which the obstruction density is a scalar analogue of the situation with the Fefferman-Graham obstruction tensor: The latter exists in even dimensions, and its linearisation can be understood, via the appropriate BGG diagrams, in a manner exactly parallel to the treatment here for the obstruction density, see [32, Section 2]. There is also an analogue of Theorem 1.3, see [40, Theorem 1.2].

6. Naturality of the obstruction density and proliferating invariants

By construction the obstruction density depends only on the data of the conformal embedding $\Sigma \hookrightarrow M$, however it remains to prove that the obstruction density is a hypersurface conformal invariant in the sense of Definition 4.1. We also establish here how the results above may be used to construct other hypersurface conformal invariants.

6.1. Hypersurface conformal invariants

To construct invariants holographically we need, as a tool, a broader class of invariants that we term coupled invariants. For Riemannian manifolds (M, g), *scalar Riemannian invariants* (as in [6]) may be thought of as pre-invariants, as defined by (i), (ii), (iii) of the Definition 2.6, if the dependence on s is required to be trivial. If s is a scalar function then we will also talk of *coupled invariants*. This means the same as a pre-invariant (again as in the Definition 2.6) where we do *not allow* the inclusion of $||\boldsymbol{d}s||_g^{-1}$ in (iii) of Definition 2.6 (but do allow polynomial dependence on the jets of s). These notions adapt easily to tensor-valued coupled invariants.

Then Riemannian invariants or coupled invariants are conformal invariants or coupled conformal invariants if we have the analogue of Definition 4.1. That is:

Definition 6.1. A *weight w coupled conformal covariant* is a coupled Riemannian invariant $P(s,g)$ with the property that $P(\Omega^u s, \Omega^2 g) = \Omega^w P(s,g)$, for any smooth positive function Ω and $u, w \in \mathbb{R}$. Any such covariant determines an invariant section of $\mathcal{E}[w]$ that we shall denote $P(\sigma, \boldsymbol{g})$, where $\boldsymbol{g}$ is the conformal metric of the conformal manifold $(M,[g])$ and $\sigma \in \Gamma(\mathcal{E}[u])$. We shall say that $P(\sigma, \boldsymbol{g})$ is a *coupled conformal invariant* of weight w, or simply a *conformal invariant* of weight w if the dependence on σ is trivial.

Now the key idea is to consider such coupled invariants when σ is a conformal unit defining density $\bar\sigma$ for a conformally embedded hypersurface $\Sigma \hookrightarrow M$. According to Theorem 4.5, the conformal unit scale $\bar\sigma$ is determined by the data $(M, \boldsymbol{c}, \Sigma)$, uniquely modulo $\mathcal{O}(\sigma^{d+1})$. Thus if, at each point, a coupled invariant $P(\bar\sigma, \boldsymbol{g})$ of $\boldsymbol{c}$ and $\bar\sigma$ depends on $\bar\sigma$ only through its d-jet, then $P(\bar\sigma, \boldsymbol{g})|_\Sigma$ is conformally invariant in that it depends only on the data $(M, \boldsymbol{c}, \Sigma)$. In fact $P(\bar\sigma, \boldsymbol{g})|_\Sigma$ is a hypersurface conformal invariant in the sense of Definition 4.1. Given its conformal invariance, to show this we only need to show that there is an expression for P, with the form described in Definition 2.6. In practice this is achieved by showing that the covariant derivatives of $\bar\sigma$ may be replaced with expressions involving the derivatives of the second fundamental form, the conormal, and the ambient Riemannian curvature, *cf.* Expression 2.14.

A technical definition is needed for the main lemma: Given a hypersurface Σ, and any defining function s for Σ, we say a linear differential operator $\mathcal{D}$ has *transverse order* at most $\ell \in \mathbb{Z}_{\geq 0}$, along Σ, if $\mathcal{D} \circ s^{\ell+1}$ acts as zero along Σ (where $s^{\ell+1}$ is viewed as a multiplication operator).

Lemma 6.2. *Suppose that $\bar\sigma$ is a conformal unit defining density for a hypersurface Σ in a conformal manifold $(M^d, \boldsymbol{c})$, with $d \geq 3$. If $g \in \boldsymbol{c}$ and $k \leq d$ is a positive integer, then the quantity*

$$\nabla^k \bar\sigma|_\Sigma, \quad \textit{where} \quad \nabla = \text{Levi-Civita of } g,$$

may be expressed as $\bar{\nabla}^{k-2}\mathrm{II}$ *plus terms involving partial contractions of the Riemannian curvature, its covariant derivatives, and covariant derivatives of* $\bar{\sigma}$ *with transverse order at most* $k-1$.

The proof of this is straightforward and so we omit it here. However the main ingredient to one effective approach is Lemma 6.4 below (which in fact yields much more).

We can now state the main result for constructing conformal hypersurface invariants from a conformal unit defining density:

Theorem 6.3. *Let* $\bar{\sigma}$ *be a conformal unit defining density for a hypersurface* $\Sigma \hookrightarrow M$. *Suppose that* P *is a weight* w *coupled conformal invariant of* $(M, \boldsymbol{c}, \bar{\sigma})$, *as in Definition 6.1, such that at each point it depends on at most the* d*-jet of* $\bar{\sigma}$. *Then the restriction of* P *to* Σ *is a conformal hypersurface invariant of weight* w.

Proof. That P depends only on the conformal embedding is immediate from Theorem 4.5. Then naturality and the other properties follow from Lemma 6.2 above combined with an obvious induction. □

6.2. Naturality of the obstruction density

Theorem 6.3 does not immediately imply that $\mathcal{B}$ is a conformal hypersurface invariant, because $\mathcal{B}$ depends on $\bar{\sigma}$ to order $d+1$. For this we need further detail from the equation defining $\bar{\sigma}$ and B, namely

$$n^2 = 1 - 2\rho\bar{\sigma} + \bar{\sigma}^d B \quad \Leftrightarrow \quad I_{\bar{\sigma}}^2 = 1 + \bar{\sigma}^d B, \tag{6.1}$$

for some smooth B, where $n := n_{\bar{\sigma}}$ is used to denote $\nabla\bar{\sigma}$, and (*cf.* (3.11))

$$[I_{\bar{\sigma}}^A] := [\widehat{D}^A\bar{\sigma}] = \begin{pmatrix} \bar{\sigma} \\ n_a \\ \rho \end{pmatrix}, \qquad \rho := \rho(\bar{\sigma}) = -\frac{1}{d}(\Delta\bar{\sigma} + \mathrm{J}\bar{\sigma}). \tag{6.2}$$

Such a defining density exists by Theorem 4.5 and is canonical to $\mathcal{O}(\bar{\sigma}^{d+1})$. The obstruction density is $\mathcal{B} = B|_\Sigma$. Differentiating equation (6.1) suitably and using the definitions and relations in (6.2) we obtain the following technical result (see [37, Section 3.1] for further detail):

Lemma 6.4. *For integers $2 \leq k \leq d$*

$$\frac{1}{2}\nabla_n^k I_{\bar\sigma}^2 + (d-k)\nabla_n^{k-1}\rho \stackrel{\Sigma}{=} -\nabla_n^{k-1}\big(\gamma^{ab}\nabla_a n_b\big) - (k-1)\big[\nabla_n^{k-2}\big(\mathrm{J}+2\rho^2\big) + (k-2)\rho\nabla_n^{k-2}\rho\big] + \text{LTOTs}, \tag{6.3}$$

where LTOTs *indicates additional terms involving lower transverse-order derivatives of $\bar\sigma$. In particular, for $2 \leq k \leq d-1$ we have*

$$\nabla_n^{k-1}\rho \stackrel{\Sigma}{=} -\frac{1}{d-k}\Big(\nabla_n^{k-1}\big(\gamma^{ab}\nabla_a n_b\big) + (k-1)\big[\nabla_n^{k-2}\big(\mathrm{J}+2\rho^2\big) + (k-2)\rho\nabla_n^{k-2}\rho\big]\Big) + \text{LTOTs}, \tag{6.4}$$

while

$$\mathcal{B} \stackrel{\Sigma}{=} -\frac{2}{d!}\Big(\nabla_n^{d-1}\big(\gamma^{ab}\nabla_a n_b\big) + (d-1)\big[\nabla_n^{d-2}\big(\mathrm{J}+2\rho^2\big) + (d-2)\rho\nabla_n^{d-2}\rho\big]\Big) + \text{LTOTs}. \tag{6.5}$$

Using this, Lemma 6.2, and a straightforward induction, then shows that the obstruction density $\mathcal{B}$ may be expressed as a linear combination of partial contractions involving $\bar\nabla^\ell \mathrm{I\!I}$ for $0 \leq \ell \leq d-1$, and the Riemannian curvature R and its covariant derivatives (to order at most $d-3$) and the undifferentiated conormal n. Given its conformal invariance by construction we thus have:

Theorem 6.5. *In each dimension $d \geq 3$ the obstruction density $\mathcal{B}$ is a conformal hypersurface invariant.*

7. Extrinsically coupled conformal Laplacians and a holographic formula for $\mathcal{B}$

In this section our main aim is to construct conformally invariant powers of the Laplacian on Σ that are canonically determined by the structure $(M, \boldsymbol{c}, \Sigma)$. In the cases where corresponding intrinsic GJMS operators exist, these differ by their dependence on the extrinsic geometry of the conformal embedding $\Sigma \hookrightarrow M$. We give holographic formulæ for these. These are then applied to construct a holographic formula for the obstruction density.

7.1. Extrinsic conformal Laplacians

We shall construct distinguished conformally invariant hypersurface operators canonically determined by $(\boldsymbol{c}, \Sigma)$. Our starting point is the Laplacian-type operators constructed in [33] where it is shown that

$$\mathcal{P}_k^\sigma : \Gamma\left(\mathcal{T}^\Phi M\left[\frac{k-d+1}{2}\right]\right) \to \Gamma\left(\mathcal{T}^\Phi M\left[\frac{-k-d+1}{2}\right]\right), \quad k \in \mathbb{Z}_{\geq 1}$$

defined by

$$\mathcal{P}_k^\sigma := \left(-\frac{1}{I_\sigma^2}\, I_\sigma . D\right)^k, \tag{7.1}$$

is tangential for *any* defining density σ. However, according to Theorem 4.5, $\bar\sigma$ is uniquely determined by $(M, \boldsymbol{c}, \Sigma)$, modulo terms of order $\bar\sigma^{d+1}$. Hence, specializing the defining density to be unit conformal, we obtain extrinsic conformal Laplace operators:

Theorem 7.1. *The operator*

$$\mathcal{P}_k^{\bar\sigma} : \Gamma\left(\mathcal{T}^\Phi M\left[\frac{k-d+1}{2}\right]\right) \to \Gamma\left(\mathcal{T}^\Phi M\left[\frac{-k-d+1}{2}\right]\right),$$

is a tangential differential operator. Moreover, for $k \leq d-1$, along Σ this is determined canonically by the data $(M, \boldsymbol{c}, \Sigma)$, and when k is even has leading term

$$(-1)^k\big((k-1)!!\big)^2\,\big(\Delta^\top\big)^{\frac{k}{2}}.$$

Thus $\mathcal{P}_k^{\bar\sigma}$ determines a differential operator

$$\mathsf{P}_k : \Gamma\left(\mathcal{T}^\Phi M\left[\frac{k-d+1}{2}\right]\right)\Big|_\Sigma \to \Gamma\left(\mathcal{T}^\Phi M\left[\frac{-k-d+1}{2}\right]\right)\Big|_\Sigma,$$

which we shall call an extrinsic conformal Laplace operator.

Proof. As stated above, $\mathcal{P}_k^\sigma$ is tangential for any σ and hence for $\bar\sigma$. Moreover, by Theorem 4.5, $\bar\sigma$ is uniquely determined by $(M, \boldsymbol{c}, \Sigma)$, modulo terms of order $\bar\sigma^{d+1}$. So we check how the operator $\bar y = y_{\bar\sigma}$ (see Definition 3.6) changes when replacing $\bar\sigma$ with $\bar\sigma + \bar\sigma^{d+1} A$, for some smooth, weight $-d$, density A.

From the formula (3.21), we have

$$\widehat{D}_B(\bar\sigma + \bar\sigma^{d+1}A) = I_B + (d+1)\bar\sigma^d I_B A + \bar\sigma^{d-1}X_B C + \mathcal{O}(\bar\sigma^{d+1}),$$

for some density C. Thus, using also expression (4.4) for I^2, we see that each operator $\bar y$ in the composition $\bar y^k$ is uniquely determined by $(M, \boldsymbol{c}, \Sigma)$ up to the addition of $\bar\sigma^{d-1}E$, for some linear operator E. Hence using the first identity of Equation (3.18), because $k \leq d-1$, it follows that the operator $\bar y^k$ is unique modulo the addition of a linear operator which vanishes along Σ. Thus $\bar y^k\big|_\Sigma$ is uniquely determined by $(M, \boldsymbol{c}, \Sigma)$ as claimed.

Finally, it follows from [33, Proposition 4.4] that when k is even, $\bar y^k$ has leading term $(-1)^k\big((k-1)!!\big)^2(\Delta^\top)^{k/2}$ (as an operator on ambient tractor fields along Σ). □

Remark 7.2. When $(M, \boldsymbol{c}, \sigma)$ is an AE structure and k is even, the operator $\mathcal{P}^\sigma_k$ gives a holographic formula for conformally invariant GJMS-type operators [33]. These take the form $(-1)^k\big((k-1)!!\big)^2\,\big(\Delta^\top\big)^{\frac{k}{2}} + LOT$, where "*LOT*" denotes lower order derivative terms. For $k \leq d-3$, these are built from the intrinsic hypersurface Levi-Civita connection and its curvature (a slightly stronger statement is available for densities [41]). If d is even and the AE structure is even (in the sense of [22]) then this holds[2] for all even k. Relaxing the AE condition as in Theorem 7.1, the terms *LOT* include extrinsic hypersurface invariants. Moreover for k odd the operators P_k are no longer trivial along Σ, in contrast to the AE case.

Because they are tangential, the extrinsic conformal Laplacians have natural formulæ involving tangential derivatives $\nabla^\top$, as recorded in the following:

Proposition 7.3. *Let $k \leq \dim\Sigma$. Then, in a given choice of scale, the extrinsic conformal Laplacian P_k has a formula*

$$\mathsf{P}_k = \mathcal{A}_0 + \sum_{j=1}^{k} \mathcal{A}^{a_1\ldots a_j}\nabla^\top_{a_1}\cdots\nabla^\top_{a_j},$$

where the scalar $\mathcal{A}_0$ and the tensors $\mathcal{A}^{a_1\ldots a_k}$ are natural formulæ given by polynomial expressions in first and second fundamental forms, and boundary Levi-Civita derivatives thereof, as well as ambient curvatures and their

[2] In [33], evenness of the AE structure for odd n was assumed but not mentioned.

ambient Levi-Civita derivatives. Moreover, when $k = \dim \Sigma$ *the scalar term* $\mathcal{A}_0 = 0$ *is absent and*

$$\mathsf{P}_{\dim \Sigma} = \mathcal{G} \circ \nabla^\top ,$$

for some tangential operator $\mathcal{G}$.

Proof. Using the fact that the operators are tangential in each case, it follows easily that there is a formula involving only tangential derivatives $\nabla^\top$. It is straightforward to check that this can be achieved using the calculus developed above; the claim concerning the natural formula then follows. For the special case $k = \dim \Sigma$, the Thomas D-operator on the right in the defining formula (7.1) factors through an ambient Levi-Civita connection on the right. Thus, when the preceding argument is applied to this case, it follows that $\mathcal{A}_0 = 0$. □

Remark 7.4. In the above, and in the proof of Theorem 7.5 below, we could equivalently trade $\nabla^\top$ for the intrinsic tractor connection coupled to the ambient tractor connection, pulled back to Σ.

Finally, in this section we show that there are extrinsic conformal Laplacian operators of all (even) order.

Theorem 7.5. *Let* $\dim(\Sigma)$ *be even and* $k \in 2\mathbb{N}$. *Then there exists a canonical differential operator on* Σ

$$\mathsf{P}_k : \Gamma\left(\mathcal{T}^\Phi M\left[\frac{k-d+1}{2}\right]\right)\Big|_\Sigma \to \Gamma\left(\mathcal{T}^\Phi M\left[\frac{-k-d+1}{2}\right]\right)\Big|_\Sigma ,$$

with leading term $(\Delta^\top)^{\frac{k}{2}}$, *determined by the data* $(M, \boldsymbol{c}, \Sigma)$.

Proof. When $k \leq \dim(\Sigma)$, the theorem is simply Theorem 7.1 specialised to even dimensional hypersurfaces. For all higher orders $k > \dim(\Sigma)$, we consider the operator

$$\mathsf{P}_k := \overset{\scriptscriptstyle\mathrm{T}}{D}_{A_1} \cdots \overset{\scriptscriptstyle\mathrm{T}}{D}_{A_\ell}\, \mathsf{P}_{\dim(\Sigma)}\, \overset{\scriptscriptstyle\mathrm{T}}{D}{}^{A_\ell} \cdots \overset{\scriptscriptstyle\mathrm{T}}{D}{}^{A_1} , \tag{7.2}$$

where $\ell := \frac{k-d+1}{2}$. Because this is built from *tangential* Thomas D-operators and the tangential operator $\mathsf{P}_{\dim(\Sigma)}$, it depends only on the data $(M, \boldsymbol{c}, \Sigma)$. Moreover, from Theorem 7.1, and the definition of $\overset{\scriptscriptstyle\mathrm{T}}{D}$, it is clear that the leading derivative term of the operator $\mathsf{P}_{\dim(\Sigma)}\, \overset{\scriptscriptstyle\mathrm{T}}{D}{}^{A_\ell} \cdots \overset{\scriptscriptstyle\mathrm{T}}{D}{}^{A_1}$ in the above display is a non-zero constant multiple of $X^{A_\ell}\cdots X^{A_1}\left(\Delta^\top\right)^{\frac{k}{2}}$, so

$$\mathsf{P}_{\dim(\Sigma)}\, D^{{\scriptscriptstyle\rm T}A_\ell}\cdots D^{{\scriptscriptstyle\rm T}A_1}\propto X^{A_\ell}\cdots X^{A_1}\left(\Delta^\top\right)^{\frac{k}{2}}+\text{LOTs},\tag{7.3}$$

where LOTs stands for some lower derivative operator.

In addition an easy calculation and induction establishes the following operator identity, valid acting on weight $-\dim(\Sigma)-\ell$ tractors:

$$D^{\scriptscriptstyle\rm T}_{A_1}\cdots D^{\scriptscriptstyle\rm T}_{A_\ell}X^{A_\ell}\cdots X^{A_1}\stackrel{\Sigma}{=}\left[\prod_{i=1}^{\ell} i(d+2i-3)\right]\,\mathrm{Id}\neq 0.$$

Hence it follows that the leading derivative term in $\mathsf{P}_{\dim(\Sigma)}\, D^{{\scriptscriptstyle\rm T}A_\ell}\cdots D^{{\scriptscriptstyle\rm T}A_1}$ produces a non-zero contribution to P_k proportional to $(\Delta^\top)^{\frac{k}{2}}$. Moreover, a weight argument shows that this is the highest possible order of derivative contribution to P_k. It only remains, therefore, to show that lower order derivative contributions LOTs involve curvatures and therefore cannot conspire in the full formula for P_k in Equation (7.2) to produce further leading order terms. For this we recall that the classification of conformal operators on the sphere S^{d-1} (*cf.* [17, 18, 24]) yields the operator identity on intrinsic, weight $-\ell$ tractors

$$\bar{\mathsf{P}}_{d-1}\bar{D}^{A_\ell}\cdots\bar{D}^{A_1}=\alpha\, X^{A_\ell}\cdots X^{A_1}\,\bar{\mathsf{P}}_{d+2\ell-1},$$

where $\bar{\mathsf{P}}_k$ denotes the usual (tractor twisted) conformal Laplacian, $\bar{D}$ is the intrinsic Thomas D-operator and α is a non-zero constant. Therefore, the lower order terms LOTs in (7.3) all involve curvatures at least linearly. □

Remark 7.6. The above proof proceeds *mutatis mutandis* if one wishes to replace the tangential Thomas D-operators in Equation (7.2) by the intrinsic Thomas D-operator twisted by the ambient tractor connection.

7.2. ASC obstruction density

We now derive a holographic formula giving the main structure of the obstruction density; in particular this shows the *rôle* of the extrinsic conformal Laplacians derived above and facilitates its computation (see [23] where the holographic formula is applied to hypersurfaces embedded in four-manifolds).

Theorem 7.7. *Let $\bar{\sigma}$ be a unit conformal defining density. Then, the ASC obstruction density $\mathcal{B}$ is given by the holographic formula*

$$(7.4)\quad \mathcal{B} = \frac{2}{d!(d-1)!}\,\bar{D}_A\Big[\Sigma^A_B\Big(\mathsf{P}_{d-1}N^B + (-1)^{d-2}\big[\bar{I}\cdot D^{d-2}(X^B K_{\rm ext})\big]\Big)\Big|_\Sigma\Big],$$

where $K_{\rm ext} := P_{AB}P^{AB}$ and $P^{AB} := \widehat{D}^A\bar{I}^B$, and N^B is any extension of the normal tractor off Σ. Also $\Sigma^A_B := \delta^A_B - N^A N_B$ denotes the projector mapping $\Gamma(\mathcal{T}M|_\Sigma) \to \Gamma(N^\perp)$.

Proof. First we use the modified Leibniz rule (3.19) to compute the following identity

$$I_\sigma\cdot DI^A_\sigma = \frac{1}{2}D^A I^2_\sigma + X^A K_{\rm ext},$$

which is valid for any defining density σ. Next we deduce from Lemma 3.11, that

$$D^A(\bar{\sigma}^d B) = -d(d-1)\bar{\sigma}^{d-2}X^A B + \mathcal{O}(\bar{\sigma}^{d-1}),$$

where the the unit conformal defining density property (4.5) has been used to replace $\bar{I}^2$ (where $\bar{I} := I_{\bar{\sigma}}$) by unity on the right hand side. Hence, along Σ we have

$$(7.5)\qquad \big(\bar{I}\cdot D^{d-2}\circ \bar{I}\cdot D\big)\,\bar{I}^A = -\frac{d(d-1)}{2}\big[\bar{I}\cdot D^{d-2}, \bar{\sigma}^{d-2}\big]X^A B + \bar{I}\cdot D^{d-2}[X^A K_{\rm ext}].$$

Here we used $\bar{I}\cdot D^{d-2}\circ\bar{\sigma}^{d-1} \stackrel{\Sigma}{=} 0$ by virtue of the algebra (3.17).

Along Σ, the above commutator in (7.5) can be replaced by

$$(-1)^{d-1}[x^{d-2}, y^{d-2}]$$

where the operators $x = \sigma$ (viewed as a multiplicative operator) and $y = -\frac{1}{I^2}I\cdot D$, introduced in Section 3.5, obey the standard $\mathfrak{sl}(2)$ algebra $[x,y] = h$, for any defining density σ, and hence in particular $\bar{\sigma}$. A simple inductive argument shows that the relation

$$[x^k, y^k] = (-1)^{k+1}k!\,h(h+1)(h+2)\cdots(h+k-1) + x\,F$$

holds in the $\mathfrak{sl}(2)$ enveloping algebra for some polynomial of F in the generators $\{x,h,y\}$. Here h is the linear operator that acts on weight w tractors by multiplying them by $d+2w$. Since $X^A B$ has weight $1-d$ we have

$hX^AB = (2-d)X^AB$. Hence, along Σ, we have

$$[\bar{I}\cdot D^{d-2}, \bar{\sigma}^{d-2}]X^AB = (d-2)!(2-d)(3-d)\cdots(-1)X^AB$$
$$= (-1)^{d-2}[(d-2)!]^2X^AB.$$

Finally, we apply the identity (3.8) (specialized to tractors along Σ) to the quantity $\bar{D}_A\big[(X^AB)|_\Sigma\big] = \bar{D}_A\big[\Sigma^A_B(X^BB)|_\Sigma\big] = \bar{D}_A(\bar{X}^AB_\Sigma)$. Elementary bookkeeping then gives the quoted result. □

Remark 7.8. From Equation (4.11), we see that the obstruction density vanishes for totally umbilic surface embeddings. The same applies for hypersurfaces embedded in Euclidean 4-space (see Equation (4.12)). Hence, it is interesting to ask, more generally, whether the obstruction density vanishes for totally umbilic hypersurfaces. However in [23], the obstruction density for general hypersurface embeddings in four-manifolds has been computed using both the above proposition and a variational method. Further it was proved there that for totally umbilic hypersurface embeddings, the obstruction is in general non-vanishing.

Appendix A. A technical lemma

The proof of Proposition 3.9 relies on the following Lemma.

Lemma A.1. *Let* $T_i \in \Gamma(\mathcal{T}^{\Phi_i}M)[w_i]$ *for* $i=1,2$ *and* $h_i := d+2w_i-2$, $h_{12} := d+2w_1+2w_2-2$. *Then*

$$-2X^A(D_BT_1)\,(D^BT_2) = h_1h_2\,D^A(T_1T_2) - h_{12}\big(h_2\,(D^AT_1)T_2 + h_1T_1(D^AT_2)\big).$$

Proof. Proving this result by verifying it in an explicit scale $g \in \boldsymbol{c}$ is extremely tedious. Instead we employ the Fefferman–Graham ambient metric construction of the standard tractor bundle [12, 31]. Our notations are those of [33, Section 6]. In particular, the Thomas D-operator is a restriction of the following operator on sections of the ambient tensor bundle:

$$\boldsymbol{D}_A = \boldsymbol{\nabla}_A(d+2\boldsymbol{\nabla}_{\boldsymbol{X}}-2) + \boldsymbol{X}_A\,\boldsymbol{\Delta}.$$

Acting on a product of ambient tensors $\tilde{T}_1\tilde{T}_2$ of homogeneities w_1 and w_2 (so that $(\boldsymbol{\nabla}_{\boldsymbol{X}}-w_1)\,\tilde{T}_1 = 0 = (\boldsymbol{\nabla}_{\boldsymbol{X}}-w_2)\,\tilde{T}_2$) we have

$$\boldsymbol{D}_A\big(\tilde{T}_1\tilde{T}_2\big) = (d+2w_1+2w_2-2)\big((\boldsymbol{\nabla}_A\tilde{T}_1)\,\tilde{T}_2 + \tilde{T}_1\,(\boldsymbol{\nabla}_A\tilde{T}_2)\big)$$
$$- \boldsymbol{X}_A\big((\boldsymbol{\Delta}\tilde{T}_1)\,\tilde{T}_2 + 2\,(\boldsymbol{\nabla}_B\tilde{T}_1)(\boldsymbol{\nabla}^B\tilde{T}_2) + \tilde{T}_1\,(\boldsymbol{\Delta}\tilde{T}_2)\big),$$

so that

$$
\begin{aligned}
&(d+2w_1-2)(d+2w_2-2)\boldsymbol{D}_A\big(\tilde T_1\tilde T_2\big)\\
&\quad=(d+2w_1+2w_2-2)\\
&\qquad\times\big((d+2w_2-2)\,(\boldsymbol{D}_A\tilde T_1)\,\tilde T_2+(d+2w_1-2)\,\tilde T_1\,(\boldsymbol{D}_A\tilde T_2)\big)\\
&\qquad+(d+2w_1+2w_2-2)\,\boldsymbol{X}_A\\
&\qquad\times\big((d+2w_2-2)\,(\boldsymbol{\Delta}\tilde T_1)\,\tilde T_2+(d+2w_1-2)\,\tilde T_1\,(\boldsymbol{\Delta}\tilde T_2)\big)\\
&\qquad-(d+2w_1-2)(d+2w_2-2)\\
&\qquad\times\boldsymbol{X}_A\big((\boldsymbol{\Delta}\tilde T_1)\,\tilde T_2+2\,(\boldsymbol{\nabla}_B\tilde T_1)(\boldsymbol{\nabla}^B\tilde T_2)+\tilde T_1\,(\boldsymbol{\Delta}\tilde T_2)\big)\\
&\quad=(d+2w_1+2w_2-2)\\
&\qquad\times\big((d+2w_2-2)\,(\boldsymbol{D}_A\tilde T_1)\,\tilde T_2+(d+2w_1-2)\,\tilde T_1\,(\boldsymbol{D}_A\tilde T_2)\big)\\
&\qquad-2\boldsymbol{X}_A\,(\boldsymbol{D}_B\tilde T_1)(\boldsymbol{D}^B\tilde T_2)+\mathcal{O}(\boldsymbol{X}^2).
\end{aligned}
$$

□

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