Regular ambitoric 4-manifolds: from Riemannian Kerr to a complete classification

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We show that the conformal structure for the Riemannian analogues of Kerr black-hole metrics can be given an ambitoric structure. We then discuss the properties of the moment maps. In particular, we observe that the moment map image is not locally convex near the singularity corresponding to the ring singularity in the interior of the black hole. We then proceed to classify regular ambitoric 4-orbifolds with some completeness assumptions. The tools developed also allow us to prove a partial classification of compact Riemannian 4-manifolds which admit a Killing 2-form.

1. Introduction

The motivating examples for this paper are the Kerr metrics [Ker63], which have Lorentzian signature and are used to model isolated rotating black

holes. In a process called Wick rotation, the Kerr metric can be transformed to have Riemannian signature by allowing time and angular momentum to take imaginary values [GH79]. One of the interesting properties of the Kerr metrics is that their Weyl curvature is algebraically special of type D. The Riemannian analogue of this type D condition is that the self-dual and anti-self-dual Weyl curvature tensors $W_\pm$ both have a unique 1-dimensional eigenspace. By the Riemannian analogue of the Goldberg-Sachs theorem [Apo98], when the metric is additionally Einstein, this condition is equivalent to the local existence of two integrable complex structures, $J_\pm$, which induce opposite orientations, are compatible with the metric, and are unique up to sign. We say that (g, J_+, J_-) is an *ambihermitian* structure. If an ambihermitian manifold (or more generally orbifold) (M, g, J_+, J_-) admits the action of a 2-torus $\mathbb{T}$ and metrics $g_\pm$ in the conformal class $[g]$ such that $(M, g_\pm, J_\pm, \mathbb{T})$ are both Kähler toric orbifolds, then $(M, [g], J_+, J_-, \mathbb{T})$ has the structure of an *ambitoric manifold* as introduced in [ACG13a]. We show that the Riemannian Kerr metrics admit a regular ambitoric structure in the sense of Definition 1.9.

Another feature of the Kerr metrics is the ring singularity, which is a circle in the interior of the black hole at each time slice where the curvature blows up to infinity. After Wick rotating, the ring singularity forms two components which act differently for each of the Kähler metrics. One of the components forms a point singularity, while the other one is infinitely distant, meaning that it's not part of the Cauchy completion. From the perspective of one of the Kähler forms, the component which the corresponding Kähler metric treats as a point singularity is a hypersurface where the form becomes degenerate. In the language introduced in [CDSGP11], the Kähler form extends as a folded symplectic form to this hypersurface, which becomes a folding hypersurface.

One of the tools for studying toric manifolds is the moment map, which in the case of compact toric manifolds identifies the space of torus orbits with a convex polytope in the dual Lie algebra of the torus [AB84]. However, in the case of the Kerr ambitoric structure, we find that the moment map images are not convex (even locally) near the folding hypersurface. This result can be understood in the context of Example 3.11 in [CDSGP11], which shows that there is no reason for the moment map to be locally convex near a folding hypersurface.

Motivated by the role that these folding hypersurfaces play in the Kerr ambitoric structure, we investigate their role in the more general setting of regular ambitoric orbifolds. Folding hypersurfaces occur as one of essentially two types of boundary that occur when taking completions. We call this type

of boundary a *fold*. We call the other type *edges*, because they occur as the boundary edges of the moment polygon for a compact toric surface. Our study of the geometry near these boundaries leads us to a classification of complete regular ambitoric orbifolds. The rest of the introduction is dedicated to presenting a precise statement of this result.

We find it convenient to work with slight generalizations of regular ambitoric orbifolds which we call regular ambitoric orbifold completions, for which the full ambitoric structure need only be defined on a dense open subset. More precisely, we call a Riemannian manifold (M, g) *completable* if its Cauchy completion M_C^g is an orbifold. A *regular ambitoric orbifold completion* is defined to be the Cauchy completion of a completable regular ambitoric manifold equipped with a free torus action. Thus every complete regular ambitoric orbifold $(M, [g], J_\pm, \mathbb{T})$ is a regular ambitoric orbifold completion, since the set of its free torus orbits, denoted by $\mathring{M}$, is an open dense submanifold.

Regular ambitoric 4-orbifolds have been locally classified on the set of orbits where the torus action is free [ACG13a]. We use this local classification to generate a set of examples of regular ambitoric 4-manifolds with free torus action, which we call *ambitoric ansatz spaces*. These examples include the free orbits of the Riemannian Kerr examples. These ambitoric ansatz spaces depend on a symmetric quadratic polynomial $q(x, y)$ and two functions of one variable $A(x)$ and $B(y)$, with the manifold given by

$$\mathcal{A}(q, A, B, \mathbb{T})$$
$$:= \left\{ (x, y, \vec{t}) \in \mathbb{R}^2 \times \mathbb{T} : A(x) > 0, B(y) > 0, q(x, y)(x - y) \neq 0 \right\},$$

equipped with the ambitoric structure given in Theorem 1.11.

Note that an ambitoric structure gives a conformal class of metrics. We will want to study metric properties, so we need to make a choice of metric in this conformal class. We pick out some distinguished metrics to study, which are unique up to homothety: the Kähler metrics $g_\pm$, and the barycentric metric g_0, which is an average of g_+ and g_-. We also study compatible metrics whose Ricci tensor is diagonal, meaning that it is invariant under both J_+ and J_-. These include the Einstein metrics, and they come in a family $\{g_p\}_{p \perp q}$, indexed by quadratic polynomials p orthogonal to the quadratic polynomial q with respect to the inner product induced by the discriminant.

The first step in our classification is to classify ambitoric ansatz spaces which are completable with respect to our chosen metric $g \in \{g_0, g_\pm\} \cup$

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$\{g_p\}_{p\perp q}$. The strategy taken is to first compare the g-Cauchy completion to the Cauchy completion with respect to a more convenient metric where the Cauchy completion is easy to compute. This comparison is done using the tools of Busemann completions, the details of which are provided in an appendix. We then decompose the convenient boundary into components which are folds, edges, or *corners* where the closures of multiple components meet. When studying g_p, we also have to study the vanishing locus of $p(x, y)$, which we denote by P. We further define a component of the convenient boundary to be *infinitely distant* if it does not lie in the g-Cauchy completion, and a fold to be *proper* if it is not an edge or a corner. We distinguish between folds being *positive/negative* in such a way that proper positive/negative folds are folding hypersurfaces for suitable extensions of the positive/negative Kähler forms. By studying the g-Cauchy completions, we get the following classification result:

Theorem 1.1. *Let $\mathcal{A}$ be an ambitoric ansatz space and $g \in \{g_\pm, g_0\} \cup \{g_p\}_{p\perp q}$. Then $\mathcal{A}$ is g-completable if and only if the following conditions hold on $\mathcal{A}_C^g$:*

- *There are no proper folds.*

- *Every edge is either infinitely distant or has a compatible normal.*

- *If an edge is a fold but not infinitely distant, then $g \in \{g_-, g_p\}$.*

- *Every corner is infinitely distant unless it is at the intersection of two edges which are not infinitely distant. If such a corner is a positive (respectively negative) fold, then $g \in \{g_+, g_p\}$ (respectively $g \in \{g_-, g_p\}$). If $g = g_p$, then such a corner is not part of P.*

Moreover, if $\mathcal{A}$ is g-completable, then its ambitoric structure extends to $\mathcal{A}_C^g$ if and only if every fold is infinitely distant.

To classify regular ambitoric orbifold completions, we first extend the local classification on the set of free torus orbits to a local classification on the Cauchy completions. This allows us to describe the boundary of the set of free orbits in terms of folds/edges/corners as was done in the case of ambitoric ansatz spaces. We find that in the case where there are no proper folds, we can apply a slight modification of the Lokal-global-Prinzip for convexity theorems [HNP94], which is a tool that can be used to prove that the moment map of a convex toric orbifold is a convex polytope. This gives us the following partial classification of regular ambitoric orbifold completions:

Theorem 1.2. *Let $(M, \mathring{M}, g, J_+, J_-, \mathbb{T})$ be a connected regular ambitoric orbifold completion, where $g \in \{g_0, g_\pm\} \cup \{g_p\}_{p \perp q}$. Then there exists an ambitoric embedding of $(\mathring{M}, [g], J_+, J_-, \mathbb{T})$ into some ambitoric ansatz space which is completable with respect to the metric induced by g and whose moment map images are affine convex polygons.*

We combine this classification with our classification of completable ambitoric ansatz spaces to obtain an explicit classification of g_0-complete regular ambitoric 4-orbifolds:

Corollary 1.3. *A regular ambitoric 4-orbifold which is complete with respect to the barycentric metric g_0 is given (uniquely up to gauge transformation) by the data of a symmetric quadratic polynomial $q(x, y)$, a pair of intervals $(x_-, x_+), (y_-, y_+) \subset \mathbb{R}$ such that $(x - y)q(x, y)$ does not vanish on $(x_-, x_+) \times (y_-, y_+)$, a lattice $\Lambda \subset \mathbb{R}^2$ and two smooth positive functions $A : (x_-, x_+) \to [0, \infty)$ and $B : (y_-, y_+) \to [0, \infty)$ satisfying for all $\epsilon > 0$ small enough:*

- *If $\int_{x_\pm}^{x_\pm \mp \epsilon} \frac{dx}{\sqrt{A(x)}}$ converges, then $-2\frac{p^{(x\pm)}}{A'(x_\pm)} \in \Lambda$,*

- *If $\int_{y_\pm}^{y_\pm \mp \epsilon} \frac{dy}{\sqrt{B(y)}}$ converges, then $-2\frac{p^{(y\pm)}}{B'(y_\pm)} \in \Lambda$,*

- *For each $\alpha, \beta \in \{\pm\}$, if the integrals from the previous conditions corresponding to x_α and y_β are both convergent, then $(x_\alpha - y_\beta)q(x_\alpha, y_\beta) \neq 0$,*

where $p^{(\gamma)} \in \mathfrak{t}$ is a convenient choice of normal to the moment map image of the level sets $\{x = \gamma\}$ and $\{y = \gamma\}$. See Definition 4.24. Note that similar results can be obtained for $g_\pm$, but the third condition which tests the corners would be more complicated.

As an application for our classification results, we consider compact Riemannian 4-manifolds which admit $*$-Killing 2-forms. These are Riemannian signature analogues of the Killing-Yano tensor in Lorentzian signature, which describes the so-called hidden symmetries of the Kerr metric. We build on the work of [GM17], who divide Riemannian 4-manifolds admitting non-parallel $*$-Killing 2-forms into 3 types. They show that one of these types is consists of regular ambitoric orbifold completions, identifying the metric with what we've been calling g_p with $p = 1$. We classify compact 4-manifolds of this type:

Theorem 1.4. *Let (M, g) be a compact connected oriented regular ambitoric orbifold completion which is a manifold admitting a non-parallel $*$-Killing 2-form. Then M is diffeomorphic to either $\mathbb{S}^4, \mathbb{CP}^2$, or a Hirzebruch surface. Conversely, each of these manifolds admit a metric with a non-parallel $*$-Killing 2-form.*

1.1. Ambitoric structures and ambitoric ansatz spaces

In this section, we introduce the basic definitions of ambitoric manifolds, as well as the local classification in four dimensions from [ACG13a].

Definition 1.5. A *Hermitian* manifold (M, g, J) is a manifold M equipped with a metric g and a compatible complex structure J. ie. $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. An *ambihermitian* manifold (M, g, J_+, J_-) is a manifold M equipped with a metric g and complex structures $J_\pm$ such that $(M, g, J_\pm)$ are both Hermitian, and the orientations induced by $J_\pm$ are opposite. If the conformal class $[g]$ of an ambihermitian manifold $(M, g, J_+, J-)$ admits metrics $g_\pm \in [g]$ such that $(M, g_\pm, J_\pm)$ are both Kähler manifolds, then $(M, [g], J_+, J_-)$ is *ambikähler*. If there is a 2-torus $\mathbb{T}$ which acts on an ambikähler manifold $(M, [g], J_+, J_-)$ by Hamiltonian isometries (with respect to both Kähler metrics), then $(M, [g], J_+, J_-, \mathbb{T})$ is an *ambitoric* manifold. Note that all of these definitions extend naturally to the case of orbifolds.

We denote the Lie algebra of $\mathbb{T}$ by $\mathfrak{t}$. The kernel of the exponential map on $\mathfrak{t}$ is the lattice of circle subgroup $\Lambda \subset \mathfrak{t}$, so that $\mathbb{T} \cong \mathfrak{t}/\Lambda$.

For an ambitoric manifold, $(M, [g], J_+, J_-, \mathbb{T})$, we denote by $\omega_\pm := g_\pm(J_\pm \cdot, \cdot)$ the Kähler forms of $(M, g_\pm, J_\pm)$. Since $g_\pm$ are in the same conformal class, there exists a positive function f such that $g_- = f^2 g_+$.

Definition 1.6. The *barycentric metric* on an ambiKähler manifold is given by $g_0 := fg_+ = f^{-1}g_-$.

Note that the metrics $g_\pm$ are uniquely chosen within their conformal class up to homothety, so that f is well-defined up to a multiplicative constant, and g_0 is well-defined up to homothety.

Since each $K \in \mathfrak{t}$ is Hamiltonian with respect to $\omega_\pm$, there exist functions $f_K^\pm \in \mathcal{C}^\infty(M)$ such that $K \lrcorner \omega_\pm = -df_K^\pm$. The map $K \mapsto f_K^\pm$ is linear, so it gives an element of $\mathfrak{t}^* \otimes \mathcal{C}^\infty(M)$. This gives a smooth map $\mu^\pm : M \to \mathfrak{t}^*$, which is the *moment map* for the toric manifold $(M, \omega_\pm, \mathbb{T})$.

Ambitoric 4-manifolds come in three families [ACG13a], which we will describe in the rest of the section.

Example 1.7. Let (Σ_1, g_1, J_1) and (Σ_2, g_2, J_2) be two (Kähler) Riemann surfaces with non-vanishing Hamiltonian Killing vector fields K_1 and K_2 respectively. Then $\Sigma_1 \times \Sigma_2$ can be given ambitoric structure

$$g := g_1 \oplus g_2,$$
$$J_\pm := J_1 \oplus \pm J_2,$$
$$\mathfrak{t} := \mathrm{Span}\{K_1, K_2\}.$$

Example 1.8. Let (M, g, J) be a Kähler surface. For any non-vanishing hamiltonian Killing vector field K, we can define an almost complex structure J_- by

$$J_- := \begin{cases} J & \text{on } \mathrm{Span}\{K, JK\} \\ -J & \text{on } \mathrm{Span}\{K, JK\}^\perp \end{cases}$$

J_- has opposite orientation to J. If (M, g, J) is conformally Kähler, then $(M, [g], J_+ := J, J_-)$ is ambikähler. Such an ambikähler manifold is said to be of *Calabi type*. In [ACG03], it is shown that the Kähler quotient of (M, g, J) using the moment map z of K is a Riemann surface $(\Sigma, (az - b)g_\Sigma, J_\Sigma)$, where a, b are constants, g_Σ and J_Σ are a metric and complex structure respectively on Σ. An ambikähler surface of Calabi type $(M, [g], J, J_-)$ is ambitoric if and only if $(\Sigma, (az - b)g_\Sigma, J_\Sigma)$ admits a Hamiltonian Killing vector field.

For an ambitoric 4-manifold $(M, [g], J_+, J_-, \mathbb{T})$, we define $\mathfrak{t}_M$ to be the subset TM spanned by the Killing vector fields $\mathfrak{t}$. On an open dense subset $\mathring{M} \subseteq M$, $\mathfrak{t}_{\mathring{M}} := \mathfrak{t}_M|_{\mathring{M}}$ is a two dimensional distribution in $T\mathring{M}$. Since the Killing vector fields $\mathfrak{t}$ are $\omega_\pm$-Hamiltonian, $\mathfrak{t}_{\mathring{M}}$ is $\omega_\pm$-Lagrangian, so that $J_+ \mathfrak{t}_{\mathring{M}} = J_- \mathfrak{t}_{\mathring{M}} = \mathfrak{t}_{\mathring{M}}^\perp$.

Since J_+ and J_- have opposite orientations, they commute. This implies that the endomorphism $-J_+ J_-$ of TM is an involution. Thus TM decomposes into ± 1 eigenbundles of $-J_+ J_-$. Let $\xi_{\mathring{M}}$ and $\eta_{\mathring{M}}$ be the intersections of $\mathfrak{t}_{\mathring{M}}$ with the $+1$ and -1 (respectively) eigenbundles of $-J_+ J_-$. Since the eigenbundles are $J_\pm$ invariant, while $\mathfrak{t}_{\mathring{M}}$ is not, $\xi_{\mathring{M}}$ and $\eta_{\mathring{M}}$ must be line bundles.

Let $\boldsymbol{K} : \mathfrak{t} \to \Gamma(\mathfrak{t}_M)$ be the function which maps a vector $X \in \mathfrak{t}$ to the associated Killing vector field on M. Let $\boldsymbol{\theta} \in \Omega^1(M, \mathfrak{t})$ be the $\mathfrak{t}$-valued one-form defined by the relations $\boldsymbol{\theta} \circ \boldsymbol{K} = \mathrm{Id}_{\mathfrak{t}}$ and $\boldsymbol{\theta}|_{\mathfrak{t}_{\mathring{M}}^\perp} = 0$. Let

$$\xi : \mathring{M} \to \mathbb{P}\mathfrak{t} : p \mapsto \boldsymbol{\theta}((\xi_{\mathring{M}})_p),$$
$$\eta : \mathring{M} \to \mathbb{P}\mathfrak{t} : p \mapsto \boldsymbol{\theta}((\eta_{\mathring{M}})_p).$$

If ξ or η is constant, then its image is the span of some Killing vector field K which realizes the ambitoric structure as being Calabi type. If neither is constant, then $d\xi \wedge d\eta$ is non-vanishing [ACG13a].

Definition 1.9. An ambitoric 4-manifold $(M, [g], J_+, J_-, \mathbb{T})$ is *regular* if $d\xi \wedge d\eta$ is non-vanishing on an open dense set.

It is clear that every ambitoric structure is either regular or of Calabi type. We will find it convenient to fix some notation for quadratic polynomials:

Definition 1.10. Let $p(z) = p_0 \, z^2 + 2p_1 \, z + p_2$ be a quadratic polynomial. We call $p(x, y) = p_0 \, xy + p_1 \, (x + y) + p_2$ the *polarization* of $p(z)$. We define a $(2, 1)$-signature inner product on the space of quadratic polynomials by

$$\langle q, p \rangle := 2q_1 p_1 - q_2 p_0 - q_0 p_2.$$

We will use extensively this local classification of regular ambitoric surfaces:

Theorem 1.11 (Theorem 3 from [ACG13a]). *Let $(M, [g_0], J_+, J_-, \mathbb{T})$ be a regular ambitoric 4-manifold with barycentric metric g_0 and Kähler metrics (g_+, ω_+) and (g_-, ω_-). Then, about any point in an open dense subset of M, there are $\mathfrak{t}$-invariant functions x, y, a quadratic polynomial $q(z)$, a $\Lambda^1(M)$-valued quadratic polynomial $d\tau(z)$ orthogonal to $q(z)$, and functions $A(z)$ and $B(z)$ of one variable with respect to which:*

$$g_0 = \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x) \left(\frac{d\tau(y)}{(x - y)q(x, y)} \right)^2 + B(y) \left(\frac{d\tau(x)}{(x - y)q(x, y)} \right)^2,$$

(1)

$$\omega_+ = \frac{dx \wedge d\tau(y) + dy \wedge d\tau(x)}{q(x, y)^2}, \qquad \omega_- = \frac{dx \wedge d\tau(y) - dy \wedge d\tau(x)}{(x - y)^2}.$$

Conversely, for any data as above, the above metric and Kähler forms do define an ambitoric Kähler structure on any simply connected open set where $\omega_\pm$ are nondegenerate and g_0 is positive definite.

The open dense subset of M where this theorem applies is the maximal open set $\mathring{M}$ where the Killing vector fields $\mathfrak{t}$ have maximal rank. The following theorem describes how various curvature conditions relate to the ansatz parameters:

Theorem 1.12 ([ACG13a]). *Let* $([g], J_\pm, \mathfrak{t})$ *be a regular ambitoric struc-* *ture. Then for any quadratic polynomial* $p(z)$ *orthogonal to* $q(z)$, *the metric* $\frac{(x-y)q(x,y)}{p(x,y)^2} g_0$ *has diagonal Ricci tensor. Any* $\mathfrak{t}$-*invariant metric in* $[g]$ *with diagonal Ricci tensor arises in this way. Such a metric has constant scalar curvature if and only if*

$$A(z) = p(z)\rho(z) + R(z),$$
$$B(z) = p(z)\rho(z) - R(z),$$

where $\rho(z)$ *is a quadratic polynomial orthogonal to* $p(z)$ *and* $R(z)$ *is a quar-* *tic polynomial orthogonal to* $q(z)p(z)$ *(equivalently* $(q, R)^{(2)} \perp p$ *or, equally,* $(p, R)^{(2)} \perp q$). *The metric is Einstein when* $\rho(z)$ *is a multiple of* $q(z)$. *Here the transvectant* $(p, R)^{(2)}$ *is the quadratic polynomial defined by*

$$(p, R)^{(2)} := p(z)R''(z) - 3p'(z)R'(z) + 6p''(z)R(z).$$

Theorem 1.11 motivates us to define a family of examples which we will call *ambitoric ansatz spaces*:

$$\mathcal{A}(q, A, B, \mathbb{T})$$
$$:= \left\{ (x, y, \vec{t}) \in \mathbb{R}^2 \times \mathbb{T} : A(x) > 0, B(y) > 0, q(x,y)(x-y) \neq 0 \right\},$$

which we equip with the ambitoric structure given by (1) while identifying $\mathfrak{t}$ with the infinitesimal vector fields of the action of $\mathbb{T}$.

Definition 1.13. A map between ambitoric oribifolds is *ambitoric* if it preserves all of the ambitoric structure. In particular, it is equivariant under the torus action, holomorphic with respect to both complex structures, and preserves both the conformal structure and the orientation.

We will use these to rephrase the above theorem in the case that the toric structure comes from a Lie group.

Corollary 1.14. *Let* $(M, [g_0], J_+, J_-, \mathbb{T})$ *be a regular ambitoric 4-manifold freely acted on by a 2-torus* $\mathbb{T}$. *Then for any point* $p \in M$, *there exists a quadratic polynomial* $q(z)$, *functions* $A(z)$ *and* $B(z)$, *a* $\mathbb{T}$-*invariant neigh-* *bourhood* U *of* p, *and an ambitoric embedding* $\phi : U \hookrightarrow \mathcal{A}(q, A, B, \mathbb{T})$.

Proof. Since the action of $\mathbb{T}$ is free, $\mathfrak{t}$ has maximal rank at each point in M. Thus we can apply the previous theorem to find local coordinates on

a neighbourhood U' of p which is naturally identified as a subset of some $\mathcal{A}(q, A, B, \mathbb{T})$. Since the ambikähler structure of M is $\mathbb{T}$-equivariant, these coordinates can naturally be extended to the orbit $U := \mathbb{T} \cdot U'$. □

1.2. Normal forms

The results in this section are taken directly from [ACG13a]. The classification from Theorem 1.11 can be further refined into 3 cases, called *parabolic*, *hyperbolic* and *elliptic* respectively if the discriminant of $q(z)$ is zero, negative, or positive respectively. Up to homothety, we can assume that the discriminant is 0 or ± 1, which leads to the normal forms described below.

1.2.1. Parabolic type. The parabolic type is characterized by $q(z) = 1$ and $d\tau_0 = 0$. This allows us to write the ambitoric structure as

$$g_0 = \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + \frac{A(x)(dt_1 + y\, dt_2)^2}{(x - y)^2} + \frac{B(y)(dt_1 + x\, dt_2)^2}{(x - y)^2},$$
$$\omega_+ = dx \wedge (dt_1 + y\, dt_2) + dy \wedge (dt_1 + x\, dt_2),$$
$$\omega_- = \frac{dx \wedge (dt_1 + y\, dt_2)}{(x - y)^2} - \frac{dy \wedge (dt_1 + x\, dt_2)}{(x - y)^2}.$$

The moment maps $\mu^\pm = (\mu_1^\pm, \mu_2^\pm) : \mathcal{A} \to \mathfrak{t}^*$ are given by

$$\mu_1^+ = x + y,$$
$$\mu_2^+ = xy,$$
$$\mu_1^- = -\frac{1}{x - y},$$
$$\mu_2^- = -\frac{x + y}{2(x - y)}.$$

1.2.2. Hyperbolic type. The hyperbolic type is characterized by $q(z) = 2z$ and $d\tau_1 = 0$. This allows us to write the ambitoric structure as

$$g_0 = \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + \frac{A(x)(dt_1 + y^2\, dt_2)^2}{(x^2 - y^2)^2} + \frac{B(y)(dt_1 + x^2\, dt_2)^2}{(x^2 - y^2)^2},$$
$$\omega_\pm = \frac{dx \wedge (dt_1 + y^2\, dt_2)}{(x \pm y)^2} \pm \frac{dy \wedge (dt_1 + x^2\, dt_2)}{(x \pm y)^2}.$$

The moment map $\mu^\pm = (\mu_1^\pm, \mu_2^\pm) : \mathcal{A} \to \mathfrak{t}^*$ is given by

$$\mu_1^\pm = -\frac{1}{x \pm y},$$
$$\mu_2^\pm = \pm\frac{xy}{x \pm y}.$$

1.2.3. Elliptic type. The elliptic type is characterized by $q(z) = 1 + z^2$ and $d\tau_0 + d\tau_2 = 0$. This allows us to write the ambitoric structure as

$$g_0 = \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + \frac{A(x)(2y\,dt_1 + (y^2 - 1)\,dt_2)^2}{(x-y)^2(1+xy)^2}$$
$$+ \frac{B(y)(2x\,dt_1 + (x^2 - 1)\,dt_2)^2}{(x-y)^2(1+xy)^2},$$
$$\omega_+ = \frac{dx \wedge (2y\,dt_1 + (y^2 - 1)\,dt_2)}{(1+xy)^2} + \frac{dy \wedge (2x\,dt_1 + (x^2 - 1)\,dt_2)}{(1+xy)^2},$$
$$\omega_- = \frac{dx \wedge (2y\,dt_1 + (y^2 - 1)\,dt_2)}{(x-y)^2} - \frac{dy \wedge (2x\,dt_1 + (x^2 - 1)\,dt_2)}{(x-y)^2}.$$

The moment maps $\mu^\pm = (\mu_1^\pm, \mu_2^\pm) : \mathcal{A} \to \mathfrak{t}^*$ are given by

$$\mu_1^+ = -\frac{1 - xy}{1 + xy},$$
$$\mu_2^+ = -\frac{x + y}{1 + xy},$$
$$\mu_1^- = -\frac{x + y}{x - y},$$
$$\mu_2^- = \frac{1 - xy}{x - y}.$$

2. Moment maps and folded symplectic structures

In this section, we will extend the symplectic structures on ambtoric ansatz spaces to folded symplectic manifolds, and describe their moment map images.

Definition 2.1. [CDSGP11] A closed 2-form ω on a $2n$-dimensional manifold M is a *folded symplectic structure* if there exists an embedded hypersurface Z of M such that ω is non-degenerate on $M \backslash Z$ and $\omega^n = 0 \neq \omega^{n-1}$ on Z. Z is called the *folding hypersurface* or *fold*.

Let

$$\mathcal{F}(q, A, B, \mathbb{T}) := \{(x, y, t_1, t_2) \in \mathbb{R}^2 \times \mathbb{T}^2 : A(x) > 0, B(y) > 0\}$$
$$\supseteq \mathcal{A}(q, A, B, \mathbb{T}).$$

To simplify notation the arguments for $\mathcal{F}$ will be tacit. Let $Z_\pm := \{f^{\mp 1} = 0\} \subseteq \mathcal{F}$ and $\mathcal{F}_\pm = \mathcal{F} \backslash Z_\mp$.

Proposition 2.2. $(\mathcal{F}_\pm, \omega_\pm)$ *is a folded symplectic manifold with fold* $Z_\pm|_{\mathcal{F}_\pm}$, *where* $\omega_\pm$ *is the 2-form defined in* (1).

Proof. By (23) of [ACG13a], $f = \frac{q(x,y)}{x-y}$. It is a routine computation to show that $\omega_\pm$ is a closed non-vanishing 2-form on $\mathcal{F}_\pm$. Also one can compute

$$\omega_\pm^2 = \frac{f^{\mp 2}}{A(x)B(y)} dx \wedge d_\pm^c x \wedge dy \wedge d_\pm^c y,$$

so that the vanishing locus of $\omega_\pm^2$ is $Z_\pm$. Noting that $\omega_\pm$ does not vanish along $Z_\pm$ then tells us that $Z_\pm$ is a folding hypersurface for $\omega_\pm$. □

Proposition 2.3. $\mu^\pm(Z_\pm)$ *is a (possibly degenerate) conic in* $\mathfrak{t}^*$, *which we will denote by* $\mathcal{C}_\pm$.

Proof. This can be done by an easy direct computation using the normal forms given in Section 1.2. In the hyperbolic case, one finds that

$$\mu^\pm(Z_\pm) = \{\mu_1^\pm \mu_2^\pm = -\tfrac{1}{4}\}.$$

In the elliptic case, one finds that

$$\mu^\pm(Z_\pm) = \{(\mu_1^\pm)^2 + (\mu_2^\pm)^2 = 1\}.$$

In the parabolic case, one finds that

$$\mu^+(Z_+) = \{(\mu_1^+)^2 = 4\mu_2^+\},$$
$$\mu^-(Z_-) = \{(0, \tfrac{1}{2}), (0, -\tfrac{1}{2})\}.$$

□

Note that the type of the conic matches the type of the ambitoric structure (aside from the degenerate conic $\mu^-(Z_-)$ in the parabolic case), so that for example a hyperbolic ambitoric structure gives a hyperbola as the image of the fold.
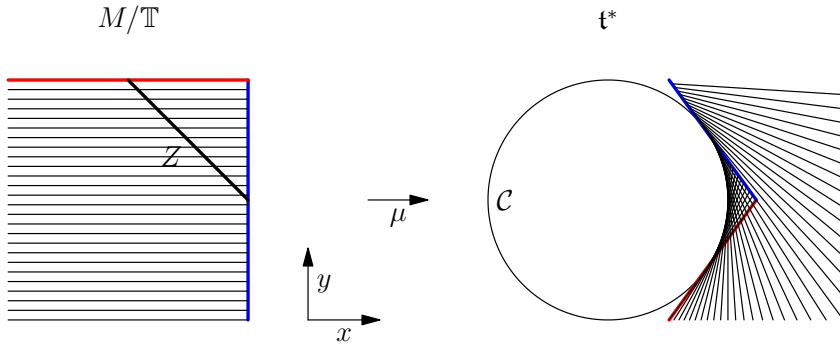
Figure 1: The moment map near a fold Z which intersects level sets of x and y where $A(x)$ and $B(y)$ (shown in blue and red respectively).

The conic (or more precisely, its dual) is used in [ACG13b] to study ambitoric compactifications. The key use made in [ACG13b] of the conic is the fact that the moment map sends level sets of x or y to (subsets of) lines tangent to the conic. If the conic is non-degenerate, this leads to moment map images near the fold that look like the example shown in Figure 1. The moment map is a $2 - 1$ cover near the conic, with the image folding along the conic and remaining in the exterior of the conic. This behaviour was noted at the end of appendix A in [ACG13b], so the previous proposition is not essentially new.
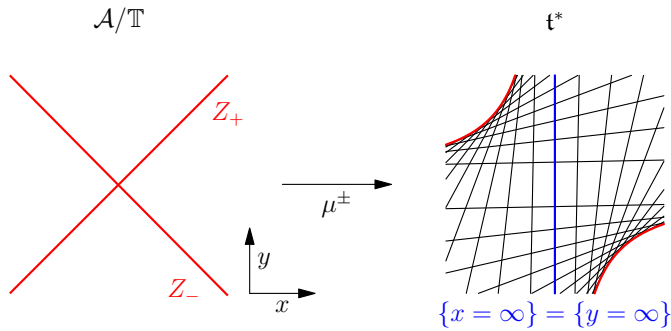

Figure 2: The moment maps of the hyperbolic normal form.

We will need the following lemma later:

Lemma 2.4. *If $p(z)$ is a quadratic polynomial orthogonal to $q(z)$, then the moment maps $\mu^{\pm}$ map the vanishing locus of $p(x,y)$ to either lines in $\mathfrak{t}^*$ or the empty set (in the case when $p(x,y)$ is non-vanishing).*

Proof. $\mathfrak{t}$ is identified with the space of quadratic polynomials orthogonal to q, so $p \in \mathfrak{t}$. The moment maps are computed in [ACG13a], which we use to deduce

$$\mu^-(x,y) \cdot p = -\frac{p(x,y)}{x-y}, \qquad \mu^+(x,y) \cdot \{q,p\} = -\frac{p(x,y)}{q(x,y)},$$

where $\cdot$ is used to denote the natural pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$. Since these vanish along $\{p(x,y) = 0\}$, we find that the vectors $p, \{q,p\} \in \mathfrak{t}$ are normals to the curves $\mu^-(\{p(x,y) = 0\})$ and $\mu^+(\{p(x,y) = 0\})$ respectively. This implies that these curves are lines as claimed. $\square$

Note that in the previous proof, the difference in the formulas for $\mu^\pm$ comes from the fact that for μ^+, $\mathfrak{t}$ is more naturally identified with the space of quadratic polynomials modulo q. The map $\{q, \cdot\}$ is an isomorphism between this identification and the one that we use. See [ACG13b] for details.

We conclude this section with diagrams indicating the behaviour of the moment maps of the three normal forms in the case where $A(x)$ and $B(y)$ are strictly positive functions.
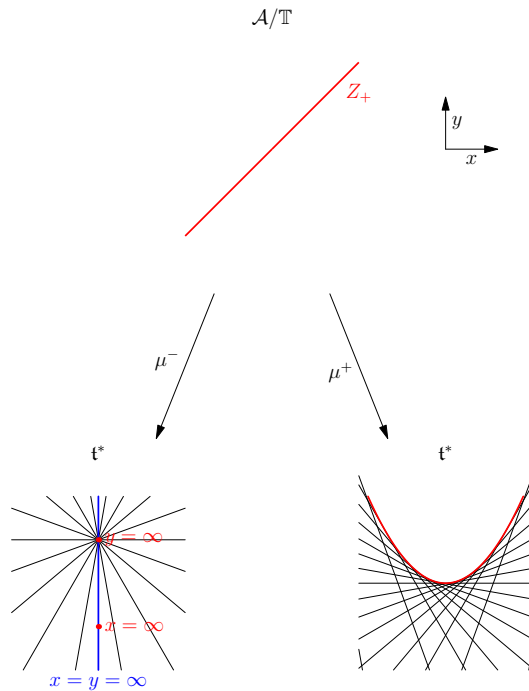

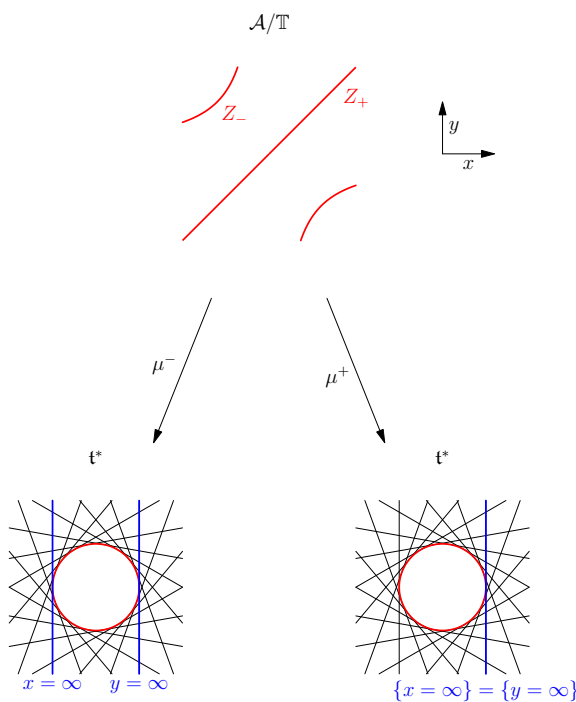Figure 3: The moment maps of the parabolic normal form.

Figure 4: The moment maps of the elliptic normal form.

3. Riemannian Kerr metrics are ambitoric

The Kerr family of metrics are 4-dimensional Lorentzian space-times which are stationary, axis-symmetric, and type D.

The stationary and axis-symmetric condition means that there are commuting Killing vector fields corresponding to the time direction and a rotation around a fixed axis. The span of these vector fields will be our Lie algebra $\mathfrak{t}$.

The type D condition tells us that the Weyl curvature tensor has two repeated principal null directions. In his work on Hermitian geometry in the Lorentzian setting, Flaherty [Fla76] shows that a metric admits compatible integrable almost complex structures $J_\pm$ with both orientations if and only if the metric is of type D. Note that in order for an almost complex structure to be compatible with a Lorentzian metric, it must be complex valued (ie, a section of $\mathrm{End}(TM) \otimes \mathbb{C}$).

Another feature of Kerr that will be useful for us is that it admits a Wick rotation. It is well known in the physics community (see for example
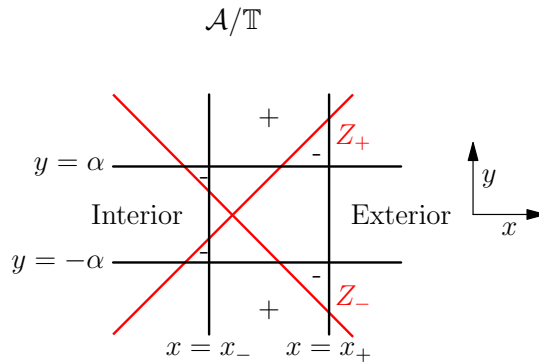
$$\mathcal{A}/\mathbb{T}$$

Figure 5: The domain of $\mathcal{A}/\mathbb{T}$ in (x, y) coordinates. The regions which we call interior and exterior are labelled, and have positive Riemannian signature. The other regions with $\pm$ Riemannian signature are labelled $\pm$. The remaining regions have signature $(2, 2)$.

[GH79]) that if one allows the time and angular momentum variables to take imaginary values instead of real ones, then the metric will still take real values. Moreover, the signature of this Wick-rotated Kerr metric is no longer Lorentzian, but either Riemannian or signature $(2, 2)$ depending on the region. We will only consider the regions which have a Riemannian signature, and call the metric in these regions *Riemannian Kerr*. The complex structures $J_\pm$ become real valued after the Wick rotation, so that the Riemannian Kerr metrics are ambihermitian.

Since Einstein ambihermitian spaces are ambikähler [ACG13a], and $\mathfrak{t}$ is the set of infinitesimal generators of a torus action, we see that the Riemannian Kerr metrics are ambitoric. The Kähler forms are given in [AS06]. They present the more general Kerr-Taub-bolt instanton, where the construction above will also hold, but for simplicity we only present the Kerr case. The more general case will correspond to adding a parameter that translates the y variable.

After a change of variables from this form, one finds that the Riemannian Kerr fits into the normal form for a hyperbolic type ambitoric manifold, with

$$(2) \qquad\qquad A(x) = x^2 - 2Mx - \alpha^2,$$
$$(3) \qquad\qquad B(y) = \alpha^2 - y^2,$$

where M and α are real parameters corresponding to the mass and imaginary angular momentum of the Wick rotated black hole. The Ricci-flat metric is given by $(x^2 - y^2)g_0$.

The roots of $A(x)$ are given by

$$x_\pm = M \pm \sqrt{M^2 + \alpha^2}.$$

Since $\alpha < M$, the domain in $x - y$ coordinates looks like the following:

The region corresponding to $x > x_+$ corresponds to the exterior of the black hole. The boundary components of this region correspond to roots of $A(x)$ or $B(y)$ or $x \to \infty$. The region at $x = \infty$ is infinitely distant, so we don't need to extend the structure there. Extending the ambitoric structure to the other boundary regions is well understood, as discussed in Lemma 4.26. In particular, this region is completable to a smooth manifold given certain periods for the angular variables (in other words, choice of lattice $\Lambda \subset \mathfrak{t}$) as discussed in [GH79].

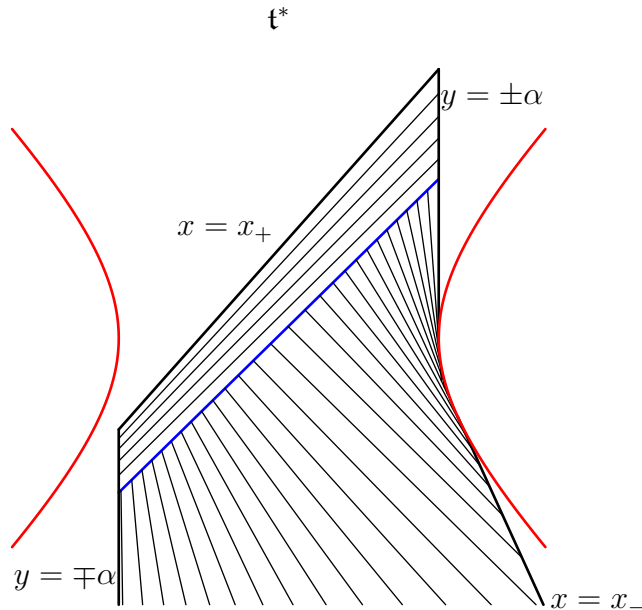

Figure 6: The moment image of Riemannian Kerr. The conic $\mathcal{C}$ is drawn in red, and $\{x = \infty\}$ is drawn in blue. The exterior region is shaded with level sets of x, while the interior region is shaded with level sets of y.

The region corresponding to $x < x_-$ corresponds to the interior of the black hole. In Lorentzian signature, there is a ring singularity in the interior

region where the curvature diverges to infinity. This singularity transforms under Wick rotation to the folding hypersurfaces $Z_\pm := \{x \mp y = 0\}$ of Riemannian Kerr. This motivates the work later in this paper where we study the geometry near these hypersurfaces. Since the curvature diverges to infinity along these hypersurfaces, this region is not completable.

The moment map image is roughly the same for both $\omega_\pm$. The main component is drawn in Figure 6. There are two triangular domains with edges given by $\{x = x_-\}, \{y = \pm\alpha\}$ and $Z_\pm$. One of these regions is near the fold, so has the behaviour depicted in Figure 1, although it is hard to see in Figure 6. The other region (not drawn) gets mapped to a cone with edges given by $\{x = x_-\}$ and $\{y = \mp\alpha\}$ oriented away from the component which was drawn.

4. Classification of complete regular ambitoric 4-orbifolds

In this section, we prove the main classification results, Theorems 1.1 and 1.2. The first goal is to prove Theorem 1.2 modulo replacing the completeness assumption with local convexity of the moment map. The idea is to focus on the open dense set where the local classification, Theorem 1.11, holds. We will first study the gauge group of regular ambitoric structures in order to understand how to glue together the local charts given by the local classification theorem. Next we discuss the asymptotics of ambitoric ansatz spaces and regular ambitoric orbifold completions. The final tool to prove our modification of Theorem 1.2 is a local-global principal for convexity, which is a slight modification of the work of [HNP94]. Finally, we classify completable ambitoric ansatz spaces, giving Theorem 1.1. The local convexity that we find from this result then combines with the other results of this section to give Theorem 1.2.

4.1. Gauge group for ambitoric ansatz spaces

The local form of a regular ambitoric 4-manifold given in Theorem 1.11 is not unique. To study the flexibility in the local form, let U be a neighbourhood of a point in a regular ambitoric 4-manifold $(M, [g_0], J_+, J_-, \mathbb{T})$ which is small enough so that the theorem applies to find the data

$$\{x, y, q(z), d\tau_0, d\tau_1, d\tau_2, A(z), B(z)\}.$$

Another application of the theorem could give a different set of data

$$\{\tilde{x}, \tilde{y}, \tilde{q}(z), d\tilde{\tau}_0, d\tilde{\tau}_1, d\tilde{\tau}_2, \tilde{A}(z), \tilde{B}(z)\}$$

over U.

Lemma 4.1. *$\tilde{x}$ is a rational-linear function of x and $\tilde{y}$ is a rational-linear function of y.*

Proof. From the way that the coordinates are constructed in Theorem 1.11, dx and $d\tilde{x}$ are sections of the line bundle ξ^*, so that $\tilde{x}$ is a function of x. Similarly $\tilde{y}$ is a function of y. From [ACG13a], we know that $f(x, y) = \frac{q(x,y)}{x-y}$. Seeing how f is represented in these two different coordinates gives

$$\frac{q(x, y)}{x - y} = \frac{\tilde{q}(\tilde{x}, \tilde{y})}{\tilde{x} - \tilde{y}}.$$

Fix a real number y_0. Then we have that

$$\frac{q(x, y_0)}{x - y_0} = \frac{\tilde{q}(\tilde{x}, \tilde{y}(y_0))}{\tilde{x} - \tilde{y}(y_0)}.$$

The left hand side of this equality is a rational-linear function of x, since $q(x, y_0)$ is a linear function of x. Similarly, the right hand side is a rational-linear function of $\tilde{x}$. Thus we can solve the equation to find that $\tilde{x}$ is a rational-linear function of x. A similar argument shows that $\tilde{y}$ is rational-linear function of y. □

Lemma 4.2.

$$\frac{(\tilde{x} - \tilde{y})^2}{(x - y)^2} = \frac{d\tilde{x}}{dx} \frac{d\tilde{y}}{dy}, \qquad \frac{d\tilde{x}}{dx} = \frac{\tilde{q}(\tilde{x})}{q(x)}, \qquad \frac{d\tilde{y}}{dy} = \frac{\tilde{q}(\tilde{y})}{q(y)}.$$

Proof. One can compute

$$df = \frac{dx\big((x - y)(q_0 y + q_1) - q(x, y)\big) + dy\big((x - y)(q_0 x + q_1) + q(x, y)\big)}{(x - y)^2}$$

$$= \frac{-q(y)dx + q(x)dy}{(x - y)^2} = \frac{-\tilde{q}(\tilde{y})d\tilde{x} + \tilde{q}(\tilde{x})d\tilde{y}}{(\tilde{x} - \tilde{y})^2}.$$

Since $dx \wedge d\tilde{x} = 0 = dy \wedge d\tilde{y}$, the above equation can be rearranged to form

$$\frac{(\tilde{x} - \tilde{y})^2}{(x - y)^2} = \frac{d\tilde{x}}{dx} \frac{\tilde{q}(\tilde{y})}{q(y)} = \frac{d\tilde{y}}{dy} \frac{\tilde{q}(\tilde{x})}{q(x)}.$$

Note that the terms in this equation are expressible as the product of functions of x and y. Thus

$$0 = \frac{\partial^2}{\partial x \partial y} \log \left(\frac{(\tilde{x} - \tilde{y})^2}{(x - y)^2} \right) = \frac{2}{(\tilde{x} - \tilde{y})^2} \frac{d\tilde{x}}{dx} \frac{d\tilde{y}}{dy} - \frac{2}{(x - y)^2}.$$

The two above equations can be combined to find

$$\frac{dx}{q(x)} = \frac{d\tilde{x}}{\tilde{q}(\tilde{x})}, \qquad \frac{dy}{q(y)} = \frac{d\tilde{y}}{\tilde{q}(\tilde{y})}.$$

$\square$

Lemma 4.3. *The rational-linear transformations $\tilde{x}(x)$ and $\tilde{y}(y)$ are the same, in the sense that if $\tilde{x} = \frac{ax+b}{cx+d}$, then $\tilde{y} = \frac{ay+b}{cy+d}$.*

Proof. Let $\tilde{x} = \frac{ax+b}{cx+d}$ and $\tilde{y} = \frac{a'y+b'}{c'y+d'}$. We compute

$$\frac{d\tilde{x}}{dx} = \frac{ad - bc}{(cx + d)^2}, \qquad \frac{d\tilde{y}}{dy} = \frac{a'd' - b'c'}{(c'y + d')^2}.$$

From Lemma 4.2, $\frac{(\tilde{x}-\tilde{y})^2}{(x-y)^2} = \frac{d\tilde{x}}{dx} \frac{d\tilde{y}}{dy}$. Plugging in the expressions for $\tilde{x}(x), \tilde{y}(x)$ and their derivatives yields

$$\left(\frac{(ax + b)(c'y + d') - (cx + d)(a'y + b')}{(x - y)(cx + d)(c'y + d')} \right)^2 = \frac{(ad - bc)(a'd' - b'c')}{(cx + d)^2(c'y + d')^2}.$$

Without loss of generality, we may assume that $ad - bc$ and $a'd' - b'c'$ are ± 1. The above equation then yields

$$ad - bc = a'd' - b'c',$$
$$(ax + b)(c'y + d') - (cx + d)(a'y + b') = \pm(x - y),$$

which implies

$$ac' = a'c, \qquad ad' - a'd = b'c - bc', \qquad bd' = bd'.$$

These relations imply that $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $\tilde{y} = \frac{a'y+b'}{c'y+d'} = \frac{ay+b}{cy+d}$.

$\square$

The above lemmas allow us to identify the gauge group of coordinate transformations which preserve the regular ambitoric structure on the set

of generic torus orbits with $\mathbb{P}\,\mathrm{SL}_2\,\mathbb{R}$, where an element $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \mathbb{P}\,\mathrm{SL}_2\,\mathbb{R}$ induces the coordinate transformation $\tilde{x} = \frac{ax+b}{cx+d}, \tilde{y} = \frac{ay+b}{cy+d}$. Note that the gauge group was already known in [ACG13a] to be $\mathbb{P}\,\mathrm{SL}_2\,\mathbb{R}$ by construction, but there it is not expressed in local coordinates.

Example 4.4. Consider the ambitoric ansatz space

$$\mathcal{A} := \mathcal{A}(q, x^4 + 1, y^4 + 1, \mathbb{T}).$$

Consider the change of gauge $\tilde{x} = -\frac{1}{x}, \tilde{y} = -\frac{1}{y}$ which transforms $\mathcal{A}$ to

$$\tilde{\mathcal{A}} := \mathcal{A}(\tilde{q}, \tilde{A}(\tilde{x}), \tilde{B}(\tilde{y})).$$

Since $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial \tilde{x}}$ span the same line bundle in $T\mathcal{A}$ (namely ξ), we can write g_0 restricted to this line bundle in both gauges as

$$g_0|_\xi = \frac{dx^2}{A(x)} = \frac{d\tilde{x}^2}{\tilde{A}(\tilde{x})}.$$

This allows us to compute

$$\tilde{A}(\tilde{x}) = A(x) \left(\frac{d\tilde{x}}{dx}\right)^2 = (x^4 + 1) \left(\frac{-1}{x^2}\right)^2 = \tilde{x}^4 + 1.$$

Similarly, one can compute $\tilde{B}(\tilde{y}) = \tilde{y}^4 + 1$. The image of this gauge transformation is $\{\tilde{x}\tilde{y} \neq 0\} \cap \tilde{\mathcal{A}}$, but our computations show that $\tilde{A}(\tilde{x})$ and $\tilde{B}(\tilde{y})$ extend to smooth positive functions on $\{\tilde{x}\tilde{y} = 0\}$. Thus the ambitoric structure extends to the points in $\{\tilde{x}\tilde{y} = 0\}$ where $(\tilde{x} - \tilde{y})\tilde{q}(\tilde{x}, \tilde{y})$ does not vanish. In terms of the original gauge, we find that $\{x = \infty\}$ and $\{x = -\infty\}$ (respectively $\{y = \infty\}$ and $\{y = \infty\}$) glue together, with the ambitoric structure induced by the inverse gauge transformation from $\{\tilde{x}\tilde{y} = 0\}$.

Now we will apply this understanding of the gauge group to get a gluing result. We can use the following lemma to glue together the locally defined embeddings from Corollary 1.14 into an immersion:

Lemma 4.5. *Let* $(M, [g_0], J_+, J_-, \mathbb{T})$ *be a regular ambitoric 4-manifold with a free* $\mathbb{T}$-*action such that* $M/\mathbb{T}$ *is contractible. Then there exists a* $\mathbb{T}$-*equivariant immersion* $\phi : M \to (\mathbb{RP}^1)^2 \times \mathbb{T}$ *such that for any* $p \in M$, *there exists an ambitoric embedding* $\phi_p : U_p \hookrightarrow \mathcal{A}_p$ *of a neighbourhood* U_p *of* p *into an ambitoric ansatz space* $\mathcal{A}_p$ *such that the following diagram commutes:*

$$M \xrightarrow{\ \phi\ } (\mathbb{RP}^1)^2 \times \mathbb{T}$$

$$\begin{array}{ccc}
M & \xrightarrow{\ \phi\ } & (\mathbb{RP}^1)^2 \times \mathbb{T} \\
\uparrow & & \uparrow \\
U_p & \xhookrightarrow{\ \phi_p\ } & \mathcal{A}_p
\end{array}$$

Moreover, if ϕ is injective and for each $z \in \mathbb{RP}^1$, $\phi(M) \cap \big(\{z\} \times \mathbb{RP}^1 \times \mathbb{T}\big)$ and $\phi(M) \cap \big(\mathbb{RP}^1 \times \{z\} \times \mathbb{T}\big)$ are connected or empty, then ϕ is an ambitoric embedding into some ambitoric ansatz space $\mathcal{A}$.

Proof. From Corollary 1.14, M is covered by $\mathbb{T}$-invariant charts $\{\phi_\alpha : U_\alpha \to \mathcal{A}_\alpha\}_\alpha$ from M into ambitoric ansatz spaces $\{\mathcal{A}_\alpha\}_\alpha$. From Section 4.1, if $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is given by some element of $\mathbb{P}\,\mathrm{SL}_2\,\mathbb{R}$. Thus the maps $\{\phi_\alpha\}_\alpha$ induce a Čech co-cycle C in $H^1(M, \mathbb{P}\,\mathrm{SL}_2\,\mathbb{R})$. Since the ϕ_α are $\mathbb{T}$-equivariant, C is $\mathbb{T}$ invariant. Thus C is equivalent to a Čech co-cycle $\bar{C}$ in $H^1(M/\mathbb{T}, \mathbb{P}\,\mathrm{SL}_2\,\mathbb{R})$. Since $M/\mathbb{T}$ is contractible, $\bar{C}$, and hence C, must be trivial. This means that the gauge can be chosen globally, so that the $\phi_\alpha s$ can be glued to form a map $\phi : M \to (\mathbb{RP}^1)^2 \times \mathbb{T}$.

Now let $p \in M$. Then $p \in U_\alpha$ for some alpha. Transforming the gauge of $U_\alpha \hookrightarrow \mathcal{A}_\alpha$ to match the global gauge used to construct ϕ results in an embedding $U_p \hookrightarrow \mathcal{A}_p$ which has the desired properties by construction.

To prove the *moreover*, assume that ϕ has the required properties. We need to show that there exist functions $A(x)$, $B(y)$ and $q(x,y)$ on the image of ϕ which agree with those given by the local embeddings $\{\phi_p\}_{p \in M}$. For each $x_0 \in \mathbb{RP}^1$, since $\phi(M) \cap \big(\{x_0\} \times \mathbb{RP}^1 \times \mathbb{T}\big)$ is connected, the $\{\phi_p\}_{p \in M}$ must agree on the value of $A(x_0)$. Since x_0 was arbitrary, it follows that $A(x)$ is uniquely determined on $\phi(M)$. Similarly $B(y)$ is uniquely determined on $\phi(M)$. The fact that $q(x,y)$ is uniquely determined follows from the fact that it is a quadratic function, thus determined on a connected set by its value on any open subset. □

4.2. Local asymptotics

We will first describe the asymptotics for an ambitoric ansatz space $\mathcal{A}$ with respect to a given metric g. We will use the appendix to relate $\mathcal{A}_C^g$ to the Cauchy completion with respect to a more convenient metric. In particular,

consider the embedding $\mathcal{A} \hookrightarrow (\mathbb{RP}^1)^2 \times \mathbb{T}$ formed by identifying $\mathbb{RP}^1$ with $\mathbb{R} \cup \{\infty\}$ for both the x and y coordinate axes of $\mathcal{A}$. Let g_f be a flat metric on $(\mathbb{RP}^1)^2 \times \mathbb{T}$. Since

$$\mathcal{A} = \left\{ (x, y, \vec{t}) \in (\mathbb{R})^2 \times \mathbb{T} : A(x) > 0, B(y) > 0, q(x, y)(x - y) \neq 0 \right\},$$

we find that

$$\mathcal{A}_C^{g_f} = \overline{\{x \in \mathbb{R}P^1 : A(x) > 0\}} \times \overline{\{y \in \mathbb{R}P^1 : B(y) > 0\}} \times \mathbb{T},$$

where $\overline{\cdot}$ is the closure in $\mathbb{RP}^1$ with the flat topology. Note that since $(x - y)q(x, y)$ is a rational function on $\mathcal{A}$, it naturally extends to $\mathcal{A}_C^{g_f}$. We see that $\partial^{g_f} \mathcal{A} := \mathcal{A}_C^{g_f} \backslash \mathcal{A}$ can be decomposed into components of two types: level sets of x or y, or components of the vanishing locus of $(x - y)q(x, y)$. Since the moment map sends level sets of x or y to lines, we call these components *edges*. We call components of the vanishing locus of $(x - y)q(x, y)$ *folds*. We will call a fold negative if $f = \frac{q(x,y)}{x-y}$ vanishes on it, and positive if $\frac{1}{f}$ vanishes on it, analogous to the folding hypersurfaces $Z_\pm$ from Section 2. A $\mathbb{T}$-orbit where a pair of edges or folds meet is called a *corner*. A fold which is not an edge or a corner is called a *proper fold*. Since we are treating corners separately, when we refer to an edge or a proper fold we will mean its interior and not include the adjacent corners.

Since $\mathcal{A}_C^{g_f}$ is compact, by Proposition A.4 in the appendix, every point p_1 in $\mathcal{A}_C^g$ can be represented by a point p_2 in $\mathcal{A}_C^{g_f}$, in the sense that there exists a curve in $\mathcal{A}$ which converges to p_1 in $\mathcal{A}_C^g$ and p_2 in $\mathcal{A}_C^{g_f}$. Note that this correspondence between $p_1 \in \mathcal{A}_C^g$ and $p_2 \in \mathcal{A}_C^{g_f}$ is not bijective. For example, let's consider the case where $U = \mathcal{A}$. If p_2 lies on $x = \infty$, then p_2 may be represented by two curves $c_\pm(t)$, with $\lim_{t\to\infty} x \circ c_\pm(t) = \pm\infty$. Then $c_\pm$ could not both represent the same point in $\mathcal{A}_C^g$, where $x = \infty$ and $x = -\infty$ are not identified. As another example, if $\mathbb{T}$ does not act freely at p_1, then there are multiple choices for p_2, since $\mathbb{T}$ acts freely on $\mathcal{A}_C^{g_f}$. We will use this correspondence to describe points in U_C^g by the coordinates on $\mathcal{A}_C^{g_f}$, keeping in mind that on U_C^g these may not be coordinate functions and may be multi-valued.

The correspondence between $\mathcal{A}_C^g$ and $\mathcal{A}_C^{g_f}$ allows us to decompose $\partial^g \mathcal{A}$ into components corresponding to the components of $\partial^{g_f} \mathcal{A}$, and we will use the same language to refer to them: folds, edges, and corners. Note that there may be components of $\mathcal{A}_C^{g_f}$ which are not represented in $\mathcal{A}_C^g$. We will refer to these components as *infinitely distant* with respect to g. Formally these lie in the Busemann completion (see Lemma A.7), but we will not

dwell on this fact. In the case where $g = g_p$, the vanishing locus of $p(x, y)$ forms an additional boundary component, which we will refer to as P. The *proper part* of P is the largest subset of P which does not intersect with any folds or edges.

Let $(M, [g_0], J_+, J_-, \mathbb{T})$ be a regular ambitoric 4-manifold with free $\mathbb{T}$-orbits. We will now show that the asymptotic behaviour of M is essentially local. Let $\{\lambda_1, \lambda_2\}$ be a set of generators of Λ.

$$\lambda_1^2 + \lambda_2^2 \in \operatorname{Sym}^2(\mathfrak{t}) \cong \operatorname{Sym}^2(\mathfrak{t}^*)^*$$

is a Riemannian metric on $\mathfrak{t}^*$, identifying $\mathfrak{t}^*$ with Euclidean space. The moment map $\mu^+ : M \to \mathfrak{t}^*$ is $\mathbb{T}$-invariant, so it descends to an *orbital moment map* $\bar{\mu}^+ : M/\mathbb{T} \to \mathfrak{t}^*$, which is an immersion. Thus $\bar{g}_E := (\bar{\mu}^+)^*(\lambda_1^2 + \lambda_2^2)$ is a Riemannian metric on $M/\mathbb{T}$. Since $\mathfrak{t}$ is the tangent space to $\mathbb{T}$, $(\lambda_1^2 + \lambda_2^2)^{-1}$ is a Riemannian metric on $\mathbb{T}$. Thus

$$g_E := (\mu^+)^*(\lambda_1^2 + \lambda_2^2) + (\lambda_1^2 + \lambda_2^2)^{-1}$$

is a $\mathbb{T}$-invariant metric on M which descends to the metric $\bar{g}_E$ on $M/\mathbb{T}$. Note that in the case of an ambitoric ansatz space $\mathcal{A}$, $\mathcal{A}_C^{g_E}$ can be naturally identified with the points in $\mathcal{A}_C^{g_f}$ which are not contained in positive folds: since μ^+ sends such points to infinity, they are infinitely distant with respect to g_E.

Lemma 4.6. *Let $(M, \mathring{M}, g, J_+, J_-, \mathbb{T})$ be a regular ambitoric orbifold completion, where $g \in \{g_0, g_\pm, g_E\} \cup \{g_p\}_{p \perp q}$. Then every point a in $\partial^{g_E} \mathring{M}$ has a neighbourhood in $\mathring{M}_C^{g_E}$ which can be identified with an open set in $\mathcal{A}_C^{g_f}$ for some ambitoric ansatz space $\mathcal{A}$ such that a is identified with a point in $\partial^{g_f} \mathcal{A}$.*

Proof. We first treat the case when $\mathbb{T} \cdot a$ admits a neighbourhood U in $\mathring{M}_C^{g_E}/\mathbb{T}$ such that $U \cap (\mathring{M}/\mathbb{T})$ is contractible. Let $\hat{U} \subseteq \mathring{M}$ be the union of the orbits $U \cap (\mathring{M}/\mathbb{T})$. By Lemma 4.5, by shrinking U if necessary, we can find an ambitoric embedding $\phi : \hat{U} \to \mathcal{A}$ of $\hat{U}$ into some ambitoric ansatz space $\mathcal{A}$. It is easy to see that $\phi^* g_f$ induces the same topology as g_E on $\hat{U}_C^{g_E}$. Thus we find that ϕ naturally extends to an injection $\bar{\phi} : \hat{U}_C^{g_E} \to \mathcal{A}_C^{g_f}$. Since $a \in \partial^{g_E} \hat{U}$, we have $\bar{\phi}(a) \in \partial^{g_f} \phi(\hat{U})$.

If $\bar{\phi}(a) \notin \partial^{g_f} \mathcal{A}$, then $\bar{\phi}(a) \in \mathcal{A}$. Without loss of generality, by shrinking U, we may assume that $\bar{\phi}(\hat{U}_C^{g_E}) \subset \mathcal{A}$. Since $g|_{\mathring{M}}$ is one of the special metrics $\{g_0, g_\pm, g_p\}$ of the ambitoric structure on $\mathring{M}$, $\phi_*(g|_{\mathring{M}})$ must be homothetic to one of the special metrics of the ambitoric structure on $\phi(\hat{U}) \subseteq \mathcal{A}$. Since this metric induces the same topology as g_f on $\mathcal{A}$, we can identify $\partial^{\phi_*(g|_{\mathring{M}})} \phi(\hat{U})$

with $\partial^{g_f}\phi(\hat{U}) = \bar{\phi}(\partial^{g_E}\hat{U}) \subset \mathcal{A}$, so that $\bar{\phi}(a) \in \partial^{\phi_*(g|_{\mathring{M}})}\phi(\hat{U})$. Moreover, since $\hat{U}$ is $\mathbb{T}$-invariant, so is $\phi(\hat{U})$, and thus $\partial^{\phi_*(g|_{\mathring{M}})}\phi(\hat{U})$. Thus the free orbit $\mathbb{T} \cdot \bar{\phi}(a)$ lies in $\partial^{\phi_*(g|_{\mathring{M}})}\phi(\hat{U})$. Since $\bar{\phi}$ is injective, this shows that there is a free $\mathbb{T}$-orbit $\mathcal{O} \subset \partial^g\hat{U}$ corresponding to the orbit $\mathbb{T} \cdot a \subset \partial^{g_E}\hat{U}$ via Lemma A.4. Since $\mathcal{O}$ is a free orbit, it must lie in $\mathring{M}$ by definition. But on $\mathring{M}$, the topologies induced by g_E and g are equivalent. Thus $a \in \mathring{M}$. This contradicts $a \in \partial^{g_E}\mathring{M}$, so that $\bar{\phi}(a) \in \partial^{g_f}\mathcal{A}$ as claimed.

We now treat the case where every neighbourhood U of $\mathbb{T} \cdot a$ satisfies the condition that $U \cap (\mathring{M}/\mathbb{T})$ is not contractible. We will show that this case is not possible, essentially because for an ambitoric ansatz space $\mathcal{A}$, $\partial^{g_f}\mathcal{A}$ does not have isolated $\mathbb{T}$-orbits. Since $M_C^{g_E}/\mathbb{T}$ is 2-dimensional, we can find a neighbourhood U of $\mathbb{T} \cdot a$ such that $U \cap (\mathring{M}/\mathbb{T})$ is homeomorphic to a punctured disc. Let $V \subset U \cap (\mathring{M}/\mathbb{T})$ be a contractible set obtained by cutting a line L between the two boundary components of the puctured disc $U \cap (\mathring{M}/\mathbb{T})$. Let $\hat{V} \subseteq \mathring{M}$ be the union of $\mathbb{T}$-orbits V. We can then use Lemma 4.5 to get an immersion $\phi : \hat{V} \to (\mathbb{RP}^1)^2 \times \mathbb{T}$. Arguing as in the previous case, ϕ extends to an immersion $\bar{\phi} : \hat{V}_C^{g_E} \to (\mathbb{RP}^1)^2 \times \mathbb{T}$. Still using Lemma 4.5, since $\hat{V}_C^{g_E}$ is compact, we can cover $\hat{V}$ with finitely many $\{\hat{V}_\alpha\}_\alpha$ such that each $\phi|_{\hat{V}_\alpha}$ is an embedding into some ambitoric ansatz space $\mathcal{A}_\alpha \subset (\mathbb{RP}^1)^2 \times \mathbb{T}$. Let S be the set of α such that $a \in \partial^{g_E}\hat{V}_\alpha$. We can repeat the argument from the previous case for each α to deduce that $\bar{\phi}(a) \in \partial^{g_f}\mathcal{A}_\alpha$ for each $\alpha \in S$. But the $\{\partial^{g_f}\mathcal{A}_\alpha\}_\alpha$ are (possibly singular) hypersurfaces which must locally agree, so there must be some neighbourhood W of $\bar{\phi}(a)$ and some hypersurface H in W passing through $\bar{\phi}(a)$ which such that $\partial^{g_f}\mathcal{A}_\alpha \cap W = H$. We treat the case where H is not singular at $\bar{\phi}(a)$ for clarity. By shrinking V small enough so that $\bar{\phi}(V) \subset W$ we find that $\phi(\hat{V})$ lies on one side of H in W. It follows that the ray in $T_{\mathbb{T} \cdot a}(V_C^{g_E})$ corresponding to L must cover the line $(T_{\bar{\phi}(a)}H)/\mathfrak{t} \subset T_{\mathbb{T} \cdot \bar{\phi}(a)}(\mathbb{RP}^1)^2$ by the map induced by $\bar{\phi}$. This is impossible, since a linear map cannot send a ray to a line. $\qquad\square$

We use the previous lemma to decompose $\partial^{g_E}\mathring{M}$ into components corresponding to the decomposition of $\partial^{g_f}\mathcal{A}$ into folds, edges, P, and corners. We will use this language to discuss the components of $\partial^{g_E}\mathring{M}$. As in the discussion at the beginning of this sub-section, we will use Proposition A.4 to decompose $\partial^g\mathring{M}$ into components, which we will call folds, edges, P, and corners likewise. We call components of $\partial^{g_E}\mathring{M}$ which are not in $\partial^g\mathring{M}$ *infinitely distant*.

The following example is a variation on example 2.8 from [KL15], and shows that the orbital moment map is not always a global embedding:

Example 4.7. Consider the elliptic type ambitoric ansatz space

$$\mathcal{A} := \mathcal{A}\left(xy + 1, x^4 + 1, y^4 + 1, \mathbb{T}\right).$$

Consider the space $\bar{\mathcal{A}}$ obtained by gluing the edges $\{x = \infty\}$ and $\{x = -\infty\}$ together, as well as the edges $\{y = \infty\}$ and $\{y = -\infty\}$ together. As in Example 4.4, the ambitoric structure from $\mathcal{A}$ extends to $\bar{\mathcal{A}}$. Since the functions $A(x)$ and $B(y)$ are positive on $\bar{\mathcal{A}}$, $\bar{\mathcal{A}}$ has no edges. It follows that $\bar{\mathcal{A}}$ has two connected components. Let $\bar{\mathcal{A}}_0$ be one of the connected components of $\bar{\mathcal{A}}$.

From Section 2, we find that the moment map $\mu^+ : \bar{\mathcal{A}}_0 \to \mathfrak{t}^*$ has an image Δ_+ which is the exterior of a conic $\mathcal{C}_+$. Since $\mathcal{A}$ is elliptic type, $\mathcal{C}_+$ is an ellipse. Moreover, the μ^+-fibre over any point in Δ_+ is a single $\mathbb{T}$-orbit in $\bar{\mathcal{A}}_0$, allowing us to identify $\bar{\mathcal{A}}_0/\mathbb{T} \cong \Delta_+$. Since Δ_+ is not simply connected, its universal cover $\pi : \tilde{\Delta}_+ \to \Delta_+$ is not injective. Since $\mu^+ : \bar{\mathcal{A}}_0 \to \Delta_+$ is a principal $\mathbb{T}$-bundle, we can use π to construct the pull-back bundle $\tilde{\mathcal{A}} := \pi^* \bar{\mathcal{A}}_0$ over $\tilde{\Delta}_+$. Since π is a covering map, it extends to a covering map from $\tilde{\mathcal{A}}$ to $\bar{\mathcal{A}}_0$, which can be used to induce the ambitoric structure from $\bar{\mathcal{A}}_0$ to $\tilde{\mathcal{A}}$.

We find that $\tilde{\mathcal{A}}$ is an ambitoric manifold with orbit space $\tilde{\mathcal{A}}/\mathbb{T} \cong \tilde{\Delta}_+$ and orbital moment map $\pi : \tilde{\Delta}_+ \to \Delta_+ \subset \mathfrak{t}^*$ which is not injective.

The main feature of the previous example was the existence of a proper fold. This motivates us to consider only regular ambitoric orbifold completions without proper folds, since we want to rule out the possibility that the orbital moment map is not injective. If $(M, \mathring{M}, g, J_+, J_-, \mathbb{T})$ is a regular ambitoric orbifold completion without proper folds, then the boundary $\partial^{g_E} \mathring{M}$ consists of only edges and the corners where they meet. The orbital moment map naturally extends to a map $(\mathring{M}/\mathbb{T})_C^{\bar{g}_E} \to \mathfrak{t}^*$, which is locally convex since $\partial^{g_E} \mathring{M}$ consists of only edges and corners, which each get locally sent to lines and convex cones respectively. As in the case of compact toric orbifolds, we wish to use this local convexity to prove global convexity, which will give us control over the topology of M. The tool to do this is the Lokal-global-Prinzip for convexity theorems [HNP94], although it requires that the moment map is proper. Since $(\mathring{M}/\mathbb{T})_C^{\bar{g}_E}$ may not be compact a priori, we have to consider the case that the moment map is not proper. However, in the next section we will see that completeness is actually enough for the proof of Lokal-global-Prinzip to hold.

4.3. Lokal-global-Prinzip

We begin this section by recalling some definitions from [HNP94], and then stating their version of the Lokal-global-Prinzip. We will then state and prove the slight modification of this result which we will need. Note that it is likely that what we prove here is only a case of the general treatment of the Lokal-global-Prinzip in [RS12] (where we got the idea to use Hopf-Rinow), but for our purposes we find it convenient to prove the result.

Definition 4.8. A continuous map $\Psi : X \to V$ from a connected Hausdorff topological space X to a finite dimensional vector space V is *locally fibre connected* if every point in X admits an arbitrarily small neighbourhood U such that $\Psi^{-1}(\Psi(u)) \cap U$ is connected for each $u \in U$.

Definition 4.9. A locally fibre connected map $\Psi : X \to V$ has *local convexity data* if every $x \in X$ admits an arbitrarily small neighbourhood U_x and a closed convex cone $C_x \subseteq V$ with vertex $\Psi(x)$ satisfying:

- $\Psi(U_x)$ is a neighbourhood of $\Psi(x)$ in C_x,
- $\Psi|_{U_x} : U_x \to C_x$ is open,
- $\Psi^{-1}(\Psi(u)) \cap U_x$ is connected for each $u \in U_x$,

where the topology on C_x is the subspace topology induced from V.

These definitions allow us to state the theorem:

Theorem 4.10 (Lokal-global-Prinzip for Convexity Theorems [HNP94]). *Let $\Psi : X \to V$ be a proper locally fibre connected map with local convexity data. Then $\Psi(X)$ is a closed locally polyhedral convex subset of V, the fibres $\Psi^{-1}(v)$ are all connected, and $\Psi : X \to \Psi(X)$ is a open mapping.*

Since we are essentially reproducing this theorem, we need to continue defining the constructions used in the proof.

Consider the equivalence relation on X defined by $x \sim y$ if and only if x and y are both contained in the same connected component of a fibre of Ψ. The quotient space $\tilde{X} := X/\sim$ is called the Ψ-*quotient* of X. Let $\tilde{\Psi} : \tilde{X} \to V$ be the map induced by Ψ on $\tilde{X}$.

Let d_V be the usual Euclidean distance on V with respect to some fixed basis. We can use this to define a distance on $\tilde{X}$ as follows:

$$d : \tilde{X} \times \tilde{X} \to [0, \infty) : (\tilde{x}, \tilde{y}) \mapsto \inf_{\gamma \in \Gamma(\tilde{x}, \tilde{y})} \text{length}_{d_V}(\tilde{\Psi} \circ \gamma),$$

where $\Gamma(\tilde{x}, \tilde{y})$ is the set of curves γ connecting $\tilde{x}$ to $\tilde{y}$ such that $\Psi \circ \gamma$ is piecewise differentiable. d is called the *metric induced on* $\tilde{X}$ by $\tilde{\Psi}$.

Now we can prove our version of the Lokal-global-Prinzip for convexity theorems. Note that the proof is the same as the relevant parts of the previous theorem aside from the replacement of the use of properness with an appeal to the Hopf-Rinow theorem.

Theorem 4.11. *Let* $\Psi : X \to V$ *be a locally fibre connected map with local convexity data. Moreover, assume that* $\tilde{X}$ *is a complete locally compact length space with respect to the metric induced by* $\tilde{\Psi}$*. Then* $\phi(X)$ *is a convex subset of* V *and the fibres* $\Psi^{-1}(v)$ *are all connected.*

Proof. Let $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$ with $c := d(\tilde{x}_0, \tilde{x}_1)$. For each $n \in \mathbb{N}$, we can find some $\gamma_n \in \Gamma(\tilde{x}_0, \tilde{x}_1)$ such that $\text{length}_{d_V}(\tilde{\Psi} \circ \gamma) \leq c + \frac{1}{n}$. For each $n \in \mathbb{N}$, let $\tilde{x}_{\frac{1}{2}}^{(n)}$ be the midpoint of γ_n. Then $\left(\tilde{x}_{\frac{1}{2}}^{(n)}\right)_{n=1}^{\infty}$ is a sequence in the closed ball $B_{c+1}(\tilde{x}_0)$ of radius $c + 1$ about $\tilde{x}_0$. Since X is a complete locally compact length space, the Hopf-Rinow theorem [BH99] tells us that $B_{c+1}(\tilde{x}_0)$ is compact. Thus $\left(\tilde{x}_{\frac{1}{2}}^{(n)}\right)_{n=1}^{\infty}$ admits a subsequence which converges to some $\tilde{x}_{\frac{1}{2}} \in B_{c+1}(\tilde{x}_0)$, which must satisfy

$$d\left(\tilde{x}_0, \tilde{x}_{\frac{1}{2}}\right) = d\left(\tilde{x}_{\frac{1}{2}}, \tilde{x}_1\right) = \frac{c}{2}.$$

Repeating the argument for the pairs of points $(\tilde{x}_0, \tilde{x}_{\frac{1}{2}})$ and $(\tilde{x}_{\frac{1}{2}}, \tilde{x}_1)$, we construct points $\tilde{x}_{\frac{1}{4}}$ and $\tilde{x}_{\frac{3}{4}}$ satisfying

$$(4) \qquad\qquad d\left(\tilde{x}_{\frac{n}{2^m}}, \tilde{x}_{\frac{p}{2^q}}\right) = c\left|\frac{n}{2^m} - \frac{p}{2^q}\right|$$

for all suitable n, m, p, q. Inductively we can repeat the argument to construct points $\tilde{x}_{\frac{n}{2^m}}$ for $n, m \in \mathbb{N}$ with $0 \leq n \leq 2^m$ satisfying (4). We can then extend the function $\frac{n}{2^m} \mapsto \tilde{x}_{\frac{n}{2^m}}$ to a continuous function $\gamma : [0, 1] \to \tilde{X}$ satisfying $d\big(\gamma(t), \gamma(t')\big) = c|t - t'|$ for all $t, t' \in [0, 1]$. This means that locally

$$d_V\big(\tilde{\Psi} \circ \gamma(t), \tilde{\Psi} \circ \gamma(t')\big) = c|t - t'|,$$

which can only happen if $\tilde{\Psi} \circ \gamma$ is a straight line segment. This line segment connects $\tilde{\Psi}(\tilde{x}_0)$ and $\tilde{\Psi}(\tilde{x}_1)$. Varying $\tilde{x}_0$ and $\tilde{x}_1$, we find that $\Psi(X) = \tilde{\Psi}(\tilde{X})$ is convex.

To prove that the fibres are all connected, let $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$ such that $\tilde{\Psi}(\tilde{x}_0) = \tilde{\Psi}(\tilde{x}_1) =: v$. Let $\gamma : [0, 1] \to \tilde{X}$ be the curve constructed above. Then $\tilde{\Psi} \circ \gamma$ is a line segment containing v. Assume that $\tilde{\Psi} \circ \gamma$ is not the constant function v. Then $\tilde{\Psi} \circ \gamma$ is a loop through v as well as a straight line segment. Thus there exists a turning point $v_0 = \tilde{\Psi} \circ \gamma(t_0)$ such that $\tilde{\Psi} \circ \gamma\big([t_0 - \epsilon, t_0]\big) = \tilde{\Psi} \circ \gamma\big([t_0, t_0 + \epsilon]\big)$ for all $\epsilon > 0$ sufficiently small. Thus $\tilde{\Psi}$ is not locally injective near $\gamma(t_0)$. This contradicts Ψ being locally fibre connected. Thus $\tilde{\Psi} \circ \gamma$ is the constant function v. This implies that $d(\tilde{x}_0, \tilde{x}_1) = 0$, so that $\tilde{x}_0 = \tilde{x}_1$ as required. $\qquad \square$

We recall proposition 3.25 from [BH99]:

Proposition 4.12. *Let X be a length space and $\tilde{X}$ be a Hausdorff topological space. Let $p : \tilde{X} \to X$ be a continuous local homeomorphism, and d be the metric induced on $\tilde{X}$ by p.*

1) *If one endows $\tilde{X}$ with the metric d, then p becomes a local isometry.*

2) *d is a length metric.*

3) *d is the unique metric on $\tilde{X}$ that satisfies properties (1) and (2).*

Combining this proposition with the previous theorem gives:

Corollary 4.13. *Let $\Psi : X \to V$ be a locally fibre connected map with local convexity data. Moreover, assume that $\tilde{\Psi}$ is a local homeomorphism, and that $\tilde{X}$ is complete with respect to the metric $\tilde{d}$ given by the pull-back with respect to $\tilde{\Psi}$ of the Euclidean metric on V. Then $\phi(X)$ is a convex subset of V and the fibres $\Psi^{-1}(v)$ are all connected.*

Proof. By the previous theorem, it suffices to show that $(\tilde{X}, d)$ is a complete locally compact length space. Since $\tilde{\Psi} : \tilde{X} \to V$ is a local homeomorphism and V is locally compact (being a vector space), $\tilde{X}$ is locally compact. By the previous proposition, $(\tilde{X}, d)$ is a length space. Note that $\tilde{d}$ and d both satisfy properties (1) and (2) in the previous proposition. Thus by uniqueness, we have that $d = \tilde{d}$. Since $(\tilde{X}, \tilde{d})$ is complete, $(\tilde{X}, d)$ is complete. $\qquad \square$

Now we can start applying the Lokal-global-Prinzip to our setting.

Lemma 4.14. *Let $\mathcal{A}_0$ be a connected component of some ambitoric ansatz space without proper folds. Then the orbital moment map $\bar{\mu}^+ : \mathcal{A}_0/\mathbb{T} \to \mathfrak{t}^*$ extends to $(\mathcal{A}_0/\mathbb{T})_C^{\bar{g}_E}$ as a local homeomorphism with local convexity data.*

Proof. $\bar{\mu}^+$ is a rational map from $\mathcal{A}_0/\mathbb{T} \subset (\mathbb{RP}^1)^2$ to $\mathfrak{t}^*$. Thus $\bar{\mu}^+$ extends naturally to a map on $(\mathbb{RP}^1)^2$, and hence on $(\mathcal{A}_0/\mathbb{T})_C^{\bar{g}_E} \subset (\mathbb{RP}^1)^2$, which we will also denote by $\bar{\mu}^+$. Since $\mathcal{A}_0/\mathbb{T}$ is connected and $\bar{\mu}^+|_{\mathcal{A}_0/\mathbb{T}}$ is a homeomorphism, so is $\bar{\mu}^+|_{(\mathcal{A}_0/\mathbb{T})_C^{\bar{g}_E}}$.

Since there are no proper folds, $\partial^{\bar{g}_E}(\mathcal{A}_0/\mathbb{T})$ consists of finitely many edges and components of P, which $\bar{\mu}^+$ sends to lines tangent to the conic $\mathcal{C}_+$. We find that for every corner $c \in \partial^{\bar{g}_E}(\mathcal{A}_0/\mathbb{T})$, arbitrarily small neighbourhoods of c get mapped by $\bar{\mu}^+$ to neighbourhoods of $\bar{\mu}^+(c)$ in the convex cone with vertex $\bar{\mu}^+(c)$ formed by the rays passing through $\mathcal{C}_+$. By similarly describing neighbourhoods of points on edges, on P, and in $\mathcal{A}_0$, we find that $\bar{\mu}^+ : (\mathcal{A}_0/\mathbb{T})_C^{\bar{g}_E} \to \mathfrak{t}^*$ has local convexity data. $\square$

We can now prove our weakened classification of regular ambitoric orbifold completions:

Theorem 4.15. *Let $(M, \mathring{M}, [g], J_+, J_-, \mathbb{T})$ be a connected regular ambitoric orbifold completion without proper folds. Then there exists an ambitoric embedding of $(\mathring{M}, [g], J_+, J_-, \mathbb{T})$ into some ambitoric ansatz space whose moment map images are (possibly unbounded) polygons.*

Proof. By Lemma 4.6, $(\mathring{M}/\mathbb{T})_C^{\bar{g}_E}$ is locally covered by charts which each have an image of the form $(\mathcal{A}/\mathbb{T})_C^{\bar{g}_E}$ for some ambitoric ansatz space $\mathcal{A}$. By assumption, these charts cannot have proper folds. Applying the previous lemma to these local charts, we find that $\bar{\mu}^+$ naturally extends to a local homeomorphism $\bar{\mu}^+ : (\mathring{M}/\mathbb{T})_C^{\bar{g}_E} \to \mathfrak{t}^*$ with local convexity data.

Since $\bar{g}_E$ is the pull-back of a Euclidean metric on $\mathfrak{t}^*$ by $\bar{\mu}^+$, we can apply Corollary 4.13 with $X = \tilde{X} = (\mathring{M}/\mathbb{T})_C^{\bar{g}_E}$, $V = \mathfrak{t}^*$ and $\Psi = \tilde{\Psi} = \bar{\mu}^+$. This tells us that $\bar{\mu}^+$ is injective with convex image. In particular $\mathring{M}/\mathbb{T}$ is contractible. We can then apply Lemma 4.5 to construct a $\mathbb{T}$-equivariant immersion $\phi : \mathring{M} \to (\mathbb{RP}^1)^2 \times \mathbb{T}$. Let $\bar{\phi} : \mathring{M}/\mathbb{T} \to (\mathbb{RP}^1)^2$ be the map induced by ϕ. We have the following commutative diagram:

$$
\begin{array}{ccc}
\left(\mathbb{RP}^1\right)^2 \times \mathbb{T} & \longrightarrow & \left(\mathbb{RP}^1\right)^2 \xrightarrow{\ \bar{\mu}^+\ } \mathfrak{t}^* \\
{\scriptstyle \phi}\uparrow & & {\scriptstyle \bar{\phi}}\uparrow \quad \nearrow{\scriptstyle \bar{\mu}^+} \\
\mathring{M} & \longrightarrow & \mathring{M}/\mathbb{T}
\end{array}
$$

Since $\bar{\mu}^+ : \mathring{M}/\mathbb{T} \to \mathfrak{t}^*$ is injective, $\bar{\phi}$ must be injective. Thus ϕ is injective. It also follows that since $\mu^+(\mathring{M})$ is convex, $\phi(\mathring{M})$ intersects each level set of x or y in at most one connected component. We are now able to apply the full force of Lemma 4.5 to deduce that the image of ϕ can be given the structure of an ambitoric ansatz space. Since $\mathring{M}$ has no proper folds, its boundary components are all either edges or parts of P. Since the moment maps $\mu^\pm$ send each of these boundary components to edges, the moment maps of M must be polygons. □

4.4. Asymptotic analysis of ambitoric ansatz spaces

Let $\mathcal{A}$ be an ambitoric ansatz space, equipped with one of its distinguished metrics $g \in \{g_0, g_\pm, g_p\}$. We will now see what we can deduce from the assumption that $\mathcal{A}$ is g-completable, discussing each type of boundary component in turn.

4.4.1. Folds.

Lemma 4.16. *If $\mathcal{A}$ is g-completable, then every proper fold is infinitely distant.*

Proof. Let F be a proper fold which is not infinitely distant. The $\mathbb{T}$-orbits in F must have well-defined g-volume. In [ACG13b], the g_+ volume of the fibre at (x, y) is shown to be proportional to $\frac{A(x)B(y)}{q(x,y)^4}$. The volumes (up to a constant) of such a fibre with respect to each of the candidate metrics for g are given in the following table:

g	g_+	g_0	g_-	g_p
$\mathrm{Vol}_g\left(\mathbb{T}_{(x,y)}\right)$	$\frac{A(x)B(y)}{q(x,y)^4}$	$\frac{A(x)B(y)}{(x-y)^2 q(x,y)^2}$	$\frac{A(x)B(y)}{(x-y)^4}$	$\frac{A(x)B(y)}{p(x,y)^4}$

If g is g_0, we find that $\frac{A(x)B(y)}{(x-y)^2 q(x,y)^2}$ must be well-defined on F. But F is not a corner or an edge, so $A(x)$ and $B(y)$ are positive functions on F. Thus we must have that $(x-y)^2 q(x,y)^2$ is non-vanishing on F. This contradicts F being a fold, so that g cannot be g_0. Similar arguments show that if F is positive, then $g \neq g_-$, and if F is negative, then $g \neq g_+$. It follows that if one writes $g = \phi(x,y)g_0$, then $\phi(x,y)$ vanishes on F.

Now we will show that the g-distance between $\mathbb{T}$-orbits in F is zero. To see this, let $(x_1, y_1, \vec{t}_0)$ and $(x_2, y_2, \vec{t}_0)$ be two different points in $\mathcal{A}_C^{g_f} \cap F$. Consider the curve γ with image in $\mathcal{A}_C^{g_f} \cap F \cap \{\vec{t} = \vec{t}_0\}$ connecting $(x_1, y_1, \vec{t}_0)$

to $(x_2, y_2, \vec{t_0})$. Let I_x and I_y be the ranges of the functions x and y respectively restricted to the image of γ. Since F is not a corner or an edge, we can find a $\delta > 0$ small enough such that $A(x)$ and $B(y)$ are positive functions on the δ-neighbourhoods (with respect to the metrics induced by g_f) I_x^δ and I_y^δ of I_x and I_y respectively. We will also choose δ small enough such that neither I_x^δ nor I_y^δ are dense subsets of $\mathbb{RP}^1$. Consider the set

$$V := \{(x, y, \vec{t}) \in \mathcal{A} : x \in I_x^\delta, y \in I_y^\delta, \vec{t} = \vec{t_0}\}.$$

Since neither I_x^δ nor I_y^δ are dense subsets of $\mathbb{RP}^1$, the metric induced by g_f on V is uniformly equivalent to $dx^2 + dy^2$. The metric induced by g_0 on V is $\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)}$. Since $A(x)$ and $B(y)$ are both positive on V, g_0 is uniformly equivalent to $dx^2 + dy^2$, and hence g_f, on V. Thus there exists some $C > 0$ such that $g_0 < Cg_f$ on V.

Let $\epsilon > 0$. Since $\phi(x, y)$ vanishes on F, we can approximate γ with a curve γ_ϵ connecting $(x_1, y_1, \vec{t_0})$ to $(x_2, y_2, \vec{t_0})$ with the interior of the image contained in V satisfying $\left|\text{length}_{g_f}(\gamma) - \text{length}_{g_f}(\gamma_\epsilon)\right| < \epsilon$ and $\phi(x, y) \circ \gamma^\epsilon < \epsilon$. Since $g = \phi(x, y)g_0$, we find

$$d_g\left(\mathbb{T}_{(x_1, y_1)}, \mathbb{T}_{(x_2, y_2)}\right) \leq \text{length}_g(\gamma_\epsilon) < \epsilon \, \text{length}_{g_0}(\gamma_\epsilon)$$
$$< C\epsilon \, \text{length}_{g_f}(\gamma_\epsilon) < C\epsilon \left(\text{length}_{g_f}(\gamma) + \epsilon\right).$$

Taking $\epsilon \to 0$, we find that the g-distance between the $\mathbb{T}$-fibres $\mathbb{T}_{(x_1, y_1)}$ and $\mathbb{T}_{(x_2, y_2)}$ is zero. This means that F represents only one $\mathbb{T}$-orbit in $\mathcal{A}_C^g$. In order for the g-volume of this orbit to be well-defined, we must have $\text{Vol}_g(\mathbb{T}_{(x, y)})$ constant. In the case when $g = g_+$, this means that $q(x, y)^4 \propto A(x)B(y)$. But it is only possible to write $q(x, y)^4$ as a product of a function of x and a function of y in the parabolic case, where F is an edge. Similarly, we rule out the case where $g = g_-$ and $g = g_p$ unless p is a constant.

If $g = g_p$ for a constant p, then $A(x)$ and $B(y)$ are constants, say A and B. We will rule out this case by a curvature computation. Since $(\mathcal{A}, g_-, \omega_-)$ is Kähler, we have that the anti-self-dual part of the Weyl curvature tensor of $[g_-]$ is given by

$$W_- \propto s_-(\omega_- \otimes \omega_-)_0^{\sharp_-},$$

where $(\cdot)_0$ is the projection onto the trace-free part of $\text{Sym}^2(\Lambda^2 \mathcal{A})$ and s_- is the scalar curvature of s_-. Using $g_p = (x - y)^2 g_-$, we estimate

$$\|\text{Rm}_{g_p}\|_{g_p} \geq \|W_-\|_{g_p} = \frac{p^2 \|W_-\|_{g_-}}{(x - y)^2} \propto \frac{|s_-|}{(x - y)^2}.$$

In [ACG13a], s_- is computed to be

$$s_- = -\frac{((x-y)^2, A(x))^{(2)} + ((x-y)^2, B(y))^{(2)}}{(x-y)q(x,y)},$$

where for arbitrary functions $f_1(z, w)$ and $f_2(z)$,

$$(f_1(z,w), f_2(z))^{(2)} = f_1^2 \frac{\partial}{\partial z}\left(f_1 \frac{\partial}{\partial z}\left(\frac{f_2}{f_1^2}\right)\right) = f_1 \frac{d^2 f_2}{dz^2} - 3\frac{\partial f_1}{\partial z}\frac{df_2}{dz} + 6\frac{\partial^2 f_1}{\partial z^2}f_2.$$

In particular, since $A(x)$ and $B(y)$ are constants, we find that $s_- = -2\frac{A+B}{(x-y)q(x,y)}$, so that $\|\operatorname{Rm}_{g_p}\|_{g_p} \gtrsim \frac{A+B}{(x-y)q(x,y)}$ on $\mathcal{A}$. Since $\|\operatorname{Rm}_{g_p}\|_{g_p}$ must be well-defined at the fold F, where $(x-y)q(x,y)$ vanishes, we must have $A + B = 0$. This contradicts $A(x)$ and $B(y)$ being positive functions. $\square$

We can use the following lemma to test the infinitely distant criterion on proper folds.

Lemma 4.17. *Let g be a metric on a dense set of $\mathcal{A}$. Let F be the vanishing locus of some function ϕ on $\mathcal{A}_C^{g_f}$. Assume that there exists some $r \in \mathbb{R}$ such that each point of F has a neighbourhood where $\|d\phi\|_g \asymp \phi^r$. In other words, there exists some $C > 0$ such that $\frac{1}{C}\phi^r < \|d\phi\|_g < C\phi^r$ on the neighbourhood. Then F is infinitely distant with respect to g if and only if $r \geq 1$.*

Proof. Let $\gamma : [0, 1) \to \mathcal{A}$ be a curve limiting to F. Since $\lim_{t\to 1} \phi \circ \gamma(t) = 0$, we can find some $T \in [0, 1)$ such that $\gamma|_{[T,1)}$ is transverse to the level sets of ϕ. Thus the function $\phi \circ \gamma|_{[T,1)} : [T, 1) \to \mathbb{R}$ is invertible with image $(0, \epsilon]$, where $\epsilon = \phi \circ \gamma(T)$. Then $\tilde{\gamma} := \gamma \circ (\phi \circ \gamma)^{-1} : (0, \epsilon] \to \mathcal{A}$ is a reparametrization of a tail of γ which satisfies $\phi \circ \tilde{\gamma}(\tau) = \tau$. Taking the derivative of this relation, we find that $d\phi \circ \dot{\tilde{\gamma}} = 1$. We can choose ϵ small enough such that the $\frac{1}{C}\phi^r < \|d\phi\|_g < C\phi^r$ holds. We compute

$$(5) \quad \operatorname{length}_g(\gamma) \geq \operatorname{length}_g(\tilde{\gamma}) = \int_0^\epsilon \|\dot{\tilde{\gamma}}(\tau)\|_g d\tau \geq \int_0^\epsilon g\left(\frac{\nabla^g \phi}{\|\nabla^g \phi\|_g}, \dot{\tilde{\gamma}}\right)_{\tilde{\gamma}(\tau)} d\tau$$

$$= \int_0^\epsilon \left(\frac{d\phi \circ \dot{\tilde{\gamma}}}{\|d\phi\|_g}\right)_{\tilde{\gamma}(\tau)} d\tau \asymp \int_0^\epsilon \frac{d\tau}{\phi^r \circ \tilde{\gamma}(\tau)} = \int_0^\epsilon \frac{d\tau}{\tau^r}.$$

In the case when $r \geq 1$, this integral is infinite, so that the g-length of g is infinite. Since γ is an arbitrary curve limiting to F, we find that F is g-infinitely distant as claimed.

Conversely, consider the case when $r < 1$. One can use the g-gradient flow of the function ϕ to construct a curve γ limiting to F which satisfies $\dot{\gamma} =$

($\nabla^g \phi$). Clearly γ admits a parametrization by some parameter τ satisfying $\phi \circ \gamma(\tau) = \tau$. Let $\tilde{\gamma}$ be the restriction of γ to a neighbourhood of $\lim_{\tau \to 0} \gamma(\tau)$ such that $\|d\phi\|_g \asymp \phi^r$. Without loss of generality, we can assume that $\gamma = \tilde{\gamma}$. Then (5) holds after replacing the inequalities with equalities. Since $r < 1$, we find that the g-length of γ is finite, so that F is not g-infinitely distant. $\qquad \square$

Lemma 4.18. *If $g \in \{g_0, g_\pm\} \cup \{g_p\}_{p \perp q}$, then all proper folds are not infinitely distant with respect to g.*

Proof. First we consider the case of a negative fold F. This is the vanishing locus of $f = \frac{q(x,y)}{x-y}$. Up to a change of gauge, we may assume that $q(x, y)$ is not a constant function. This allows us to identify F as the vanishing locus of $q(x, y)$. It is a simple computation with the normal forms to verify that $dq(x, y)$ and $q(x, y)$ do not share any vanishing points. This, combined with the form of g_0 from (1) allow us to deduce that $\|dq(x,y)\|_{g_0}$ extends to a positive function on F. This allows us to apply the previous lemma with $\phi = q(x, y)$ and $r = 0$ to deduce that F is not infinitely distant with respect to g_0. Since $g_\pm = f^{\mp 1} g_0$, $\|dq(x,y)\|_{g_\pm} = f^{\pm \frac{1}{2}} \|dq(x,y)\|_{g_0}$. Since $(x - y)$ does not vanish on F, we can then apply the previous lemma with $r = \pm \frac{1}{2}$ to deduce that F is not infinitely distant with respect to $g_\pm$.

Similarly, if $p \neq q$, then applying the previous lemma with $r = 0$ gives the analogous result for g_p. If $p = q$, then $g_p = g_+$, so we've already seen that F is not g_p-infinitely distant. The case of positive folds follows similarly by switching the roles of $q(x, y)$ and $(x - y)$. $\qquad \square$

Lemma 4.19. *The proper part of P is infinitely distant with respect to g_p.*

Proof. If p is constant, then P is empty, so there is nothing to show. As we saw in Lemma 2.4, $dp(x, y) = \frac{1}{2} p'(y) dx + \frac{1}{2} p'(x) dy$, which only vanishes along P in the case when $< p, p >= 0$. In this case, there is some $\gamma \in \mathbb{R}$ so that $p(z) = (z - \gamma)^2$, and $\{dp(x, y) = 0\} \cap P = \{x = y = \gamma\}$. This is a $\mathbb{T}$-orbit in $\mathcal{A}_C^{g_f}$ which is a corner where P meets the fold $\{x = y\}$. Since we are treating corners separately, we find that $dp(x, y)$ is well-defined and non-vanishing on the proper part of P. It follows that $\|dp(x,y)\|_{g_0}$ extends to a positive function on the proper part of P. Since $\|dp(x,y)\|_{g_p} = p(x,y)\|dp(x,y)\|_{g_0}$, the result follows by applying Lemma 4.17 with $\phi = p(x, y)$ and $r = 1$. $\qquad \square$

4.4.2. Edges. We start our investigation of the edges by building criteria to test the infinitely distance condition. The following lemma lies at the heart of the argument:

Lemma 4.20. *Let N be a manifold and $T > 0$. Consider the Riemannian manifold*

$$(\hat{N}, g) := \big((0, T) \times N, a(t)^2 dt^2 + g_N(t)\big),$$

where $a(t)$ is a smooth positive function on $(0, T)$ and $\{g_N(t)\}_{t \in (0,T)}$ is a smooth family of metrics on N. Then any smooth curve $\gamma : [0, 1) \to \hat{N}$ satisfying $\lim_{\tau \to 1} t \circ \gamma(\tau) = 0$ has finite g-length if and only if there exist some $\epsilon > 0$ such that $\int_0^\epsilon a(t)dt$ converges.

Proof. First assume that any curve $\gamma : (0, 1) \to \tilde{M}$ with $\lim_{\tau \to 0} t \circ \gamma(\tau) = 0$ has infinite length. In particular, for any $p \in N$ and $\epsilon \in (0, T)$, the curve $\gamma(\tau) := (\epsilon\tau, p)$ has infinite length. But this length is

$$\int_0^1 \sqrt{g(\dot{\gamma}(\tau), \dot{\gamma}(\tau))}d\tau = \int_0^1 \epsilon a(\epsilon\tau)d\tau = \int_0^\epsilon a(t)dt,$$

so that this integral cannot converge as claimed.

Conversely, suppose that any curve $\gamma : (0, 1) \to \tilde{M}$ with $\lim_{\tau \to 0} t \circ \gamma(\tau) = 0$ has infinite length. Fix such a gamma. We may reparametrize γ so that for some $\epsilon > 0$, $t \circ \gamma(\tau) = \tau$ for all $\tau \in (0, \epsilon)$. We then have

$$\infty > \int_0^1 \sqrt{g(\dot{\gamma}(\tau), \dot{\gamma}(\tau))}d\tau \geq \int_0^\epsilon \sqrt{g(\dot{\gamma}(\tau), \dot{\gamma}(\tau))}d\tau$$

$$\geq \int_0^\epsilon \sqrt{a(t)^2 dt^2(\dot{\gamma}(\tau), \dot{\gamma}(\tau))}d\tau = \int_0^\epsilon a(\tau)d\tau.$$

This proves our claim. $\qquad\qquad\square$

We now move on to consider the case $g = g_0$:

Lemma 4.21. *If an edge E is given by $\{x = x_0\}$ (respectively $\{y = y_0\}$), then E is infinitely distant with respect to g_0 if and only if there does not exist an $\epsilon > 0$ such that $\int_{x_0}^{x_0+\sigma\epsilon} \frac{dx}{\sqrt{A(x)}}$ (respectively $\int_{y_0}^{y_0+\sigma\epsilon} \frac{dy}{\sqrt{B(y)}}$) converges. Here $\sigma \in \{\pm 1\}$ is chosen so that $x_0 + \sigma\epsilon$ or $y_0 + \sigma\epsilon$ lies in the range of x or y on $\mathcal{A}$. Also x_0 and y_0 are assumed to be finite, which can be arranged after a gauge transformation.*

Proof. We will prove the case for x_0 with $\sigma = 1$. Consider a $\mathbb{T}$-orbit $\mathcal{O}$ on E. It has coordinates (x_0, y_0) for some $y_0 \in \mathbb{R}$. We can find a neighbourhood I_y of y_0 and some $T > 0$ such that $(x_0, x_0 + T) \times I_y \times \mathbb{T} \subset \mathcal{A}$. Let $N = I_y \times \mathbb{T}$.

Consider the following coordinate transformation

$$\phi : M \to (0, T) \times N : (x, y, \vec{t}) \mapsto (x - x_-, (y, \vec{t})).$$

It is easy to see that $\phi_* g_0$ is of the form $a(t)^2 dt^2 + g_N(t)$, where $\phi^* a(t) = \frac{1}{\sqrt{A(x)}}$. By the previous lemma, we find that $\mathcal{O}$ is in the g_0-Cauchy completion of M if and only if there exists some $\epsilon > 0$ such that the integral

$$\int_0^\epsilon a(t) dt = \int_{x_-}^{x_- + \epsilon} \frac{dx}{\sqrt{A(x)}}$$

converges. Since $\mathcal{O}$ was an arbitrary $\mathbb{T}$-orbit on E, this gives the desired result. $\square$

To understand how the condition of g-completable applies to the edges, we will borrow from the work of [ACG13b]. They define:

Definition 4.22. An *ambitoric compactification* is a compact connected oriented 4-orbifold M with an effective action of 2-torus $\mathbb{T}$ such that on the (dense) union $\mathring{M}$ of the free $\mathbb{T}$-orbits, there is an ambitoric structure $(g_\pm, J_\pm, \mathbb{T})$ for which at least one of the Kähler metrics extends smoothly to a toric Kähler metric on $(M, \mathbb{T})$. An ambitoric compactification is *regular* if the ambitoric structure on $\mathring{M}$ is regular with (x, y)-coordinates that are globally defined.

We will add some definitions:

Definition 4.23. A connected component U of an ambitoric ansatz space $\mathcal{A}$ is *box-type* if all of its folds are also edges or corners. It's clear that such a U must be of the form $(x_-, x_+) \times (y_-, y_+) \times \mathbb{T} \subseteq \mathcal{A}$, for some intervals $(x_-, x_+), (y_-, y_+) \subset \mathbb{R}$.

Definition 4.24. Let E be an edge given by $\{x = x_0\}$ (respectively $\{y = y_0\}$). We say E has a *compatible normal* if $A(x_0) = 0$ (respectively $B(y_0) = 0$) and the normal vector to $\mu^\pm(E)$ given by $-2\frac{p^{(x_0)}}{A'(x_0)}$ (respectively $2\frac{p^{(y_0)}}{B'(y_0)}$) lies in the lattice $\Lambda \subset \mathfrak{t}$, where for each $\gamma \in \mathbb{R}$, $p^{(\gamma)} \in \mathfrak{t}$ corresponds to the polynomial $p^\gamma(x, y) := \frac{1}{2}(x - \gamma)q(y, \gamma) + \frac{1}{2}q(x, \gamma)(y - \gamma)$ under the identification of $\mathfrak{t}$ with symmetric quadratic polynomials orthogonal to $q(x, y)$. See [ACG13a] for details on this identification.

Note that the above definition is given for the case when x_0 or y_0 is finite. In the infinite case, one can reduce to the finite case by performing a gauge transformation (for example $z \mapsto -\frac{1}{z}$).

We can then paraphrase the classification of ambitoric compactifications (proposition 3 in [ACG13b]):

Proposition 4.25. *A box-type component of an ambitoric ansatz space is the interior of an ambitoric compactification if and only if each of its edges has a compatible normal.*

This will allow us to prove:

Lemma 4.26. *Let $g \in \{g_0, g_\pm\} \cup \{g_p\}_{p \perp q}$. If $\mathcal{A}$ is g-completable, then each edge is either infinitely distant, a fold, part of P, or has compatible normal.*

Proof. Let E be an edge which is neither a fold nor infinitely distant nor part of P. Since $\mathcal{A}$ is is g-completable, E lies in $\mathcal{A}_C^g$ as a set of points with at worst orbifold singularities, and g extends to E as a smooth metric. Let $\phi(x,y)$ be the function such that $g_+ = \phi(x,y)g$. Since E is not a fold, $\phi(x,y)$ is a positive function along E. This implies that g_+ also extends to E as a smooth metric.

We want to apply the previous proposition here. The proof of that proposition is local in nature, so we can apply it locally. More explicitly, if both g_+ and ω_+ extend to E, then E looks like a piece of an ambitoric compactification, so we can deduce that E has a compatible normal. We have shown that g_+ extends, but not yet ω_+. However, the proof of the previous proposition does not actually require that ω_+ extends, although it certainly is necessary by the converse. To see this, let p be a point on E. Without loss of generality, by switching the roles of x and y if necessary, E is of the form $\{x = x_0\}$ and the $\mathbb{T}$-orbit $\mathbb{T} \cdot p$ is given by $\{x = x_0, y = y_0\}$. Let $\mathcal{A}_{y=y_0} = \{(x, y, \vec{t}) \in \mathcal{A} : y = y_0\}$. Then the normal bundle to $\mathbb{T} \cdot p$ in $T_p \mathcal{A}_{y=y_0}$ is a vector space which coincides with the space V_p in the proof of proposition 1 of [ACGTF04]. The construction of V_p was the only part of that direction of that proof which used the symplectic structure, so we can use it to deduce boundary conditions which are shown in [ACG13b] to be equivalent to the condition that E has a compatible normal. $\qquad\square$

Lemma 4.27. *Let $g \in \{g_0, g_\pm\} \cup \{g_p\}_{p \perp q}$. If $\mathcal{A}$ is g-completable, then every edge which is also a fold but neither infinitely distant nor part of P must have a compatible normal and $g \in \{g_-, g_p\}$.*

Proof. Let E be such an edge. An edge can only be a fold in the parabolic case, and the fold must be negative. We will change the gauge so that $q(x, y) = xy$. We consider the case when E is given by $\{x = 0\}$, while the case $\{y = 0\}$ can be treated similarly.

To rule out $g \in \{g_0, g_+\}$, we use the fact that the volume of a $\mathbb{T}$-fibre on E must be well-defined, as in Lemma 4.16. First consider the case $g = g_0$. We saw in Lemma 4.16 that the g_0 volume of the $\mathbb{T}$-fibre at (x, y) is given by $\frac{A(x)B(y)}{(x-y)^2 x^2 y^2}$. For this to be defined at $x = 0$, $A(x)$ must vanish to at least second order at 0. By Lemma 4.21, this implies that E is infinitely distant with respect to g_0, contradicting the definition of E.

Similarly for $g = g_+$, we find that $\frac{A(x)B(y)}{x^4 y^4}$ must be well-defined, implying that $A(x)$ vanishes to order 4 at 0. A similar argument to Lemma 4.21 shows that if $q(x, y) = xy$, then E is infinitely distant with respect to g_+ if and only if there does not exist an $\epsilon > 0$ such that the integral $\int_0^\epsilon \frac{x \, dx}{\sqrt{A(x)}}$ converges. This is the case when $A(x)$ vanishes to order 4 at 0, contradicting the definition of E as not infinitely distant with respect to g_+.

The fact that Ee must have a compatible normal follows the same argument as the previous lemma, replacing g_+ and ω_+ with g_- and ω_-. $\square$

Lemma 4.28. *If $\mathcal{A}$ is g_p-completable, then every edge which is a part of P is infinitely distant with respect to g_p.*

Proof. Let E be an edge which is part of P which is not infinitely distant with respect to g_p. We will treat the case where E is given by $\{x = \gamma\}$ for some $\gamma \in \mathbb{R}$. The case where $E = \{y = \gamma\}$ can be treated similarly, and the case when $\gamma = \infty$ can be reduced to the finite case by a gauge transformation. Since E is part of P, there exist some $c \in \mathbb{R}$ such that $p(x, y) = c(x - \gamma)(y - \gamma)$. Since $\mathcal{A}$ is g_p-completable and E is not infinitely distant, we know that the volume of a $\mathbb{T}$-fibre on E must be well defined. Again referring to the table from Lemma 4.16, this volume is $\frac{A(x)B(y)}{p(x,y)^4}$. In order for this to be well-defined, $A(x)$ must vanish to order at least 4 at γ.

Now consider the function $\phi = x - \gamma$ on $\mathcal{A}$. We compute

$$\|d\phi\|_{g_p} = \frac{p(x, y)\sqrt{A(x)}}{\sqrt{q(x, y)(x - y)}}.$$

Since $p(x, y)$ vanishes to order 1 on E and $A(x)$ vanishes to order at least 4, while $q(x, y)$ vanishes to order 1 or 0 (if $q = p$ or $q \neq p$ respectively), we find that $\|d\phi\|_{g_p} \gtrsim \phi^{\frac{3}{2}}$. We now apply Lemma 4.17 to deduce the result. $\square$

4.4.3. Corners.

Lemma 4.29. *If $\mathcal{A}$ is g-completable, then every corner where an infinitely distant edge or fold meets another edge or fold must be infinitely distant.*

Proof. Assume that two infinitely distant edges or folds meet at a corner C which is not infinitely distant. We can find a small neighbourhood of this corner of the form

$$((\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\})) \cup C,$$

where C is glued into $(0,0)$ as a subgroup of $\mathbb{T}$. This is never an orbifold, contradicting $\mathcal{A}$ being g-completable.

Similarly, let C' be a corner where an infinitely distant edge or fold E_∞ meets an edge or fold E_f which is not infinitely distant. By Lemmas 4.16, 4.26 and 4.27, E_f must be an edge with a compatible normal. It follows that points of E_f have neighbourhoods of the form $\mathbb{C}/\Gamma \times \mathbb{C}$, where Γ is the orbifold covering group. It follows that C has a neighbourhood of the form

$$((\mathbb{C}/\Gamma) \times (\mathbb{C}\backslash\{0\})) \cup C,$$

where C is glued into $(0,0)$ as a subgroup of $\mathbb{T}$. This is never an orbifold, contradicting $\mathcal{A}$ being g-completable. $\qquad\square$

Lemma 4.30. *Let $g \in \{g_0, g_\pm\} \cup \{g_p\}_{p\perp q}$. If $\mathcal{A}$ is g-completable and C is a corner at the intersection of two edges which are not infinitely distant, then if C is a positive fold, then $g \in \{g_+, g_p\}$, while if C is a negative fold, then $g \in \{g_-, g_p\}$. If $g = g_p$, then C cannot be part of P.*

Proof. We will prove the case that C is a positive fold. The case when C is a negative fold or part of P is treated similarly. By Lemmas 4.26 and 4.27, the edges adjacent to C must have compatible normals. This implies that $A(x)$ and $B(y)$ both vanish to order 1 at C. For topological reasons, C cannot be a free $\mathbb{T}$ fibre with respect to g. In particular, the volume of the fibre at C must vanish. Again we refer to the table of fibre volumes in Lemma 4.16 to deduce that $g \in \{g_+, g_p\}$. $\qquad\square$

4.5. Classification results

Now we can combine all of our asymptotic analysis of the previous section to prove our classification of completable ambitoric ansatz spaces. This then combines with Theorem 4.15 to give our main classification results.

Proof of Theorem 1.1. First assume that $\mathcal{A}$ is g completable. By Lemma 4.16, every proper fold must be infinitely distant, but by Lemma 4.18 no

proper fold is infinitely distant. Thus $\mathcal{A}$ can have no proper folds, which is the first statement. The next two statements are the result of Lemmas 4.26 and 4.27.

By Lemma 4.30, every corner is infinitely distant unless it is at the intersection of two folds or edges which are not infinitely distant. However, we've seen that a fold or an edge which is not infinitely distant must be an edge with a compatible normal. This added to Lemma 4.30 gives the last condition.

Conversely, assume that the stated conditions hold. The first condition combined with Lemma 4.19 allows us to decompose $\mathcal{A}_C^g$ into edges and corners.

The second condition implies that each edge which is not infinitely distant has a compatible normal. Let E be such an edge. Since the proof of Proposition 4.25 is local in nature, it allows us to deduce that for one of the metrics $g_\pm$, E lies in $\mathcal{A}_C^{g_\pm}$ as a set of points with at worst orbifold singularities, and $g_\pm$ extends to E as a smooth metric. In the case that E is not a fold, then the conformal factor relating g to $g_\pm$ extends to a positive function on E, so that g also extends to E as a smooth metric. If E is a fold, then the choice of $g_\pm$ must have been g_- by the third condition. The third condition also ensures that $g = \eta(x,y)g_-$, for some $\eta(x,y) \in \{1, \frac{(x-y)^2}{p(x,y)^2}\}$. Since $\eta(x,y)$ is positive on E, g extends to E as a smooth metric.

Finally, let C be a corner which is not infinitely distant. By the last condition, C must lie at the intersection of two edges which are not infinitely distant, and thus have compatible normals. As in the previous paragraph, we can locally apply Proposition 4.25 to deduce that for one of $g_\pm$, C lies in $\mathcal{A}_C^{g_\pm}$ as a point with at worst an orbifold singularity. The conditions that we put on g if C is a fold ensures that the conformal factor relating g to $g_\pm$ on $\mathcal{A}$ extends to $\mathcal{C}$ as a positive function, ensuring that g extends smoothly to C.

We've shown that every point in $\mathcal{A}_C^g$ has at worst orbifold singularities, and that g extends to a smooth metric on $\mathcal{A}_C^g$. Thus $\mathcal{A}$ is g-completable as claimed.

To prove the *moreover*, first note that if the ambitoric structure extends to $\mathcal{A}_C^g$, then the function f is a conformal factor between the metrics g_+ and g_0 on $\mathcal{A}_C^g$. In particular, f must be a smooth positive function on $\mathcal{A}_C^g$. It follows that $\mathcal{A}_C^g$ has no folds. In other words, every fold is infinitely distant.

Conversely, if every fold is infinitely distant, then the functions $q(x,y)$ and $(x-y)$ extend to smooth positive functions on $\mathcal{A}_C^g$. We can use these functions to construct the conformal factor between $g_\pm$ and g_- on $\mathcal{A}_C^g$, so that $g_\pm$ both extend as smooth metrics to $\mathcal{A}_C^g$. Similarly, by considering the

expressions for $\omega_\pm$ in (1), these also extend to $\mathcal{A}_C^g$, so that $\mathcal{A}_C^g$ is ambitoric. $\square$

Proof of Theorem 1.2. By Theorem 1.1, M has no proper folds. The result then follows from Theorem 4.15. $\square$

Proof of Corollary 1.3. A regular ambitoric 4-orbifold M which is complete with respect to g_0 is a regular ambitoric orbifold completion with respect to g_0. We can then apply Theorem 1.2 to embed the set of free orbits $\mathring{M}$ of M in the Cauchy completion (with respect g_0) of an ambitoric ansatz space $\mathcal{A}(q, A, B, \mathbb{T})$ which has polygonal moment map images. Moreover, since $\mathcal{A}$ is completable, it has no proper folds by Theorem 1.1. Since $g \in \{g_0, g_\pm\}$, there is no P boundary component. Combining these facts, we find that $\mathcal{A}$ is of box-type. In particular there exist $x_\pm, y_\pm \in \mathbb{R}$ such that $\mathring{M} \cong (x_-, x_+) \times (y_-, y_+) \times \mathbb{T} \subset \mathcal{A}(q, A, B, \mathbb{T})$, and that $(x - y)q(x, y)$ does not vanish on $(x_-, x_+) \times (y_-, y_+)$.

By Theorem 1.1, each edge is either infinitely distant or has compatible normal, and each fold is infinitely distant. The condition on the edges gives the constraints on $A(x)$ and $B(y)$, using Lemma 4.21. By an argument similar to Lemma 4.21, each corner is infinitely distant with respect to g_0 if and only if either of the adjacent edges is. This implies that if two edges which are not infinitely distant meet at a corner, then that corner cannot be a fold. This gives the last condition.

Conversely, given the data described in corollary, on can construct a torus $\mathbb{T} = \mathbb{R}^2/\Lambda$, and an ambitoric ansatz space $\mathcal{A}(q, A, B, \mathbb{T})$. We find that $\mathcal{A}_0 : (x_-, x_+) \times (y_-, y_+) \times \mathbb{T} \subset \mathcal{A}(q, A, B, \mathbb{T})$ is a connected component of an ambitoric ansatz space with no proper folds and each of its four edges being either infinitely distant or having compatible normals. Then Theorem 1.1 tells us that $(\mathcal{A}_0)_C^{g_0}$ is a regular ambitoric 4-orbifold which is complete with respect to g_0. $\square$

5. Application: Compact 4-manifolds admitting Killing 2-forms

In this section, we apply our results to the setting of 4-manifolds admitting Killing 2-forms. This will build on the work in [GM17], which we will summarize now.

Definition 5.1. A differential form ψ on a Riemannian manifold (M, g) is *conformally Killing* if its covariant derivative $\nabla \psi$ is of the form

$$\nabla_X \psi = \alpha \wedge X^\flat + X \lrcorner \beta, \qquad \forall X \in \Gamma(TM),$$

for some differential forms α and β. Moreover, ψ is *Killing* if $\alpha = 0$ and *∗-Killing* if $\beta = 0$.

Note that if M is oriented and $*$ is the Hodge star operator, then ψ is Killing if and only if $*\psi$ is $*$-Killing.

Proposition 5.2 (2.1 in [GM17]). *Let (M, g) be a connected oriented Riemannian 4-manifold admitting a non-parallel ∗-Killing 2-form ψ. Then on an open dense subset M_0 of M, the pair (g, ψ) gives rise to an am-biKähler structure $(g_+, J+, \omega_+), (g_-, J_-, \omega_-)$, where $g_\pm = f_\pm^{-2} g$. Here $f_\pm := \frac{|\psi_\pm|}{\sqrt{2}}$, where $\psi_\pm$ are the self-dual and anti-self-dual parts of ψ.*

Define vector fields K_1 and K_2 on such an M by

$$K_1 := -\frac{1}{2} \alpha^\sharp,$$

$$K_2 := \frac{1}{2} (*\psi)^\sharp (K_1).$$

These can be used to form a rough classification:

Proposition 5.3 (3.3 in [GM17]). *Any connected oriented 4-dimensional Riemannian manifold (M, g) admitting a non-parallel ∗-Killing 2-from ψ fits into one of the following three exclusive possible cases:*

1) *The vector fields K_1 and K_2 are Killing and independent on a dense open set $\mathcal{U}$ of M.*

2) *The vector fields K_1 and K_2 are Killing and $K_2 = cK_1$ for some $c \in \mathbb{R} \backslash 0$.*

3) *$f_+ = f_-$ on all of M, $K_2 = 0$, and K_1 is not a Killing vector field in general.*

The authors go on to show that in the first case, $\mathcal{U}$ admits a regular ambitoric structure $(\mathcal{U}, [g], J_+, J_-)$ of hyperbolic type, where g is identified with the metric g_1 (g_p when $p = 1$). In the language that we have developed, $(M, \mathcal{U}, g = g_1, J_+, J_-)$ is a regular ambitoric manifold completion. We can then apply our classification result to obtain our claimed result:

Proof of Theorem 1.4. By Theorem 1.2, $\mathcal{U}$ can be embedded into some ambitoric ansatz space $\mathcal{A}$ whose moment map images are polygonal. Since $P = \{1 = p(x, y) = 0\} = \emptyset$, we can argue as in the proof of Corollary 1.3 that $\mathcal{U}$ is embedded as a box-type subset of $\mathcal{A}$. It follows that the action of 2-torus on $\mathcal{U}$ extends to M with at most 4 fixed points, corresponding to the corners of the box. We can then apply the work of [OR70], which classifies oriented 4-manifolds equipped with a 2-torus action. Since the number of fixed points t is at most 4, this classification tells us that M is (up to orientation reversal) diffeomorphic to one of $\{\mathbb{S}^4, \mathbb{CP}^2, \mathbb{CP}^2 \# \mathbb{CP}^2, \mathbb{S}^2 \times \mathbb{S}^2, \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\}$.

Each Hirzebruch surface is diffeomorphic to either $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, so it suffices to rule out $\mathbb{CP}^2 \# \mathbb{CP}^2$. To see this, we note that the convexity of $\mu_\pm(\mathcal{U})$ implies that we can ensure that the $\{\epsilon_i\}_{i=1}^t$ used in [OR70] are all equal to 1 by choosing the generators of the isotropy subgroups to correspond to inward normals for the sides of $\mu_\pm(\mathcal{U})$. This rules out the case of $\mathbb{CP}^2 \# \mathbb{CP}^2$, since this case has $t = 4$ and $\epsilon_1 \epsilon_4 = -\epsilon_2 \epsilon_3$.

Conversely, we must show that $\mathbb{S}^4$, $\mathbb{CP}^2$ and the Hirzebruch surfaces all admit metrics which admit non-parallel $*$-Killing 2-forms. The case of $\mathbb{S}^4$ with the usual constant scalar curvature metric is explored in detail in [GM17]. For $\mathbb{CP}^2$ and the Hirzebruch surfaces, we will show that they are regular ambitoric orbifold completions of hyperbolic type. The converse of Proposition 5.2 [GM17] then provides the claimed $*$-Killing 2-form. We will use the hyperbolic normal form from Section 1.2.2, so that $q(x, y) = x + y$ and the conic is given by $\mathcal{C}_\pm = \{4\mu_1^\pm \mu_2^\pm = -1\}$. We will take Λ to be the standard basis for $\mathfrak{t}$ in the coordinates given in the normal form.

We can construct lines tangent to $\mathcal{C}_\pm$ with normals $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, as in Figure 7. Each pair of normals generates Λ, so that the moment map image is a Delzant triangle. It is well known that the Delzant construction applied to a Delzant triangle is $\mathbb{CP}^2$.

We can construct lines tangent to $\mathcal{C}_\pm$ with normals $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -k-1 \\ k \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, as in Figure 8. As in the triangle case, it is easy to check that the trapezoid Δ formed by these lines is Delzant. Moreover the equation $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -k-1 \\ k \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ implies that Δ is the moment-map image of the kth Hirzebruch surface H_k (see [KKP07]). Note that to get all of the diffeomorphism types, it suffices to only construct H_1 and H_2, since the odd (respectively even) order Hirzebruch surfaces are all diffeomorphic
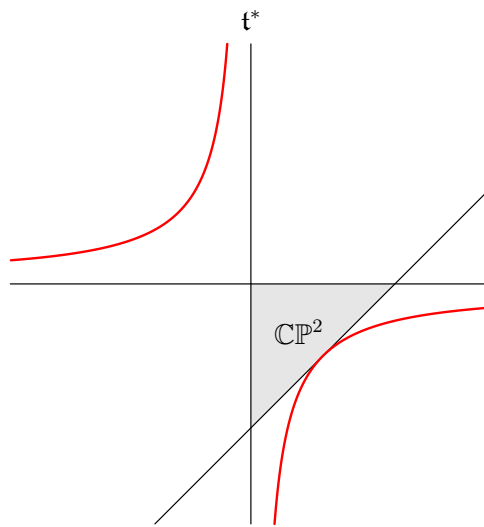
Figure 7: The moment map image of $\mathbb{CP}^2$ with a hyperbolic ambitoric structure.

(see for example [KKP07]). Moreover, $\mathbb{S}^2 \times \mathbb{S}^2$ is diffeomorphic to H_2, so it is covered by this construction as well.
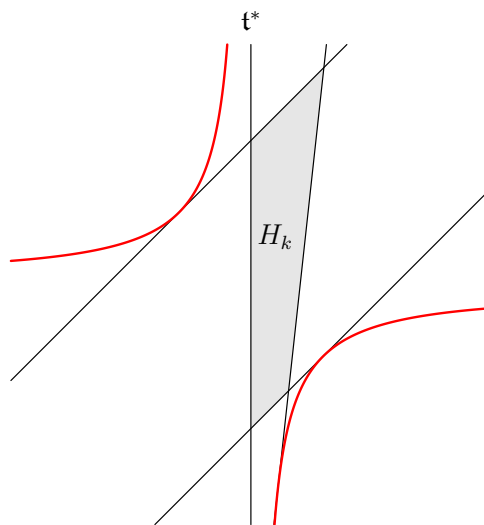

Figure 8: The moment map image of the kth Hirzebruch surface H_k with a hyperbolic ambitoric structure.

□

Acknowledgments

Appendix A. Busemann completions

The Busemann completion of a Riemannian manifold is a set which includes the Cauchy completion as well as elements corresponding to directions at infinity. Our goal is to relate the Busemann completions of different metrics on the same manifold. We follow the work of [FHS13], although we make some cosmetic changes of definitions to suit our goals.

Definition A.1. Let (M, g) be a Riemannian manifold. Let

$$C(M) := \big\{c : [0, \infty) \to M \text{ piece-wise smooth curve}\big\}/ \sim,$$

where $\sim$ is equivalence of curves by reparametrization. We will refer to a class of curves $[c] \in C(M)$ simply by c, thinking of it as an unparametrized curve. For $c \in C(M)$, we define the *Busemann function*

$$b_c^g : M \to \mathbb{R} \cup \{\infty\} : x \mapsto \lim_{s \to \infty} \left(\int_0^s \|\dot{c}(t)\|_g dt - d_g\big(x, c(s)\big) \right),$$

where d_g is the distance with respect to g. b_c is well defined, since it is clearly invariant under reparametrization of c.

We reproduce some facts from [FHS13]:

Proposition A.2 (Proposition 4.15 in [FHS13]).

- *If $b_c^g(x) = \infty$ for some $x \in M$, then $b_c^g \equiv \infty$.*

- *If $c \in C(M)$ has finite length, then there exists some $\bar{x}$ in the Cauchy completion M_C^g of M such that*

(A.1) $$b_c^g(x) = \text{length}_g(c) - d_g(x, \bar{x}).$$

Conversely, for every $\bar{x} \in M_C^g$, there exists a $c \in C(M)$ with finite length satisfying (A.1).

Definition A.3. Let $B(M)^g := \{b_c\}_{c \in C(M)} \backslash \{\infty\}$ be the set of *finite Busemann functions*. The *Busemann completion* of (M, g) is given by $M_B^g := B(M)^g / \mathbb{R}$, where $\mathbb{R}$ acts by the addition of constant functions.

Note that the second part of Proposition A.2, we can identify the Cauchy completion M_C^g as a subset of the Busemann completion M_B^g via the map $\bar{x} \mapsto [-d_g(\bar{x}, \cdot)]$.

We define an equivalence relation on $C(M)$ by

$$c_1 \sim_g c_2 \iff \exists r \in \mathbb{R} : b_{c_2}^g = b_{c_1}^g + r.$$

The map $[c] \mapsto [b_c]$ gives us an isomorphism

$$C(M)/\sim_g \; \cong M_B^g \cup \{[c \in C(M) : b_c^g \equiv \infty]\}.$$

This allows us to interpret M_B^g as a set of equivalence classes of curves. This will help us understand how M_B depends on the metric since the curves are independent of the metric; only the equivalence relation changes.

Let g_1 and g_2 be two Riemannian metrics on M. The rest of the appendix will be dedicated to proving the following proposition:

Proposition A.4. *If $M_C^{g_2}$ is compact, then every point $\bar{x} \in M_C^{g_1}$ can be represented by a point $\bar{x}' \in M_C^{g_2}$, in the sense that there is a curve $c \in C(M)$ such that $[c]_{g_1} = \bar{x}$ and $[c]_{g_2} = \bar{x}'$.*

The following lemma will do most of the work for us:

Lemma A.5. *Let $(x_n)_{n=1}^{\infty}$ be a sequence in M which g_1-converges to some $\bar{x} \in M_C^{g_1}$. Then there exists a g_2-geodesic curve c such that $\bar{x}$ lies in the g_1-closure of the image of c.*

Proof. Let $p \in M$. For each $n \in \mathbb{N}$, let $c_n \in C(M)$ be a g_2-geodesic ray starting from x_0 and passing through x_n. Each c_n is generated by the exponential mapping from a direction $u_n \in \mathbb{S}(T_pM)$ in the unit sphere bundle at p. Since $\mathbb{S}(T_pM)$ is compact, there exists a subsequence $(u_{n_k})_{k=1}^{\infty}$ of $(u_n)_{n=1}^{\infty}$ converging to some u_∞. Let c be the geodesic generated by u_∞.

Let $\epsilon > 0$. Since $\lim_{k \to \infty} u_{n_k} = u_\infty$, there exists a $k_0 \in \mathbb{N}$ such that $u_{n_k} \in B_\epsilon(u_\infty)$ for all $k > k_0$, where $B_\epsilon(u_\infty)$ is the ϵ-ball around u_∞ in $\mathbb{S}(T_pM)$.

Consider the set

$$S_\epsilon := \{\exp_p^{g_2}(tu) : t \in [0, \infty), u \in B_\epsilon(N_\infty), tu \in \text{domain}(\exp_p^{g_2})\} \subseteq M.$$

For each $k > k_0$,

$$S_\epsilon \supset \text{Image}(c_{n_k}) \ni x_{n_k}.$$

Thus

$$\overline{S_\epsilon} \supset \overline{\{x_{n_k}\}_{k > k_0}} \ni \bar{x},$$

where $\bar{\cdot}$ indicates the closure in $M_C^{g_1}$. Since $\epsilon > 0$ was arbitrarily chosen, we find that

$$\bar{x} \in \bigcap_{\epsilon > 0} \bar{S}_\epsilon = \overline{\text{Image}(c)}.$$

$\square$

Lemma A.6. *Let g_3 be a third metric on M. Let $(x_n)_{n=1}^\infty$ be a sequence in M which g_1-converges to some $\bar{x}_1 \in M_C^{g_1}$ and g_3-converges to some $\bar{x}_3 \in M_C^{g_3}$. Then there exists a g_2-geodesic curve c such that $\bar{x}_j$ lies in the g_j-closure of the image of c for each $j \in \{1, 3\}$.*

Proof. This follows from applying the previous lemma twice (once using g_3 in place of g_1), and noting from the proof that resulting curve would be the same, since it doesn't depend on g_1. $\square$

We would like to build an equivalence relation which includes the information of both $\sim_{g_1}$ and $\sim_{g_2}$. We define the relation $\sim_{g_2}^{g_1}$ by

$$c_1 \sim_{g_2}^{g_1} c_2 \iff c_1 \sim_{g_1} c_2 \text{ or } c_1 \sim_{g_2} c_2.$$

Note that $\sim_{g_2}^{g_1}$ is not an equivalence relation in general, since it may not be transitive. To see this, consider the following example. Let $M = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$. Let g_1 (respectively g_2) be a metric which contracts the first (respectively second) $\mathbb{S}^1$ factor to a point at the origin of $\mathbb{R}$. For example, using coordinates $(\theta_1, \theta_2, r) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$,

$$g_1 = r^2 d\theta_1^2 + d\theta_2^2 + dr^2,$$
$$g_2 = d\theta_1^2 + r^2 d\theta_2^2 + dr^2.$$

Let $\{t_1, t_2, t_3\}$ be different points in $\mathbb{S}^1$. Consider the constant curves on M :

$$c_1 = (t_1, t_2, 0),$$
$$c_2 = (t_3, t_2, 0),$$
$$c_3 = (t_3, t_1, 0).$$

We see that $c_1 \sim_{g_1} c_2 \sim_{g_2} c_3$, but $c_1 \not\sim_{g_2}^{g_1} c_3$.

Consider the equivalence relation $\approx$ on $C(M)$ defined by $c_1 \approx c_2$ if and only if there exists a finite set of curves $\{C_i\}_{i=1}^m \subset C(M)$ such that

$$c_1 \sim_{g_2}^{g_1} C_1 \sim_{g_2}^{g_1} C_2 \sim_{g_2}^{g_1} \cdots \sim_{g_2}^{g_1} C_m \sim_{g_2}^{g_1} c_2.$$

In other words, $\approx$ is the transitive closure of $\sim_{g_2}^{g_1}$. Define

$$M_B^{\widetilde{\approx}} := (C(M)/\approx) \backslash \{[c \in C(M) : b_c^{g_1} = g_c^{g_2} \equiv \infty]\}.$$

Let $\pi_i : M_B^{g_i} \to M_B^{\widetilde{\approx}}$ be the natural quotient map for each $i \in \{1, 2\}$.

Lemma A.7. $\pi_1(M_C^{g_1}) \subseteq \pi_2(M_B^{g_2})$.

Proof. Assume that there exists $\bar{x} \in M_C^{g_1}$ such that $\pi_1(\bar{x}) \notin \pi_2(M_B^{g_2})$. Unravelling the definitions, this means that for each curve $c \in C(M)$ g_1-converging to $\bar{x}$, $b_c^{g_2} \equiv \infty$. Fix $c(t)$ a parametrized curve g_1-converging to $\bar{x}$. Since $b_c^{g_2} \equiv \infty$, c has infinite g_2-length by Proposition A.2.

Claim: The image of c is unbounded with respect to g_2.

Proof of claim. Assume that the claim is false. Then the image of c lies in some closed set $B \subset M_C^{g_2}$ which is bounded with respect to g_2. Thus there exists a sequence $(t_n)_{n=1}^\infty$ in $\mathbb{R}$ such that $(c(t_n))_{n=1}^\infty$ converges in g_2 to some $\bar{x}' \in B$. Applying Lemma A.6 we find a g_2 geodesic curve c' which g_1-converges to $\bar{x}$ and g_2 converges to $\bar{x}'$. Since c' g_2-converges to $\bar{x}'$ it has finite g_2-length. Thus $b_{c'}^{g_2} \not\equiv \infty$, contradicting the assumption on $\bar{x}$ since c' g_1-converges to $\bar{x}$. $\qquad\square$

The claim allows us to find a sequence $(t_n)_{n=1}^\infty$ in $\mathbb{R}$ such that for any $x_0 \in M$, $\lim_{n \to \infty} d_{g_2}(x_0, c(t_n)) = \infty$. Applying Lemma A.5 to this sequence gives a

g_2-geodesic curve c' which g_1-converges to $\bar{x}$. From [Sak96] the function

$$t : \mathbb{S}(T_{x_0}M) \to (0, \infty] : u \mapsto \sup\left\{t > 0 : d_{g_2}(x_0, \exp_{x_0} tu) = t\right\}$$

is continuous. Thus

$$t(u_\infty) = \lim_{k \to \infty} t(u_{n_k}) \geq \lim_{k \to \infty} d_{g_2}\big(x_0, c(t_{n_k})\big) = \infty,$$

so that $t(u_\infty) = \infty$, where $(u_{n_k})_{k=1}^\infty$ is the sequence constructed in Lemma A.5. In other words, the geodesic c' generated by u_∞ is g_2-distance minimizing. Thus

$$b_{c'}^{g_2}(x_0) = \lim_{s \to \infty} \left(\int_0^s \|\dot{c}_\infty(t)\|_{g_2} dt - d_{g_2}(x_0, c_\infty(s)) \right) = \lim_{s \to \infty} 0 = 0,$$

so that $b_{c'}^{g_2} \not\equiv \infty$. This contradicts the assumption on $\bar{x}$, since c' g_1-converges to $\bar{x}$. $\qquad\square$

Lemma A.8. *If M_C^g is compact, then $M_C^g = M_B^g$.*

Proof. Let $x_0 \in M$. Since M_C^g is compact, the diameter of M is bounded, so that the function $d_g(x_0, \cdot)$ is bounded. This implies that $b_c^g(x_0) = \infty$ for all infinite length curves c. The result then follows from Proposition A.2. $\qquad\square$

Combining the previous two lemmas produces a proof of Proposition A.4.

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