Asymptotically flat Einstein-Maxwell fields are inheriting

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We prove that Maxwell fields of asymptotically flat solutions of the
Einstein-Maxwell equations inherit the stationarity of the metric.

1. Introduction

In the study of static or stationary Einstein-Maxwell solutions of Einstein's equations, including black hole solutions, it is frequently assumed that the Maxwell field is also static or stationary, in the sense that the Lie derivative of the Maxwell tensor F_{ij} along the time-translation Killing vector K^i is zero. Evidently one needs the energy-momentum tensor T_{ij} to be static or stationary in this sense for the Einstein equations to be consistent, but this does not actually require that F_{ij} be static or stationary — there could be a duality rotation of the electromagnetic field as one moves along the Killing vector, a transformation that is known to leave the energy-momentum tensor invariant. It is customary (see e.g. [16]) to say that F_{ij} does not *inherit* the symmetry if T_{ij} is static or stationary but F_{ij} is not, and then one can consider non-inheriting solutions. Such solutions are well known, we review them in Section 2 below.

The question was raised in [30] whether non-inheriting, static or stationary Einstein-Maxwell solutions could be asymptotically-flat.[1] Arguments were presented that under stronger conditions (analyticity up to and including the horizon) there could be no strictly non-inheriting static Einstein-Maxwell black holes but it was left open whether this undesirable analyticity requirement could be dispensed with. (It is known that analyticity holds away from the horizon, as in the inheriting case [29].) We return to the question in this paper and show that the Maxwell fields of asymptotically-flat solutions of Einstein-Maxwell equations inherit the stationarity of the metric. Hence, neither strictly stationary ("soliton") solutions nor black hole solutions which are asymptotically flat and non-inheriting exist. In fact, the mere existence of an asymptotically flat end on a spacelike hypersurface suffices to prove inheritance, without any further global conditions.

This paper is organised as follows: In Section 2 we review some non-inheriting solutions. In Section 3 we outline the derivation of the equations at hand. In Section 4 we show that the metrically-static solutions must be inheriting using integration-by-parts arguments. In Section 5 we present a proof which covers the metrically-strictly-stationary solutions, adapting and extending the arguments in [31]. While the arguments in Section 4 are hand-tailored for the problem at hand, the ones in Section 5 provide a general result which applies to a wide class of equations.

It is conceivable that the results of [29, 30] together with the unique continuation results of [18] can be used to exclude inheriting solutions with bifurcate Killing horizons, independently of asymptotic conditions, but we have not explored further this line of thought.

2. Non-inheriting solutions

Given a Killing vector field K and a Maxwell tensor F with dual $F^\star$ and energy-momentum tensor $T^{(M)}$ one says that the Maxwell field does not *inherit* the symmetry determined by K if $\mathcal{L}_K T^{(M)} = 0$ but $\mathcal{L}_K F \neq 0$. Here the symmetry can be timelike or spacelike but the main interest lies with timelike. It is straightforward to see that, in this setting, $\mathcal{L}_K F = -a\, F^\star$ for some a which need not be constant. This is a duality rotation of the Maxwell field along the Killing vector.

Some historical comments about the problem at hand are in order. Non-inheritance of symmetry is discussed in [27, Section 11.1]. Another good reference in the static case is [16]. These authors reference earlier literature

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[1]Something close to this was also asked in [22].

and reduce the static, cylindrically-symmetric Einstein-Maxwell equations, when the staticity is not inherited by the Maxwell field, to a set of coupled ordinary differential equations. They conclude that such solutions therefore exist, at least locally, contradicting a conjecture then current. Among their references is [19, Equation (5.2)], with the solution metric

$$\mathbf{g} = -(dt - br^2 d\varphi)^2 + e^{b^2 r^2}(dz^2 + dr^2) + r^2 d\varphi^2,$$

with real constant b, while the Maxwell potential is

$$A = \cos(2bz)(dt - br^2 d\varphi).$$

The metric has Killing vectors $\partial/\partial t, \partial/\partial \varphi$ and $\partial/\partial z$ and the last of these is not inherited by the Maxwell potential. The Maxwell field undergoes a duality rotation when Lie-dragged along $\partial/\partial z$, so that in this case $a = -2b$. The metric is not asymptotically-flat, and in fact it is not orthogonally-transitive with respective to either of the two isometry groups generated respectively by $\langle \partial/\partial t, \partial/\partial z \rangle$ or $\langle \partial/\partial \varphi, \partial/\partial z \rangle$.

A simple explicit solution given in [27, equation (24.46)], and discussed in [29], which illustrates some other possibilities identified in [30], is provided by the (conformally-flat) plane-wave metric

$$\mathbf{g} = -2du(dv + b^2 \zeta \bar{\zeta} du) + 2d\zeta d\bar{\zeta},$$

with real constant b and Maxwell field

$$F = be^{-\mathrm{i}f(u)}du \wedge d\bar{\zeta} + c.c.,$$

with arbitrary real $f(u)$ (which does not appear in the metric). Here the bar denotes complex conjugation. This is Einstein-Maxwell for any $f(u)$, and the (timelike or null) Killing vector $K = \partial/\partial u$ is not inherited: one has

$$\mathcal{L}_K F = -aF^\star \text{ with } a = f'(u),$$

where $F^\star$ is the dual of F. (There are several more symmetries of this metric.) In this example the Maxwell field is null, as it has to be if a is to be non-constant, and the metric also admits a twist-free, shear-free null geodesic congruence, here tangent to $\partial/\partial v$, which again is necessary for non-constant a — see [30] for a proof. Choosing $f(u)$ non-analytic, one obtains an example of a stationary but non-analytic solution of the Einstein-Maxwell equations.

3. The equations

For completeness we review the derivation of the key equations in both static and stationary cases.

3.1. Static case

Following [30], we assume a static metric, which we write in the form

$$(3.1) \qquad \mathbf{g} = -V^2 dt^2 + g_{ij}(x^k)dx^i dx^j,$$

where the Killing vector is $K = \partial/\partial t$, thus $\mathbf{g}(K, K) = -V^2$. The non-inheriting conditions on the Maxwell field tensor F_{ij} and its dual $F_{ij}^\star$ are

$$(3.2) \qquad \mathcal{L}_K F_{ij} = -aF_{ij}^\star, \quad \mathcal{L}_K F_{ij}^\star = aF_{ij}.$$

In the static case one needs

$$T_{ij}K^j = fK_i,$$

for some real, non-negative f, where T_{ij} is the Maxwell stress tensor, since the momentum constraint requires $T_{0i} = 0$ which implies this. This in turn prevents F_{ij} from being null except where it vanishes (if it ever vanishes). Now the source-free Maxwell equations impose, in form language:

$$da \wedge F = 0 = da \wedge F^\star,$$

and non-nullness of F imposes $da = 0$ and so a is a real, nonzero constant. In the region where $V > 0$ the electric and magnetic field vectors are defined by

$$E_i = V^{-1}K^j F_{ji}; \quad B_i = V^{-1}K^j F_{ji}^\star.$$

It was shown in [30] that F_{ij} vanishes at the bifurcation surface of a bifurcate horizon, so that E_i and B_i end up finite on such a horizon, but in any case this will not be an issue since our proof proceeds by unique continuation from infinity.

Then non-inheriting can be shown to imply

$$(3.3) \qquad E_i = W_i \sin(at), \quad B_i = -W_i \cos(at),$$

for some real W_i orthogonal to K^a, and a as before. The Einstein-Maxwell equations become

$$(3.4) \qquad -\Delta V = \frac{1}{2}(g^{ij}W_iW_j)V,$$

$$(3.5) \qquad \epsilon_i{}^{jk}\nabla_j(VW_k) = -aW_i,$$

$$(3.6) \qquad R_{ij} - \frac{1}{2}Rg_{ij} = V^{-1}\nabla_i\nabla_jV - W_iW_j,$$

where these are all 3-dimensional quantities: R_{ij}, R are 3-dimensional Ricci tensor and scalar, ∇_i is 3-dimensional Levi-Civita covariant derivative, $\Delta = -g^{ij}\nabla_i\nabla_j$ etc.

Simplify (3.5) by redefining and rescaling:

$$\omega_i := VW_i, \quad \widehat{g}_{ij} := V^{-2}g_{ij},$$

for then

$$\widehat{\epsilon}_{ijk} = V^{-3}\epsilon_{ijk}, \quad \widehat{\epsilon}_i{}^{jk} = V\epsilon_i{}^{jk},$$

and (3.5) becomes

$$\widehat{\epsilon}_i{}^{jk}\partial_j\omega_k = -a\omega_i$$

indices raised by $\widehat{g}$, and in form language this is

$$(3.7) \qquad * d\omega + a\omega = 0,$$

where $*$ is the 3-dimensional Hodge dual. Trace (3.6) and use (3.4) to obtain

$$R = \frac{1}{2}g^{ij}W_iW_j.$$

The hypothesis that the metric is asymptotically flat and has a well defined and finite energy leads to the condition

$$(3.8) \qquad W \in L^2(M_{\text{ext}}).$$

where M_{ext} denotes the asymptotically flat region $\{|x| \geq R\}$, which translates to the same requirement for ω.

3.2. Stationary case

We mostly follow [29], but with some different conventions. We still have (3.2), but a does not have to be constant anymore: as we saw in the preceding section, non-trivial (and not asymptotically flat) solutions with a non-constant a have been found. From the method of proof of Theorem 1.3 of [30] it also follows that:

Theorem 3.1. *Let $(M, \mathbf{g}, F)$ be an Einstein-Maxwell space-time with a Killing vector field K_i such that $\mathscr{L}_K F_{ij} = aF^\star_{ij}$. If a is not constant then F is a null Maxwell field and the space-time admits a non-twisting, shear-free null geodesic congruence.*

The metric as in Theorem 3.1 lies in the Robinson-Trautman class if the expansion is non-zero, or the Kundt class if the expansion is zero, or is a pp-wave if the generator of the congruence can be chosen to be parallel. Such solutions are unlikely to be asymptotically flat, and therefore will not be considered any longer here. In fact, it is known that pp-waves cannot be asymptotically flat by [2] (see also [13]), but it should be admitted that the remaining cases are not entirely clear (compare [4, 17]). For the sake of completeness it would be of interest to settle this.

In our analysis below we will assume that a is a nonzero, real constant and take the metric to be

$$\mathbf{g} = -V^2(dt + \theta_i dx^i)^2 + g_{ij}dx^i dx^j.$$

Introduce E_i and B_i as before (these are not now closed). We find

$$E_i + \mathrm{i}B_i = V^{-1}\zeta_i e^{\mathrm{i}at},$$

where ζ_i, the counterpart of ω_i considered in the static case, is now complex but still with $\mathscr{L}_K \zeta_i = 0$. The relevant equation (unnumbered in [29] but just before (32) there) turns out to be

$$(3.9) \qquad \epsilon_i{}^{jk}(\partial_j\zeta_k - \mathrm{i}a\theta_j\zeta_k) = -aV^{-1}\zeta_i,$$

or in form notation

$$(3.10) \qquad *(d\zeta - \mathrm{i}a\theta \wedge \zeta) = -aV^{-1}\zeta.$$

This is similar in form to the static case (the V^{-1} factor on the right-hand side can be absorbed into a conformal rescaling of the spatial metric) but with the exterior derivative 'twisted' by θ_i.

Equation (3.9) implies, away from the zeros of V,

$$(3.11) \qquad \nabla_i\zeta^i = \big((\ln V)_{,i} + iV\,\epsilon_i{}^{jk}\partial_j\theta_k - ia\theta_i\big)\zeta^i,$$

$$(3.12) \qquad (-\Delta + a^2)\zeta_\ell = a^2(1 - V^{-2})\zeta_\ell + a\epsilon_\ell{}^{ij}(\nabla_i(V^{-1}) + i\theta_i)\zeta_j$$
$$+ \nabla_\ell\left(i\epsilon^{ijk}\nabla_i(\theta_j\zeta_k) - \zeta^i\nabla_i(V^{-1})\right)$$
$$- ia\nabla^k(\theta_\ell\zeta_k - \theta_k\zeta_\ell).$$

Note that the second derivatives of ζ appearing at the right-hand side of the last equation can be replaced by lower-order ones using (3.9). This provides a homogeneous second-order equation for ζ with diagonal principal part to which our uniqueness analysis of Section 5 applies, leading to the vanishing of ζ for field configurations satisfying the asymptotic flatness conditions.

4. Static case

We return to (3.7), where the lapse function V has been absorbed into a redefinition of the metric. Henceforth we consider a complete three dimensional Riemannian manifold (M^3, g) with or without boundary and with a one-form ω satisfying

$$(4.1) \qquad *d\omega = a\,\omega,$$

where $a \in \mathbb{R} \setminus \{0\}$. Note that (4.1) implies that

$$(4.2) \qquad \delta\omega = 0.$$

From (4.1) and (4.2), the Hodge Laplacian of ω is found to be

$$(4.3) \qquad \Delta_H\omega = \delta d\omega + d\delta\omega = *d(*d\omega) = a^2\omega.$$

Thus, by the Weitzenbock formula,

$$(4.4) \qquad \nabla^i\nabla_i\omega_j = -\Delta_H\omega_j + R_{ij}\omega^i = -a^2\omega_j + R_{ij}\omega^i.$$

We will say that (M, g) contains an asymptotically flat end $M_{\text{ext}} = \{|x| \geq R\}$ for some R, if it holds that

$$(4.5) \qquad |g_{ij}(x) - \delta_{ij}| + |x||\partial g_{ij}(x)| + |x|^2|\partial^2 g_{ij}(x)| \leq C_* |x|^{-\delta}$$

for some $C_* > 0$ and $\delta > 0$, and where $|x|$ denotes the Euclidean norm.

In the remainder of this section we will prove:

Theorem 4.1. *If (M, g) contains an asymptotically flat end M_{ext} with $\omega \in L^2(M_{\mathrm{ext}})$ then $\omega \equiv 0$.*

In fact, we will prove a slightly stronger statement, see Proposition 4.10. The proof is a unique continuation argument which uses a suitable monotone Almgren-type frequency function; see [1, 8], compare [3] for a recent application of this method to geometric problems. See also [26, Sections 9-10] for a systematic treatment of related integral quantities and ODEs in the context of the closely related limiting absorption principle.

In a nutshell, our argument examines the decay rate of the normalized L^2-norm of ω on large 'spheres' (i.e. the quantity $X(r)$ defined below in (4.9), where r is a suitably defined distance function). The Almgren-type frequency $F(r)$ is then recognized as a perturbation of $\frac{-rX'(r)}{2X(r)}$. We will show that F is non-increasing, which implies that the decay rate of X is not faster than polynomial unless $\omega \equiv 0$. A more quantitative estimate for the derivative of F shows in fact that X cannot decay faster than r^{-3} unless $\omega \equiv 0$. But then the same estimate, together with the assumption that $\omega \in L^2$, implies that X must decay at least as fast as r^{-3}, which concludes our argument.

4.1. Preliminaries about distance functions

We collect here some facts about the distance function which we will use later on. For large $R \gg 1$, let $B_R(0)$ denote the coordinate ball of radius R and define

$$d_R(x) := \mathrm{dist}(x, B_R(0)).$$

We first show that, for large R, d_R is smooth outside of $B_R(0)$. We will use the following weighted Poincaré inequality, whose proof is left to the readers:

Lemma 4.2. *Fix $\ell > 0$ and $\tau > 0$. For any $\varphi \in C^1([0, \ell])$ with $\varphi(\ell) = 0$, there holds*

$$\int_0^\ell \frac{\varphi^2(t)}{(R+t)^{2+\delta}} \, dt \leq \frac{4}{(1+\delta)^2 R^{2\delta}} \int_0^\ell |\varphi'(t)|^2 \, dt.$$

We also have:

Lemma 4.3. *Assume that the asymptotic flatness condition* (4.5) *holds. There exists some large $R_0 > 0$ such that, for any $R > R_0$, the distance function d_R is smooth on $M \setminus B_R(0)$.*

Proof. By the asymptotic flatness condition (4.5), we have for large R that the shape operator S_R of $\partial B_R(0)$ satisfies

$$S_R(X) = \frac{1}{R}X + O(R^{-1-\delta}|X|_g).$$

We hence assume in the proof that R is sufficiently large so that

$$g(S_R(X), X) > 0 \quad \text{on} \ \ \partial B_R(0).$$

Arguing by contradiction, assume that there are a point $p \in \partial B_R(0)$, a normalized geodesic $\gamma \subset M \setminus B_R(0)$ emanating from p and perpendicular to $\partial B_R(0)$, and some point $q = \gamma(\ell) \in M \setminus \overline{B}_R(0)$, $\ell > 0$, which is the first focal point of $\partial B_R(0)$ along γ. Then there exists a non-trivial Jacobi field V along γ such that $V(0) \in T_p(\partial B_R(0))$, $V'(0) = S_p(V(0))$ and $V(\ell) = 0$, where S_p is the shape operator of $\partial B_R(0)$ at p. Note that $V(0) \neq 0$ by non-triviality.

By the asymptotic flatness condition (4.5), we have

$$I[V] \geq \int_0^\ell [|V'|_g^2 - \frac{C}{(R+t)^{2+\delta}}|V|_g^2]\, dt$$
$$\geq \int_0^\ell [(\frac{d}{dt}|V|_g)^2 - \frac{C}{(R+t)^{2+\delta}}|V|_g^2]\, dt,$$

where C denotes some positive constant which depends only on the constant in (4.5). Thus, by Lemma 4.2, there exists $R_0 > 0$ such that if $R > R_0$, then $I[V] \geq 0$. On the other hand, by the Jacobi equation, we have

$$I[V] := \int_0^\ell [|V'|_g^2 - R(\gamma', V, \gamma', V)]\, dt = -g(S_p(V(0)), V(0)) < 0,$$

which yields a contradiction. The proof is complete.   □

We will also use the following Hessian estimate, which follows from a standard ODE comparison argument, see e.g. [25, p. 175].

Lemma 4.4. *Assume that the asymptotic flatness condition* (4.5) *holds for some $\delta \in (0, 1)$. There exist $C_1 > 0$ and $R_1 > 0$ such that, for all $R > R_1$,*

the Hessian of d_R satisfies

$$\left(\frac{1}{R + d_R} - \frac{C_1}{(R + d_R)^{1+\delta}} \right) h \leq \nabla^2 d_R(x) \leq \left(\frac{1}{R + d_R} + \frac{C_1}{(R + d_R)^{1+\delta}} \right) h,$$

where h is the metric induced by g on the level sets of d_R.

4.2. Unique continuation

Recall that we aim to show that any solution of (4.1) satisfying

$$(4.6) \qquad\qquad\qquad \omega \in L^2(M).$$

must be identically zero.

Without loss of generality, we may assume that the boundary of M is some large coordinate sphere S_0 of radius R_0 near infinity. Let r denote the g-distance function to S_0, which, by Lemma 4.3, is a smooth function on the exterior of S_0. For $t > 0$, let S_t and $\Omega_{t,\infty}$ denote respectively the set $\{r = t\}$ and $\{r > t\}$.

In the sequel, unless otherwise stated, C will denote some positive constant which varies from line to line, but depends only on R_0 and the constant in the asymptotic flatness condition (4.5).

Note that, by applying elliptic estimates to the PDE (4.4) on any Euclidean unit ball in the asymptotic region, it is readily seen that (4.6) implies

$$(4.7) \qquad |\nabla \omega(x)|_g \leq C \sup_{B(x,1/2)} |\omega|_g \leq C \|\omega\|_{L^2(B(x,1))} \text{ for } x \in \Omega_{1,\infty},$$

where the constant C depends only on a, R_0 and the constant in (4.5). This implies in particular that

$$(4.8) \qquad\qquad\qquad \nabla \omega \in L^2(M).$$

Define

$$(4.9) \qquad X(r) = \frac{1}{(r + R_0)^2} \int_{S_r} |\omega|_g^2 \, d\sigma_g,$$

$$(4.10) \qquad E(r) = \frac{1}{(r + R_0)^2} \int_{\Omega_{r,\infty}} \left[|\nabla \omega|_g^2 - a^2 |\omega|_g^2 + \mathrm{Ric}(\omega^\sharp, \omega^\sharp) \right] dv_g.$$

Note that E is well defined thanks to (4.6) and (4.8). Note also that, by (4.4),

$$E(r) = \frac{1}{2(r+R_0)^2} \int_{\Omega_{r,\infty}} \nabla^i \nabla_i |\omega|_g^2 \, dv_g = -\frac{1}{(r+R_0)^3} \int_{S_r} g(\beta, \omega) \, d\sigma_g,$$

where

$$\beta_j = (r+R_0) \nabla^i r \, \nabla_i \omega_j.$$

In addition, there exists $C_2 > 0$ depending only on R_0 and the constant in (4.5) such that

$$(4.11) \qquad \left| \frac{d}{dr} X(r) + 2E(r) \right| \leq \frac{C_2 X(r)}{(r+R_0)^{1+\delta}}.$$

Lemma 4.5. *Assume that the asymptotic flatness condition (4.5) and the L^2 condition (4.6) hold. There exist $k_0 > 0$ such that for $k > k_0$ one can find some $r_1 > 0$ so that for $r > r_1$,*

$$(4.12) \quad \frac{d}{dr} \left[(r+R_0) \exp\left(\frac{k}{(r+R_0)^\delta} \right) E(r) \right]$$

$$\leq -\frac{2 \exp\left(\frac{k}{(r+R_0)^\delta} \right)}{(r+R_0)^3} \int_{S_r} |\beta|_g^2 \, d\sigma_g - \frac{a^2 \exp\left(\frac{k}{(r+R_0)^\delta} \right)}{(r+R_0)^2} \int_{\Omega_{r,\infty}} |\omega|_g^2 \, dv_g.$$

In particular, $E(r) \geq 0$ for $r > r_1$.

Proof. For $m \gg R_0$, let ζ_m be a cut-off function such that $0 \leq \zeta_m \leq 1$ in M, $\zeta_m \equiv 1$ in $\{r < m\}$, $\zeta_m \equiv 0$ in $\{r > 2m\}$, and $|\nabla \zeta_m|_g \leq \frac{C}{m}$ in M, where here and below C denotes some constant which is independent of m.

Keeping in mind the asymptotic flatness and the Hessian estimate Lemma 4.4, we compute for $0 < s \ll m$,

$$\int_{\Omega_{s,\infty}} \nabla^i \nabla_i \omega_j \, \beta^j \, \zeta_m \, dv_g = \int_{\Omega_{s,\infty}} (r+R_0) \, \nabla^i \nabla_i \omega_j \, \nabla^k r \, \nabla_k \omega^j \, \zeta_m \, dv_g$$

$$\leq -\int_{\Omega_{s,\infty}} \nabla_i \omega_j \left[\nabla^i ((r+R_0) \nabla^k r) \, \nabla_k \omega^j + (r+R_0) \nabla^k r \, \nabla^i \nabla_k \omega^j \right] \zeta_m \, dv_g$$

$$-\frac{1}{s+R_0} \int_{S_s} |\beta|_g^2 \, d\sigma_g + C \int_{\Omega_{m,\infty}} |\nabla \omega|_g^2 \, dv_g$$

$$\leq -\int_{\Omega_{s,\infty}} \left[1 - \frac{C}{(r+R_0)^\delta} \right] |\nabla \omega|_g^2 \, \zeta_m dv_g$$

$$- \int_{\Omega_{s,\infty}} (r+R_0) \left[\frac{1}{2} \nabla^k r \, \nabla_k |\nabla \omega|_g^2 - \nabla^k r \, \nabla^i \omega^j \, R_{kijl} \omega^l \right] \zeta_m \, dv_g$$

$$- \frac{1}{s+R_0} \int_{S_s} |\beta|_g^2 \, d\sigma_g + C \int_{\Omega_{m,\infty}} |\nabla \omega|_g^2 \, dv_g$$

$$\leq \int_{\Omega_{s,\infty}} \left[\frac{1}{2} |\nabla \omega|_g^2 + \frac{C}{(r+R_0)^\delta} (|\nabla \omega|_g + |\omega|_g) |\nabla \omega|_g \right] \zeta_m dv_g$$

$$- \frac{1}{s+R_0} \int_{S_s} |\beta|_g^2 \, d\sigma_g + \frac{1}{2}(s+R_0) \int_{S_s} |\nabla \omega|_g^2 \, d\sigma_g + C \int_{\Omega_{m,\infty}} |\nabla \omega|_g^2 \, dv_g.$$

On the other hand, by the PDE (4.4) and Lemma 4.4,

$$\int_{\Omega_{s,\infty}} \nabla^i \nabla_i \omega_j \, \beta^j \, \zeta_m \, dv_g$$

$$= \int_{\Omega_{s,\infty}} \left[-\frac{1}{2} a^2 (r+R_0) \nabla^k r \nabla_k |\omega|_g^2 \right.$$

$$\left. + (r+R_0) R_{ij} \omega^i \, \nabla^k r \, \nabla_k \omega_j \right] \zeta_m dv_g$$

$$\geq \int_{\Omega_{s,\infty}} \left[\frac{3}{2} a^2 |\omega|_g^2 - \frac{C}{(r+R_0)^\delta} |\omega|_g (|\nabla \omega|_g + |\omega|_g) \right] dv_g$$

$$+ \frac{1}{2}(s+R_0) \int_{S_s} a^2 |\omega|_g^2 \, d\sigma_g - C \int_{\Omega_{m,\infty}} |\omega|_g^2 \, dv_g.$$

In addition, we have

$$\int_{S_s} \text{Ric}(\omega^\sharp, \omega^\sharp) \, d\sigma_g \geq \int_{S_s} \frac{C}{(r+R_0)^{2+\delta}} |\omega|_g^2 \, d\sigma_g$$

$$= \int_{\Omega_{s,\infty}} \nabla^i \left(\frac{C}{(r+R_0)^{2+\delta}} |\omega|_g^2 \, \zeta_m \, \nabla_i r \right) dv_g$$

$$\geq -\int_{\Omega_{s,\infty}} \frac{C}{(r+R_0)^\delta} |\omega|_g (|\nabla \omega|_g + |\omega|_g) \, dv_g.$$

Combining the last three estimates and sending $m \to \infty$ (using (4.6) and (4.8)), we can find some $C_1 > 0$ depending only on R_0 and the constant

in (4.5) such that

$$\int_{S_s} \left[|\nabla\omega|_g^2 - a^2|\omega|_g^2 + \mathrm{Ric}(\omega^\sharp, \omega^\sharp) \right] d\sigma_g$$

$$\geq \frac{2}{(s+R_0)^2} \int_{S_s} |\beta|_g^2 \, d\sigma_g - (s+R_0)E(s)$$

$$+ \frac{1}{s+R_0} \int_{\Omega_{s,\infty}} \left[2a^2|\omega|_g^2 - \frac{C_1}{(r+R_0)^\delta}(|\nabla\omega|_g^2 + |\omega|_g^2) \right] dv_g.$$

It follows that, for all sufficiently large s,

$$(r+R_0)\exp\left(-\frac{k}{(r+R_0)^\delta} \right) \frac{d}{dr} \left[(r+R_0)\exp\left(\frac{k}{(r+R_0)^\delta} \right) E(r) \right] \Big|_{r=s}$$

$$= -(s+R_0)\left(1 + \frac{k\delta}{(s+R_0)^\delta} \right) E(s)$$

$$- \int_{S_s} \left[|\nabla\omega|_g^2 - a^2|\omega|_g^2 + \mathrm{Ric}(\omega^\sharp, \omega^\sharp) \right] d\sigma_g$$

$$\leq -\frac{2}{(s+R_0)^2} \int_{S_s} |\beta|_g^2 \, d\sigma_g$$

$$- \frac{1}{s+R_0} \int_{\Omega_{s,\infty}} \left[\frac{k\delta}{(s+R_0)^\delta}|\nabla\omega|_g^2 + \frac{2(s+R_0)^\delta - k\delta}{(s+R_0)^\delta}a^2|\omega|_g^2 \right] dv_g$$

$$+ \frac{1}{s+R_0} \int_{\Omega_{s,\infty}} \frac{1}{(r+R_0)^\delta} \left[C_1|\nabla\omega|_g^2 + (C+C_1)|\omega|_g^2 \right] dv_g.$$

The conclusion is readily seen. □

Let

$$(4.13) \qquad\qquad F(r) = \frac{(r+R_0)\exp\left(\frac{2k}{(r+R_0)^\delta} \right) E(r)}{X(r)},$$

wherever $X(r) > 0$. In view of (4.11), F is readily seen as a perturbation of $\frac{-rX'}{2X}$, and so bears some resemblance to the Almgren frequency function [1] for harmonic functions. The following lemma establishes the monotonicity of F.

Lemma 4.6. *Assume that the asymptotic flatness condition* (4.5) *and the* L^2 *condition* (4.6) *hold. There exist* $k > 0$ *and* $r_1 > 0$ *such that*

$$(4.14) \qquad\qquad \frac{d}{dr}F(r) \leq -\frac{a^2 \exp\left(\frac{2k}{(r+R_0)^\delta} \right)}{(r+R_0)^2 X(r)} \int_{\Omega_{r,\infty}} |\omega|_g^2 \, dv_g$$

for all $r > r_1$ satisfying $X(r) > 0$. In particular, we have the dichotomy

- *either $\omega \equiv 0$ in M,*

- *or $X(r) > 0$ for all $r > r_1$ and the Almgren-type frequency function $F(r)$ is non-increasing for $r \geq r_1$.*

Proof. Let k_0 be as in Lemma 4.5. Fix some $k > \max(k_0, C_2)$. By (4.12) and (4.11), we have for large r that

$$
\frac{X^2(r) \exp\left(-\frac{2k}{(r+R_0)^\delta}\right)}{r + R_0} \frac{d}{dr} F(r)
$$

$$
\leq -\frac{1}{(r + R_0)^6} \left(2 \int_{S_r} |\beta|_g^2 \, d\sigma_g + a^2(r + R_0) \int_{\Omega_{r,\infty}} |\omega|_g^2 \, dv_g \right) \int_{S_r} |\omega|_g^2 \, d\sigma_g
$$

$$
- E(r) \left(\frac{d}{dr} X(r) + \frac{k\delta}{(r + R_0)^{1+\delta}} X \right)
$$

$$
\leq -\frac{1}{(r + R_0)^6} \left(2 \int_{S_r} |\beta|_g^2 \, d\sigma_g + a^2(r + R_0) \int_{\Omega_{r,\infty}} |\omega|_g^2 \, dv_g \right) \int_{S_r} |\omega|_g^2 \, d\sigma_g
$$

$$
- E(r) \left(-2E(r) + \frac{k\delta - C_2}{(r + R_0)^{1+\delta}} X(r) \right)
$$

$$
= -\frac{2}{(r + R_0)^6} \left\{ \int_{S_r} |\beta|_g^2 \, d\sigma_g \int_{S_r} |\omega|_g^2 \, d\sigma_g - \left(\int_{S_r} g(\beta, \omega) \, dv_g \right)^2 \right\}
$$

$$
- \frac{a^2 X(r)}{(r + R_0)^3} \int_{\Omega_{r,\infty}} |\omega|_g^2 \, dv_g - \frac{k\delta - C_2}{(r + R_0)^{1+\delta}} E(r) \, X(r).
$$

As $k > C_2$ and $E(r) \geq 0$ for large r (thanks to Lemma 4.5), estimate (4.14) follows from Cauchy-Schwarz' inequality.

We turn to the proof of the stated dichotomy. Assume by contradiction that $\omega \not\equiv 0$ in M but $X(r_2) = 0$ for some $r_2 > r_1$. By unique continuation, X cannot be identically zero in a non-empty open subinterval of (r_1, r_2). Thus, the set $\{r \in (r_1, r_2) : X(r) \neq 0\}$ is a union of pairwise disjoint open subintervals of (r_1, r_2). Let (r_-, r_+) be a connected component of this set, so that $X(r_+) = 0$. ($X(r_-)$ is also zero unless $r_- = r_1$, but we will not need this fact.) In this interval, the function F is well-defined and is non-increasing thanks to (4.14).

Define

$$
F_+ = \lim_{r \to r_+} F(r) \geq 0.
$$

Then, for $r \in (r_-, r_+)$, we have $F(r) \geq F_+$ and so

$$E(r) \geq \frac{F_+}{r + R_0} \exp\left(-\frac{2k}{(r + R_0)^\delta}\right) X(r) \geq \frac{F_+}{r + R_0}\left(1 - \frac{2k}{(r + R_0)^\delta}\right) X(r).$$

Recalling (4.11), we have

$$\frac{d}{dr} X(r) \leq -2E(r) + \frac{C_2 X(r)}{(r + R_0)^{1+\delta}} \leq \left[-\frac{2F_+}{r + R_0} + \frac{C_2 + 4kF_+}{(r + R_0)^{1+\delta}}\right] X(r),$$

which is equivalent to

$$\frac{d}{dr}\left[(r + R_0)^{2F_+} \exp\left(\frac{C_2 + 4kF_+}{\delta(r + R_0)^\delta}\right) X(r)\right] \leq 0 \text{ for } r \in (r_-, r_+),$$

But this is impossible as $X(r_+) = 0$ and $X(r) > 0$ for $r \in (r_-, r_+)$. This contradiction proves the desired dichotomy. $\qquad\square$

Assume that $\omega \not\equiv 0$. Then $X(r) > 0$ for large r. Let F be the Almgren frequency function defined by (4.13) and set

(4.15) $$F(\infty) := \lim_{r \to \infty} F(r).$$

The next two statements show that, roughly speaking, X decays like $r^{-2F(\infty)}$.

Corollary 4.7. *Assume that the asymptotic flatness condition (4.5) and the L^2 condition (4.6) hold and that $\omega \not\equiv 0$ in M. Then, there exists some $C > 0$ such that*

$$X(r) \leq C(r + R_0)^{-2F(\infty)} \text{ for } r > 0.$$

Proof. By Lemma 4.6, there exists some $r_1 > 0$ such that $F(r) \geq F(\infty)$ for all $r > r_1$. We can then follow the argument in the second half of the proof of Lemma 4.6 (with F_+ replaced by $F(\infty)$) to show that

$$\frac{d}{dr}\left[(r + R_0)^{2F(\infty)} \exp\left(\frac{C_2 + 4kF(\infty)}{\delta(r + R_0)^\delta}\right) X(r)\right] \leq 0 \text{ for } r > r_1.$$

The conclusion follows. $\qquad\square$

Corollary 4.8. *Assume that the asymptotic flatness condition (4.5) and the L^2 condition (4.6) hold and that $\omega \not\equiv 0$ in M. Then, for any $\gamma > 2F(\infty)$,*

$$\liminf_{r \to \infty} r^\gamma X(r) = \infty.$$

Proof. Assume by contradiction that $\liminf_{r\to\infty} r^\gamma X(r) < \infty$. Replacing γ by a smaller number, which is still larger than $2F(\infty)$, if necessary, we may assume without loss of generality that $\liminf_{r\to\infty} r^\gamma X(r) = 0$.

Let k and r_1 be as in Lemma 4.6 and fix some $s > r_1$ such that $2F(s) < \gamma$. Then $F(r) \leq F(s) < \frac{1}{2}\gamma$ for all $r \geq s$. In view of (4.11), we have for $r \geq s$ that

$$
\begin{aligned}
\frac{X'(r)}{X(r)} &\geq -\frac{2\exp\left(-\frac{2k}{(r+R_0)^\delta}\right)}{r+R_0}F(r) - \frac{C_2}{(r+R_0)^{1+\delta}} \\
&\geq -\frac{2}{r+R_0}F(s) - \frac{C_2}{(r+R_0)^{1+\delta}},
\end{aligned}
$$

and so

$$
\frac{d}{dt}\left[X(r)\,(r+R_0)^{2F(s)}\exp\left(-\frac{C_2}{\delta(r+R_0)^\delta}\right)\right] \geq 0.
$$

As $2F(s) < \gamma$, we have by assumption that the function in the square bracket tends to zero along a sequence $r_i \to \infty$. It follows that $X(r) = 0$ for all $r > s$, which contradicts the fact that $X(r) > 0$ for large r. $\qquad\square$

We next show that, under the L^2 assumption on ω, X decays faster than r^{-3}.

Corollary 4.9. *Assume that the asymptotic flatness condition* (4.5) *and the L^2 condition* (4.6) *hold and that $\omega \not\equiv 0$ in M. Then $F(\infty) \geq \frac{3}{2}$.*

Proof. By Corollary 4.8, for any $\varepsilon > 0$, there exists r_ε such that

$$
X(r) \geq (r+R_0)^{-2F(\infty)-\varepsilon} \text{ for } r > r_\varepsilon.
$$

Thus, by (4.6), we have

$$
\infty > \int_{r_\varepsilon}^\infty (r+R_0)^2\,X(r)\,dr \geq \int_{r_\varepsilon}^\infty (r+R_0)^{2-2F(\infty)-\varepsilon}\,dr.
$$

This implies that $2F(\infty) + \varepsilon > 3$, from which the conclusion follows. $\qquad\square$

We now wrap up the argument, by showing, using the estimate for F' of Lemma 4.6, that X cannot decay faster than r^{-3}, unless $\omega \equiv 0$.

Proposition 4.10. *Let (M^3, g) be an asymptotically flat three dimensional Riemannian manifold with or without boundary satisfying the asymptotic flatness condition* (4.5)*. If a one-form $\omega \in L^2(M_{\text{ext}})$ satisfies* (4.4) *on M for some non-zero $a \in \mathbb{R} \setminus \{0\}$, then $\omega \equiv 0$.*

Proof. Assume by contradiction that $\omega \not\equiv 0$. By Lemma 4.6, there exists some r_1 such that $X(r) > 0$ for all $r > r_1$.

Let $F(\infty)$ be given by (4.15). By Corollary 4.9, $F(\infty) \geq \frac{3}{2}$.

Fix some $\varepsilon > 0$. By Corollaries 4.7 and 4.8, there exists some $r_\varepsilon > 0$ such that

$$(r + R_0)^{-2F(\infty)-\varepsilon} \leq X(r) \leq C(r + R_0)^{-2F(\infty)} \text{ for } r > r_\varepsilon,$$

which implies that

$$\frac{1}{X(r)} \int_{\Omega_{r,\infty}} |\omega|_g^2 \, dv_g = \frac{1}{X(r)} \int_r^\infty (r + R_0)^2 \, X(r) \, dr$$

$$\geq \frac{1}{C}(r + R_0)^{3-\varepsilon} \text{ for } r > r_\varepsilon.$$

Returning to the estimate (4.14) in Lemma 4.6, we obtain

$$\frac{d}{dr} F(r) \leq -\frac{a^2}{C}(r + R_0)^{1-\varepsilon} \text{ for } r > \max(r_1, r_\varepsilon).$$

As F is bounded, this is impossible for small ε. This contradiction finishes the proof. $\square$

5. Stationary case

In this section we will prove non-existence of stationary non-inheriting solutions. This will follow from a general result concerning L^2-solutions of PDEs arising from elliptic second order Laplacian-type operators, Corollary 5.5 below. As already mentioned, the results here also cover the static case of Section 4 where, however, the result was obtained by a more elementary argument. We believe that both proofs have interest in their own.

Similarly to (4.5), we assume that there exists a constant $\delta > 0$ such that

$$(5.1) \qquad g_{ij} - \delta_{ij} = O_k(r^{-\delta}), \quad V - 1 = O_k(r^{-\delta}), \quad \theta_i = O_k(r^{-\delta}),$$

for some k large enough, which we leave unspecified. Here we write $f = O_k(r^{-\delta})$ if for all multi-indices α with $0 \leq |\alpha| \leq k$ we have

$$r^{|\alpha|} \partial^\alpha f = O(r^{-\delta})$$

for large r.

Assuming (5.1) and $V > 0$ one readily checks that the stationary inheriting equations (3.12) satisfy the hypotheses of Corollary 5.5 below. Indeed,

expanding the right hand side of (3.12) each term has a coefficient which is either $\theta, 1 - V$ or their derivative (which in the latter case means just a derivative of V), in front of metric, thus scattering (a notion discussed below), derivatives of the unknown ζ. This leads to:

Theorem 5.1. *There are no asymptotically flat solutions of the Maxwell-Einstein equations with a stationary metric and non-inheriting Maxwell fields.*

Remark 5.2. In fact, our proof applies to the more general class of metrics (5.3)–(5.4) below.

We emphasise that no global hypotheses on the solution are imposed other than the existence of an asymptotically flat end.

The operators covered by our analysis below include those of the form $\Delta_g - \lambda$, $\lambda > 0$, where

$$(5.2) \qquad\qquad \Delta_g = -g^{ij}\nabla_i\nabla_j$$

is the Laplacian of an asymptotically Euclidean metric, and their modifications by decaying lower order terms. This includes operators acting on sections of vector bundles which behave like the Laplacian both in terms of the highest derivatives and in terms of the asymptotic behavior of the coefficients at the boundary, i.e. at metric infinity. We show that such solutions in fact vanish identically.

The methods are Carleman-type estimates, phrased in a way that was used in [32] to analyze geometric N-body problems, showing unique continuation at infinity for all second order perturbations of the Laplacian. These in turn were motivated by the closely related works of Froese and Herbst [6] and the unique continuation theorems at infinity discussed in [12] and [11, Theorem 17.2.8]. The key estimates arise from a positive commutator estimate for the exponentially conjugated Hamiltonian, which is closely related to Hörmander's solvability condition for PDE's [5, 9, 10]; see [35] for a discussion, including the relationship to numerical computation

The operators H that we consider below will be acting on sections of a vector bundle E which is equipped with a Hermitian metric. We assume that

$$H = \Delta \otimes Id_E + \mathbb{W}$$

where $\Delta \equiv \Delta_g$ is as in (5.2), and recall that g satisfies

$$g - g_0 \in S^{-\delta}(\overline{M}; {}^{\mathrm{sc}}T^*\overline{M}^{\otimes_s^2})$$

for some $\delta > 0$, and $\mathbb{W} \in \mathrm{Diff}^{1,\delta}_{\mathrm{scc}}(\overline{M}; E) = S^{-\delta} \, \mathrm{Diff}^1_{\mathrm{sc}}(\overline{M}; E)$ is formally self-adjoint; see Section 5.1 for notation.

We prove the following results: The first theorem states that positive energy (eigenvalue) eigenfunctions of H decay superexponentially. There is a simple modification of the proof to show exponential decay for negative energy eigenfunctions, at any rate α where $\alpha^2 < -\lambda$, $\lambda < 0$ the eigenvalue; this corresponds to the N-body result in which case the decay rate is given by the square root of the distance to the next threshold above λ, which is 0 if $\lambda < 0$, and non-existent (can be considered as $+\infty$) if $\lambda > 0$. This fact also explains why it is natural to consider the unique continuation theorem separately, namely why super-exponential decay assumptions are natural there; unique continuation, given superexponential decay, holds for *arbitrary* eigenvalues.

In fact, we show a version of the aforementioned results by only assuming that the equation holds on a collar neighborhood of the boundary at infinity. So let M_{ext} be a collar neighborhood $\{0 \le x \le x_1\}$, for some $x_1 > 0$, of the boundary at infinity $\partial \overline{M} = \{x = 0\}$. In the case of asymptotically flat metrics we have $x = 1/r$, and the interior of M_{ext} coincides with the asymptotically flat region $\{r > R_1 := 1/x_1\}$. We have (compare [32, Theorem 3.1], [31, Proposition B.2] and [6, Theorem 2.1]):

Theorem 5.3. *Let $\lambda > 0$, and suppose that $\psi \in L^2_{sc}(M_{\mathrm{ext}})$ satisfies $H\psi = \lambda\psi$. Then $e^{\alpha/x}\psi \in L^2_{sc}(M_{\mathrm{ext}})$ for all $\alpha \in \mathbb{R}$.*

The unique continuation theorem at infinity is the following; note that λ is now arbitrary (see [32, Theorem 4.1], [31, Proposition B.3] and [6, Theorem 3.1] for related results):

Theorem 5.4. *Let $\lambda \in \mathbb{R}$. If $H\psi = \lambda\psi$, $\exp(\alpha/x)\psi \in L^2_{sc}(M_{\mathrm{ext}})$ for all α, then $\psi = 0$.*

As an immediate corollary we deduce the absence of positive eigenvalues for first order perturbations of Δ_g:

Corollary 5.5. *Let $\lambda > 0$. Suppose that $H\psi = \lambda\psi$, $\psi \in L^2_{sc}(M_{\mathrm{ext}})$. Then $\psi \equiv 0$.*

5.1. Definitions

Recall from [20] the notion of a (short-range) *scattering metric*. Namely, with $\overline{M}$ is a manifold with boundary and x is a boundary defining function on

$\overline{M}$, a (short-range) scattering metric g_0 is a Riemannian metric on $M := \overline{M}^\circ$ which is of the form

$$(5.3) \qquad g_0 = x^{-4}\,dx^2 + x^{-2}h$$

near $\partial\overline{M}$, where h is a symmetric two-covariant tensor that restricts to a metric on $\partial\overline{M}$. In this work we allow manifolds (M, g) which might have several boundary components and asymptotic ends, with unspecified behaviour there except for one end where the metric is asymptotically flat, cf. (5.3). The discussion that follows applies only to such asymptotic regions.

As such, the metric g_0 as in (5.3) is a smooth (on $\overline{M}$) section of the second symmetric power of the scattering cotangent bundle $^{\mathrm{sc}}T^*\overline{M}$, with some additional product structure. Indeed, near $\partial\overline{M}$, a general smooth section of $^{\mathrm{sc}}T^*\overline{M}$ is a linear combination, with $\mathcal{C}^\infty(\overline{M})$-coefficients, of $\frac{dx}{x^2}, \frac{dy_j}{x}$, where y_j are local coordinates on ∂M, thus a general smooth section of this bundle if a linear combination of $\frac{dx^2}{x^4}, \frac{dx\,dy_j}{x^3}, \frac{dy_i dy_j}{x^2}$; these short range metrics thus have $1 + O(x^2)$ for the coefficient of $\frac{dx^2}{x^4}$, and $O(x)$ for the coefficient of $\frac{dx\,dy_j}{x^3}$.

Let Δ_{g_0} be the Laplacian of this metric. This is a typical element of $\mathrm{Diff}_{\mathrm{sc}}(\overline{M})$, the algebra of scattering differential operators. Here "sc" stands for "scattering". The latter is generated, over $\mathcal{C}^\infty(\overline{M})$, by the vector fields $\mathcal{V}_{\mathrm{sc}}(\overline{M}) = x\mathcal{V}_{\mathrm{b}}(\overline{M})$; $\mathcal{V}_{\mathrm{b}}(\overline{M})$ being the Lie algebra of $\mathcal{C}^\infty$ vector fields on $\overline{M}$ that are tangent to $\partial\overline{M}$. Thus, in local coordinates, and over $\mathcal{C}^\infty(\overline{M})$ (locally), $\mathcal{V}_{\mathrm{sc}}(\overline{M})$ is spanned by $x^2\partial_x$ and $x\partial_{y_j}$, while $\mathcal{V}_{\mathrm{b}}(\overline{M})$ is spanned by $x\partial_x$ and ∂_{y_j}. We work with what might be called a (very) long-range scattering metric, namely we assume that g is a conormal to ∂M, or in other words symbolic, real section of $^{\mathrm{sc}}T^*\overline{M}^{\otimes^2_s}$, of order 0 with the extra property that

$$(5.4) \qquad g - g_0 \in S^{-\delta}(\overline{M}; {}^{\mathrm{sc}}T^*\overline{M}^{\otimes^2_s}), \text{ for some } 0 < \delta \leq 1.$$

Recall that, in this compactified notation, a symbol, $a \in S^\alpha$, of "symbolic order" α, i.e. growth rate, is one satisfying that for all $P \in \mathrm{Diff}_{\mathrm{b}}(\overline{M})$ (equivalently, for all finite, possibly empty, products P of elements of $\mathcal{V}_{\mathrm{b}}(M)$), $x^\alpha P a \in L^\infty(M)$; thus our very long range metrics in particular allow $O(x^\delta)$ coefficients for $\frac{dx\,dy_j}{x^3}$, and $1 + O(x^\delta)$ for the coefficients for $\frac{dx^2}{x^4}$. (The 'long-range' metrics would have smooth $1 + O(x) \frac{dx^2}{x^4}$ terms, i.e. would only have a product structure *exactly at* ∂M, and would include metrics with asymptotic behaviour similar to the Schwarzschild one; our 'very long-range scattering metrics' satisfy even weaker conditions.) We write

$$S^m \mathrm{Diff}^k_{\mathrm{sc}}(\overline{M}) = \mathrm{Diff}^{k,-m}_{\mathrm{scc}}(\overline{M}) = S^m \otimes \mathrm{Diff}^k_{\mathrm{sc}}(\overline{M})$$

for scattering differential operators with such conormal/symbolic coefficients.

Classical symbols in S^α are symbols that have a one-step polyhomogeneous expansion, i.e., $\sum_{j\in\mathbb{N}} x^{j-\alpha} f_j(y)$, considered as an asymptotic summation. See [20] for more details; we follow closely the notations there. In particular we write $L^2_{\mathrm{sc}}(\overline{M})$ for $L^2(M)$ with the measure induced from the asymptotically flat metric.

The operator space $S^m \operatorname{Diff}^k_{\mathrm{sc}}(\overline{M})$ form a filtered algebra, so

$$A \in S^m \operatorname{Diff}^k_{\mathrm{sc}}(\overline{M}), \ B \in S^{m'} \operatorname{Diff}^{k'}_{\mathrm{sc}}(\overline{M}) \implies AB \in S^{m+m'} \operatorname{Diff}^{k+k'}_{\mathrm{sc}}(\overline{M}),$$

which is in fact commutative to leading order both in terms of the differentiability and the growth orders, so

$$(5.5) \qquad\qquad [A, B] \in S^{m+m'-1} \operatorname{Diff}^{k+k'-1}_{\mathrm{sc}}(\overline{M}).$$

It may help the reader if we explain why the Euclidean setting is a particular example of this setup. There M is a vector space with a metric g_0, which can hence by identified with $\mathbb{R}^n$. Moreover, $\overline{M}$ is the radial (or geodesic) compactification of $\mathbb{R}^n$ to a ball. Explicitly, this arises by considering 'inverse' polar coordinates, and writing $w \in M$ as $w = r\omega = x^{-1}\omega$, $\omega \in \mathbb{S}^{n-1}$, so $x = |w|^{-1}$, e.g. in $|w| \geq 1$, and attaching $x = 0$, i.e. $\{0\}_x \times \mathbb{S}^{n-1}$ to $(0,1]_x \times S^n$ by simply extending the range of x. In particular, $\partial \overline{M}$ is given by $x = 0$, i.e. it is just $\mathbb{S}^{n-1}$. The metric g_0 then has the form $dr^2 + r^{-2}h_0 = x^{-4}dx^2 + x^2 h_0$, where h_0 is the standard metric on $\mathbb{S}^{n-1}$, so $(\overline{M}, g_0)$ fits exactly into this framework. Then ${}^{\mathrm{sc}}T\overline{M}, {}^{\mathrm{sc}}T^*\overline{M}$ are trivial vector bundles over $\overline{M}$; namely ${}^{\mathrm{sc}}T^*\overline{M} = \overline{M} \times M^*$, M^* being the dual vector space of M. Thus, in terms of the coordinates w_j, ∂_{w_j} span $\mathcal{V}_{\mathrm{sc}}(\overline{M})$ over $\mathcal{C}^\infty(\overline{M})$, i.e. with classical symbolic coefficients of order 0 on M. This in particular shows that the scattering differential operator algebra is the geometric generalization of the algebras considered by Parenti [23] and Shubin [28]. On the other hand, in the complement of the origin, say in the region where $w_n > \epsilon|w_j|$, $j \neq n$, $\epsilon > 0$, $w_n \partial_{w_i}$, $i = 1, \ldots, n$, span $\mathcal{V}_{\mathrm{b}}(\overline{M})$ in the similar sense, which shows why the above description of symbolic regularity is *exactly* the standard one on the vector space M.

5.2. Sketch of proofs

It might be helpful to the reader to provide an extended outline of proofs; the details will follow. The rough idea is to conjugate by exponential weights e^F, where F is a symbol of growth order 1, for example $F = \alpha/x$ for small

x. If ψ is an eigenfunction of H of eigenvalue λ, then $\psi_F = e^F \psi$ solves

$$P\psi_F = 0 \text{ where } P = H(F) - \lambda = e^F H e^{-F} - \lambda.$$

Now let $\operatorname{Re} P := (P + P^*)/2$, where $*$ denotes formal adjoint relative to L^2 is given by $H - \alpha^2 - \lambda$, while $\operatorname{Im} P := (P - \operatorname{Re} P)/\mathrm{i}$ is given by $-2\alpha(x^2 D_x)$, modulo $x \operatorname{Diff}_{\mathrm{sc}}(\overline{M})$; the notation is justified by the principal symbol of $\operatorname{Re} P$ being the real part of the principal symbol of P, etc. Here, and throughout this section,

$$D_k = \frac{1}{\mathrm{i}} \partial_k,$$

with $\mathrm{i} = \sqrt{-1}$. By elliptic regularity, using $P\psi_F = 0$, $\|\psi_F\|_{x^p H^k_{\mathrm{sc}}(\overline{M})}$ is bounded by $C_{k,p} \|\psi_F\|_{x^p L^2_{\mathrm{sc}}(\overline{M})}$, so the order of various differential operators is irrelevant for the purpose of norm estimates, while the weight is important. Since

$$P^* P = (\operatorname{Re} P)^2 + (\operatorname{Im} P)^2 + \mathrm{i}(\operatorname{Re} P \operatorname{Im} P - \operatorname{Im} P \operatorname{Re} P),$$

so

(5.6) $0 = (\psi_F, P^* P \psi_F)$
$$= \|\operatorname{Re} P\psi_F\|^2 + \|\operatorname{Im} P\psi_F\|^2 + (\psi_F, \mathrm{i}[\operatorname{Re} P, \operatorname{Im} P]\psi_F).$$

Now, being a commutator, $[\operatorname{Re} P, \operatorname{Im} P] \in x \operatorname{Diff}_{\mathrm{sc}}(\overline{M})$, i.e. has an extra order of vanishing, which shows that

$$\|\operatorname{Re} P\psi_F\| \leq C_1 \|x^{1/2}\psi_F\|, \quad \|\operatorname{Im} P\psi_F\| \leq C_1 \|x^{1/2}\psi_F\|.$$

Here, and elsewhere, $\|\cdot\|$ denotes the L^2-norm with respect to the standard measure associated with the metric g, and $(\cdot, \cdot)$ the associated scalar product. Due to the extra factor of $x^{1/2}$, this can be interpreted roughly as saying that ψ_F is, in an asymptotic (decay) sense, 'almost' in the nullspace of both $\operatorname{Re} P$ and of $\operatorname{Im} P$, hence both of $H - \lambda - \alpha^2$ and $x^2 D_x$.

If, moreover, $(\psi_F, i[\operatorname{Re} P, \operatorname{Im} P]\psi_F)$ is positive, with a lower bound by $c\|x^{1/2}\psi_F\|^2$, $c > 0$, modulo terms which are multiples (of appropriate orders) of $\operatorname{Re} P$ or $\operatorname{Im} P$, see for instance (5.31), which can be absorbed in the squares in (5.6), and terms of the form $(\psi_F, R\psi_F)$, $R \in x^{1+\delta} \operatorname{Diff}^{*,0}_{\mathrm{scc}}(\overline{M})$ (with $*$ showing that the differential order is irrelevant due to elliptic regularity), which are thus bounded by $C_2\|x^{(1+\delta)/2}\psi_F\|^2$, then the presence of the factor $x^{(1+\delta)/2}$ (which has an extra $x^{\delta/2}$ compared to $\|x^{1/2}\psi_F\|$, and thus a small constant) yields easily a bound for $\|x^{1/2}\psi_F\|$ in terms of $\|\psi\|$.

This gives estimates for the norm $\|x^{1/2}\psi_F\|$, uniform both in F and in ψ. A regularization argument in F then gives the exponential decay of ψ.

The positivity of $(\psi_F, \mathrm{i}[\mathrm{Re}\,P, \mathrm{Im}\,P]\psi_F)$, in the sense described above, is easy to see if we replace $\mathrm{i}[\mathrm{Re}\,P, \mathrm{Im}\,P]$ by $\mathrm{i}[H - \lambda - \alpha^2, -2\alpha x^2 D_x]$: this commutator is a standard one considered in N-body scattering, although the even more usual one would be $\mathrm{i}[H - \lambda - \alpha^2, -2xD_x]$, whose local positivity in the spectrum of H is the Mourre estimate [7, 21, 24]. Indeed, the latter commutator is the one considered by Froese and Herbst in Euclidean N-body potential scattering, and we could adapt their argument (though we would need to deal with numerous error terms) to our setting. However, the argument presented here is more robust, especially in the high energy sense discussed below, in which their approach would not work in the generality considered here. There is one exception: for $\alpha = 0$, $\mathrm{Im}\,P$ degenerates, and in this case we need to 'rescale' the commutator argument, and consider $\mathrm{i}[H - \lambda - \alpha^2, -2xD_x]$ directly.

We next want to let $\alpha \to \infty$. Since most of the related literature considers "semiclassical problems", we let $h = \alpha^{-1}$, and replace P above by $P_h = h^2 P$, which is a semiclassical differential operator, $P_h \in \mathrm{Diff}^2_{\mathrm{scc},h}(\overline{M})$. Here $\mathrm{Diff}_{\mathrm{sc},h}(\overline{M})$ is the algebra of semiclassical scattering differential operators discussed, for example, in [34] in this setting (see [36] for a general introduction to semiclassical analysis), and $\mathrm{Diff}_{\mathrm{scc},h}(\overline{M})$ is its conormal/symbolic coefficient version. The space $\mathrm{Diff}_{\mathrm{sc},h}(\overline{M})$ is generated by $h\mathcal{V}_{\mathrm{sc}}(\overline{M})$ over $\mathcal{C}^\infty(\overline{M} \times [0,1)_h)$. In this semiclassical sense, the first and zeroth order terms in H do not play a role in P_h: their contribution is in $h\,\mathrm{Diff}^1_{\mathrm{sc},h}(\overline{M})$, hence their contribution to the commutator $\mathrm{i}[\mathrm{Re}\,P_h, \mathrm{Im}\,P_h]$ is in $xh^2\,\mathrm{Diff}_{\mathrm{sc},h}(\overline{M})$. Moreover, at infinity $\mathrm{i}[\mathrm{Re}\,P_h, \mathrm{Im}\,P_h]$ is close to the corresponding commutator with P_h replaced by $h^2(e^F \Delta_{g_0} e^{-F} - \lambda)$. Since in the latter case the commutator is positive, modulo terms than can be absorbed in the two squares in (5.6), $\mathrm{i}[\mathrm{Re}\,P_h, \mathrm{Im}\,P_h]$ is also positive for g near g_0, which automatically holds near infinity (where this is relevant). This gives an estimate as above, from which the vanishing of ψ near $x = 0$ follows easily.

We remark that the estimates we use are related to the usual proof of unique continuation at infinity on $\mathbb{R}^n$ (i.e. not in the N-body setting), see [11, Theorem 17.2.8], and to Hörmander's solvability condition for PDE's in terms of the real and imaginary parts of the principal symbol. Indeed, although in [11, Theorem 17.2.8] various changes of coordinates are used first, which change the nature of the PDE at infinity, ultimately the necessary estimates also arise from a commutator of the kind $\mathrm{i}[\mathrm{Re}\,P, \mathrm{Im}\,P]$. However, even in that setting, the proof we present appears more natural from the

point of view of scattering than the one presented there, which is motivated by unique continuation at points in $\mathbb{R}^n$. We remark that related estimates, obtained by different techniques, form the backbone of the unique continuation results of Jerison and Kenig [14, 15].

The true flavor of our arguments is most clear in the proof of the unique continuation theorem, Theorem 5.16. The reason is that on the one hand there is no need for regularization of F, since we are assuming super-exponential decay, on the other hand the positivity of $i[\operatorname{Re} P_h, \operatorname{Im} P_h]$ is easy to see.

There are a number of differences between [32] and Section 5 of the present paper. The main one is that slower decay of the coefficients to their limit at infinity is assumed; in [32] the decay rate used is $\delta = 1$. While this slower decay does not ultimately affect the argument in a major way, some details need to be revisited, especially since commutators (which give the positivity here) in general gain an order of decay relative to products. In addition, we replace the N-body type Mourre estimates, which involve spectral cutoffs, by more elementary positive commutator computations, e.g. in Proposition 5.8. Part of the point is to remove the overhead of N-body scattering since it is irrelevant to this paper. A side effect of this more accessible approach is that the very first step, demonstrating superpolynomial decay, is also explicitly done, unlike in [32] where background microlocal analysis results were used; we note that the work [6] of Froese and Herbst starts the proof with an analogous discussion.

The structure of this part of our work is the following. In Section 5.3 we discuss various preliminaries, including the structure of the conjugated Hamiltonian and a Mourre-type global positive commutator estimate. In Section 5.4 we prove polynomial, and then in Section 5.5 the exponential decay of eigenfunctions with $\lambda > 0$. In Section 5.6, we prove the unique continuation theorem at infinity. We emphasize that the presence of bundles such as E makes no difference in the discussion, hence they are ignored in order to keep the notation manageable.

5.3. Preliminaries

We first remark that, for the metrics g under consideration, the Riemannian measure density takes the form

$$(5.7) \quad dg = \sqrt{\det(g_{ij})}\, dx\, dy = \widetilde{g}\, \frac{dx\, dy}{x^{n+1}}, \ \ n = \dim M,\ \widetilde{g} \in \mathcal{C}^\infty(\overline{M}) + S^{-\delta}(\overline{M}).$$

By our conditions (5.4) on the form of g, the Laplacian takes the following form

$$(5.8) \qquad \Delta_g = (x^2 D_x)^2 + \sum_j b_j x^2 P_j + x^\delta R$$

with $b_j \in \mathcal{C}^\infty(\overline{M})$, $P_j \in \mathrm{Diff}^2(\partial \overline{M})$, $R \in S^0 \mathrm{Diff}^2_{\mathrm{sc}}(\overline{M})$, and with all sums over finite sets of indices. (Recall that $D_x = \frac{1}{i}\partial_x$.) Hence, $H = \Delta_g + \mathbb{W}$ takes the form

$$H = (x^2 D_x)^2 + \sum_j b'_j x^2 P'_j + x^\delta R',$$

with $b'_j \in \mathcal{C}^\infty(\overline{M})$, $P'_j \in \mathrm{Diff}^2(\partial \overline{M})$, $R' \in \mathrm{Diff}^2_{\mathrm{sc}}(\overline{M})$.

Below we consider the conjugated Hamiltonian $H(F) = e^F H e^{-F}$, where F is a symbol of growth order 1. The exponential weights will facilitate exponential decay estimates, and eventually the proof of unique continuation at infinity. Let $x_0 = \sup_{\overline{M}} x$. By altering x in a compact subset of M, we may assume that $x_0 < 1/2$; we do this for the convenience of notation below. We let $S^m([0,1)_x)$ be the space of all symbols F of order (growth rate) m on $[0,1)$, which satisfy $F \in \mathcal{C}^\infty((0,1))$, vanish on $(1/2, 1)$, and for which $\sup_{x \in (0,1)} |x^{m+k} \partial_x^k F| < \infty$ for all k. The topology of $S^m([0,1))$ is given by the seminorms $\sup |x^{m+k} \partial_x^k F|$. Also, as already mentioned, the spaces $S^m(\overline{M})$ of symbols is defined similarly, i.e. it is given by seminorms $\sup_{\overline{M}} |x^m PF|$, $P \in \mathrm{Diff}^k_\mathrm{b}(\overline{M})$.

We have:

Lemma 5.6. *Suppose* $\lambda \in \mathbb{R}$, $H\psi = \lambda \psi$, $\psi \in L^2_{sc}(\overline{M})$. *Then with* $F \in S^1([0,1))$, $F \le \alpha/x + \beta |\log x|$ *for some* β, $\mathrm{supp}\, F \subset [0, 1/2)$, $\psi_F = e^F \psi$,

$$P \equiv P(F) := e^F (H - \lambda) e^{-F} = H(F) - \lambda, \ \ H(F) = H + e^F [H, e^{-F}],$$

we have

$$(5.9) \qquad P(F)\psi_F = 0,$$
$$(5.10) \qquad P(F) = H - 2(x^2 D_x F)(x^2 D_x)$$
$$+ (x^2 D_x F)^2 - \lambda + x^\delta R_1, \quad R_1 \in \mathrm{Diff}^2_{scc}(\overline{M}),$$

with

$$(5.11) \qquad \begin{aligned} \mathrm{Re}\, P(F) &= H + (x^2 D_x F)^2 - \lambda + x^\delta R_2, \\ \mathrm{Im}\, P(F) &= 2(x^2 \partial_x F)(x^2 D_x) + x^\delta R_3, \end{aligned}$$

$R_2, R_3 \in \mathrm{Diff}^2_{scc}(\overline{M})$, R_j bounded as long as $x^2 \partial_x F$ is bounded in $S^0([0,1))$, hence as long as F is bounded in $S^1([0,1))$.

The coefficients of the terms $x^\delta R_2$, $x^\delta R_3$ are in fact polynomials with vanishing constant term, in $(x^2 \partial_x)^{m+1} F$, $0 \le m \le 1$.

Furthermore,

$$(5.12) \quad \mathrm{i}[\mathrm{Re}\, P(F), \mathrm{Im}\, P(F)] = \mathrm{i}[H + (x^2 D_x F)^2, 2(x^2 \partial_x F)(x^2 D_x)] + x^{1+\delta} R_4,$$

where $R_4 \in \mathrm{Diff}^2_{scc}(\overline{M})$ is bounded as long as $x^2 \partial_x F$ is bounded in $S^0([0,1))$.

Remark 5.7. The presence of bundles E leaves (5.11) unaffected, hence (5.12) holds as well.

Proof. First note that

$$(5.13) \qquad [x^2 D_x, e^{-F}] = -(x^2 D_x F)e^{-F}, \qquad x^2 D_x F \in S^0([0,1)),$$

so $e^F[H, e^{-F}] \in \mathrm{Diff}^1_{scc}(\overline{M})$ and indeed expanding H in terms of $(x^2 D_x)^2$, $(x^2 D_x)(x D_{y_j})$, etc., we have

$$(5.14) \qquad e^F[H, e^{-F}] = -(x^2 D_x F)B_1 - B_2(x^2 D_x F),$$

with

$$B_j - (x^2 D_x) \in x^\delta \mathrm{Diff}^1_{scc}(\overline{M}).$$

The dependence of the terms of $P(F)$ on F thus comes from $x^2 D_x F$, and its commutators via commuting it through other vector fields (as in rewriting $(x^2 D_x)(x^2 D_x F)$ as $(x^2 D_x F)(x^2 D_x)$ plus a commutator term), hence through $(x^2 D_x)^{m+1} F$, $0 \le m \le 1$. Notice that writing H as

$$(x^2 D_x)a_{00}(x^2 D_x) + \sum_j \left((x^2 D_x)a_{0j}(x D_{y_j}) + (x D_{y_j})a_{0j}(x^2 D_x) \right)$$

modulo terms without factors of $x^2 D_x$ and modulo $x \, \mathrm{Diff}^1_{scc}(\overline{M})$, where all commutator terms end up after the rearrangements, we find

$$B_1 = a_{00}(x^2 D_x) + a_{0j}(x D_{y_j}) + x R_1',$$

with R_1' bounded in $\mathrm{Diff}^1_{scc}(\overline{M})$, and thus with a_{00}, a_{0j} being real, with

$$(5.15) \qquad\qquad B_1 - B_1^* \in x \, \mathrm{Diff}^1_{scc}(\overline{M}),$$

and similarly for B_2.

We use

$$(5.16) \qquad \operatorname{Im} P(F) = \frac{1}{2i}(P(F) - P(F)^*)$$
$$= \frac{1}{2i}(e^F[H, e^{-F}] + [H, e^{-F}]e^F)$$
$$= 2\left((x^2\partial_x F)\widetilde{B} + \widetilde{B}^*(x^2\partial_x F)\right),$$

with

$$\widetilde{B} - x^2 D_x \in x^\delta \operatorname{Diff}^1_{\mathrm{scc}}(\overline{M}), \quad \widetilde{B} - \widetilde{B}^* \in x \operatorname{Diff}^1_{\mathrm{scc}}(\overline{M}),$$

with $\widetilde{B} = \frac{1}{2}(B_1 + B_2^*)$, and

$$\operatorname{Re} P(F) = \frac{1}{2}(P(F) + P(F)^*) = H - \lambda + \frac{1}{2}(e^F[H, e^{-F}] - [H, e^{-F}]e^F)$$
$$= H - \lambda + \frac{1}{2}[e^F, [H, e^{-F}]]$$

to prove (5.11) (note that only the $(x^2 D_x)^2$ terms in H gives a non-vanishing contribution to the double commutator).

To prove (5.12), set

$$Q = \operatorname{Re} P(F) - \underbrace{(H + (x^2 D_x F)^2 - \lambda)}_{=:Q_1} \in x^\delta \operatorname{Diff}^2_{\mathrm{sc}}(\overline{M}),$$
$$Q' = \operatorname{Im} P(F) - \underbrace{2(x^2\partial_x F)(x^2 D_x)}_{=:Q_1'} \in x^\delta \operatorname{Diff}^2_{\mathrm{sc}}(\overline{M}).$$

We can write

$$[\operatorname{Re} P(F), \operatorname{Im} P(F)] \equiv [Q_1 + Q, Q_1' + Q']$$
$$= [Q_1, Q_1'] + [Q_1, Q'] + [Q, Q_1' + Q'].$$

Equation (5.5) shows that both last terms can be put in R_4. $\qquad \square$

In light of (5.12), we need a positivity result for $\mathrm{i}[x^2 D_x, H]$. Such a result follows directly from a Poisson bracket computation. Let $\chi \in \mathcal{C}_c^\infty([0, 1))$ be supported near 0, identically 1 on a smaller neighborhood of 0, and let

$$(5.17) \qquad B = \frac{1}{2}(\chi(x)x^2 D_x + (\chi(x)x^2 D_x)^*)$$

be the symmetrization of the radial vector field. Here the formal adjoint is taken with respect to the metric measure on M, which deserves some

comments: Indeed, $x^2 D_x$ is formally self-adjoint with respect to the measure $\frac{dx\,dy}{x^2}$, and if C is formally self-adjoint with respect to a density dg' then its adjoint with respect to $\alpha\,dg'$, α smooth real-valued, is $\alpha^{-1}C\alpha = C + \alpha^{-1}[C, \alpha]$. In the notation of (5.7), using $xD_x(x^{-n+1}\widetilde{g}) \in x^{-n+1}(\mathcal{C}^\infty(\overline{M}) + S^{-\delta}(\overline{M}))$, we find

$$ib := x^{-1}(B - \chi(x)x^2 D_x) \in \mathcal{C}^\infty(\overline{M}) + S^{-\delta}(\overline{M})$$

and b is real-valued. It is easy to check that $b|_{\partial M} = \frac{n-1}{2}$, where $n = \dim M$.

For the next theorem we also introduce

$$(5.18) \qquad A = \frac{1}{2}(\chi(x)xD_x + (\chi(x)xD_x)^*).$$

Recall from [20, Equation (4.2)] that the space $\Psi_{scc}^{s,r}(\overline{M})$ of scattering pseudo-differential operators is locally defined using quantisations of product-type symbols satisfying estimates

$$(5.19) \qquad |\partial_z^\alpha \partial_\zeta^\beta a(z, \zeta)| \leq C(\alpha, \beta)(1 + |z|)^{r-|\alpha|}(1 + |\zeta|)^{s-|\beta|},$$

where z is thought as a local coordinate on the manifold and ζ the momentum variable. *Notice that we have the negative of the convention of [20] for the growth order r, so for us $r > 0$ means growing coefficients.* (Parenti [23] and Shubin [28] introduced this class earlier on $\mathbb{R}^n$.) Here 'locally' means a coordinate identification of an open set on the manifold with boundary with an open set on the radial compactification $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$ discussed at the end of Section 5.1. Near $\partial\overline{\mathbb{R}^n}$, since in that region $\overline{\mathbb{R}^n}$ is identified with $[0, \epsilon)_x \times \mathbb{S}^{n-1}$, the coordinate identification is thus with the closure of an asymptotically conic subset of $\mathbb{R}^n$.

On the other hand, $\Psi_{sc}^{s,r}(\overline{M})$ is the subspace of $\Psi_{scc}^{s,r}(\overline{M})$ consisting of classical pseudodifferential operators, see [20, Equation (4.7)]. Here classical means, in terms of the local description above, that the symbol a has a one-step (i.e., with powers in the expansions stepping by one) polyhomogeneous expansion both in terms of the defining function of spatial infinity, $|z|^{-1}$, and the defining function of momentum infinity, $|\zeta|^{-1}$, cf. the discussion at the end of Section 5.1. This joint behavior can again be encoded via a compactification. We compactify the momentum variable ζ similarly to the position variable z, so the amplitude a is considered as a function on the interior $\mathbb{R}_z^n \times \mathbb{R}_\zeta^n$ of $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$. For $s = r = 0$, classicality means that a extends (necessarily uniquely) as a smooth function to $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$; for general

s, r, the statement is the analogous extendability for

$$(1 + |z|^2)^{-r/2}(1 + |\zeta|^2)^{-s/2}a;$$

the expression $(1 + |\cdot|^2)^{1/2}$ is used in place of $1 + |\cdot|$ to ensure smoothness at the origin.

Taking into account that differential operators have amplitudes that are polynomial, thus classical, in the momentum variable, these definitions are consistent with those of $\mathrm{Diff}^{s,-r}_{\mathrm{sc}}(\overline{M}) \subset \Psi^{s,r}_{\mathrm{sc}}(\overline{M})$ (notice the change in sign!) and $S^r \mathrm{Diff}^s_{\mathrm{sc}}(\overline{M}) = \mathrm{Diff}^{s,-r}_{\mathrm{scc}}(\overline{M}) \subset \Psi^{s,r}_{\mathrm{scc}}(\overline{M})$, for s a non-negative integer.

Proposition 5.8. *Let A and B be given by (5.17)–(5.18). There exist $R \in x\,\Psi^{0,0}_{\mathrm{sc}}(\overline{M})$ and $K \in x^{1+\delta}\,\Psi^{2,0}_{\mathrm{scc}}(\overline{M})$ such that*

$$(5.20) \qquad \mathrm{i}[B, H] = 2\lambda x - 2BxB + (H - \lambda)R + R^*(H - \lambda) + K.$$

In addition, there exist $\widetilde{R} \in \Psi^{0,0}_{\mathrm{sc}}(\overline{M})$, $\widetilde{K} \in x^\delta\,\Psi^{2,0}_{\mathrm{scc}}(\overline{M})$, such that

$$(5.21) \qquad \mathrm{i}[A, H] = 2\lambda + (H - \lambda)\widetilde{R} + \widetilde{R}^*(H - \lambda) + \widetilde{K}.$$

Remark 5.9. Again, the presence of bundles makes no difference in this proof.

Proof. We show (5.20); the proof of (5.21) is entirely analogous.

Adding terms to H which differ from it by an element of

$$S^{-\delta}\,\mathrm{Diff}^2_{\mathrm{sc}}(\overline{M}) = \mathrm{Diff}^{2,\delta}_{\mathrm{scc}}(\overline{M})$$

results in a term in the commutator in $x^{1+\delta}\,\mathrm{Diff}^2_{\mathrm{sc}}(\overline{M})$ that can be absorbed into K, and similarly for adding a term in $\mathrm{Diff}^{1,\delta}_{\mathrm{scc}}(\overline{M})$ to B. Thus, all of the terms arising from $S^{-\delta}$ terms in either g or V can be ignored. A straightforward principal symbol computation (which recall is modulo one order gain both in derivatives and decay) then gives that (modulo these $S^{-\delta}$ terms which give K-absorbable results) the principal symbol of the left hand side is the same as that of $xH + Hx - 2BxB$, so that of

$$2x\lambda - 2BxB + x(H - \lambda) + (H - \lambda)x,$$

proving the proposition since the agreement of the principal symbols means that the operators agree modulo $x^2\,\mathrm{Diff}^{2,0}_{\mathrm{sc}}(\overline{M})$, which again is absorbable into K. $\qquad\square$

We also need a somewhat more general computation. For this, let $k > 0$, and we note first that, with $t \in (0, 1]$ a parameter, $(1 + t/x)^{-k} = (x/(x + t))^k$, as a function of x, is a symbol of order $-k$ for $t > 0$, and is uniformly bounded in symbols of order 0. Indeed, this follows from

$$x\partial_x(1 + t/x)^{-k} = k(t/x)(1 + t/x)^{-k-1} = k(1 + t/x)^{-k} - k(1 + t/x)^{-k-1},$$

so iterative regularity under derivatives is easy to check, and from $(1 + t/x)^{-k} \leq 1$ and for $t > 0$, $(1 + t/x)^{-k} \leq t^{-k}x^k$. Thus, $(1 + t/x)^{-k}x^2D_x$ is in $S^{-k} \operatorname{Diff}^1_{\mathrm{scc}}(\overline{M})$ for $t > 0$, and is uniformly bounded in $S^0 \operatorname{Diff}^1_{\mathrm{scc}}(\overline{M})$, converging to x^2D_x in $S^{\delta'} \operatorname{Diff}^1_{\mathrm{scc}}(\overline{M})$ for $\delta' > 0$. This proves the first part of the following proposition:

Proposition 5.10. *For $t \in [0, 1]$, let*

$$B_{s,k,t} = \frac{1}{2}((\chi(x)x^{-s}(1 + t/x)^{-k}x^2D_x) + (\chi(x)x^{-s}(1 + t/x)^{-k}x^2D_x)^*).$$

Then for $t > 0$, $B_{s,k,t} \in S^{s-k} \operatorname{Diff}^1_{sc}(\overline{M})$, and $\{B_{s,k,t} : t \in [0,1]\}$ is uniformly bounded in $S^s \operatorname{Diff}^1_{sc}(\overline{M})$, with $B_{s,k,t} \to B_{s,k,0}$ in $S^{s+\delta'} \Psi^{1,0}_{scc}(\overline{M})$, $\delta' > 0$ arbitrary, as $t \to 0$.

Furthermore, with $\widehat{R}_{s,k,t}$ uniformly bounded in $S^{s-1} \operatorname{Diff}^{0,0}_{sc}(\overline{M})$, $\widehat{K}_{s,k,t}$, $\widehat{K}'_{s,k,t}$ uniformly bounded in $S^{s-1-\delta} \operatorname{Diff}^{2,0}_{sc}(\overline{M})$,

$$
\begin{aligned}
\mathrm{i}[B_{s,k,t}, H] =~& 2x^{1-s}(1 + t/x)^{-k}\left(\left(s - k\frac{t/x}{1 + t/x}\right)(x^2D_x)^2 + \Delta_h\right) + \widehat{K}'_{s,k,t} \\
=~& 2x^{1-s}(1 + t/x)^{-k}\lambda + 2(x^2D_x)^*x^{1-s}(1 + t/x)^{-k} \\
& \times \left(s - 1 - k\frac{t/x}{1 + t/x}\right)(x^2D_x) \\
& + (H - \lambda)\widehat{R}_{s,k,t} + \widehat{R}^*_{s,k,t}(H - \lambda) + \widehat{K}_{s,k,t}.
\end{aligned}
$$

(5.22)

Thus, if $s - k \geq 1$,

$$\mathrm{i}[B_{s,k,t}, H] \geq 2x^{1-s}(1 + t/x)^{-k}\lambda + (H - \lambda)\widehat{R}_{s,k,t} + \widehat{R}^*_{s,k,t}(H - \lambda) + \widehat{K}_{s,k,t}.$$

Remark 5.11. The main point of this computation is that on the one hand $s - k\frac{t}{x}\left(1 + \frac{t}{x}\right)^{-1} \geq s - k$, thus is positive with a positive lower bound if $k < s$, on the other hand converges to s in any growing symbol space, $S^{\delta'}$, $\delta' > 0$, as $t \to 0$.

Proof. It only remains to do the commutator calculation. Again, this is a principal symbol computation in which all of the terms arising from $S^{-\delta}$ terms in either g or V can be ignored. $\qquad\square$

5.4. Super-polynomial decay

As a starting point, we show that L^2 solutions decay superpolynomially. From a microlocal analysis perspective, this follows from propagation results in the framework of scattering pseudodifferential operators as in [20], cf. also the arguments preceding [33, Proposition 4.13], but we give a more elementary argument (under, however, stronger assumptions than those of [20]).

We recall the scattering Sobolev spaces $H^{s,r}_{\mathrm{sc}}(\overline{M}) = x^r H^{s,0}_{\mathrm{sc}}(\overline{M})$, which are modelled, via local coordinate identification much as for the scattering pseudodifferential operators, on the standard weighted Sobolev spaces

$$H^{s,r}(\mathbb{R}^n) = \langle z \rangle^{-r} H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \ \langle z \rangle^r u \in H^s(\mathbb{R}^n)\},$$

where $\langle z \rangle = (1 + |z|^2)^{1/2}$. Correspondingly, the space $H^{s,r}_{\mathrm{sc}}(\overline{M})$ is the space of tempered distributions u on $\overline{M}$ (dual of $\dot{\mathcal{C}}^\infty(\overline{M})$, consisting of $\mathcal{C}^\infty$ functions vanishing with all derivatives at ∂M) such that for all $A \in \Psi^{s,r}_{\mathrm{scc}}(\overline{M})$, $Au \in L^2_{\mathrm{sc}}(\overline{M})$. This in turn is equivalent to requiring $Au \in L^2_{\mathrm{sc}}(\overline{M})$ for a *single fully elliptic* $A \in \Psi^{s,r}_{\mathrm{scc}}(\overline{M})$; full ellipticity means that for the local coordinate amplitude a, $|a|$ has a lower bound $c(1 + |z|)^r(1 + |\zeta|)^s$, $c > 0$, for $|z| + |\zeta|$ sufficiently large. For instance, with Δ the positive Laplacian of a scattering metric, $(1 + \Delta)^{s/2}$ is elliptic in $\Psi^{s,0}_{\mathrm{scc}}(\overline{M})$, and thus $x^{-r}(1 + \Delta)^{s/2}$ is elliptic in $\Psi^{s,r}_{\mathrm{scc}}(\overline{M})$. Elliptic regularity in the differential order sense, i.e. for an operator A with principal symbol satisfying $|a(z,\zeta)| \geq c(1 + |z|)^r(1 + |\zeta|)^s$, $c > 0$, for $|\zeta|$ sufficiently large (but not necessarily if $|z|$ is large and ζ is in a bounded region), is the statement that $Au \in H^{k-s,p-r}_{\mathrm{sc}}(\overline{M})$ and $u \in H^{k',p}_{\mathrm{sc}}$ for some k' implies $u \in H^{k,p}_{\mathrm{sc}}(\overline{M})$, with a corresponding estimate on u. In particular, there is a constant $C = C_{k,k',p}$ such that if $Au = 0$ then $\|u\|_{H^{k,p}_{\mathrm{sc}}(\overline{M})} \leq C_{k,k',p}\|u\|_{H^{k',p}_{\mathrm{sc}}(\overline{M})}$; note that there is an improvement in the differentiable, but not in the decay order of the space.

As a first step it is useful to observe the following:

Lemma 5.12. *Suppose that $Q \in S^{1+2\beta}\,\Psi^{1,0}_{sc}(\overline{M})$ and $(H - \lambda)\psi = 0$, $x^{-\beta}\psi \in L^2_{sc}(M)$. Then*

$$([Q, H]\psi, \psi) = 0.$$

Here the commutator is understood to be defined on Schwartz functions; the issue is the lack of sufficient decay of the individual products in terms of applicability to, and pairability with, ψ.

Proof. As $(H - \lambda)\psi = 0$, by elliptic regularity

$$(5.23) \qquad \|\psi\|_{x^p H^k_{sc}(\overline{M})} \leq \widetilde{b}_{1,k,p} \|x^p \psi\|,$$

Thus, by elliptic regularity, the differential order of operators below never matters.

Formally the lemma follows from

$$([Q, H]\psi, \psi) = ([Q, H - \lambda]\psi, \psi) = ((H - \lambda)\psi, Q^*\psi) + (Q\psi, (H - \lambda)\psi) = 0,$$

but care needs to be taken as the pairing on left hand side is *only* defined (as the dual pairing between $x^{-\beta} L^2_{sc}(\overline{M})$ and $x^\beta L^2_{sc}(\overline{M})$) due to the fact that $[Q, H] \in S^{2\beta} \Psi^{2,0}_{sc}(\overline{M})$, while

$$Q(H - \lambda), \quad (H - \lambda)Q \in S^{2\beta+1} \Psi^{3,0}_{sc}(\overline{M}),$$

so the pairing $((H - \lambda)Q\psi, \psi)$ is not a priori defined, with the lack of sufficient decay being the issue.

To remedy this, one simply regularizes; here we use $(1 + t/x)^{-1} = \frac{x}{x+t}$, $t > 0$, as a regularizer since it also plays a role below, this is uniformly bounded as $t \to 0$ (by 1), and is $O(x)$ for $t > 0$, removing the pairing issue. Namely, we have, with the pairing being the dual pairing,

$$(5.24) \quad ([Q, H]\psi, \psi) = \lim_{t \to 0}((1 + t/x)^{-1}[Q, H]\psi, \psi)$$
$$= \lim_{t \to 0}((1 + t/x)^{-1}(Q(H - \lambda) - (H - \lambda)Q)\psi, \psi)$$
$$= \lim_{t \to 0}\left(((1 + t/x)^{-1}Q(H - \lambda)\psi, \psi) - ((1 + t/x)^{-1}(H - \lambda)Q\psi, \psi)\right)$$
$$= -\lim_{t \to 0}\left(([(1 + t/x)^{-1}, H - \lambda] + (H - \lambda)(1 + t/x)^{-1})Q\psi, \psi\right)$$
$$= -\lim_{t \to 0}\left(([(1 + t/x)^{-1}, H - \lambda]Q\psi, \psi) + ((H - \lambda)(1 + t/x)^{-1}Q\psi, \psi)\right),$$

where the penultimate equality used that $(H - \lambda)\psi = 0$.

Now, $[(1 + t/x)^{-1}, (H - \lambda)]$ is uniformly bounded (as $t \to 0$) in the space $S^{-1} \mathrm{Diff}^{1,0}_{sc}(\overline{M})$, and indeed converges to 0 in $S^{-1+\delta'} \mathrm{Diff}^{1,0}_{sc}(\overline{M})$, $\delta' > 0$, so $[(1 + t/x)^{-1}, (H - \lambda)]Q$ is uniformly bounded in $S^0 \mathrm{Diff}^{2,0}_{sc}(\overline{M})$ and converges to 0 in $S^{\delta'} \mathrm{Diff}^{2,0}_{sc}(\overline{M})$. This implies that $[(1 + t/x)^{-1}, (H - \lambda)]Q\psi$ converges to 0 in $H^{-1,0}_{sc}(\overline{M})$ as $t \to 0$, as can be seen that if $\psi_j \to \psi$ in $L^2_{sc}(\overline{M})$ with $\psi_j \in$

$\dot{\mathcal{C}}^\infty(\overline{M})$, then $[(1 + t/x)^{-1}, (H - \lambda)]Q\psi$ converges to 0 as $t \to 0$ in $\dot{\mathcal{C}}^\infty(\overline{M})$ thus in $H_{\text{sc}}^{-1,0}(\overline{M})$, while given any $\epsilon > 0$, $[(1 + t/x)^{-1}, (H - \lambda)]Q(\psi - \psi_j)$ is $< \epsilon$ in $H_{\text{sc}}^{-1,0}(\overline{M})$ for sufficiently large j. Thus,

$$
\begin{aligned}
([Q, H]\psi, \psi) = -\lim_{t \to 0} \Big(&([(1 + t/x)^{-1}, H - \lambda]Q\psi, \psi) \\
&+ ((H - \lambda)(1 + t/x)^{-1}Q\psi, \psi) \Big) \\
= -\lim_{t \to 0} &((H - \lambda)(1 + t/x)^{-1}Q\psi, \psi) \\
= -\lim_{t \to 0} &((1 + t/x)^{-1}Q\psi, (H - \lambda)\psi) = 0,
\end{aligned}
$$

proving the lemma.   □

Proposition 5.13. *Let* $\lambda > 0$, *and suppose that* $\psi \in L_{sc}^2(M_{\text{ext}})$ *satisfies* $H\psi = \lambda\psi$. *Then for all* $\beta \in \mathbb{R}$, $x^{-\beta}\psi \in L_{sc}^2(M_{\text{ext}})$.

Remark 5.14. In the proof we will assume for clarity that (M, g) is complete with $\psi \in L_{\text{sc}}^2(\overline{M})$. If we merely assume $x^{-\beta}\psi \in L_{\text{sc}}^2(M_{\text{ext}})$, then all integrations by parts, such as e.g. those involved in the proof of Lemma 5.12, should be carried-out on M_{ext}. This will introduce controlled boundary terms at the inner boundary $\{x = x_1\}$ which will not affect the argument. Equivalently, in the proof the eigenfunction ψ can be multiplied by a cut-off function which vanishes near $\{x = x_1\}$ and equals one near $\{x = 0\}$, leading to error terms in the equations which can be estimated by $C\|\psi\|$, and resulting in the same conclusions.

Proof. Let

$$
\beta_1 = \sup\{\beta \in [0, \infty) : \ x^{-\beta}\psi \in L_{\text{sc}}^2(\overline{M})\},
$$

and assume β_1 is finite. Let $\beta \in (\beta_1, \beta_1 + \delta/2) > 0$. Take $s = 1 + 2\beta$ and $0 < k \leq \min(1, \delta) = \delta$ with $\beta - k/2 \in [0, \beta_1)$ if $\beta_1 > 0$, $\beta - k/2 = 0$ if $\beta_1 = 0$. Thus, $s - k = 1 + 2(\beta - k/2) \geq 1$, and $\frac{s-1-k}{2} = \beta - k/2$ is either 0 or $< \beta_1$, so in either case

$$
x^{-(s-k-1)/2}\psi \in L_{\text{sc}}^2(M),
$$

and $\frac{s-1-\delta}{2} \leq \frac{s-1-k}{2}$ as well, so

$$
x^{-(s-1-\delta)/2}\psi \in L_{\text{sc}}^2(M).
$$

Apply Proposition 5.10 with this s, k, using that $\widehat{K}_{s,k,t}$ uniformly bounded in $S^{s-1-\delta} \text{Diff}_{\text{sc}}^{2,0}(\overline{M})$, so in view of elliptic regularity, (5.23),

$$
|(\widehat{K}_{s,k,t}\psi, \psi)| \leq \widetilde{C}_s \|x^{-(s-1-\delta)/2}\psi\|^2,
$$

and $\widehat{R}_{s,k,t}$ is in $S^{s-k-1}\operatorname{Diff}_{\mathrm{sc}}^{0,0}(\overline{M})$ for $t > 0$ (not uniformly bounded in t though!), to conclude, using $(H - \lambda)\psi = 0$, that

$$
\begin{aligned}
(\mathrm{i}[B_{s,k,t}, H]\psi, \psi) &\geq 2\lambda\|(1 + t/x)^{-k/2}x^{-\beta}\psi\|^2 + ((H - \lambda)\widehat{R}_{s,k,t}\psi, \psi) \\
&\quad + (\widehat{R}_{s,k,t}^*(H - \lambda)\psi, \psi) + (\widehat{K}_{s,k,t}\psi, \psi) \\
&\geq 2\lambda\|(1 + t/x)^{-k/2}x^{-\beta}\psi\|^2 - \widetilde{C}_s\|x^{-(s-1-\delta)/2}\psi\|^2.
\end{aligned}
$$

On the other hand, by Lemma 5.12,

$$
(\mathrm{i}[B_{s,k,t}, H]\psi, \psi) = 0.
$$

Thus, we conclude that

$$
2\lambda\|(1 + t/x)^{-k/2}x^{-\beta}\psi\|^2
$$

is uniformly bounded as $t \to 0$, thus $x^{-\beta}\psi \in L^2$, contradicting the definition of β.

This shows that β_1 is not finite, hence proves the proposition. □

5.5. Superexponential decay

Using the global positive commutator estimate, Proposition 5.8, we can now prove a decay faster-than-any-exponential of non-threshold eigenfunctions. For this part of the paper, we could adapt the proof of Froese and Herbst [6] in Euclidean potential scattering, as was done in [31] in the geometric potential scattering setting. However, we focus on the approach that will play a crucial role in the proof of unique continuation at infinity. Nonetheless, a modification of the Froese-Herbst commutator will play a role when $\alpha = 0$ (in the notation of Lemma 5.6), where conjugated Hamiltonian is close to being self-adjoint (in fact, it is, if $F = 0$), so we will use a modification of xD_x, more precisely a rescaling of $\operatorname{Im} P$, for a commutator estimate in place of $\operatorname{Im} P$.

Theorem 5.15. *Let $\lambda > 0$, and suppose that $\psi \in L_{sc}^2(M_{\mathrm{ext}})$ satisfies $H\psi = \lambda\psi$. Then for all $\alpha \in \mathbb{R}$, $e^{\alpha/x}\psi \in L_{sc}^2(M_{\mathrm{ext}})$.*

Proof. We start by pointing-out that Remark 5.14 applies again.

The proof is by contradiction. First note that

$$\psi \in \dot{\mathcal{C}}^\infty(\overline{M})$$

by Proposition 5.13 and the usual weighted elliptic estimates. (Recall that $\dot{\mathcal{C}}^\infty(\overline{M})$ denotes the space of smooth functions which decay to infinite order at the boundary.)

Let

$$\alpha_1 = \sup\{\alpha \in [0, \infty) : \exp(\alpha/x)\psi \in L^2_{\rm sc}(\overline{M})\}.$$

If $\alpha_1 = 0$, then let $\alpha = 0$, otherwise suppose that $\alpha < \alpha_1$, and in either case $\gamma \in (0, 1]$ will be a constant satisfying $\alpha + \gamma > \alpha_1$. These two cases will require separate treatment.

The $\alpha_1 > 0$ case is more representative of the proof of the unique continuation result, so we will start with that. We show in this case that for sufficiently small γ (depending only on α_1) $\exp((\alpha + \gamma)/x)\psi \in L^2_{\rm sc}(\overline{M})$, which contradicts our assumption on α_1 if α is close enough to α_1.

We start with a general discussion, so we do not yet make assumptions on α_1.

Below we use two positivity estimates, namely (5.12) and the Mourre-type estimate, Proposition 5.8, at energy $\lambda + \alpha_1^2$ (i.e. with λ replaced by this in the statement of the proposition), with $B = \chi(x)x^2 D_x + (\chi(x)x^2 D_x)^*$. That is, since $\lambda + \alpha_1^2 > 0$, there exists $c_0 > 0$, $R \in \Psi^{0,0}_{\rm sc}(\overline{M})$, $K \in \Psi^{2,0}_{\rm sc}(\overline{M})$, such that for $\widetilde{\psi} \in L^2_{\rm sc}(\overline{M})$,

$$(5.25) \quad (\widetilde{\psi}, {\rm i}[B, H]\widetilde{\psi})$$
$$\geq c_0 \|x^{1/2}\widetilde{\psi}\|^2 - 2\,{\rm Re}(\widetilde{\psi}, x(x^2 D_x)^2\widetilde{\psi})$$
$$+ {\rm Re}((H - \lambda - \alpha_1^2)\widetilde{\psi}, xR\widetilde{\psi}) + {\rm Re}(x^{(1+\delta)/2}\widetilde{\psi}, Kx^{(1+\delta)/2}\widetilde{\psi}).$$

We will apply this with $\widetilde{\psi} = \psi_F$, with F given by (5.26) below.

We first note that we certainly have for all $\beta \in \mathbb{R}$, $\exp(\alpha/x)x^\beta\psi \in L^2_{\rm sc}(\overline{M})$, due to our choice of α. We apply Lemma 5.6 with $\beta \geq 1$ and with

$$(5.26) \qquad\qquad F \equiv F_\beta := \frac{\alpha}{x} + \beta \log\left(1 + \frac{\gamma}{\beta x}\right),$$

and let

$$\psi_\beta := \psi_F \equiv e^F \psi.$$

The reason for the choice (5.26) is that on the one hand $F(x) \to (\alpha + \gamma)/x$ as $\beta \to \infty$, so in the limit we will obtain an estimate on $e^{(\alpha+\gamma)/x}\psi$, and

on the other hand $F(x) \leq \frac{\alpha}{x} + \beta|\log x|$, so e^{F_β} is bounded by $x^\beta e^{\alpha/x}$, for all values of β, i.e. e^{F_β} provides a 'regularization' (in terms of growth) of $e^{(\alpha+\gamma)/x}$, so that Lemma 5.6 can be applied.

Note that $F = F_\beta \in S^1([0,1))$, and F_β is uniformly bounded in $S^1([0,1))$ for $\beta \in [1,\infty)$, $\alpha \in [0,\alpha_1)$ (or $\alpha = \alpha_1$ if $\alpha_1 = 0$), $\gamma \in [0,1]$. Indeed,

$$0 \leq -x^2\partial_x F = \alpha + \gamma\left(1 + \frac{\gamma}{\beta x}\right)^{-1} \leq \alpha + \gamma,$$

and in general $(x\partial_x)^m(1 + \frac{\gamma}{\beta x})^{-1} = (-r\partial_r)^m(1+r)^{-1}$, $r = \frac{\gamma}{\beta x}$, so the uniform boundedness of F follows from $(1+r)^{-1}$ being a symbol in the usual sense on $[0,\infty)$. In particular, all symbol norms of $-x^2\partial_x F - \alpha$ are $O(\gamma)$.

Below, when $\alpha = 0$, we will need to consider

$$(-x^2\partial_x F)^{-1}(x^2\partial_x)^m(-x^2\partial_x F),$$

with $m \geq 0$. By the Leibniz rule, this can be written as

$$\sum_{j \leq m} c_j x^m (-x^2\partial_x F)^{-1}(x\partial_x)^j(-x^2\partial_x F).$$

In terms of r, $(-x^2\partial_x F)^{-1}(x\partial_x)^j(-x^2\partial_x F)$ with $\alpha = 0$ takes the form

$$(1+r)(-r\partial_r)^j(1+r)^{-1},$$

hence it is still bounded on $[0,\infty)$, so in fact

(5.27) $\qquad x^{-m}(-x^2\partial_x F)^{-1}(x^2\partial_x)^m(-x^2\partial_x F), \quad m \geq 0,$

is uniformly bounded on $[0,\infty)$. In fact, (5.27) is uniformly bounded in $S^0([0,1))$, since applying $x\partial_x$ to it gives rise to additional factors such as

$$(-x^2\partial_x F)^{-k}(x\partial_x)^k(-x^2\partial_x F),$$

which we have just seen to be uniformly bounded on $[0,\infty)$.

We remark first that $P(F)\psi_F = 0$, so by elliptic regularity,

$$\|\psi_F\|_{x^p H^k_{\mathrm{sc}}(\overline{M})} \leq b_{1,k,p}\|x^p\psi_F\|,$$

with $b_{1,k,p}$ independent of F as long as α is bounded. This follows from the fact that the estimates on the derivatives of F, as needed for controlling $b_{1,k,p}$, are uniform in $\alpha \in [0,\alpha_1]$, $\gamma \in [0,1]$ and $\gamma/\beta \in [0,1]$. *In general, below*

b_j denote positive constants that are independent of α, β, γ in these intervals, and R_j denote operators which are uniformly bounded in $\mathrm{Diff}^2_{scc}(\overline{M})$, or on occasion in $\Psi^{m,0}_{scc}(\overline{M})$, for some m. (Note again that, by elliptic regularity, the differential order never matters.)

As already pointed out, the proof is slightly different in the cases $\alpha > 0$ and $\alpha = 0$ since in the latter case the usually dominating term, $-2\alpha x^2 D_x$, of $\mathrm{Im}\, P$ vanishes.

When $\alpha > 0$, the key step in the proof of this theorem arises from considering, with $P \equiv P_\beta := H(F) - \lambda$,

$$P^*P = (\mathrm{Re}\, P)^2 + (\mathrm{Im}\, P)^2 + \mathrm{i}(\mathrm{Re}\, P \, \mathrm{Im}\, P - \mathrm{Im}\, P \, \mathrm{Re}\, P),$$

so

$$\begin{aligned}
(5.28) \qquad 0 &= (\psi_F, P^*P\psi_F) \\
&= \|\mathrm{Re}\, P\psi_F\|^2 + \|\mathrm{Im}\, P\psi_F\|^2 + (\psi_F, \mathrm{i}[\mathrm{Re}\, P, \mathrm{Im}\, P]\psi_F).
\end{aligned}$$

The first two terms on the right-hand side are non-negative, so the key issue is the positivity of the commutator. Note that

$$\begin{aligned}
(5.29) \qquad \mathrm{Re}\, P &= H - \alpha^2 - \lambda + \gamma R_1 + x^\delta R_2, \\
\mathrm{Im}\, P &= -2\alpha x^2 D_x + \gamma R_3 + x^\delta R_4.
\end{aligned}$$

By (5.12),

$$(5.30) \qquad \mathrm{i}[\mathrm{Re}\, P, \mathrm{Im}\, P] = 2\alpha \mathrm{i}[x^2 D_x, H] + x\gamma R_5 + x^{1+\delta} R_6.$$

Hence, from (5.28) and (5.25), with the $\mathrm{Re}\, P$ and $\mathrm{Im}\, P$ terms in the scalar products arising from (5.25),

$$\begin{aligned}
(5.31) \quad 0 \geq{}& \|\mathrm{Re}\, P\psi_F\|^2 + \|\mathrm{Im}\, P\psi_F\|^2 + 2\alpha c_0\|x^{1/2}\psi_F\|^2 + \gamma(\psi_F, xR_{11}\psi_F) \\
&+ (\psi_F, xR_{12}\,\mathrm{Re}\, P\psi_F) + (\psi_F, \mathrm{Re}\, PxR_{13}\psi_F) \\
&+ (\psi_F, xR_{14}\,\mathrm{Im}\, P\psi_F) + (\psi_F, \mathrm{Im}\, PxR_{15}\psi_F) + (\psi_F, x^{1+\delta}R_{16}\psi_F).
\end{aligned}$$

Now, terms such as $|(\psi_F, x^{1+\delta}R_{16}\psi_F)|$ can be estimated by $b_2\|x^{(1+\delta)/2}\psi_F\|^2$, while $\gamma|(\psi_F, xR_{11}\psi_F)|$ may be estimated by $\gamma b_3\|x^{1/2}\psi_F\|^2$, while

$$\begin{aligned}
|(\psi_F, xR_{12}\,\mathrm{Re}\, P\psi_F)| &\leq b_4\|x\psi_F\|\|\mathrm{Re}\, P\psi_F\| \leq b_4(\epsilon^{-1}\|x\psi_F\|^2 + \epsilon\|\mathrm{Re}\,\psi_F\|^2), \\
|(\psi_F, xR_{14}\,\mathrm{Im}\, P\psi_F)| &\leq b_5\|x\psi_F\|\|\mathrm{Im}\, P\psi_F\| \leq b_5(\epsilon^{-1}\|x\psi_F\|^2 + \epsilon\|\mathrm{Im}\,\psi_F\|^2),
\end{aligned}$$

with similar estimates for the remaining terms. Putting this together, (5.31) yields

$$
(5.32) \qquad 0 \geq (1 - b_6\epsilon)\|\operatorname{Re} P\psi_F\|^2 + (1 - b_7\epsilon)\|\operatorname{Im} P\psi_F\|^2
$$
$$
+ (2\alpha c_0 - \gamma b_8)\|x^{1/2}\psi_F\|^2 - b_9(\epsilon)\|x^{(1+\delta)/2}\psi_F\|^2.
$$

For $\widetilde{\delta} > 0$, in $x \geq \widetilde{\delta}$, $x|\psi_F| = xe^F|\psi| \leq b_{10}(\widetilde{\delta})|\psi|$, so

$$
\|x^{(1+\delta)/2}\psi_F\|^2 = \|x^{(1+\delta)/2}\psi_F\|^2_{x \leq \widetilde{\delta}} + \|x^{(1+\delta)/2}\psi_F\|^2_{x \geq \widetilde{\delta}}
$$
$$
\leq \widetilde{\delta}^\delta \|x^{1/2}\psi_F\|^2_{x \leq \widetilde{\delta}} + b_{10}(\widetilde{\delta})\|\psi\|^2_{x \geq \widetilde{\delta}}
$$
$$
\leq \widetilde{\delta}^\delta \|x^{1/2}\psi_F\|^2 + b_{10}(\widetilde{\delta})\|\psi\|^2.
$$

Thus, (5.32) yields that

$$
(5.33) \qquad 0 \geq (1 - b_6\epsilon)\|\operatorname{Re} P\psi_F\|^2 + (1 - b_7\epsilon)\|\operatorname{Im} P\psi_F\|^2
$$
$$
+ (2\alpha c_0 - \gamma b_8 - b_9(\epsilon)\widetilde{\delta}^\delta)\|x^{1/2}\psi_F\|^2 - b_{10}(\widetilde{\delta})\|\psi\|^2.
$$

Hence, choosing $\epsilon > 0$ sufficiently small so that $b_6\epsilon < 1$, $b_7\epsilon < 1$, then choosing $\gamma_0 > 0$ sufficiently small so that $b_{11} = 2\alpha c_0 - \gamma_0 b_8 > 0$, we deduce that for $\gamma < \gamma_0$,

$$
(5.34) \qquad b_{10}(\widetilde{\delta})\|\psi\|^2 \geq (b_{11} - b_9\widetilde{\delta}^\delta)\|x^{1/2}\psi_F\|^2.
$$

But, for $\widetilde{\delta}^\delta \in (0, \frac{b_{11}}{b_9})$, this shows that $\|x^{1/2}\psi_F\|^2$ is uniformly bounded as $\beta \to \infty$. Noting that F is an increasing function of β and ψ_F converges to $e^{(\alpha+\gamma)/x}\psi$ pointwise, we deduce from the monotone convergence theorem that

$$
x^{1/2}e^{(\alpha+\gamma)/x}\psi \in L^2_{\mathrm{sc}}(\overline{M}),
$$

so for $\gamma' < \gamma$, $e^{(\alpha+\gamma')/x}\psi \in L^2_{\mathrm{sc}}(\overline{M})$.

We pass now to the case $\alpha = 0$. Then (5.30) becomes

$$
(5.35) \qquad i[\operatorname{Re} P, \operatorname{Im} P] = \gamma x R_{16} + x^{1+\delta} R_{17}.
$$

The calculations so far lead instead to

$$
\|\operatorname{Re} P\psi_F\| \leq b_{12}\|x^{1/2}\psi_F\|, \quad \|\operatorname{Im} P\psi_F\| \leq b_{12}\|x^{1/2}\psi_F\|.
$$

This implies that

$$(5.36) \qquad \|(H - \lambda)\psi_F\| \leq \gamma b_{13}\|\psi_F\| + b_{14}\|x^\delta \psi_F\|,$$
$$(5.37) \qquad \|(x^2 \partial_x F)x^2 D_x \psi_F\| \leq b_{15}\|x^\delta \psi_F\|.$$

To continue, instead of the degenerating commutator $[\operatorname{Re} P, \operatorname{Im} P]$ (where the term $x^2 \partial_x F$ loses its leading order contribution from α), we recall from (5.16) that

$$\operatorname{Im} P = x(x^2 \partial_x F)\widetilde{A} + \widetilde{A}^* x(x^2 \partial_x F),$$
$$\widetilde{A} = xD_x + x^{-1+\delta}\widetilde{R}_1 = A + x^{-1+\delta}\widetilde{R}_2, \ \widetilde{A} - \widetilde{A}^* \in \operatorname{Diff}^1_{\mathrm{scc}}(\overline{M})$$

with $\widetilde{R}_1, \widetilde{R}_2$ uniformly bounded in $\operatorname{Diff}^1_{\mathrm{scc}}(\overline{M})$, and consider $P^*\widetilde{A} - \widetilde{A}P$. Note that

$$\operatorname{Im} P = (\operatorname{Im} P)^* = 2x(x^2 \partial_x F)\widetilde{A} + [\widetilde{A}, x(x^2 \partial_x F)] + (\widetilde{A}^* - \widetilde{A})x(x^2 \partial_x F)$$
$$= 2x(x^2 \partial_x F)\widetilde{A}^* + x\gamma \widetilde{R}_3 = \widetilde{A}^* 2x(x^2 \partial_x F) + x\gamma \widetilde{R}_4$$

with $\widetilde{R}_3, \widetilde{R}_4$ uniformly bounded in $\operatorname{Diff}^1_{\mathrm{scc}}(\overline{M})$.
    Now,

$$(5.38) \qquad i(\widetilde{A}P - P^*\widetilde{A}) = \mathrm{i}[\widetilde{A}, \operatorname{Re} P] - (\operatorname{Im} P\widetilde{A} + \widetilde{A}\operatorname{Im} P)$$
$$= \mathrm{i}[A, H - \lambda] + x^\delta R_{18} - \widetilde{A}^* 2x(x^2 \partial_x F)\widetilde{A}$$
$$- \widetilde{A}2x(x^2 \partial_x F)\widetilde{A}^* + \gamma R_{21}.$$

Thus,

$$0 = (\psi_F, i(\widetilde{A}P - P^*\widetilde{A})\psi_F)$$
$$\geq (\psi_F, \mathrm{i}[A, H]\psi_F) + 2\|x^{1/2}(-x^2 \partial_x F)^{1/2}\widetilde{A}\psi_F\|^2$$
$$+ 2\|x^{1/2}(-x^2 \partial_x F)^{1/2}\widetilde{A}^*\psi_F\|^2 - b_{16}\|x^{\delta/2}\psi_F\|^2 - b_{18}\gamma\|\psi_F\|^2.$$

Using the Mourre estimate (5.21) in Proposition 5.8, we deduce that, with $c_0' \equiv 2\lambda > 0$,

$$0 \geq c_0'\|\psi_F\|^2 - b_{19}\|(H - \lambda)\psi_F\|\|\psi_F\| - b_{20}\|x^{\delta/2}\psi_F\|^2 - b_{18}\gamma\|\psi_F\|^2.$$

Using (5.36)–(5.37) we deduce, as above, that

$$
\begin{aligned}
0 &\geq c_0'\|\psi_F\|^2 - \gamma b_{21}\|\psi_F\|^2 - b_{22}\|\psi_F\|\|x^\delta\psi_F\| - b_{23}\|x^{\delta/2}\psi_F\|^2 \\
&\geq (c_0' - \gamma b_{21} - \epsilon_1 b_{22})\|\psi_F\|^2 - b_{22}\epsilon_1^{-1}\|x^\delta\psi_F\|^2 - b_{23}\|x^{\delta/2}\psi_F\|^2 \\
&\geq (c_0' - \gamma b_{21} - \epsilon_1 b_{22} - (b_{22}\epsilon_1^{-1}\widetilde{\delta}^\delta + b_{23}\widetilde{\delta}^{\delta/2}))\|\psi_F\|^2 - b_{24}(\widetilde{\delta})\epsilon_1^{-1}\|\psi\|^2.
\end{aligned}
$$

Again, we fix first $\epsilon_1 > 0$ so that $c_0' - \epsilon_1 b_{22} > 0$, then $\gamma_0 > 0$ so that

$$
c_0' - \gamma_0 b_{21} - \epsilon_1 b_{22} > 0,
$$

finally $\widetilde{\delta} > 0$ so that

$$
c_0' - \gamma_0 b_{21} - \epsilon_1 b_{22} - (b_{22}\epsilon_1^{-1}\widetilde{\delta}^\delta + b_{23}\widetilde{\delta}^{\delta/2}) > 0.
$$

Now letting $\beta \to \infty$ gives that $e^{\gamma/x}\psi \in L^2_{sc}(\overline{M})$ for $\gamma < \gamma_0$, as above. $\square$

5.6. Absence of positive eigenvalues — high energy estimates

We next prove that faster than exponential decay of an eigenfunction of H implies that it vanishes. This was also the approach taken by Froese and Herbst. However, we use a different, more robust, approach to deal with our much larger error terms. The proof is based on conjugation by $\exp(\alpha/x)$ and letting $\alpha \to +\infty$. Correspondingly, we require positive commutator estimates at high energies. In such a setting first order terms are irrelevant, i.e. V does not play a significant role below. Indeed, we work "semiclassically" (writing $h = \alpha^{-1}$), and the key fact we use is that the commutator of the real and imaginary parts of the conjugated Hamiltonian has the correct sign on its characteristic variety.

We start by recalling (see [36] for a general introduction to semiclassical analysis, [34] for a specific discussion in the scattering setting) that semiclassical scattering vector fields $V \in \mathcal{V}_{sc,h}(\overline{M})$ are simply h-dependent families of vector fields of the form hV', $V' \in \mathcal{C}^\infty([0,1)_h; \mathcal{V}_{sc}(\overline{M}))$, i.e. h times scattering vector fields smoothly depending on h. (Note that one may simply choose to have bounded, not smooth, families of vector fields V' in $\mathcal{V}_{sc}(\overline{M})$, analogously to how we define pseudodifferential operators — since h is a parameter, differentiation in it is not an issue.) The corresponding differential operators, $P \in \mathrm{Diff}^m_{sc,h}(\overline{M})$ are finite sums of up to m-fold products of these, with $\mathcal{C}^\infty(\overline{M} \times [0,1))$ coefficients. Thus, in local coordinates, such an

operator P is of the form

$$\sum_{j+|\alpha|\leq m} a_{j,\alpha}(x,y,h)(hx^2 D_x)^j (hx D_y)^\alpha.$$

Ellipticity of such an operator in the usual, differential, sense is the statement that

$$\left| \sum_{j+|\alpha|=m} a_{j,\alpha}(x,y,h)\xi^j \eta^\alpha \right| \geq c(|\xi|+|\eta|)^m, \qquad |\xi|+|\eta| > R$$

for some $c > 0$ and $R > 0$. Note that the h factors appearing in front of the derivatives are regarded as parts of the expression, i.e. it is $hx^2 D_x$ that is turned into ξ, etc. One defines $\text{Diff}_{\text{scc},h}$ analogously, by allowing symbolic (rather than just smooth) coefficients, smoothly depending on h. Note that if $A \in \text{Diff}_{\text{sc},h}^{s,r}(\overline{M})$, $B \in \text{Diff}_{\text{sc},h}^{s',r'}(\overline{M})$ then

$$[A,B] \in h\,\text{Diff}_{\text{sc},h}^{s+s'-1,r+r'-1}(\overline{M}),$$

i.e. in addition to the gain in the two orders, there is also an extra h gained; there is a similar statement for $\text{Diff}_{\text{scc},h}$.

If $\overline{M}$ is the radial compactification of $\mathbb{R}^n$, obtained by "adding a sphere at $r = \infty$" (cf., e.g., [20]; not to be confused with a one-point compactification familiar to general relativists), as discussed in Section 5.1, this means that P is of the form

$$\sum_{|\beta|\leq m} b_\beta(z,h)(hD_z)^\beta,$$

where b_β are classical symbols smoothly depending on h.

The semiclassical Sobolev norms $\|.\|_{H_{\text{sc},h}^{s,r}(\overline{M})}$, for $s \geq 0$ integer, are, for h-dependent families of functions in $H_{\text{sc}}^{s,r}(\overline{M})$, supported in a coordinate chart,

$$\|u\|^2_{H_{\text{sc},h}^{s,r}(\overline{M})} = \sum_{j+|\alpha|\leq s} \|x^{-r}(hx^2 D_x)^j(hxD_y)^\alpha u\|^2_{L^2_{\text{sc}}(\overline{M})},$$

and in general via a partition of unity. In the case of the radial compactification of $\mathbb{R}^n$, to which the general case locally reduces, this is equivalent to

$$\|u\|^2_{H_{\text{sc},h}^{s,r}(\overline{M})} = \sum_{|\beta|\leq s} \|\langle z\rangle^r (hD_z)^\beta u\|^2_{L^2},$$

i.e. they are like standard weighted Sobolev spaces but with an h appearing in front of each derivative.

Similarly, recall that semiclassical scattering pseudodifferential operators $A \in \Psi_{\mathrm{scc},h}^{s,r}(\overline{M})$ reduce to semiclassical pseudodifferential operators on $\mathbb{R}^n$ resulting from semiclassical quantizations

$$Au(z) = (2\pi h)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(z-z')\cdot\zeta/h} a(z,\zeta,h) u(z')\, dz'\, d\zeta$$

of symbols satisfying estimates of the kind

$$|D_z^\alpha D_\zeta^\beta a(z,\zeta,h)| \leq C(\alpha,\beta)(1+|z|)^{r-|\alpha|}(1+|\zeta|)^{s-|\beta|},$$

i.e. uniform (in h) families of scattering symbols. Notice the factor $1/h$ appearing in the exponent; one could change variables to $\zeta' = \zeta/h$ in the integral, then a would be evaluated at $(z, h\zeta', h)$, explaining the appearance of h in front of derivatives in the differential operator discussion above when a is a polynomial in ζ. The standard results, in particular elliptic estimates, hold, so if A is elliptic in the scattering sense in this semiclassical context, meaning that $|a|$ has a comparable positive lower bound for $|z| + |\zeta|$ large, then elliptic estimates

$$\|u\|_{H_{\mathrm{sc},h}^{k,p}(\overline{M})} \leq C_{k,k',p,p'}(\|Au\|_{H_{\mathrm{sc},h}^{k-s,p-r}(\overline{M})} + \|u\|_{H_{\mathrm{sc},h}^{k',p'}(\overline{M})})$$

hold. Again, there is a version when ellipticity only holds in the differential order sense, namely if $|\zeta|$ is large; this assumes that $u \in H_{\mathrm{sc},h}^{k',p}(\overline{M})$ (i.e. $p = p'$) and then is of the form

$$\|u\|_{H_{\mathrm{sc},h}^{k,p}(\overline{M})} \leq C_{k,k',p}(\|Au\|_{H_{\mathrm{sc},h}^{k-s,p-r}(\overline{M})} + \|u\|_{H_{\mathrm{sc},h}^{k',p}(\overline{M})});$$

cf. the discussion at the beginning of Section 5.4.

Theorem 5.16. *Let $\lambda \in \mathbb{R}$. If $H\psi = \lambda\psi$, $\exp(\alpha/x)\psi \in L_{sc}^2(M_{\mathrm{ext}})$ for all α, then $\psi \equiv 0$.*

Proof. As in the previous proofs, for clarity of the argument we assume first that (M, g) is complete and $\psi \in L_{\mathrm{sc}}^2(\overline{M})$. We will present the, essentially notational, changes arising when $\psi \in L_{\mathrm{sc}}^2(M_{\mathrm{ext}})$ at the end of the proof.

Let

$$F = F_\alpha = \varphi(x)\frac{\alpha}{x},$$

where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ is supported near 0, identically 1 in a smaller neighborhood of 0, and let $\psi_F := e^F \psi$. Then with $h = \alpha^{-1}$, $H_h = h^2 H(F)$ and $P_h := H_h - h^2\lambda$ are elliptic semiclassical differential operators, elliptic in the

usual sense of differentiable order (i.e. the lower bound for the absolute value of the principal symbol holds for $|\zeta|$ large, as discussed above), and

$$P_h \psi_h = 0, \ \psi_h \equiv \psi_F,$$

so by elliptic regularity,

$$(5.39) \qquad \|\psi_h\|_{x^p H^k_{\mathrm{sc},h}(\overline{M})} \leq C_1 \|\psi_h\|_{x^p L^2_{\mathrm{sc}}(\overline{M})},$$

C_1 independent of $h \in (0,1]$ (but depends on k and p). In general, below the C_j denote constants independent of $h \in (0,1]$ (and $\delta > 0$).

The key step in the proof of this theorem arises from considering

$$P_h^* P_h = (\operatorname{Re} P_h)^2 + (\operatorname{Im} P_h)^2 + \mathrm{i}(\operatorname{Re} P_h \operatorname{Im} P_h - \operatorname{Im} P_h \operatorname{Re} P_h)$$

so

$$
\begin{aligned}
(5.40) \qquad 0 &= (\psi_h, P_h^* P_h \psi_h) \\
&= \|\operatorname{Re} P_h \psi_h\|^2 + \|\operatorname{Im} P_h \psi_h\|^2 + (\psi_h, \mathrm{i}[\operatorname{Re} P_h, \operatorname{Im} P_h]\psi_h).
\end{aligned}
$$

The first two terms on the right-hand side are non-negative, so the key issue is the positivity of the commutator. More precisely, we need that there exist operators R_j bounded in $\operatorname{Diff}^{2,0}_{\mathrm{sc},h}(\overline{M})$ such that

$$
\begin{aligned}
(5.41) \qquad (\psi_h, &\mathrm{i}[\operatorname{Re} P_h, \operatorname{Im} P_h]\psi_h) \\
&\geq (\psi_h, (xh + \operatorname{Re} P_h xh R_1 + xh R_2 \operatorname{Re} P_h \\
&\quad + \operatorname{Im} P_h xh R_3 + xh R_4 \operatorname{Im} P_h + xh^2 R_5 + x^{1+\delta} h R_6)\psi_h).
\end{aligned}
$$

The important point is that when ignoring both $\operatorname{Re} P_h$ and $\operatorname{Im} P_h$ in the right-hand side, the commutator is estimated from below by a positive multiple of xh, plus terms $O(xh^2)$ and $O(x^{1+\delta}h)$.

We first prove (5.41), and then show how to use it to prove the theorem. First, modulo terms that will give contributions that are in the error terms, $\operatorname{Re} P_h$ may be replaced by $h^2 \Delta_g - 1 - h^2 \lambda$, and indeed $h^2 \Delta_g - 1$, while $\operatorname{Im} P_h$ may be replaced by $-2h(x^2 D_x)$ (compare Lemma 5.6). Now, by a principal symbol calculation, see Proposition 5.8,

$$
\begin{aligned}
\mathrm{i}[h^2 \Delta_{g_0} - 1, -2h(x^2 D_x)] &= \mathrm{i}[h^2 \Delta_{g_0}, -2h(x^2 D_x)] \\
&= xh(4h^2 \Delta_{g_0} - 4h^2(x^2 D_x)^2 + R_7), \ R_7 \in x^\delta \operatorname{Diff}^2_{\mathrm{sc},h}(\overline{M}) \\
&= xh(4 + 4\operatorname{Re} P_h - 4\operatorname{Im} P_h^2 + R_7'), \ R_7' \in x^\delta \operatorname{Diff}^2_{\mathrm{sc},h}(\overline{M})
\end{aligned}
$$

which is of the desired form.

We now show how to use (5.41) to show unique continuation at infinity. Let $x_0 = \sup_{\overline{M}} x$. We first remark that

$$|(\psi_h, xhR_2 \operatorname{Re} P_h\psi_h)| \leq C_2 h \|x\psi_h\| \|\operatorname{Re} P_h\psi_h\|$$
$$\leq C_2 h \|x\psi_h\|^2 + C_2 h \|\operatorname{Re} P_h\psi_h\|^2,$$
$$|(\psi_h, xhR_4 \operatorname{Im} P_h\psi_h)| \leq C_3 h \|x\psi_h\| \|\operatorname{Im} P_h\psi_h\|$$
$$\leq C_3 h \|x\psi_h\|^2 + C_3 h \|\operatorname{Im} P_h\psi_h\|^2,$$

with similar expressions for the R_1 and R_3 terms in (5.41). Next,

$$|(\psi_h, xh^2 R_5\psi_h)| \leq C_4 h^2 \|x^{1/2}\psi_h\|^2,$$
$$|(\psi_h, x^{1+\delta}hR_6\psi_h)| \leq C_5 h \|x^{(1+\delta)/2}\psi_h\|^2.$$

For $\widetilde{\delta} > 0$, in $x \geq \widetilde{\delta}$, $|\psi_h| = e^{1/xh}|\psi| \leq e^{1/(\widetilde{\delta}h)}|\psi|$, so

$$\|x^{(1+\delta)/2}\psi_h\|^2 = \|x^{(1+\delta)/2}\psi_h\|^2_{x\leq\widetilde{\delta}} + \|x^{(1+\delta)/2}\psi_h\|^2_{x\geq\widetilde{\delta}}$$
$$\leq \widetilde{\delta}^\delta \|x^{1/2}\psi_h\|^2_{x\leq\widetilde{\delta}} + x_0^{1+\delta}e^{2/(\widetilde{\delta}h)}\|\psi\|^2_{x\geq\widetilde{\delta}}$$
$$\leq \widetilde{\delta}^\delta \|x^{1/2}\psi_h\|^2 + x_0^{1+\delta}e^{2/(\widetilde{\delta}h)}\|\psi\|^2.$$

Thus,

$$\|(\psi_h, x^{1+\delta}hR_6\psi_h)| \leq C_5 h\widetilde{\delta}^\delta \|x^{1/2}\psi_h\|^2 + C_5 x_0^{1+\delta}he^{2/(\widetilde{\delta}h)}\|\psi\|^2.$$

Hence, we deduce from (5.40)–(5.41) that

$$(5.42) \qquad 0 \geq (1 - C_6 h)\|\operatorname{Re} P_h\psi_h\|^2 + (1 - C_7 h)\|\operatorname{Im} P_h\psi_h\|^2$$
$$+ h(1 - C_8 h - C_9\widetilde{\delta}^\delta)\|x^{1/2}\psi_h\|^2 - C_{10}he^{2/(\widetilde{\delta}h)}\|\psi\|^2.$$

Hence, there exists $h_0 > 0$ such that for $h \in (0, h_0)$,

$$(5.43) \qquad C_{10}he^{2/(\widetilde{\delta}h)}\|\psi\|^2 \geq h(\frac{1}{2} - C_9\widetilde{\delta}^\delta)\|x^{1/2}\psi_h\|^2$$

Now suppose that $\widetilde{\delta} \in \left(0, \min((\frac{1}{4C_9})^{1/\delta}, \frac{1}{h_0})\right)$ and $\operatorname{supp}\psi \cap \{x \leq \frac{\widetilde{\delta}}{4}\}$ is non-empty. Since $xe^{2/xh} = h^{-1}f(xh)$ where $f(t) = te^{2/t}$, and f is decreasing on $(0, 2)$ (its minimum on $(0, \infty)$ is assumed at 2), we deduce that for

$x \leq \widetilde{\delta}/2$, $xe^{2/xh} \geq \frac{\widetilde{\delta}}{2}e^{4/(\widetilde{\delta}h)}$, so

$$\|x^{1/2}\psi_h\|^2 \geq C_{11}\widetilde{\delta}e^{4/(\widetilde{\delta}h)}, \ C_{11} > 0.$$

Thus, we conclude from (5.43) that

$$(5.44) \qquad C_{10}\|\psi\|^2 \geq (\frac{1}{2} - C_9\widetilde{\delta}^{\delta})C_{11}\widetilde{\delta}e^{2/(\widetilde{\delta}h)}.$$

But letting $h \to 0$, the right-hand side goes to $+\infty$, providing a contradiction.

Thus, ψ vanishes for $x \leq \widetilde{\delta}/4$, hence vanishes identically on $\overline{M}$ by the usual Carleman-type unique continuation theorem [11, Theorem 17.2.1], and when $\psi \in L^2_{\mathrm{sc}}(\overline{M})$ we are done.

As already hinted at, the result for $\psi \in L^2_{\mathrm{sc}}(M_{\mathrm{ext}})$ follows by change of notation. Namely, let now $\widetilde{\psi}_h$ denote the original eigenfunction, thus we have $P_h\widetilde{\psi}_h = 0$. Let $\psi_h = \chi\widetilde{\psi}_h$, where $\chi \equiv 1$ on a neighborhood of supp φ, i.e. supp F, but still supported in a collar neighborhood of the Euclidean end. We have

$$P_h\psi_h = \chi P_h\widetilde{\psi}_h + [P_h, \chi]\widetilde{\psi}_h = [P_h, \chi]\widetilde{\psi}_h,$$

and by construction, namely on supp $d\chi$ where the weight F vanishes, we have

$$P_h\psi_h = [P_h, \chi]\widetilde{\psi}.$$

In particular all semiclassical Sobolev norms of $P_h\psi_h$ are bounded by a constant C (independent of $h \in (0, 1]$, of course depending on the norm).

We repeat the argument above. The left hand side of (5.40) is not zero anymore, rather $\|P_h\psi_h\|^2$, which is bounded by C^2. The rest of the computation is unchanged until (5.42), where the left-hand side becomes C^2 instead of 0. Hence, (5.43) also has a C^2 added to the left-hand side, and then (5.44) becomes

$$(5.45) \qquad C_{10}\|\psi\|^2 + C^2h^{-1}e^{-2\widetilde{\delta}/h} \geq \left(\frac{1}{2} - C_9\widetilde{\delta}^{\delta}\right)C_{11}\widetilde{\delta}e^{2/(\widetilde{\delta}h)}.$$

Since the new term also goes to 0 as $h \to 0$, the final step of the argument is unchanged, whence ψ vanishes for small x.

We conclude again that our original eigenfunction $\widetilde{\psi}$ vanishes for small x, and the usual elliptic unique continuation result finishes the proof. $\qquad\square$

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