Communications in
Analysis and Geometry
Volume 28, Number 8, 1807–1846, 2020

Isotropic curve flows

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Dedicated to the 75th Birthday of Professor Karen Uhlenbeck

A smooth curve γ in $\mathbb{R}^{n+1,n}$ is *isotropic* if $\gamma, \gamma_x, \ldots, \gamma_x^{(2n)}$ are linearly independent and the span of $\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}$ is isotropic. We construct two hierarchies of isotropic curve flows on $\mathbb{R}^{n+1,n}$, whose differential invariants are solutions of Drinfeld-Sokolov's KdV type soliton hierarchies associated to the affine Kac-Moody algebra $\hat{B}_n^{(1)}$ and $\hat{A}_{2n}^{(2)}$. For example, the $\hat{B}_1^{(1)}$-KdV is the KdV hierarchy and the $\hat{A}_2^{(2)}$-KdV hierarchy is the Kupershmidt-Kaup (KK) hierarchy. Hence we our study gives geometric interpretations of the KdV and KK equations as the curvature flows of natural geometric curve flows on the light cone of $\mathbb{R}^{2,1}$. Bi-Hamiltonian structures and conservation laws for isotropic curve flows on $\mathbb{R}^{n+1,n}$ are also given.

*Research supported in part by NSF of China Grant no. 11401327 and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

1. Introduction

Let $\mathbb{R}^{n+1,n}$ be the vector space $\mathbb{R}^{2n+1}$ equipped with the index n, non-degenerate bilinear form

$$\langle X, Y\rangle = X^t\rho_n Y, \quad \text{where } \rho_n = \sum_{i=1}^{2n+1}(-1)^{n+i-1}e_{i,2n+2-i}. \tag{1.1}$$

Let $O(n+1,n)$ denote the group of linear isomorphisms on $\mathbb{R}^{n+1,n}$ preserving $\langle\ ,\ \rangle$.

A subspace $\mathcal{I}\subset\mathbb{R}^{n+1,n}$ is called *isotropic* if $\langle X, Y\rangle = 0$ for all $X, Y\in\mathcal{I}$. We note that a maximal isotropic subspace in $\mathbb{R}^{n+1,n}$ has dimension n. A smooth curve $\gamma:\mathbb{R}\to\mathbb{R}^{n+1,n}$ is *isotropic* if $\gamma, \gamma_s, \ldots, \gamma_s^{(2n)}$ are linearly independent and the span of $\gamma, \gamma_s, \ldots, \gamma_s^{(n-1)}$ is a maximal isotropic subspace of $\mathbb{R}^{n+1,n}$ for all $s\in\mathbb{R}$. Note that a curve being isotropic is independent of the choice of parameter. It is easy to see that there is an orientation preserving parameter x (unique up to translation) for an isotropic curve such that $\langle\gamma_x^{(n)}, \gamma_x^{(n)}\rangle\equiv 1$. We call such x the *isotropic parameter* of γ.

Set

$$\mathcal{M}_{n+1,n} = \{\gamma:\mathbb{R}\to\mathbb{R}^{n+1,n} \mid \gamma \text{ is isotropic}, \langle\gamma_x^{(n)}, \gamma_x^{(n)}\rangle\equiv 1\}.$$

We prove that given $\gamma\in\mathcal{M}_{n+1,n}$, there exists a unique smooth map $g:\mathbb{R}\to O(n+1,n)$ such that the i-th column is $\gamma_x^{(i-1)}$ for $1\le i\le n+1$ and $g^{-1}g_x$ is of the form

$$g^{-1}g_x = b + \sum_{i=1}^{n} u_i\beta_i$$

for some $u_i\in C^\infty(\mathbb{R},\mathbb{R})$, where

$$b = \sum_{i=1}^{2n} e_{i+1,i}, \quad \beta_i = e_{n+1-i,n+i} + e_{n+2-i,n+1+i}. \tag{1.2}$$

We call g and $u = \sum_{i=1}^{n} u_i\beta_i$ the *isotropic moving frame* and the *isotropic curvature* along γ respectively.

Let $\Psi:\mathcal{M}_{n+1,n}\to C^\infty(\mathbb{R}, V_n)$ be the *isotropic curvature map* defined by

$$\Psi(\gamma) = u = g^{-1}g_x - b = \sum_{i=1}^{n} u_i\beta_i, \tag{1.3}$$

where g and u are the isotropic moving frame and the isotropic curvature along γ and

$$V_n = \oplus_{i=1}^n \mathbb{R}\beta_i \tag{1.4}$$

the *isotropic curvature space*.

In [3], Drinfeld and Sokolov constructed

(i) a G-hierarchy of soliton equations for each affine Kac-Moody algebra G, and

(ii) a G-KdV hierarchy on a cross section of certain gauge action by pushing down the G-hierarchy along gauge orbits to the cross section.

Note that G-KdV hierarchies constructed from two different cross sections look different but they are gauge equivalent.

Let $\mathcal{B}_n^+$ and $\mathcal{B}_n^-$ denote the subalgebras of upper and lower triangular matrices in $o(n+1,n)$ respectively, and $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ the subalgebras of strictly upper and strictly lower triangular matrices in $o(n+1,n)$, and $B_n^\pm$ and $N_n^\pm$ the corresponding connected subgroups of $O(n+1,n)$ with Lie algebra $\mathcal{B}_n^\pm$ and $\mathcal{N}_n^\pm$ respectively.

In this paper, we prove the following results:

1) We show that $C^\infty(\mathbb{R}, V_n)$ is a cross section of the gauge action of $C^\infty(\mathbb{R}, N_n^+)$ on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, where V_n is the isotropic curvature space for $\mathcal{M}_{n+1,n}$ defined by (1.4).

2) Two Poisson structures $\{\,,\}_1$ and $\{\,,\}_2$ are *compatible* if $c_1\{\,,\}_1 + c_2\{\,,\}_2$ is again a Poisson structure for any constants $c_1, c_2 \in \mathbb{R}$. A bi-Hamiltonian structure for a soliton hierarchy is a pair of compatible Poisson structures $(\{\,,\}_1, \{\,,\}_2)$ on the phase space such that the flows in the soliton hierarchy are Hamiltonian with respect to both Poisson structures. The $\hat{B}_n^{(1)}$-KdV and $\hat{A}_{2n}^{(2)}$-KdV hierarchies have bi-Hamiltonian structures and they share the same Poisson structure $\{\,,\}_2$ defined by (5.20). We study the Hamiltonian theory of

$$\mathcal{M}_{n+1,n}(S^1) = \{\gamma \in \mathcal{M}_{n+1,n} \mid \gamma(x+2\pi) = \gamma(x), \forall\, x \in \mathbb{R}\}$$

with respect to the pull back $\{\,,\}_2^\wedge$ of the Poisson structure $\{\,,\}_2$ on $C^\infty(S^1, V_n)$ to $\mathcal{M}_{n+1,n}(S^1)$ by the isotropy curvature map Ψ.

3) We call the commuting Hamiltonian flows on $\mathcal{M}_{n+1,n}(S^1)$ with respect to $\{\,,\}_2^\wedge$ obtained from the commuting Hamiltonians for the $\hat{B}_n^{(1)}$-KdV

and $\hat{A}^{(2)}_{2n}$-KdV hierarchies the isotropic curve flows of type B and A respectively. We show that there is a correspondence between solutions of isotropic curve flows of type B (type A resp.) and solutions of the $\hat{B}^{(1)}_n$-KdV ($\hat{A}^{(2)}_{2n}$-KdV resp.) flows.

In particular, for $n = 1$ we prove the following results:

(a) A map $\gamma : \mathbb{R} \to \mathbb{R}^{2,1}$ lies in $\mathcal{M}_{2,1}$ if and only if γ is a space-like curve parametrized by the arc-length in the light cone of $\mathbb{R}^{2,1}$ and the isotropic curvature of such curve is the standard curvature in differential geometry.

(b) The KdV equation

$$q_t = q_{xxx} - 3qq_x, \tag{1.5}$$

and the Kupershmidt-Kaup (KK) equation

$$q_t = -\frac{1}{9}(q^{(5)} - 10qq_{xxx} - 25q_xq_{xx} + 20q^2q_x) \tag{1.6}$$

are the third $\hat{B}^{(1)}_1$-KdV flow and the fifth $\hat{A}^{(2)}_2$-KdV flow for $u = q\beta_1$ respectively.

(c) The bi-Hamiltonian structure $(\{\,,\}_1, \{\,,\}_2)$ for the $\hat{B}^{(1)}_1$-KdV hierarchy on $C^\infty(S^1, \mathbb{R})$ is the standard bi-Hamiltonian structure for the KdV hierarchy.

(d) The third isotropic curve flow of B-type and the fifth isotropic curve flow of A-type on $\mathcal{M}_{2,1}(S^1)$ are

$$\gamma_t = q_x\gamma - q\gamma_x, \tag{1.7}$$

$$\gamma_t = -\frac{1}{9}(q_{xxx} - 8qq_x)\gamma + \frac{1}{9}(q_{xx} - 4q^2)\gamma_x, \tag{1.8}$$

which are the Hamiltonian flows for

$$F_3(\gamma) = -\oint q^2 \mathrm{d}x, \tag{1.9}$$

$$G_5(\gamma) = -\frac{1}{9}\oint \left(q_x^2 + \frac{8}{3}q^3\right) \mathrm{d}x, \tag{1.10}$$

with respect to $\{\,,\}^\wedge_2$ respectively, where $u = q\beta_1$ is the isotropic curvature of γ.

(e) If γ is a solution of (1.7) or (1.8), then the isotropic curvature q is a solution of the KdV (1.5) and KK (1.6) respectively.

(f) We use the solution of the periodic Cauchy problem for the KdV (KK resp.) to solve the periodic Cauchy problem for the isotropic curve flow (1.7) ((1.8) resp.).

We construct Darboux transforms for the $\hat{B}_n^{(1)}$ and $\hat{A}_{2n}^{(2)}$-hierarchies and use them to construct explicit soliton solutions for the isotropic curve flows of type B and type A in [18] and [19] respectively. In particular, we show in [18] that given a constant $\alpha \in \mathbb{R}\backslash 0$,

$$\gamma_{\alpha^2,0}(x,t) = \begin{pmatrix} 1 - \frac{\alpha s_\alpha(x,t)}{c_\alpha(x,t)+1}x + \frac{\alpha^2(c_\alpha(x,t)-1)}{4(c_\alpha(x,t)+1)}x^2 \\ \frac{2s_\alpha(x,t)}{\alpha(c_\alpha(x,t)+1)} - \frac{c_\alpha(x,t)-1}{c_\alpha(x,t)+1}x \\ \frac{2(c_\alpha(x,t)-1)}{\alpha^2(c_\alpha(x,t)+1)} \end{pmatrix}$$

is a solution of the third isotropic flow (1.7) of type B on $\mathcal{M}_{2,1}$ and its isotropic curvature is the 1-soliton solution

$$q = -\alpha^2 \mathrm{sech}^2\left(\frac{\alpha}{2}x + \frac{\alpha^3}{2}t\right)$$

of the KdV, where $c_\alpha(x,t) = \cosh(\alpha x + \alpha^3 t)$ and $s_\alpha(x,t) = \sinh(\alpha x + \alpha^3 t)$. We show in [19] that given a constant $r \in \mathbb{R}\backslash 0$,

$$\gamma_r(x,t) = \begin{pmatrix} \frac{1-\cosh(\sqrt{3}(rx-r^5t))}{2+\cosh(\sqrt{3}(rx-r^5t))} \\ \frac{r^{-1}(\cosh(\sqrt{3}(rx-r^5t))-1)x-\sqrt{3}\sinh(\sqrt{3}(rx-r^5t))}{2+\cosh(\sqrt{3}(rx-r^5t))} \\ \frac{r^{-2}(1-\cosh(\sqrt{3}(rx-r^5t)))x^2+2\sqrt{3}\sinh(\sqrt{3}(rx-r^5t))x-3(1+\cosh(\sqrt{3}(rx-r^5t)))}{4+2\cosh(\sqrt{3}(rx-r^5t))} \end{pmatrix}$$

is a solution of the fifth isotropic curve flow on $\mathcal{M}_{2,1}$ of type A with the 1-soliton solution of the KK equation,

$$q_r(x,t) = -\frac{9r^2}{2}\left(\frac{1+2\cosh(\sqrt{3}(rx-r^5t))}{2+\cosh(\sqrt{3}(rx-r^5t))^2}\right),$$

as its isotropic curvature.

Note that the relation between central affine curve flows on $\mathbb{R}^n\backslash 0$ and the soliton theory of the $\hat{A}_{n-1}^{(1)}$-KdV hierarchy were considered in [12] and [16] for $n = 2$, in [1] for $n = 3$, and for general n in [17]. General methods for constructing integrable curve flows on homogeneous spaces can be found in Ovsienko and Khesin [11], Mari-Beffa ([6]–[10]), and in Terng [13].

The organization of this paper is as follows: We prove the existence of isotropic parameters and construct isotropic moving frames and curvatures for $\gamma \in \mathcal{M}_{n+1,n}$ in Section 2, and give description of the tangent space of $\mathcal{M}_{n+1,n}$ at γ in Section 3. In Section 4, we study the Hamiltonian flows on $\mathcal{M}_{n+1,n}(S^1)$ with respect to Poisson structure $\{\,,\}_2^\wedge$ and their Cauchy problems. We give the construction and some basic results of the $\hat{B}_n^{(1)}$-KdV and $\hat{A}_{2n}^{(2)}$-KdV hierarchies in Sections 5 and 6 respectively. In the last section, we explain the relation between solutions of the $\hat{B}_n^{(1)}$-KdV ($\hat{A}_{2n}^{(2)}$-KdV resp.) flows and isotropic curve flows of B-type (A-type resp.).

2. Moving frames along isotropic curves

In this section, we prove the existence of isotropic parameter and construct isotropic moving frames and curvatures along isotropic curves.

Note that the Lie algebra of $O(n+1,n)$ is

$$\begin{aligned} o(n+1,n) &= \{A \in sl(2n+1,\mathbb{R}) \mid A^t\rho + \rho A = 0\} \\ &= \{(A_{ij}) \mid A_{ij} + (-1)^{i-j} A_{2n+2-j,2n+2-i} = 0, \quad 1 \leq i \leq 2n+1\}. \end{aligned}$$

A direct computation implies that $A = (A_{ij}) \in o(n+1,n)$ if and only if

(i) A_{ij}'s are symmetric (skew-symmetric resp.) with respect to the skew diagonal line $i+j = 2n+2$ if $i+j$ is odd (even resp.),

(ii) $A_{ij} = 0$ if $i+j = 2n+2$.

Let

$$\mathcal{G}_i = o(n+1,n) \cap \operatorname{span}\{e_{j,j+i} \mid 1 \leq i+j \leq 2n+1\}. \tag{2.1}$$

Then we have the following gradation:

$$o(n+1,n) = \oplus_{i=-2n}^{2n} \mathcal{G}_i, \quad \mathcal{G}_{-2n} = \mathcal{G}_{2n} = \mathbf{0}, \quad [\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}.$$

A basis $\{v_1, \ldots, v_{2n+1}\}$ of $\mathbb{R}^{2n+1}$ is called an *isotropic basis* if $\langle v_i, v_j \rangle = \rho_{ij}$, where $\rho_n = (\rho_{ij})$ is the matrix defined by (1.1), or equivalently, the matrix $(v_1, \ldots, v_{2n+1})$ is in $O(n+1,n)$.

Proposition 2.1.

(i) *The $O(n+1,n)$-action on the space of ordered isotropic bases of $\mathbb{R}^{n+1,n}$ defined by $g \cdot (v_1, \ldots, v_{2n+1}) = (gv_1, \ldots, gv_{2n+1})$ is transitive.*

(ii) *The dimension of a maximal isotropic subspace of* $\mathbb{R}^{n+1,n}$ *is* n.

Proof. (i) follows from linear algebra. To proves (ii), first let $\{e_i, 1 \leq i \leq 2n+1\}$ denote the standard basis of $\mathbb{R}^{2n+1}$. Then $A = \text{span}\{e_1, e_2, \ldots, e_n\}$ is an isotropic subspace in $\mathbb{R}^{n+1,n}$.

Let $V = \text{span}\{v_1, \ldots, v_n\}$ be another n-dimension isotropic subspace, $g_1 = (e_1, \ldots, e_n)$, and $g_2 = (v_1, \ldots, v_n)$. We claim that there exists $C \in O(n+1, n)$ such that $g_2 = Cg_1$. From linear algebra, we can extend $\{v_1, \ldots, v_n\}$ to a basis $\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n+1}\}$ in $\mathbb{R}^{n+1,n}$ and denote $\tilde{g}_2 = (v_1, \ldots, v_{2n+1}) \in O(n+1, n)$. Then choose $C = \tilde{g}_2$.

Suppose $B = \text{span}\{w_1, \ldots, w_{n+1}\}$ is an isotropic subspace in $\mathbb{R}^{n+1,n}$ of dimension n+1. According to (i), there exists $C \in O(n+1, n)$, such that $(w_1, \ldots, w_n) = C(e_1, \ldots, e_n)$. Therefore, we may assume $w_i = e_i, 1 \leq i \leq n$. Then from $\langle e_i, w_{n+1} \rangle = 0$ for $1 \leq i \leq n$ and $\langle w_{n+1}, w_{n+1} \rangle = 0$, we have $w_{n+1} = 0$, which is a contradiction. This proves (ii). □

Proposition 2.2. *If* $\gamma(s)$ *is isotropic in* $\mathbb{R}^{n+1,n}$ *for all* $s \in \mathbb{R}$*, then there exists an orientation preserving parameter* $x = x(s)$ *unique up to translation such that* $\langle \gamma_x^{(n)}, \gamma_x^{(n)} \rangle = 1$*, i.e.,* x *is the isotropic parameter of* γ.

Proof. Since γ is isotropic,

$$\langle \gamma_s^{(n-1)}, \gamma_s^{(i)} \rangle = 0, \quad 0 \leq i \leq n-1.$$

Take the derivative with respect to s of both sides to get

$$\langle \gamma_s^{(n-1)}, \gamma_s^{(i)} \rangle_s = \langle \gamma_s^{(n)}, \gamma_s^{(i)} \rangle + \langle \gamma_s^{(n-1)}, \gamma_s^{(i+1)} \rangle = 0$$

So $\langle \gamma_s^{(n)}, \gamma_s^{(i)} \rangle = 0$ for any $0 \leq i \leq n-2$. But $\langle \gamma_s^{(n-1)}, \gamma_s^{(n-1)} \rangle = 0$ implies that

$$\langle \gamma_s^{(n)}, \gamma_s^{(n-1)} \rangle = 0. \tag{2.2}$$

This shows that $\langle \gamma_s^{(n)}, \gamma_s^{(i)} \rangle = 0$ for $0 \leq i \leq n-1$. Since the span of $\{\gamma, \ldots, \gamma_s^{(n-1)}\}$ is a maximal isotropic subspace, $\langle \gamma_s^{(n)}, \gamma_s^{(n)} \rangle \neq 0$.

We claim that $\langle \gamma_s^{(n)}, \gamma_s^{(n)} \rangle > 0$ for all $s \in \mathbb{R}$. To see this, we first note that from Proposition 2.1 (ii), there exists $C \in O(n+1, n)$ such that

$$C(\gamma, \ldots, \gamma_s^{(n-1)}) = (e_1, e_2, \ldots, e_n),$$

where e_i is the i-th standard basis of $\mathbb{R}^{2n+1}$. Let $\mathbf{c} = (c_1, c_2, \ldots, c_{2n+1})^t = C\gamma_s^{(n)}$. For $0 \le i \le n-1$, we use (2.2) to see that

$$\langle C\gamma_s^{(n)}, C\gamma_s^{(i)}\rangle = \langle \mathbf{c}, e_{i+1}\rangle = (-1)^{n+i} c_{2n+1-i} = \langle \gamma_s^{(n)}, \gamma_s^{(i)}\rangle = 0$$

for $1 \le i \le n$. So $c_{2n+2-i} = 0$ for $1 \le i \le n$. This implies that

$$\langle \gamma_x^{(n)}, \gamma_x^{(n)}\rangle = \langle C\gamma_x^{(n)}, C\gamma_x^{(n)}\rangle = \mathbf{c}^t \rho \mathbf{c} = c_{n+1}^2.$$

But $\langle \gamma_s^{(n)}, \gamma_s^{(n)}\rangle \neq 0$. This proves the claim.

Choose x such that $\frac{dx}{ds} = \langle \gamma_s^{(n)}, \gamma_s^{(n)}\rangle^{1/2n}$ and the proposition follows. $\square$

Next we want to construct moving frames and a complete set of differential invariants for $\gamma \in \mathcal{M}_{n+1,n}$. First note that if $\langle \gamma, \gamma_x^{(i)}\rangle = 0$ for $0 \le i \le n-1$, then $\langle \gamma_x^{(i)}, \gamma_x^{(j)}\rangle = 0$ for $0 \le i, j \le n-1$. So $\gamma \in \mathcal{M}_{n+1,n}$ is determined by $n+1$ independent conditions

$$\langle \gamma_x^{(n)}, \gamma_x^{(n)}\rangle = 1, \quad \langle \gamma, \gamma_x^{(i)}\rangle = 0, \quad 0 \le i \le n-1.$$

Hence we expect there should be n differential invariants for $\gamma \in \mathcal{M}_{n+1,n}$.

Recall that the Frenet frame g for curves in the Euclidean space $\mathbb{R}^n$ satisfies $A = (a_{ij}) := g^{-1}g_x$ in $o(n)$, where all entries of A are zero except $a_{i+1,i} = k_i = -a_{i,i+1}$ for $1 \le i \le n-1$. Motivated by this, we seek a moving frame g for $\gamma \in \mathcal{M}_{n+1,n}$ satisfying $g^{-1}g_x$ lies in $b + V_n$, where V_n is defined by (1.4). We will first give detailed constructions of such g for $\gamma \in \mathcal{M}_{2,1}$ and $\mathcal{M}_{3,2}$ so that the construction for general $\mathcal{M}_{n+1,n}$ is easier to follow.

Example 2.3 (The isotropic moving frame for $\gamma \in \mathcal{M}_{2,1}$). Assume that there exists p_3 such that (γ, γ_x, p_3) is in $O(2,1)$ and satisfies

$$(\gamma, \gamma_x, p_3)_x = (\gamma, \gamma_x, p_3)\begin{pmatrix} 0 & q & 0 \\ 1 & 0 & q \\ 0 & 1 & 0 \end{pmatrix} \tag{2.3}$$

for some q. Note that the first column of (2.3) is automatically true. The second column of (2.3) holds if and only if $\gamma_{xx} = q\gamma + p_3$. So we can choose

$$p_3 = q\gamma - \gamma_{xx}.$$

To find q, we first compute

$$\langle \gamma, \gamma_{xx}\rangle = (\langle \gamma, \gamma_x\rangle)_x - \langle \gamma_x, \gamma_x\rangle = 0 - 1 = -1.$$

So we have $\langle \gamma, p_3 \rangle = -1$ and $\langle \gamma_x, p_3 \rangle = 0$. We note that

$$\langle p_3, p_3 \rangle = \langle \gamma_{xx}, \gamma_{xx} \rangle - 2q \langle \gamma, \gamma_{xx} \rangle = \langle \gamma_{xx}, \gamma_{xx} \rangle + 2q$$

which is zero if and only if

$$q = -\frac{1}{2} \langle \gamma_{xx}, \gamma_{xx} \rangle.$$

This implies that (γ, γ_x, p_3) is in $O(2,1)$ and (2.3) holds.

Note that a smooth curve $\gamma : \mathbb{R} \to \mathbb{R}^{2,1}$ lies in $\mathcal{M}_{2,1}$ if and only if γ is a smooth space-like curve in the null cone $\Sigma = \{y \in \mathbb{R}^{2,1} \mid \langle y, y \rangle = 0\}$ parameterized by its arc-length. The isotropic moving frame and isotropic curvature are the standard moving frame and curvature of γ in Σ in differential geometry.

Example 2.4 (The isotropic moving frame for $\gamma \in \mathcal{M}_{3,2}$). Suppose $g = (\gamma, \gamma_x, \gamma_{xx}, p_4, p_5) : \mathbb{R} \to O(3,2)$ satisfies

$$(\gamma, \gamma_x, \gamma_{xx}, p_4, p_5)_x = (\gamma, \gamma_x, \gamma_{xx}, p_4, p_5) \begin{pmatrix} 0 & 0 & 0 & u_2 & 0 \\ 1 & 0 & u_1 & 0 & u_2 \\ 0 & 1 & 0 & u_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.4}$$

for some u_1, u_2. Set $p_i = \gamma_x^{(i-1)}$ for $1 \leq i \leq 3$. Note that $\rho = (\rho_{ij}) = e_{33} - e_{24} - e_{42} + e_{15} + e_{51}$. We need to choose p_4 and p_5 so that $\langle p_i, p_j \rangle = \rho_{ij}$ and (2.4) holds. It follows from

$$\langle \gamma_x^{(i)}, \gamma_x^{(j)} \rangle = 0, \quad \text{for } 0 \leq i, j \leq 1, \quad \langle \gamma_{xx}, \gamma_{xx} \rangle = 1 \tag{2.5}$$

that

$$\langle \gamma, \gamma_{xx} \rangle = \langle \gamma_x, \gamma_{xx} \rangle = 0, \quad \langle \gamma_x, \gamma_{xx} \rangle = -1. \tag{2.6}$$

So $\langle p_i, p_j \rangle = \rho_{ij}$ for $1 \leq i, j \leq 3$. The first two columns of (2.4) are true. The third column of (2.4) holds if and only if $\gamma_{xxx} = u_1 \gamma_x + p_4$, i.e.,

$$p_4 = \gamma_{xxx} - u_1 \gamma_x. \tag{2.7}$$

We need to find u_1 such that $\langle p_i, p_4 \rangle = \rho_{i4}$ for $1 \leq i \leq 4$. It follows from (2.5) and (2.6) that $\langle p_i, p_4 \rangle = 0$ for $1 \leq i \leq 3$. So

$$\langle p_4, p_4 \rangle = \langle \gamma_{xxx}, \gamma_{xxx} \rangle - 2u_1 \langle \gamma_x, \gamma_{xxx} \rangle = \langle \gamma_{xxx}, \gamma_{xxx} \rangle + 2u_1.$$

This implies that if we choose

$$u_1 = \frac{1}{2}\langle \gamma_{xxx}, \gamma_{xxx}\rangle,$$

then $\langle p_i, p_j\rangle = \rho_{ij}$ for $1 \le i, j \le 4$. Next we construct p_5. The fourth column of equation (2.4) gives $(p_4)_x = u_2\gamma + u_1\gamma_{xx} + p_5$. Hence

$$p_5 = (p_4)_x - u_2\gamma - u_1\gamma_{xx}.$$

Note that

$$\begin{aligned}
\langle \gamma, (p_4)_x\rangle &= (\langle \gamma, p_4\rangle)_x - \langle \gamma_x, p_4\rangle = -\rho_{24} = 1,\\
\langle \gamma_x, (p_4)_x\rangle &= (\langle \gamma_x, p_4\rangle)_x - \langle \gamma_{xx}, p_4\rangle = -\rho_{34} = 0,\\
\langle \gamma_{xx}, (p_4)_x\rangle &= (\langle \gamma_{xx}, p_4\rangle)_x - \langle \gamma_{xxx}, p_4\rangle = -\langle \gamma_{xxx}, p_4\rangle\\
&= -\langle u_1\gamma_x + p_4, p_4\rangle = -u_1\rho_{24} = u_1.
\end{aligned}$$

These equalities imply that $\langle p_i, p_5\rangle = \rho_{i5}$ for $1 \le i \le 4$. Note that

$$\langle p_5, p_5\rangle = \langle (p_4)_x, (p_4)_x\rangle + u_1^2 + 2u_2$$

is zero if we choose

$$u_2 = -\frac{1}{2}(\langle (p_4)_x, (p_4)_x\rangle + u_1^2).$$

This proves that $(p_1, \ldots, p_5)$ is in $O(3,2)$ and satisfies (2.4).

Theorem 2.5. *Given $\gamma \in \mathcal{M}_{n+1,n}$, then there exists a unique smooth map $g = (p_1, \ldots, p_{2n+1}) : \mathbb{R} \to O(n+1, n)$ such that $p_i = \gamma_x^{(i-1)}$ for $1 \le i \le n+1$ and*

$$g_x = g\left(b + \sum_{i=1}^{n} u_i\beta_i\right) \tag{2.8}$$

for some n smooth functions $u_1, \ldots, u_n$, where b and β_i's are given in (1.2). *Moreover,*

$$p_i = \gamma_x^{(i-1)} + \sum_{j=0}^{i-3} r_{ij}(u)\gamma_x^{(j)} \tag{2.9}$$

for some differential polynomials $r_{ij}(u)$ in u for $n+2 \le i \le 2n+1$.

Proof. Set $p_i = \gamma_x^{(i-1)}$ for $1 \leq i \leq n+1$. We need to find $p_{n+2}, \ldots, p_{2n+1}$ such that $(p_1, \ldots, p_{2n+1})$ is in $O(n+1, n)$ and satisfies (2.8).

(i) We claim that $\langle p_i, p_j \rangle = \rho_{ij}$ for $1 \leq i, j \leq n+1$, where $\rho = (\rho_{ij})$ as defined by (1.1). Since $\gamma \in \mathcal{M}_{n+1,n}$, $\langle p_i, p_j \rangle = 0$ for $1 \leq i, j \leq n$ and $\langle p_{n+1}, p_{n+1} \rangle = 1$. For $1 \leq i \leq n-1$,

$$\langle p_i, p_{n+1} \rangle = \langle \gamma_x^{(i-1)}, \gamma_x^{(n)} \rangle = \langle \gamma_x^{(i-1)}, \gamma_x^{(n-1)} \rangle_x - \langle \gamma_x^{(i)}, \gamma_x^{(n-1)} \rangle = 0.$$

And $\langle p_n, p_{n+1} \rangle = \langle \gamma_x^{(n-1)}, \gamma_x^{(n)} \rangle = \frac{1}{2} \langle \gamma_x^{(n-1)}, \gamma_x^{(n-1)} \rangle_x = 0$.

This proves the claim.

(ii) The $(n+1)$-th column of (2.8) gives $\gamma_x^{(n+1)} = u_1 \gamma_x^{(n-1)} + p_{n+2}$. We need to determine u_1. From $\langle \gamma_x^{(n-1)}, \gamma_x^{(n)} \rangle = 0$, we get

$$\langle \gamma_x^{(n-1)}, \gamma_x^{(n+1)} \rangle = -\langle \gamma_x^{(n)}, \gamma_x^{(n)} \rangle = -1.$$

And the condition $\langle p_{n+2}, p_{n+1} \rangle = 0$ implies that

$$u_1 = -\frac{1}{2} \langle \gamma_x^{(n+1)}, \gamma_x^{(n+1)} \rangle = -\frac{1}{2} \langle (p_{n+1})_x, (p_{n+1})_x \rangle. \tag{2.10}$$

Then we have

$$\begin{cases} \langle p_i, p_{n+2} \rangle = 0, \quad 0 \leq i \leq n+2, \quad i \neq n, \\ \langle p_n, p_{n+2} \rangle = -1. \end{cases} \tag{2.11}$$

(iii) The $(n+2)$-th column of (2.8) gives $(p_{n+2})_x = u_2 p_{n-1} + u_1 p_{n+1} + p_{n+3}$, i.e.,

$$p_{n+3} = (p_{n+2})_x - u_2 p_{n-1} - u_1 p_{n+1}.$$

For $1 \leq i \leq n-2$ and $i = n$,

$$\langle p_i, p_{n+3} \rangle = \langle \gamma_x^{(i-1)}, (p_{n+2})_x \rangle = \langle \gamma_x^{(i-1)}, p_{n+2} \rangle_x - \langle \gamma_x^{(i)}, p_{n+2} \rangle = 0.$$

Moreover, $\langle p_{n-1}, p_{n+3} \rangle = \langle \gamma_x^{(n-2)}, p_{n+2} \rangle_x - \langle \gamma_x^{(n-1)}, p_{n+2} \rangle = 1$.

From (2.10) and (2.11), we have

$$\begin{aligned} \langle p_{n+1}, p_{n+3} \rangle &= \langle \gamma_x^{(n)}, (p_{n+2})_x \rangle - u_1 \langle \gamma_x^{(n)}, \gamma_x^{(n)} \rangle \\ &= \langle \gamma_x^{(n)}, p_{n+2} \rangle_x - \langle \gamma_x^{(n+1)}, p_{n+2} \rangle - u_1 \\ &= -\langle \gamma_x^{(n+1)}, \gamma_x^{(n+1)} - u_1 \gamma_x^{(n-1)} - u_1 \rangle = 0. \\ \langle p_{n+2}, p_{n+3} \rangle &= \langle p_{n+2}, (p_{n+2})_x \rangle = \frac{1}{2} \langle p_{n+2}, p_{n+2} \rangle_x = 0. \end{aligned}$$

A direct computation shows that $\langle p_{n+3}, p_{n+3}\rangle = 0$ if we choose

$$u_2 = \frac{1}{2}(\langle (p_{n+2})_x, (p_{n+2})_x\rangle + u_1^2).$$

(iv) Suppose we have already found $p_{n+2}, \ldots, p_{n+j}$ and $u_1, \ldots, u_{j-1}$ for $j \geq 3$ satisfying

$$\begin{cases} \langle p_i, p_{n+j}\rangle = 0, \quad 1 \leq i \leq n+j, \quad i \neq n+2-j, \\ \langle p_{n+2-j}, p_{n+j}\rangle = (-1)^{j-1}, \\ (p_{n-1+j})_x = p_{n+j} + u_{j-1}p_{n+2-j} + u_{j-2}p_{n+4-j}. \end{cases} \tag{2.12}$$

Set

$$\begin{cases} u_j = \frac{(-1)^j}{2}\langle (p_{n+j})_x, (p_{n+j})_x\rangle, \\ p_{n+j+1} = (p_{n+j})_x - u_j p_{n+1-j} - u_{j-1}p_{n+3-j}. \end{cases}$$

For $1 \leq i \leq n+1$ and $i \neq n+1-j$,

$$\begin{aligned} \langle p_i, p_{n+j+1}\rangle &= \langle \gamma_x^{(i-1)}, p_{n+j+1}\rangle \\ &= \langle \gamma_x^{(i-1)}, (p_{n+j})_x - u_j p_{n+1-j} - u_{j-1}p_{n+3-j}\rangle \\ &= \langle \gamma_x^{(i-1)}, p_{n+j}\rangle_x - \langle \gamma_x^{(i)}, p_{n+j}\rangle = 0. \end{aligned}$$

And $\langle p_{n+1-j}, p_{n+j+1}\rangle = -\langle \gamma_x^{(n+1-j)}, p_{n+j}\rangle = (-1)^j$.

For $n+1 \leq i \leq n+j$ and $i \neq n+j-1$,

$$\begin{aligned} \langle p_i, p_{n+j+1}\rangle &= \langle p_i, (p_{n+j})_x - u_j p_{n+1-j} - u_{j-1}p_{n+3-j}\rangle \\ &= \langle p_i, p_{n+j}\rangle_x - \langle (p_i)_x, p_{n+j}\rangle \quad \text{by (2.12)} \\ &= 0. \end{aligned}$$

From $u_{j-1} = \frac{(-1)^{j-1}}{2}\langle (p_{n+j-1})_x, (p_{n+j-1})_x\rangle$, we have $\langle p_{n+j-1}, p_{n+j}\rangle = 0$. And u_j can be solved from $\langle p_{n+j}, p_{n+j}\rangle = 0$.

(v) The uniqueness follows from the construction. □

Example 2.6. Isotropic curves in $\mathbb{R}^{n+1,n}$ with zero isotropic curvatures are of the form

$$\gamma = c_0 e^{bx} e_1 = c_0 \left(1, x, \frac{x^2}{2!}, \ldots, \frac{x^{2n-1}}{(2n-1)!}, \frac{x^{2n}}{(2n)!}\right)^t, \tag{2.13}$$

where $c_0 \in O(n+1, n)$ is a constant and

$$e_1 = (1, 0, \ldots, 0)^t \in \mathbb{R}^{n+1,n}. \tag{2.14}$$

If follows from the Existence and Uniqueness Theorem of ordinary differential equations that we have the following.

Proposition 2.7. *Let V_n be the isotropic curvature space defined by* (1.4), *and $\Psi : \mathcal{M}_{n+1,n} \to C^\infty(\mathbb{R}, V_n)$ the isotropic curvature map defined by* (1.3). *Then Ψ is onto and $\Psi^{-1}(\Psi(\gamma))$ is the $O(n+1,n)$-orbit at γ.*

Hence $\{u_1, \ldots, u_n\}$ is a complete set of differential invariants for $\gamma \in \mathcal{M}_{n+1,n}$ under the group $O(n+1,n)$.

Remark 2.8. Let $g = (\gamma, \ldots, \gamma_x^{(n)}, p_{n+2}, \ldots, p_{2n})$ denote the isotropic frame along $\gamma \in \mathcal{M}_{n+1,n}$. If $\delta\gamma$ is tangent to $\mathcal{M}_{n+1,n}$ at γ, then $\delta(\gamma_x^{(i)}) = (\delta\gamma)_x^{(i)}$. So we can use (2.9) to write down δg in terms of $\delta\gamma$.

The following Proposition follows from a straight forward computation.

Proposition 2.9. *Let $\Psi : \mathcal{M}_{n+1,n} \to C^\infty(\mathbb{R}, V_n)$ be the isotropic curvature map. Then the differential of Ψ at γ is*

$$d\Psi(\delta\gamma) = \delta u = [\partial_x + b + u, g^{-1}\delta g], \tag{2.15}$$

where g, u, and δg are the isotropic moving frame, isotropic curvature, and the variation of g when we vary γ by $\delta\gamma$ respectively.

Proof. It follows from $g^{-1}g_x = b + u$ that we have

$$\delta u = -g^{-1}\delta g g^{-1} g_x + g^{-1}(\delta g)_x = -g^{-1}\delta g(b+u) + g^{-1}(\delta g)_x.$$

On the other hand,

$$(g^{-1}\delta g)_x = -g^{-1}g_x g^{-1}\delta g + g^{-1}(\delta g)_x = -(b+u)g^{-1}\delta g + g^{-1}(\delta g)_x.$$

Therefore

$$\begin{aligned} \delta u &= -g^{-1}\delta g(b+u) + (g^{-1}\delta g)_x + (b+u)g^{-1}(\delta g) \\ &= [\partial_x + b + u, g^{-1}\delta g]. \end{aligned}$$

□

3. The tangent space of $\mathcal{M}_{n+1,n}$ at γ

In this section, we

(i) give descriptions of the tangent space $T_\gamma \mathcal{M}_{n+1,n}$ and show that it is isomorphic to $C^\infty(\mathbb{R}, \mathbb{R}^n)$,

(ii) construct linear differential operator $P_u : C^\infty(\mathbb{R}, V_n^t) \to C^\infty(\mathbb{R}, o(n+1,n))$ that is needed for the $\hat{B}_n^{(1)}$-KdV and $\hat{A}_{2n}^{(2)}$-KdV hierarchies for $u \in C^\infty(\mathbb{R}, V_n)$.

We have seen in Proposition 2.9 that if $\delta\gamma = \sum_{i=1}^{2n+1} \xi_i p_i \in T_\gamma \mathcal{M}_{n+1,n}$ then $[\partial_x + b + u, g^{-1}\delta g] \in C^\infty(\mathbb{R}, V_n)$, where $g = (p_1, \ldots, p_{2n+1})$ and u are the isotropic moving frame and isotropic curvature along γ, and δg is variation of g when we vary γ. Below we show that the converse is also true.

Proposition 3.1. *Let g and u be the isotropic moving frame and isotropic curvature along $\gamma \in \mathcal{M}_{n+1,n}$ respectively.*

1) *If $C : \mathbb{R} \to O(n+1,n)$ satisfies*

$$[\partial_x + b + u, C] \in C^\infty(\mathbb{R}, V_n), \tag{3.1}$$

then $\xi(\gamma) = gCe_1$ is tangent to $\mathcal{M}_{n+1,n}$ at γ, where e_1 is defined by (2.14).

2) *If $\delta\gamma$ is tangent to $\mathcal{M}_{n+1,n}$ at γ, then $C := g^{-1}\delta g$ satisfies* (3.1) *and $\delta\gamma = gCe_1$.*

Proof. (1) It follows from the definition of $\mathcal{M}_{n+1,n}$ that $\delta\gamma$ is tangent to $\mathcal{M}_{n+1,n}$ at γ if

$$\begin{cases} \langle (\delta\gamma)_x^{(i)}, \gamma_x^{(j)} \rangle + \langle \gamma_x^{(i)}, (\delta\gamma)_x^{(j)} \rangle = 0, & 0 \le i, j \le n-1, \\ \langle (\delta\gamma)_x^{(n+1)}, \gamma_x^{(n)} \rangle = 0. \end{cases} \tag{3.2}$$

Let η_j denote the j-th column of gC for $1 \le j \le 2n+1$. To prove gCe_1 is tangent to $\mathcal{M}_{n+1,n}$, it suffices to prove that η_1 satisfies (3.2). Let $\rho = [\partial_x + b + u, C]$. A direct computation gives

$$(gC)_x = g_x C + gC_x = gC(b+u) + g\rho.$$

Since the first n columns of ρ are zero, the first $n+1$ columns of gC are related by

$$\eta_2 = (\eta_1)_x, \ldots, \eta_{n+1} = (\eta_1)_x^{(n)}.$$

Hence, for $0 \leq i, j \leq n-1$, we have

$$\begin{aligned}\langle (\eta_1)_x^{(i)}, \gamma_x^{(j)} \rangle + \langle \gamma_x^{(i)}, (\eta_1)_x^{(j)} \rangle &= \langle gCe_{i+1}, ge_{j+1} \rangle + \langle ge_{i+1}, gCe_{j+1} \rangle \\ &= \langle Ce_{i+1}, e_{j+1} \rangle + \langle e_{i+1}, Ce_{j+1} \rangle \\ &= e_{i+1}^t (C^t \rho + \rho C^t) e_{j+1} \\ &= 0.\end{aligned}$$

Since $C = (C_{ij}) \in o(n+1, n)$, $\langle (\eta_1)_x^{(n)}, \gamma_x^{(n)} \rangle = C_{n+1,n} = 0$. So $\xi(\gamma) = \eta_1$ is tangent to $\mathcal{M}_{n+1,n}$ at γ.

(2) Note that Proposition 2.9 implies that $C := g^{-1}\delta g$ satisfies (3.1). By definition, $\delta\gamma = g(g^{-1}\delta g)e_1 = gCe_1$. □

Next we prove that if $C = (C_{ij})$ satisfying (3.1) then C is determined by $\{C_{n+i,n+1-i}, 1 \leq i \leq n\}$ or $\{C_{2i,1}, 1 \leq i \leq n\}$.

Theorem 3.2. *Let $u \in C^\infty(\mathbb{R}, V_n)$, $C = (C_{ij}) \in C^\infty(\mathbb{R}, o(n+1, n))$, $v_i := C_{n+i,n+1-i}$ for $1 \leq i \leq n$, and $v = \sum_{i=1}^n v_i \beta_i^t$. Assume that*

$$[\partial_x + b + u, C] \in C^\infty(\mathbb{R}, V_n). \tag{3.3}$$

Then there exist differential polynomials $\phi_{ij}(u, v)$ for $1 \leq i, j \leq 2n+1$ satisfying the following conditions:

(i) *$C_{ij} = \phi_{ij}(u, v)$ for all $i, j \leq 2n+1$ and $\phi_{n+i,n+1-i}(u, v) = v_i$ for $1 \leq i \leq n$.*

(ii) *$\phi_{2n-2i,1}(u, v) = v_{n-i} + \phi_i(u, v_{n+1-i}, \ldots, v_n)$ for $0 \leq i \leq n-1$.*

(iii) *There exist differential polynomials h_{2i+1} such that*

$$C_{2i+1,1} = h_{2i+1}(u, C_{2i+2,1}, \ldots, C_{2n,1}), \quad 0 \leq i \leq n-1,$$

(iv) *C_{ij}'s are differential polynomials of $u, C_{21}, C_{41}, \ldots, C_{2k,1}, \ldots, C_{2n,1}$.*

Proof. Since $C \in o(n+1, n)$, $C_{n+1+i,n+2-i} = C_{n+i,n+1-i} = v_i$. Let $v = \sum_{i=1}^n v_i \beta_i^t \in V_n^t$, where V_n and β_i's are defined in (1.4). Let $\mathcal{G}_k$ be as in (2.1). Then $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$. For $\xi \in o(n+1, n)$, let $\xi_{\mathcal{G}_i}$ denote the $\mathcal{G}_i$-component of ξ w.r.t. $o(n+1, n) = \oplus_{i=1-2n}^{2n-1} \mathcal{G}_i$.

Suppose $[\partial_x + b + u, C] = \sum_{i=1}^{n} \eta_i \beta_i$. Write $C = \sum_{i=1-2n}^{2n-1} C_i$ with $C_i \in \mathcal{G}_i$. Then

$$(3.4)\qquad C_j' + [b, C_{j+1}] + [u, C]_{\mathcal{G}_j} = \begin{cases} \eta_i\beta_i, & j = 2i-1, \\ 0, & j \neq 2i-1. \end{cases}$$

We claim that C_j are differential polynomials in v and u. For $j = 1-2n$, we have $C_{2n,1} = C_{2n+1,2} = v_n$. For $j < 0$, if j is even, $ad(b) : \mathcal{G}_j \to \mathcal{G}_{j-1}$ is a bijection. If j is odd, then $\dim(\mathrm{Im}(ad(b)(\mathcal{G}_j))) = \dim(\mathcal{G}_{j-1}) = \dim(\mathcal{G}_j) - 1$. Then from (3.4), for both cases, entries of C_j are differential polynomials in $v_n, \ldots, v_{-[\frac{j}{2}]}$. Then by induction, the claim is true for $j < 0$.

Note that $ad(b)$ is a bijection from $\mathcal{G}_0$ to $\mathcal{G}_{-1}$, and we have the $\mathcal{G}_j$ component $[u, C]_{\mathcal{G}_j}$ depends only on u, $v_1, \ldots, v_n$. So C_0 is a differential polynomial in u and v.

For $j > 0$, we see that when j is odd, $ad(b) : \mathcal{G}_j \to \mathcal{G}_{j-1}$ is again a bijection. When j is even, we have $\dim(\mathrm{Im}(ad(b)(\mathcal{G}_j))) = \dim(\mathcal{G}_j) = \dim(\mathcal{G}_{j-1}) - 1$. Therefore, in both cases, C_j can be solved uniquely from C_{j-1} and η_i's are differential polynomials in entries of C_{2i-1}. By induction, the claim is true for $j > 0$. This proves the statement (i).

To prove (ii), let $j = 2i + 1 - 2n$ in (3.4). Then the linear system implies that $C_{2n-2i,1} = v_{n-i} + \phi_i, 0 \leq i \leq n-1$, where ϕ_i is a differential polynomial in $u, v_{n+1-i}, \ldots, v_n$.

Statement (iii) and (iv) are consequence from (i) and (ii). □

Let

$$V_n^t = \{v \in o(n+1, n) \mid v^t \in V_n\},$$

and $\pi_0 : o(n+1, n) \to V_n^t$ the natural projection onto V_n^t, i.e.,

$$(3.5)\qquad \pi_0(\xi) = \sum_{i=1}^{2n} \xi_{n+i,n-i+1}(e_{n+i,n+1-i} + e_{n+i+1,n-i+2})$$

for $\xi = (\xi_{ij}) \in o(n+1, n)$.

The proof of Theorem 3.2 implies the converse of Theorem 3.2 is true.

Theorem 3.3. *Let $u \in C^\infty(\mathbb{R}, V_n)$, $v \in C^\infty(\mathbb{R}, V_n^t)$, and $\phi_{ij}(u, v)$ the differential polynomials given in Theorem 3.2. Let $C = (C_{ij}) \in C^\infty(\mathbb{R}, o(n+1, n))$ defined by $C_{ij} = \phi_{i,j}(u, v)$ (so $\pi_0(C) = v$). Then C satisfies* (3.3).

Corollary 3.4. *Let $g = (p_1, \ldots, p_{2n+1})$ and u denote the isotropic moving frame and isotropic curvature along $\gamma \in \mathcal{M}_{n+1,n}$. Then $\xi = \sum_{i=1}^{2n} \xi_i p_i$ is tangent to $\mathcal{M}_{n+1,n}$ at γ if and only if*

$$\xi_{2i+1} = h_{2i+1}(u, \xi_{2i+2}, \ldots, \xi_{2n}), \quad 0 \leq i \leq n-1,$$

where h_{2i+1}'s are the differential polynomials given in Theorem 3.2. In particular, we identify $T_\gamma \mathcal{M}_{n+1,n}$ as $C^\infty(\mathbb{R}, \mathbb{R}^n)$.

Corollary 3.5. *If $u \in C^\infty(\mathbb{R}, V_n)$, then there is a unique linear differential operator*

$$P_u : C^\infty(\mathbb{R}, V_n^t) \to C^\infty(\mathbb{R}, o(n+1, n))$$

satisfying

1) $\pi_0(P_u(v)) = v$,
2) $[\partial_x + b + u, P_u(v)] \in C^\infty(\mathbb{R}, V_n)$.

Moreover, the coefficients of the linear differential operator P_u are differential polynomials of u.

It follows from Proposition 3.1 and Theorem 3.2 that we have the following.

Corollary 3.6. *Let g and u be the isotropic moving frame and the isotropic curvature along $\gamma \in \mathcal{M}_{n+1,n}$ respectively. Then the following statements are equivalent for $C : \mathbb{R} \to o(n+1, n)$:*

1) $[\partial_x + b + u, C] \in C^\infty(\mathbb{R}, V_n)$.
2) *$\delta\gamma := gCe_1$ is tangent to $\mathcal{M}_{n+1,n}$ at γ and $C = g^{-1}\delta g$, where g is the isotropic moving frame along $\gamma = ge_1$, δg is the variation of g when we vary γ by $\delta\gamma$.*
3) $C = P_u(\pi_0(C))$,

where P_u is the differential operator defined in Corollary 3.5.

Note that the proof of Theorem 3.2 gives an algorithm to compute $P_u(v)$. We write down the operator P_u and $T_\gamma \mathcal{M}_{n+1,n}$ for $n = 1, 2$ in the two examples below.

Example 3.7. When $n=1$, we have $b=e_{21}+e_{32}$, $\beta_1=e_{12}+e_{23}$, and $V_1=\mathbb{R}\beta_1$. Let $u=q\beta_1$, and $v=v_1\beta_1^t$. Use the algorithm given in the proof of Theorem 3.2 to compute $P_u(v)$ and obtain

$$P_u(v)=\begin{pmatrix} -(v_1)_x & -(v_1)_{xx}+qv_1 & 0 \\ v_1 & 0 & -(v_1)_{xx}+qv_1 \\ 0 & v_1 & (v_1)_x \end{pmatrix}. \tag{3.6}$$

Corollary 3.6 implies that all tangent vectors of $\mathcal{M}_{2,1}$ at γ is of the form $gP_u(v)e_1$. So

$$T_\gamma\mathcal{M}_{2,1}=\{-\xi_x\gamma+\xi\gamma_x \mid \xi\in C^\infty(\mathbb{R},\mathbb{R})\}.$$

Example 3.8. For $n=2$, we have $b=\sum_{i=1}^4 e_{i+1,i}$, $\beta_1=e_{23}+e_{34}$, $\beta_2=e_{14}+e_{25}$, and $V_2=\mathbb{R}\beta_1\oplus\mathbb{R}\beta_2$. Let $u=u_1\beta_1+u_2\beta_2$, and $v=v_1\beta_1^t+v_2\beta_2^t$. Use the algorithm given in the proof of Theorem 3.2 to get

$$P_u(v)=\begin{pmatrix} \eta & * & * & * & 0 \\ \xi & a & * & 0 & * \\ -(v_2)_x & v_1 & 0 & * & * \\ v_2 & 0 & v_1 & -a & * \\ 0 & v_2 & (v_2)_x & \xi & -\eta \end{pmatrix}, \tag{3.7}$$

where

$$\begin{aligned} \xi &= v_1+(v_2)_{xx}-u_1v_2, \\ \eta &= (v_2)_x^{(3)}-2\xi_x-(u_1)_xv_2, \\ a &= (v_2)_x^{(3)}-\xi_x-(u_1v_2)_x. \end{aligned}$$

By Corollary 3.6, we have $\delta\gamma\in T_\gamma\mathcal{M}_{3,2}$ if and only if

$$\delta\gamma=((v_2)_x^{(3)}-2\xi_x-(u_1)_xv_2)\gamma+\xi\gamma_x-(v_2)_x\gamma_{xx}+v_2p_4,$$

for some $\xi, v_2\in C^\infty(\mathbb{R},\mathbb{R})$, where $g=(\gamma,\gamma_x,\gamma_{xx},p_4,p_5)$ is the isotropic moving frame along γ.

4. Hamiltonian isotropic curve flows

The gradient $\nabla F(u)$ of a functional $F: C^\infty(S^1,V_n)\to\mathbb{R}$ is the unique map in $C^\infty(S^1,V_n^t)$ satisfying

$$\mathrm{d}F_u(v)=\oint\langle\nabla F(u),v\rangle\mathrm{d}x$$

for all $v \in C^\infty(S^1, V_n)$. Note that

$$\{F_1, F_2\}(u) = -\langle [\partial_x + b + u, P_u(\nabla F_1(u))], \nabla F_2(u) \rangle$$

is the Poisson structure $\{\,,\}_2$ defined by (5.20) in Section 5 on $C^\infty(S^1, V_n)$. So the Hamiltonian flow for $F : C^\infty(S^1, V_n) \to \mathbb{R}$ with respect to $\{\,,\}$ is

$$u_t = [\partial_x + b + u, P_u(\nabla F(u))]. \tag{4.1}$$

Given a functional F on $C^\infty(S^1, V_n)$, let

$$\hat{F} = F \circ \Psi.$$

Then $\hat{F}$ is a functional on $\mathcal{M}_{n+1,n}(S^1)$ invariant under $O(n+1, n)$. Let $\{\,,\}^\wedge$ denote the pull back of the Poisson structure $\{\,,\}$ on $C^\infty(S^1, V_n)$ to $\mathcal{M}_{n+1,n}(S^1)$ by the isotropic curvature map Ψ. Then

$$\{\hat{F}, \hat{G}\}^\wedge = \{F, G\} \circ \Psi \tag{4.2}$$

for functionals F and G on $C^\infty(S^1, V_n)$.

In this section, we write down the Hamiltonian flow on $\mathcal{M}_{n+1,n}(S^1)$ with respect to the Poisson structure $\{\,,\}^\wedge$ and study their Cauchy problems.

Recall that the Hamiltonian vector field for $\hat{F}$ is the unique vector field $X_{\hat{F}}$ satisfying

$$\{\hat{F}, \hat{G}\}^\wedge = -X_{\hat{F}}(\hat{G}) = -\mathrm{d}\hat{G}(X_{\hat{F}}),$$

and the Hamiltonian flow for $\hat{F}$ is $\gamma_t = X_{\hat{F}}(\gamma)$.

Theorem 4.1. *Let F be a functional on $C^\infty(S^1, V_n)$. Then the Hamiltonian flow of $\hat{F} = F \circ \Psi$ with respect to the pull back Poisson structure $\{\,,\}^\wedge$ defined by* (4.2) *is*

$$\gamma_t = g P_u(\nabla F(u)) e_1, \tag{4.3}$$

where g and u are the isotropic moving frame and isotropic curvature along γ respectively, P_u is the linear operator given in Corollary 3.5, and e_1 is defined by (2.14).

Proof. By definition of P_u, we have $[\partial_x + b + u, P_u(\nabla F(u))] \in C^\infty(S^1, V_n)$. It follows from Corollary 3.6 (2) that

$$\delta\gamma := gP_u(\nabla F(u))e_1 \tag{4.4}$$

is tangent to $\mathcal{M}_{n+1,n}(S^1)$ at γ. Corollary 3.6 implies that

$$g^{-1}\delta g = P_u(\nabla F(u)), \tag{4.5}$$

where δg is the variation of g when we vary γ by $\delta\gamma$. Next we compute

$$\begin{aligned} \{\hat{F}, \hat{H}\}^\wedge(\gamma) &= -\langle [\partial_x + b + u, P_u(\nabla F(u))], \nabla H(u)\rangle, \quad \text{by (4.5)}, \\ &= -\langle [\partial_x + b + u, g^{-1}\delta g], \nabla H(u)\rangle, \quad \text{by Proposition 2.9}, \\ &= -\langle \mathrm{d}\Psi(\delta\gamma), \nabla H(u)\rangle = -\mathrm{d}H(\mathrm{d}\Psi(\delta\gamma)) = -\mathrm{d}\hat{H}(\delta\gamma). \end{aligned}$$

This proves that the Hamiltonian vector field of $\hat{F}$ at γ is $gP_u(\nabla F(u))e_1$. □

Example 4.2. If $\nabla F(u) = \xi\beta_1^t$, then use (3.6) to see that

$$gP_u(\nabla F(u))e_1 = g(-\xi_x, \xi, 0)^t = -\xi_x\gamma + \xi\gamma_x.$$

By Proposition 4.1, the Hamiltonian flow for $\hat{F} = F \circ \Psi$ is

$$\gamma_t = -\xi_x\gamma + \xi\gamma_x, \quad \text{where} \quad \nabla F(u) = \xi\beta_1. \tag{4.6}$$

Recall the following elementary fact:

Proposition 4.3. *Let $A, B \in C^\infty(\mathbb{R}^2, o(n+1, n))$. Then the following linear system*

$$\begin{cases} g_x = gA, \\ g_t = gB, \end{cases}$$

is solvable for $g : \mathbb{R}^2 \to O(n+1, n)$ if and only if

$$A_t = B_x + [b + u, B] = [\partial_x + b + u, B].$$

It follows from Proposition 4.3 that we have the following.

Proposition 4.4.

1) *u is a solution of* (4.1) *if and only if the linear system*

$$\begin{cases} g_x = g(b+u), \\ g_t = gP_u(\nabla F(u)). \end{cases} \tag{4.7}$$

is solvable for $g : \mathbb{R}^2 \to O(n+1, n)$.

2) *If* $g : \mathbb{R}^2 \to O(n+1, n)$ *is a smooth solution of* (4.7), *then* u *satisfies* (4.1).

Theorem 4.5.

1) *If* γ *is a solution of* (4.3), *then the isotropic curvature* $u(\cdot, t)$ *along* $\gamma(\cdot, t)$ *is a solution of* (4.1).
2) *If* u *is a solution of* (4.1), *then given* $c \in O(n+1, n)$, *there exists a unique smooth solution* $g : \mathbb{R}^2 \to O(n+1, n)$ *for* (4.7) *with* $g(0, 0) = c$. *Moreover,* $\gamma := ge_1$ *is a solution of* (4.3) *whose isotropic curvature is* u.

Proof. (1) is true because $\{\,,\}^\wedge$ is the pull back of $\{\,,\}$ by Ψ. The existence of g follows from Proposition 4.4. Compute directly to see that $\gamma_t = (ge_1)_t = g_te_1 = gP_u(\nabla F(u))e_1$. □

Next we use Theorem 4.5 to solve the Cauchy problem for (4.3) on the line from the solution of the Cauchy problem for (4.1) on the line.

Theorem 4.6 (Cauchy problem on the line). *Let* $\gamma_0 \in \mathcal{M}_{n+1,n}$, *and* u_0, g_0 *are the isotropic curvature and isotropic moving frame along* γ_0. *Suppose* $u(x, t)$ *is the solution of* (4.1) *with initial data* $u(x, 0) = u_0(x)$, *and* g *the solution of* (4.7) *with* $g(0, 0) = g_0(0)$. *Then* $\gamma(x, t) = g(x, t)e_1$ *is a solution of* (4.3) *with* $\gamma(x, 0) = \gamma_0(x)$.

To solve the periodic Cauchy problem for the isotropic curve flow (4.3), we need to solve the period problem of (4.1).

Theorem 4.7 (Periodic Cauchy problem). *Let* $\gamma_0 \in \mathcal{M}_{n+1,n}(S^1)$, *and* g_0 *and* u_0 *the isotropic moving frame and curvature along* γ_0. *Let* $u(x, t)$ *be the solution of* (4.1) *periodic in* x *such that* $u(x, 0) = u_0(x)$, *and* $g(x, t)$ *the solution of* (4.7) *with* $g(0, 0) = g_0(0)$. *Then* $\gamma(x, t) = g(x, t)e_1$ *is a solution of* (4.3) *with* $\gamma(x, 0) = \gamma_0(x)$ *and* $\gamma(\cdot, t)$ *is periodic in* x.

Proof. By Theorem 4.6, γ is a solution of (4.3) on $\mathbb{R}\times\mathbb{R}$. We claim that $y(t) := g(2\pi, t) - g(0, t) \equiv 0$. Note that both $g(x, 0)$ and $g_0(x)$ satisfy $h^{-1}h_x = b + u_0$ with the same initial condition, so $g(x, 0) = g_0(x)$. Because g_0 is periodic, we have $y(0) = 0$. Set $B(u) = P_u(\nabla F(u))$. It follows from Proposition 4.4 that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(2\pi, t)B(u)(2\pi, t) - g(0, t)B(u)(0, t).$$

Since $u(x, t)$ is periodic in x with period 2π, $B(u)(2\pi, t) = B(u)(0, t)$. So

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(t)B(u)(0, t).$$

Note that the constant function 0 is the solution of the above linear system with $y(0) = 0$. It follows from the uniqueness of solutions of ordinary differential equations that we prove the claim $y(t) \equiv 0$. So $g(x, t)$ is periodic in x, which implies that $\gamma(x, t) = g(x, t)e_1$ is periodic in x. □

5. The $\hat{B}_n^{(1)}$- and $\hat{B}_n^{(1)}$-KdV hierarchies

In this section, we give the constructions of the $\hat{B}_n^{(1)}$- and the $\hat{B}_n^{(1)}$-KdV hierarchies and study their Hamiltonian theory (cf. [3, 17]).

First we construct the $\hat{B}_n^{(1)}$-hierarchy. Let

$$\hat{B}_n^{(1)} = \{\xi(\lambda) = \sum_{i\le n_0} \xi_i\lambda^i \mid n_0 \in \mathbb{Z},\, \xi_i \in o(n+1, n)\},$$

and

(5.1) $$J_B = \beta\lambda + b,$$

where b is defined by (1.2) and

(5.2) $$\beta = \frac{1}{2}\beta_n = \frac{1}{2}(e_{1,2n} + e_{2,2n+1}).$$

Note that

$$J_B^{2i} \notin \hat{B}_n^{(1)}, \quad J_B^{2j-1} \in \hat{B}_n^{(1)},$$
$$J_B^{2n+1}(\lambda) = \lambda J_B(\lambda).$$

Theorem 5.1. *([3]) Let $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, and J_B as in (5.1). Then there exists a unique*

$$T(q,\lambda) = \beta\lambda + \sum_{i\le 0} T_{1,i}(q)\lambda^i$$

in $\hat{B}_n^{(1)}$ satisfying

$$\begin{cases} [\partial_x + J_B(\lambda) + q, T(q,\lambda)] = 0, \\ T^{2n+1}(q,\lambda) = \lambda T(q,\lambda), \end{cases} \tag{5.3}$$

Moreover, $T_{1,i}(q)$'s are differential polynomials in q for all i and can be computed by comparing coefficients of $\mathcal{G}_i\lambda^j$ of (5.3).

Write $T^{2j-1}(q,\lambda)$ as a power series in λ:

$$T^{2j-1}(q,\lambda) = \sum_{i\le [\frac{2j-1}{2n+1}]+1} T_{2j-1,i}(q)\lambda^i. \tag{5.4}$$

It follows from Theorems 5.1 that $T_{2j-1,i}(q)$'s are differential polynomials in q and can be computed from $T_{1,k}(q)$'s.

If A, B are in an associative algebra and $[A,B]=0$, then $[A,B^j]=0$ for all j. So it follows from the first equation of (5.3) that we have

$$[\partial_x + J_B(\lambda) + q, T^{2j-1}(q,\lambda)] = 0. \tag{5.5}$$

Compare the constant term of the above equation as a power series expansion in λ to see that

$$[\partial_x + b + q, T_{2j-1,0}(q)] = [T_{2j-1,-1}(q), \beta]. \tag{5.6}$$

Since the right hand side of (5.6) is upper triangular, $[\partial_x + b + q, T_{2j-1,0}(q)] \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$. So we have

Definition 5.2. The $(2j-1)$-*th* $\hat{B}_n^{(1)}$-*flow* is the following flow

$$q_t = [\partial_x + b + u, T_{2j-1,0}(q)] \tag{5.7}$$

on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$.

Remark 5.3. It follows from (5.6) that if $q = (q_{ij})$ is a solution of (5.7), then $q_{ij}(x,t) = q_{ij}(0,0)$ for $i+j \le 2n+1$, $i \ne 1$, and $j \ne 2$. In fact, (5.7) is a flow on $C^\infty(\mathbb{R}, Y_n)$, where $Y_n = \mathrm{Im}(\mathrm{ad}(\beta)) = \{[\beta, y] \mid y \in o(n+1,n)\} \subset \mathcal{B}_n^+$.

Definition 5.4. The $(2j-1)$*-th* $\hat{B}_n^{(1)}$*-KdV flow* is the following flow on $C^\infty(\mathbb{R}, V_n)$,

$$u_t = [\partial_x + b + u, P_u(\pi_0(T_{2j-1,0}(u))], \tag{5.8}$$

where P_u is the operator defined in Corollary 3.5.

Next we discuss the gauge action and construct a cross section of this action. The group $C^\infty(\mathbb{R}, N_n^+)$ acts on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$ by gauge transformation

$$\triangle * q = \triangle(b+q)\triangle^{-1} - \triangle_x\triangle^{-1} - b, \tag{5.9}$$

where $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, $\triangle \in C^\infty(\mathbb{R}, \mathcal{N}_n^+)$. A direct computation implies that

$$\triangle(\partial_x + b + q)\triangle^{-1} = \partial_x + b + \triangle * q, \tag{5.10}$$

$$\triangle(\partial_x + J_B(\lambda) + q)\triangle^{-1} = \partial_x + J_B(\lambda) + \triangle * q. \tag{5.11}$$

Proposition 5.5. *Let* $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$*,* $T(q,\lambda)$ *defined by* (5.3)*, and* $\triangle \in C^\infty(\mathbb{R}, N_n^+)$*. Then*

$$\triangle T(q,\lambda)\triangle^{-1} = T(\triangle * q, \lambda). \tag{5.12}$$

Proof. Use (5.11) and conjugate (5.3) by $\triangle$ to see that

$$[\partial_x + \beta\lambda + b + \triangle * q, \triangle T(q,\lambda)\triangle^{-1}] = 0.$$

The Proposition follows from Theorem 5.1 . □

The next Proposition shows that $C^\infty(\mathbb{R}, V_n)$ is a cross section of this gauge action, where V_n is defined as in (1.4).

Proposition 5.6. *Given* $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$*, then there exist a unique* $\triangle \in C^\infty(\mathbb{R}, N_n^+)$ *such that* $u := \triangle * q \in C^\infty(\mathbb{R}, V_n)$*, i.e.,*

$$\triangle(\partial_x + J_B(\lambda) + q)\triangle^{-1} = \partial_x + J_B(\lambda) + u, \tag{5.13}$$

where V_n *is as in* (1.4) *and* $J_B(\lambda)$ *is given by* (5.2)*. Moreover, entries of* $\triangle$ *and* u *are differential polynomials of* q*, which can be computed from* (5.13)*.*

Proof. Let $\mathcal{G}_i$'s be as in (2.1), we write elements $m \in \mathcal{B}_n^+$ as $m = \sum_{i=0}^{2n-1} m_i$ with $m_i \in \mathcal{G}_i$. First note that $\triangle\beta = \beta\triangle$. Hence it suffices to prove that

$\triangle(\partial_x + b + q) = (\partial_x + b + u)\triangle$. From a direct computation, we have the following recursive formula:

$$q_i + \triangle_{i+1} b + \sum_{j+k=i} \triangle_j q_k = u_i + (\triangle_i)_x + b\triangle_{i+1} + \sum_{j+k=i} u_j \triangle_k.$$

Note that for each u_i, there is only one unknown term needs to be solved. Hence u_i and $\triangle_{i+1}$ can be solved uniquely from the previous solved terms. This proves the proposition. □

Example 5.7. Given $q = q_1(e_{11} - e_{33}) + q_2(e_{12} + e_{23})$, the proof of Proposition 5.6 gives a method to compute u and $\triangle$ explicitly as differential polynomials of q. For the case $n = 1$, we have

$$\triangle = \begin{pmatrix} 1 & -q_1 & \frac{1}{2}q_1^2 \\ 0 & 1 & -q_1 \\ 0 & 0 & 1 \end{pmatrix},$$
$$u = \left(q_2 + \frac{1}{2}q_1^2 + (q_1)_x \right)(e_{12} + e_{21}).$$

Proposition 5.8. *Assume that $q : \mathbb{R}^2 \to \mathcal{B}_n^+$ is a solution of the $(2j-1)$-th $\hat{B}_n^{(1)}$-flow (5.7) and $\triangle(\cdot, t)$ is the unique N_n^+-map as in Proposition 5.6 such that $u(\cdot, t) := \triangle(\cdot, t) * q(\cdot, t)$ lies in V_n. Then*

1) $u_t = [\partial_x + b + u, T_{2j-1,0}(u) - \triangle_t \triangle^{-1}]$,
2) $T_{2j-1,0}(u) - \triangle_t \triangle^{-1} = P_u(\pi_0(T_{2n-1,0}(u)))$,
3) *entries of $\triangle_t \triangle^{-1}$ are differential polynomials of u in x variable,*
4) *u is a solution of the $(2j-1)$-th $\hat{B}_n^{(1)}$-KdV flow (5.8).*

Proof. Recall that $\triangle(\partial_x + b + q)\triangle^{-1} = \partial_x + b + \triangle * q = \partial_x + b + u$. Compute directly we obtain

$$\begin{aligned} u_t &= (\partial_x + b + u)_t = (\triangle(\partial_x + b + q)\triangle^{-1})_t \\ &= \triangle q_t \triangle^{-1} + [\triangle_t \triangle^{-1}, \triangle(\partial_x + b + q)\triangle^{-1}] \\ &= \triangle[\partial_x + b + q, T_{2j-1,0}(q)]\triangle^{-1} - [\triangle(\partial_x + b + q)\triangle^{-1}, \triangle_t \triangle^{-1}] \\ &= [\triangle(\partial_x + b + q)\triangle^{-1}, \triangle T_{2j-1,0}(q)\triangle^{-1}] \\ &\quad - [\triangle(\partial_x + b + q)\triangle^{-1}, \triangle_t \triangle^{-1}], \quad \text{by (5.12)}, \\ &= [\partial_x + b + u, T_{2j-1,0}(u)] - [\partial_x + b + u, \triangle_t \triangle^{-1}]. \end{aligned}$$

This proves (1).

Since $\triangle_t\triangle^{-1}$ lies in $\mathcal{N}_n^+$,

$$\pi_0(T_{2j-1,0}(u) - \triangle_t\triangle^{-1}) = \pi_0(T_{2j-1,0}(u)).$$

Since $u(x,t) \in V_n$, $u_t \in V_n$. So (2) and (3) follow from Theorem 3.2.

(4) follows from (1). □

It follows from Proposition 5.8 that we have the following:

Theorem 5.9. *If $u \in C^\infty(\mathbb{R}, V_n)$, then*

$$\eta_j(u) := T_{2j-1,0}(u) - P_u(\pi_0(T_{2j-1,0}(u))) \tag{5.14}$$

is a $\mathcal{N}_n^+$-valued differential polynomial of u.

Corollary 5.10. *The flow on the cross section $C^\infty(\mathbb{R}, V_n)$ obtained by pushing the $(2j-1)$-th $\hat{B}_n^{(1)}$-flow along the orbit of the gauge action of $C^\infty(\mathbb{R}, N_n^+)$ is the $(2j-1)$-th $\hat{B}_n^{(1)}$-KdV flow.*

It was proved in [3] that the $\hat{B}_n^{(1)}$-flows commute. Hence the $\hat{B}_n^{(1)}$-KdV flows commute.

As a consequence of Proposition 4.3 we have the following.

Proposition 5.11.

1) *$q \in C^\infty(\mathbb{R}^2, \mathcal{B}_n^+)$ is the $(2j-1)$-th flow $\hat{B}_n^{(1)}$-flow (5.7) if and only if the following system is solvable for $h : \mathbb{R}^2 \to O(n+1, n)$,*

$$\begin{cases} h^{-1}h_x = b + q, \\ h^{-1}h_t = T_{2j-1,0}(q). \end{cases} \tag{5.15}$$

2) *$u \in C^\infty(\mathbb{R}^2, V_n)$ is a solution of the $(2j-1)$-th $\hat{B}_n^{(1)}$-KdV flow (5.8) if and only if the following linear system*

$$\begin{cases} g^{-1}g_x = b + u, \\ g^{-1}g_t = P_u(\pi_0(T_{2j-1,0}(u))), \end{cases} \tag{5.16}$$

is solvable for $g : \mathbb{R}^2 \to O(n+1, n)$.

By Propositions 5.8, 5.11, we have the following.

Proposition 5.12. *Let q, $\triangle$, and u be as in Proposition 5.8. If h is a solution of* (5.15)*, then $g := h\triangle^{-1}$ is a a solution of* (5.16)*.*

Next we discuss the bi-Hamiltonian structure. The gradient of a functional $F : C^\infty(S^1, \mathcal{B}_n^+) \to \mathbb{R}$ at q is the unique $\nabla F(q) \in C^\infty(S^1, \mathcal{B}_n^-)$ satisfying

$$\mathrm{d}F_q(\xi) = \oint \langle \nabla F(q), \xi \rangle \mathrm{d}x$$

for all $\xi \in C^\infty(S^1, \mathcal{B}_n^+)$. It was proved in [3] that

$$\{F_1, F_2\}_1^b(q) = \oint \langle [\nabla F_1(q), \beta], \nabla F_2(q) \rangle \mathrm{d}x, \tag{5.17}$$

$$\{F_1, F_2\}_2^b(q) = \oint \langle [\partial_x + b + q, \nabla F_1(q)], \nabla F_2(q) \rangle \mathrm{d}x, \tag{5.18}$$

give a bi-Hamiltonian structure on $C^\infty(S^1, \mathcal{B}_n^+)$ for the $\hat{B}_n^{(1)}$-hierarchy.

Given a functional F on $C^\infty(\mathbb{R}, V_n)$, let $\hat{F}$ be the functional on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$ defined by

$$\hat{F}(q) = F(u)$$

if $u = \triangle * q$ for some $\triangle \in C^\infty(\mathbb{R}, N_n^+)$. Since $C^\infty(\mathbb{R}, V_n)$ is a cross section, all functionals on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$ that are invariant under the gauge action of $C^\infty(\mathbb{R}, N_n^+)$ arise this way.

Lemma 5.13. *Let F_1, F_2 be functionals on $C^\infty(S^1, V_n)$, and $\hat{F}_1, \hat{F}_2$ the functional on $C^\infty(S^1, \mathcal{B}_n^+)$ given above. Then*

1) *$\{\hat{F}_1, \hat{F}_2\}_i^b$ is invariant under the gauge action of $C^\infty(S^1, N_n^+)$,*
2) *let $\{F_1, F_2\}_i =$ the restriction of $\{\hat{F}_1, \hat{F}_2\}_i^b$ to $C^\infty(S^1, V_n)$, then $\{\,,\}_i$ is a Poisson structure on $C^\infty(S^1, V_n)$ for $i = 1, 2$.*

Proof. It follows from the definition of the gradient and a direct computation that we have

$$\nabla \hat{F}(q) = \triangle^{-1} \nabla F(\triangle * q) \triangle.$$

Note also that $\triangle\beta\triangle^{-1} = \triangle\beta\rho_n\triangle^t\rho_n = \beta$. So we have

$$\{\hat{F}_1, \hat{F}_2\}_1^b(q) = \oint \langle[\triangle^{-1}\nabla F_1(\triangle * q)\triangle, \beta], \triangle^{-1}\nabla F_2(\triangle * q)\triangle\rangle \mathrm{d}x$$

$$= \oint \langle[\nabla F_1(\triangle * q), \beta], \nabla F_2(\triangle * q)\rangle \mathrm{d}x = \{F_1, F_2\}_1^b(\triangle * q),$$

$$\{\hat{F}_1, \hat{F}_2\}_2^b(q) = \oint \langle[\partial_x + b + q, \nabla \hat{F}_1(q)], \nabla \hat{F}_2(q)\rangle \mathrm{d}x$$

$$= \oint \langle[\partial_x + b + q, \triangle^{-1}\nabla F_1(\triangle * q)\triangle], \triangle^{-1}\nabla F_2(\triangle * q)\triangle\rangle \mathrm{d}x$$

$$= \oint \langle[\triangle(\partial_x + b + q)\triangle^{-1}, \nabla F_1(\triangle * q)], \nabla F_2(\triangle * q)\rangle \mathrm{d}x, \quad \text{by (5.10)},$$

$$= \oint \langle[\partial_x + b + \triangle * q, \nabla F_1(\triangle * q)], \nabla F_2(\triangle * q)\rangle \mathrm{d}x$$

$$= \{F_1, F_2\}_2^b(\triangle * q).$$

This proves (1). Statement (2) is a consequence of (1). □

The Lemma and Theorem can be proved the same way as for the $\hat{A}_n^{(1)}$ case given in [17].

Lemma 5.14. *Let H be a functional on $C^\infty(S^1, V_n)$, and $\tilde{H}$ the functional on $C^\infty(S^1, \mathcal{B}_n^+)$ defined by $\tilde{H}(q) = H(u)$ if $\triangle * q = u \in C^\infty(S^1, V_n)$ for some $\triangle \in C^\infty(S^1, N_n^+)$. Then $\nabla\tilde{H}(u) = \pi_{\mathcal{B}_n^-}(P_u(\nabla H(u)))$, where $\pi_{\mathcal{B}_n^-}$ is the projection onto $\mathcal{B}_n^-$ along $\mathcal{N}_n^+$ and P_u is the linear differential operator given in Corollary 3.5.*

Theorem 5.15. *Let $\{\,,\}_i$ be the Poisson structure on $C^\infty(S^1, V_n)$ defined in Lemma 5.13. Then we have*

$$\{F_1, F_2\}_1(u) = \oint \langle[P_u(\nabla F_1(u)), \beta], P_u(\nabla F_2(u))\rangle \mathrm{d}x, \tag{5.19}$$

$$\{F_1, F_2\}_2(u) = \oint \langle[\partial_x + b + u, P_u(\nabla F_1(u))], \nabla F_2(u)\rangle \mathrm{d}x. \tag{5.20}$$

The *Poisson operator* at u for $\{\,,\}_i$ is the operator $J_i : C^\infty(S^1, V_n^t) \to C^\infty(S^1, V_n)$ defined by

$$\{F_1, F_2\}_i(u) = \oint \langle(J_i)_u(\nabla F_1(u)), \nabla F_2(u)\rangle \mathrm{d}x.$$

So

$$(J_2)_u(v) = [\partial_x + b + u, P_u(v)].$$

For $\{\,,\}_1$, let $\xi_i = \nabla F_i(u)$. Use integration by part to get

$$\oint \langle [P_u(\xi_1), \beta], P_u(\xi_2)\rangle \mathrm{d}x = \oint \langle (J_1)_u(\xi_1), \xi_2\rangle \mathrm{d}x.$$

The Hamiltonian flows for $F : C^\infty(S^1, V_n) \to \mathbb{R}$ with respect to $\{\,,\}_i$ is

$$u_t = (J_i)_u(\nabla F(u))$$

for $i = 1, 2$.

Example 5.16. [Bi-Hamiltonian structure for $\hat{B}_1^{(1)}$-KdV]
Write $\tilde{\xi} = \nabla F_1(u) = \xi(e_{21} + e_{32})$, $\tilde{\eta} = \nabla F_2(u) = \eta(e_{21} + e_{32})$, $C = P_u(\tilde{\xi}) = (C_{ij})$ and $D = P_u(\tilde{\eta}) = (D_{ij})$. We use the formula (3.6) for $P_u(v)$ to write down C and D in terms of ξ and η respectively and compute directly to see that

$$\{F_1, F_2\}_1(u) = \langle [C, \beta], D\rangle = -2 \oint \xi_x \eta dx,$$
$$\{F_1, F_2\}_2(u) = \langle [\partial_x + b + u, C], D\rangle = -2 \oint (\xi_{xxx} - 2u_1\xi_x - (u_1)_x\xi)\eta \mathrm{d}x.$$

Since $\langle \tilde{\xi}, \tilde{\eta}\rangle = 2 \oint \xi\eta \mathrm{d}x$, we see that

$$(J_1)_u(\tilde{\xi}) = -\xi_x \beta_1,$$
$$(J_2)_u(\tilde{\xi}) = -(\xi_{xxx} - 2u_1\xi_x - (u_1)_x\xi)\beta_1,$$

where $\beta_1 = e_{12} + e_{23}$. This is the standard bi-Hamiltonian structure for the KdV-hierarchy (cf. [2]).

The commuting Hamiltonians for the $\hat{B}_n^{(1)}$-hierarchy given in [3] can be written in terms of $T_{2j-1,-1}(u)$.

Theorem 5.17. *([3]) Let $u, \beta, T(u, \lambda)$ be as in Theorem 5.1, $T_{2j-1,-1}(u)$ as in* (5.4)*, $h_{2j-1}(u) = -\mathrm{tr}(T_{2j-1,-1}(u)\beta)$, and $F_{2j-1} : C^\infty(S^1, V_n) \to \mathbb{R}$ defined by*

$$F_{2j-1}(u) = \oint h_{2j-1}(u)\mathrm{d}x = -\oint \mathrm{tr}(T_{2j-1,-1}(u)\beta)\mathrm{d}x. \tag{5.21}$$

Then we have the following:

1) $\nabla F_{2j-1}(u) = \pi_0(T_{2j-1,0}(u))$, *where π_0 is the projection onto V_n^t defined by* (3.5).

2) *The Hamiltonian equation for* F_{2j-1} *with respect to the Poisson structure* $\{\,,\}_2$ *defined by* (5.20) *is the* $(2j-1)$*-th* $\hat{B}_n^{(1)}$*-KdV flow.*

3) *The Hamiltonian flow of* F_{2j-1} *for* $j > n$ *with respect to* $\{\,,\}_1$ *defined by* (5.19) *is the* $(2(j-n)-1)$*-th* $\hat{B}_n^{(1)}$*-KdV flow.*

Example 5.18 (The third $\hat{B}_1^{(1)}$-KdV flow). For $n = 1$. we have $u = q\beta_1$. Since $J_B^3 = \lambda J_B$, $T^3(u, \lambda) = \lambda T(u, \lambda)$. So $T_{3,-1}(u) = T_{1,-2}(u)$. We compare coefficients of $\mathcal{G}_i\lambda^j$ of (5.3) to compute the $\mathcal{G}_{-1}$ component of $T_{1,-2}$ and see that $h_3(u) = -q^2$. So F_3 given in Theorem 5.17 is (1.9), i.e.,

$$F_3(q\beta_1) = -\oint q^2 \mathrm{d}x.$$

Next we compute ∇F_3. For $v = \eta\beta_1$, we have

$$\mathrm{d}(F_3)_u(v) = -2\oint q\eta \mathrm{d}x = \oint \langle -q\beta_1^t, \eta\beta_1\rangle \mathrm{d}x.$$

(Here we use the fact that $\langle \beta_1, \beta_1^t\rangle = 2$). So $\nabla F_3(u) = -q\beta_1^t$. It follows from Example 5.16 that the Hamiltonian flow for F_3 with respect to $\{\,,\}_2$ is the KdV (1.5).

Example 5.19 (The third $\hat{B}_2^{(2)}$-KdV flows). For $n = 2$, we have $\beta_1 = e_{23} + e_{14}$, $\beta_2 = e_{14} + e_{25}$, and $u = u_1\beta_1 + u_2\beta_2$. Compare coefficient of $\mathcal{G}_i\lambda^j$'s of (5.3) to obtain $T_{1,i}(u)$ for small i, then use them to compute $T_{3,0}(u)$ and the $\mathcal{G}_{-4}$ component of $T_{3,-1}(u)$. We obtain

$$h_3(u) = \frac{1}{2}u_1^2 + 2u_2.$$

For $v = v_1\beta_1 + v_2\beta_2$, we have

$$\mathrm{d}(F_3)_u(v) = \oint u_1v_1 + 2u_2v_2 \mathrm{d}x = \oint \langle \frac{1}{2}u_1\beta_1^t + \beta_2^t, v_1\beta_1 + v_2\beta_2\rangle \mathrm{d}x.$$

Thus

$$\nabla F_3(u) = \frac{1}{2}u_1\beta_1^t + \beta_2^t.$$

We use (3.7) to write down $P_u(\nabla F_3(u))$ with $\xi = -\frac{u_1}{2}$, $\eta = 0$, and $v_2 = 1$. A direct computation implies that the third $\hat{B}_2^{(1)}$-KdV flow, $u_t = [\partial_x + b +$

$u, P_u(\nabla F_3(u))]$, is

$$\begin{cases} (u_1)_t = -\frac{1}{2}u_1^{(3)} + \frac{3}{2}u_1(u_1)_x + 3(u_2)_x, \\ (u_2)_t = u_2^{(3)} - \frac{3}{2}u_1(u_2)_x. \end{cases} \tag{5.22}$$

6. The $\hat{A}_{2n}^{(2)}$- and $\hat{A}_{2n}^{(2)}$-KdV hierarchies

In this section, we give the constructions of the $\hat{A}_{2n}^{(2)}$- and the $\hat{A}_{2n}^{(2)}$-KdV hierarchies and their Hamiltonian theory.

Let θ be the involution of $sl(2n+1,\mathbb{C})$ defined by

$$\theta(y) = -\rho_n^{-1}y^t\rho_n,$$

where ρ_n is defined by (1.1). Note that $sl(2n+1,\mathbb{R})$ is invariant under θ. Let $\mathcal{K}$ and $\mathcal{P}$ be the $+1$ and -1 eigenspaces of θ on $sl(2n+1,\mathbb{R})$ respectively, i.e.,

$$\mathcal{K} = \{y \in sl(2n+1,\mathbb{R}) \mid -\rho_n^{-1}y^t\rho_n = y\} = o(n+1,n),$$
$$\mathcal{P} = \{y \in sl(2n+1,\mathbb{R}) \mid \rho_n^{-1}y^t\rho_n = y\}.$$

Then we have

$$[\mathcal{K},\mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K},\mathcal{P}] \subset \mathcal{P}, \quad [\mathcal{P},\mathcal{P}] \subset \mathcal{K}. \tag{6.1}$$

The affine Kac-Moody algebra $\hat{A}_{2n}^{(2)}$ is the algebra of $\xi(\lambda) \in sl(2n+1,\mathbb{C})$ satisfying the $\hat{A}_{2n}^{(2)}$-*reality condition*,

$$\overline{\xi(\bar{\lambda})} = \xi(\lambda), \quad \theta(\xi(-\lambda)) = \xi(\lambda). \tag{6.2}$$

We have the following simple facts.

1) $\xi(\lambda) = \sum_i \xi_\lambda^i$ lies in $\hat{A}_{2n}^{(2)}$ if and only if ξ_i is in $\mathcal{K}$ for even i and in $\mathcal{P}$ for odd i.

2) Let

$$J(\lambda) = e_{1,2n+1}\lambda + b. \tag{6.3}$$

Then J^{2j-1} are in $\hat{A}_{2n}^{(2)}$ for all $j \geq 1$ and

$$J^{2n+1}(\lambda) = \lambda \mathrm{I}_{2n+1}. \tag{6.4}$$

Theorem 6.1. *([3], [15]) Let $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, and J as in (6.3). Then there exists a unique*

$$S(q,\lambda) = e_{1,2n+1}\lambda + \sum_{i\le 0} S_{1,i}(q)\lambda^i$$

in $\hat{A}_{2n}^{(2)}$ satisfying

$$\begin{cases} [\partial_x + J + q, S(q,\lambda)] = 0, \\ S^{2n+1}(q,\lambda) = \lambda \mathrm{I}_{2n+1}. \end{cases} \tag{6.5}$$

Moreover, $S_{1,i}(q)$ is a differential polynomial of q for all i.

Write

$$S^{2j-1}(q,\lambda) = \sum_{i \le [\frac{2j-1}{2n+1}]+1} S_{2j-1,i}(q)\lambda^i. \tag{6.6}$$

Since $S(q,\cdot)$ satisfies (6.2), $S^{2j-1}(q,\cdot)$ also satisfies (6.2). So we have

$$S_{2j-1,i}(q) \in \begin{cases} \mathcal{K}, & \text{for even } i, \\ \mathcal{P}, & \text{for odd }. \end{cases} \tag{6.7}$$

It follows from $[\partial_x + J(\lambda) + q, S(q,\lambda)] = 0$ that we have

$$[\partial_x + J(\lambda) + q, S^{2j-1}(q,\lambda)] = 0. \tag{6.8}$$

Compare the constant coefficient of (6.8) as a power series in λ to get

$$[\partial_x + b + u, S_{2j-1,0}(q)] = [S_{2j-1,-1}(q), e_{1,2n+1}]. \tag{6.9}$$

It follows from (6.7) that the left hand side of (6.9) lies in $\mathcal{K} = o(n+1,n)$ and the right hand side is upper triangular. So the left hand side lies in $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$ and we have the following flows:

Definition 6.2. For $j \not\equiv 0(\mathrm{mod}(n))$, the $(2j-1)$*-th* $\hat{A}_{2n}^{(2)}$*-flow* is the following flow on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$,

$$q_t = [\partial_x + b + q, S_{2j-1,0}(q)]. \tag{6.10}$$

Remark 6.3. If $q = (q_{ij}) \in C^(\mathbb{R}^2, \mathcal{B}_n)$ is a solution of (6.10), then by (6.9) we have $q_{ij}(x,t) = q_{ij}(0,0)$ for $i + j \le 2n+1$ and $i \ne 1$. In other words, (6.10) is a flow on $C^\infty(\mathbb{R}, X_n)$, where $X_n = \{[e_{1,2n+1}, y] \mid y \in \mathcal{P}\} \subset \mathcal{B}_n^+$.

Definition 6.4. The $(2j-1)$*-th* $\hat{A}_{2n}^{(2)}$*-KdV flow* is the following flow on $C^\infty(\mathbb{R}, V_n)$,

$$u_t = [\partial_x + b + u, P_u(\pi_0(S_{2j-1,0}(u)))], \tag{6.11}$$

where P_u is the operator defined as in Corollary 3.5.

Remark 6.5. Let $\mathfrak{b}_{2n}$ denote the subalgebra of upper triangular matrices in $sl(2n+1, \mathbb{R})$, $\mathbb{N}_{2n}^+$ the subgroup of upper triangular matrices $y = (y_{ij})$ in $SL(2n+1, \mathbb{R})$ with $y_{ii} = 1$ for all $1 \le i \le 2n+1$,

$$\begin{aligned} Y_{2n} &:= \oplus_{i=1}^{2n} \mathbb{R} e_{i,2n+1}, \\ Z_{2n} &:= V_n \oplus (\oplus_{i=1}^{n} \mathbb{R} e_{i,2n+2-i}). \end{aligned}$$

It is known (cf. [3], [17]) that $C^\infty(\mathbb{R}, Y_{2n})$ is a cross section of the gauge action of $C^\infty(\mathbb{R}, \mathbb{N}_{2n}^+)$ on $C^\infty(\mathbb{R}, \mathfrak{b}_{2n}^+)$. Use a proof similar to the one given for Proposition 5.6 to see that $C^\infty(\mathbb{R}, Z_{2n})$ is also a cross section of this gauge action. So we obtain two $\hat{A}_{2n}^{(1)}$-KdV hierarchies. One on $C^\infty(\mathbb{R}, Y_{2n})$, which is the Gelfand-Dickey hierarchy. The second is a hierarchy on $C^\infty(\mathbb{R}, Z_{2n})$. They look different but are gauge equivalent. Moreover, the $\hat{A}_{2n}^{(1)}$-KdV hierarchy on $C^\infty(\mathbb{R}, Z_{2n})$ leaves $C^\infty(\mathbb{R}, V_n)$ invariant and the restriction of the $\hat{A}_{2n}^{(1)}$-KdV hierarchy to $C^\infty(\mathbb{R}, V_n)$ is the $\hat{A}_{2n}^{(2)}$-KdV hierarchy.

The following two propositions can be proved by similar argument as for the $\hat{B}_n^{(1)}$ case.

Theorem 6.6. *If* $u \in C^\infty(\mathbb{R}, V_n)$ *and* $j \ge 1$*, then*

$$\tilde{\eta}_j(u) := S_{2j-1}(u) - P_u(\pi_0(S_{2j-1,0}(u))) \tag{6.12}$$

is a $\mathcal{N}_+$*-valued differential polynomial of* u.

Proposition 6.7.

1) $q : \mathbb{R}^2 \to \mathcal{B}_n^+$ *is a solution of the* $(2j-1)$*-th* $\hat{A}_{2n}^{(2)}$*-flow if and only if the following linear system is solvable for* $h : \mathbb{R}^2 \to O(n+1, n)$,

$$\begin{cases} h^{-1}h_x = b + q, \\ h^{-1}h_t = S_{2j-1,0}(q). \end{cases} \tag{6.13}$$

2) $u : \mathbb{R}^2 \to V_n$ *is a solution of the* $(2j-1)$*-th* $\hat{A}_{2n}^{(2)}$*-KdV flow if and only if the following linear system is solvable for* $g : \mathbb{R}^2 \to O(n+1, n)$,

$$\begin{cases} g^{-1} g_x = b + q, \\ g^{-1} g_t = P_u(\pi_0(S_{2j-1,0}(u))). \end{cases} \tag{6.14}$$

Proposition 6.8. *Let* q *be a solution of the* $(2j-1)$*-th* $\hat{A}_{2n}^{(2)}$*-flow, and* $\triangle : \mathbb{R}^2 \to N_n^+$ *such that* $\triangle(\cdot, t) * q(\cdot, t)$ *is in* $C^\infty(\mathbb{R}, V_n)$ *for all* t*. Then* $u = \triangle * q$ *is a solution of the* $(2j-1)$*-KdV flow. Moreover, if* h *is a solution of* (6.13)*, then* $g = h\triangle^{-1}$ *is a solution of* (6.14).

Remark 6.9. It follows from similar arguments as for the $\hat{B}_n^{(1)}$-KdV hierarchy that we obtain two compatible Poisson structures on $C^\infty(S^1, V_n)$ for the $\hat{A}_{2n}^{(2)}$-KdV hierarchy:

$$\{F_1, F_2\}_1^a(u) = \oint \langle [P_u(\nabla F_1(u)), e_{1,2n+1}], P_u(\nabla F_2(u)) \rangle \mathrm{d}x,$$
$$\{F_1, F_2\}_2^a(u) = \oint \langle [\partial_x + b + u, P_u(\nabla F_1(u))], \nabla F_2(u) \rangle \mathrm{d}x.$$

We claim that $\{\,,\}_1^a = 0$. To see this, first note that $e_{1,2n+1} \in \mathcal{P}$, $\mathcal{K} = o(n+1, n)$, $[\mathcal{K}, \mathcal{P}] \subset \mathcal{P}$, and $\langle \mathcal{P}, \mathcal{K} \rangle = 0$. Since $P_u(\nabla F(u)) \subset V_n \subset \mathcal{K}$, we have $[P_u(\nabla F(u)), e_{1,2n+1}] \subset \mathcal{P}$. Hence

$$\{F_1, F_2\}_1^a(u) = \oint \langle [P_u(\nabla F_1(u)), e_{1,2n+1}], P_u(\nabla F_2(u)) \rangle \mathrm{d}x = 0.$$

This proves the claim.

It is known (cf. [14], [17]) that $\{\,,\}_1^a$ and $\{\,,\}_2^a$ generate a sequence of compatible Poisson structures $\{\,,\}_j$ defined by

$$\{F_1, F_2\}_j^a(u) = \oint \langle (J_j^a(\nabla F_1(u))), \nabla F_2(u) \rangle \mathrm{d}x, \quad \text{where}$$
$$J_j^a = J_2^a((J_1^a)^{-1} J_2^a)^{j-2}$$

for $j \geq 1$ (cf. [14], [17]). Although $\{\,,\}_{2i-1}^a = 0$, $\{\,,\}_{2i}^a$ defines a Poisson structure for the $\hat{A}_{2n}^{(2)}$-KdV hierarchy. Hence $(\{\,,\}_2^a, \{\,,\}_4^a)$ is a bi-Hamiltonian structure for the $\hat{A}_{2n}^{(2)}$-KdV hierarchy. Note also that $\{\,,\}_2^a$ is the same Poisson structure $\{\,,\}_2$ defined by (5.20) for the $\hat{B}_n^{(1)}$-KdV.

Theorem 6.10. *([3], [17]) Let u, and $S(u,\lambda)$ be as in Theorems 6.1, and $S_{2j-1,0}(u)$ defined by* (6.6). *Let $k_{2j-1}(u) = -\mathrm{tr}(S_{2j-1,-1}(u)\beta)$, and*

$$G_{2j-1}(u) = \oint h_{2j-1}(u)\mathrm{d}x = -\oint \mathrm{tr}(S_{2j-1,-1}(u)e_{1,2n+1})\mathrm{d}x. \tag{6.15}$$

Then

1) $\nabla G_{2j-1}(u) = \pi_0(S_{2j-1,0}(u))$, *where π_0 is the projection onto V_n^t defined by* (3.5),
2) *the Hamiltonian flow for G_{2j-1} with respect to $\{\,,\}_2$ defined by* (5.20) *is the $(2j-1)$-th $\hat{A}_{2n}^{(2)}$-KdV flow* (6.11),
3) *the Hamiltonian flow for G_{2j-1} for $j > 2n$ with respect to $\{\,,\}_4^a$ defined by* (5.20) *is the $(2(j-2n)-1)$-th $\hat{A}_{2n}^{(2)}$-KdV flow.*

Example 6.11 (The fifth $\hat{A}_2^{(2)}$-KdV flows). For the $\hat{A}_2^{(2)}$-KdV hierarchy, we have $u = q(e_{12}+e_{23})$ and $S^3(u,\lambda) = \lambda \mathrm{I}_3$. Hence $S_{5,-1}(u) = S_{2,-1}(u)$. Note that the leading term of $S^2(u,\lambda)$ is $b^t\lambda$. Write $S^2(u,\lambda) = b^t\lambda + \sum_{i\le 0} S_{2,i}(u)\lambda^i$. Compare coefficients of $\mathcal{G}_i\lambda^j$ of

$$\begin{cases} [\partial_x + b + u, b^t\lambda + \sum_{i\le 0} S_{2,i}(u)\lambda^i] = 0, \\ (b^t\lambda + \sum_{i\le 0} S_{2,i}(u)\lambda^i)^3 = \lambda^2 \mathrm{I} \end{cases}$$

to obtain $k_5(u) = -\frac{1}{9}(q_x^2 + \frac{8}{3}q^3)$ and

$$G_5(u) = -\frac{1}{9}\oint \left(q_x^2 + \frac{8}{3}q^3\right)\mathrm{d}x.$$

A direct computation implies that $\nabla G_5(u) = \frac{1}{9}(q_{xx} - 4q^2)\beta_1^t$. Use (3.6) to compute $P_u(\nabla G_5(u))$ and see that the fifth $\hat{A}_2^{(2)}$-KdV is the KK equation (1.6).

Example 6.12 (The third $\hat{A}_4^{(2)}$-KdV flows). We have $\beta_1 = e_{23} + e_{14}$, $\beta_2 = e_{14} + e_{25}$, and $u = u_1\beta_1 + u_2\beta_2$. We first compare both sides of the $\mathcal{G}_i\lambda^j$ component of (6.5) to obtain $S_{1,-i}(u)$ for $0 \le i \le 2$, then use these to compute the coefficient of $(2n+1,1)$-th entry of the coefficient of λ^{-1} of $T^3(u,\lambda)$. We see that $k_3(u) = 2u_2 + \frac{4}{5}u_1^2$ and

$$G_3(u) = \oint \left(2u_2 + \frac{4}{5}u_1^2\right)\mathrm{d}x$$

on $\mathcal{M}_{3,2}(S^1)$. Moreover,

$$\mathrm{d}(G_3)_u(v) = \oint \left(\frac{8}{5}u_1 v_1 + 2v_2\right) \mathrm{d}x = \oint \left\langle \frac{4}{5}u_1 \beta_1^t + \beta_2, v_1\beta_1 + v_2\beta_2 \right\rangle \mathrm{d}x.$$

So we have $\nabla G_3(u) = \frac{4}{5}u_1\beta_1^t + \beta_2^t$. Use (3.7) to get $P_u(\nabla G_3(u))$. Then a direct computation implies that the third $\hat{A}_4^{(2)}$-KdV flow is

$$\begin{cases} (u_1)_t = 3u_2' - 2u_1^{(3)} + \frac{12}{5}u_1 u_1', \\ (u_2)_t = u_2^{(3)} - \frac{3}{5}u_1^{(5)} + \frac{6}{5}u_1 u_1^{(3)} + \frac{3}{5}u_1' u_1'' + \frac{7}{5}u_1' u_2 - \frac{6}{5}u_1 u_2'. \end{cases}$$

7. Isotropic curve flows on $\mathcal{M}_{n+1,n}$ of type B and A

In this section, we use Sections 4, 5, and 6 to write down isotropic curve flows on $\mathcal{M}_{n+1,n}$ of type B and A respectively, and their relations to the $\hat{B}_n^{(1)}$- and $\hat{A}_{2n}^{(2)}$-flows.

It follows from Theorems 4.1, 5.17 and 6.10 that we have the following:

Theorem 7.1. *Let F_{2j-1}, G_{2j-1} be the functionals on $C^\infty(S^1, V_n)$ defined by* (5.21) *and* (6.15) *respectively, $\hat{F}_{2j-1} = F_{2j-1} \circ \Psi$, $\hat{G}_{2j-1} = G_{2j-1} \circ \Psi$, where Ψ is the isotropic curvature map from $\mathcal{M}_{n+1,n}(S^1)$ to $C^\infty(S^1, V_n)$. Then the Hamiltonian flows for $\hat{F}_{2j-1}$ and $\hat{G}_{2j-1}$ with respect to the Poisson structure $\{\,,\}^\wedge$ on $\mathcal{M}_{n+1,n}$ defined by* (4.2) *are*

$$\gamma_t = gP_u(\pi_0(T_{2j-1,0}(u)))e_1, \tag{7.1}$$

$$\gamma_t = gP_u(\pi_0(S_{2j-1,0}(u)))e_1, \tag{7.2}$$

respectively, where $g(\cdot, t)$ and $u(\cdot, t)$ are the isotropic moving frame and curvature along $\gamma(\cdot, t)$.

Definition 7.2. Equations (7.1) and (7.2) on $\mathcal{M}_{n+1,n}$ are called the $(2j-1)$-th *isotropic curve flow of type B and A* respectively.

It follows from Theorem 4.5 (1) that we have

Corollary 7.3. *If $\gamma(x,t)$ is a solution of flow* (7.1) *(*(7.2) *resp.), then its isotropic curvature $u(\cdot, t)$ is a solution of the $(2j-1)$-th $\hat{B}_n^{(1)}$-KdV ($\hat{A}_{2n}^{(2)}$-KdV resp.) flow.*

It follows from Theorems 5.9 and 6.6 that the $(2j-1)$-th isotropic curve flows on $\mathcal{M}_{n+1,n}$ of B-type and A-type can be written as

$$\gamma_t = gT_{2j-1,0}(u)e_1, \tag{7.3}$$

$$\gamma_t = gS_{2j-1,0}(u)e_1.. \tag{7.4}$$

Proposition 7.4. *If q is a solution of the $(2j-1)$-th $\hat{B}_n^{(1)}$-flow ($\hat{A}_{2n}^{(2)}$-flow resp.) and h is a solution of (5.15) ((6.13) resp.), then $\gamma := he_1$ is a solution of the (7.1) ((7.2) resp.). Moreover, let $\triangle(x,t) \in N_n^+$ such that $\triangle(\cdot,t) * q(\cdot,t) \in C^\infty(\mathbb{R}, V_n)$. Then the isotropic curvature of γ is $\triangle^{-1} * q$.*

Proof. By Proposition 5.12, $u := \triangle^{-1} * q$ is a solution of the $(2j-1)$-th $\hat{B}_n^{(1)}$-KdV flow and $g := h\triangle^{-1}$ is a solution of (5.16). It follows from Theorem 4.5 that $\gamma := ge_1$ is a solution of (7.1) with u as its isotropic curvature. Since $\triangle \in N_n^+$, $ge_1 = h\triangle^{-1}e_1 = he_1$. □

Example 7.5 (The trivial solution of isotropic curve flows). Note that $u = 0$ is a solution of the $(2j-1)$-th $\hat{B}_n^{(1)}$- ($\hat{A}_{2n}^{(2)}$- resp.) flow respectively, and $h(x,t) = \exp(bx + b^{2j-1}t)$ is a solution of (5.15). So by Proposition 7.4,

$$\gamma(x,t) = \exp(bx + b^{2j-1}t)e_1$$

is the solution of (7.1) ((7.2) resp.) with zero isotropic curvature.

Example 7.6 (The third isotropic curve flow of B-type on $\mathcal{M}_{2,1}$). For $n = 1$, we have seen in Example 5.18 that $\nabla F_3(u) = -q\beta_1^t$ for $u = q\beta_1$. Formula (3.6) implies that the first column of $P_u(\nabla F_3(u))$ is $(q_x, -q, 0)^t$. Hence the third isotropic curve flow on $\mathcal{M}_{2,1}$ is (1.7).

Example 7.7 (The third isotropic curve flow of B-type on $\mathcal{M}_{3,2}$). We have seen in Example 5.19 that for $u = u_1\beta_1 + u_2\beta_2$, we have $\nabla F_3(u) = \frac{1}{2}u_1\beta_1^t + \beta_2^t$. So (3.7) implies that the first column of $P_u(\nabla F_3(u))$ is

$$\left(0, -\frac{1}{2}u_1, 0, 1, 0\right)^t.$$

So the third isotropic curve flow of B-type on $\mathcal{M}_{3,2}$ is

$$\begin{aligned} \gamma_t &= (\gamma, \gamma_x, \gamma_x x, p_4, p_5)P_u(\nabla F_3(u))e_1 = -\frac{1}{2}u_1\gamma_x + p_4, \quad \text{by (2.7)}, \\ &= -\frac{3}{2}u_1\gamma_x + \gamma_{xxx}. \end{aligned}$$

Example 7.8 (The fifth isotropic curve flow of A-type on $\mathcal{M}_{2,1}$). For $u = q\beta_1$, we have seen in Example 6.11 that $\nabla G_5(u) = \frac{1}{9}(q_{xx} - 4q^2)\beta_1^t$. Use (3.6) to see that the first column of $P_u(\nabla G_5(u))$ is $(-\xi_x, \xi, 0)^t$, where $\xi = \frac{1}{9}(q_{xx} - 4q^2)$. Hence the third isotropic curve flow of A-type on $\mathcal{M}_{2,1}$ is (1.8).

Example 7.9 (The third isotropic curve flow of A-type on $\mathcal{M}_{3,2}$). For $u = u_1\beta_1 + u_2\beta_2$, we have seen in Example 6.12 that $\nabla G_3(u) = \frac{4}{5}u_1\beta_1^t + \beta_2^t$. The formula (3.7) implies that the first column of $P_u(\nabla G_3(u))$ is

$$\left(-\frac{3}{5}(u_1)_x, -\frac{1}{5}u_1, 0, 1, 0\right)^t.$$

So the third isotropic curve flow of A-type on $\mathcal{M}_{3,2}$ is

$$\gamma_t = -\frac{3}{5}(u_1)_x\gamma - \frac{1}{5}u_x\gamma_x + p_4.$$

Use formula (2.7) for p_4 to see that it can be written as

$$\gamma_t = -\frac{3}{5}(u_1)_x\gamma - \frac{6}{5}u_1\gamma_x + \gamma_x^{(3)}.$$

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Received June 14, 2018
Accepted October 14, 2019